

Projection operator
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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The outer product $|\psi_1\rangle\langle\psi_2|$ is not a number. It is an operator. A special case is the outer product of a ket with its corresponding bra vector, $|\psi\rangle\langle\psi|$. Assuming that $|\psi\rangle$ is normalized, this operator is called a projection operator. The projection operator can be said to project a vector onto the direction defined by $|\psi\rangle$. We discuss the properties of the projection operator.

1. Adjoint operator

The adjoint operator of

$$\hat{A} = |\psi_1\rangle\langle\psi_2|,$$

is given by

$$\hat{A}^+ = |\psi_2\rangle\langle\psi_1|.$$

((Proof))

From the definition, we have

$$\langle\beta|\hat{A}^+|\alpha\rangle = \langle\alpha|\hat{A}|\beta\rangle^*.$$

Here we note that

$$\langle\alpha|\hat{A}|\beta\rangle^* = (\langle\alpha|\psi_1\rangle\langle\psi_2|\beta\rangle)^* = \langle\psi_2|\beta\rangle^*\langle\alpha|\psi_1\rangle^* = \langle\beta|\psi_2\rangle\langle\psi_1|\alpha\rangle.$$

Then we have

$$\hat{A}^+ = |\psi_2\rangle\langle\psi_1|.$$

2. Properties of projection operator

The projection operator is defined as

$$\hat{P}_1 = |\psi_1\rangle\langle\psi_1|,$$

where $|\psi_1\rangle$ is normalized; $\langle\psi_1|\psi_1\rangle=1$

(a)

$$\hat{P}_1^+ = |\psi_1\rangle\langle\psi_1| = \hat{P}_1,$$

which means that \hat{P}_1 is the Hermitian operator.

$$\hat{P}_1|\psi_1\rangle = |\psi_1\rangle\langle\psi_1|\psi_1\rangle = |\psi_1\rangle.$$

(b)

$$\hat{P}_1^2 = \hat{P}_1|\psi_1\rangle\langle\psi_1| = |\psi_1\rangle\langle\psi_1| = \hat{P}_1,$$

or

$$\hat{P}_1^2 = \hat{P}_1.$$

In general, we have

$$\hat{P}_1 = |\psi_1\rangle\langle\psi_1|, \quad \hat{P}_1|\psi_1\rangle = |\psi_1\rangle.$$

It follows that for any vector $|\psi\rangle$

$$\hat{P}_1^2|\psi\rangle = \hat{P}_1|\psi_1\rangle\langle\psi_1|\psi\rangle = |\psi_1\rangle\langle\psi_1|\psi\rangle = \hat{P}_1|\psi\rangle,$$

or

$$\hat{P}_1^2 = \hat{P}_1.$$

The identity operator is a simple example of a projection operator, since

$$\hat{I}^+ = \hat{I}, \quad \hat{I}^2 = \hat{I}.$$

(c)

$$\hat{P}_i = |\psi_i\rangle\langle\psi_i|, \quad \hat{P}_j = |\psi_j\rangle\langle\psi_j|.$$

where $i \neq j$. Hence

$$\hat{P}_i \hat{P}_j |\psi\rangle = \hat{P}_i |\psi_j\rangle\langle\psi_j|\psi\rangle = 0,$$

$$\hat{P}_j \hat{P}_i |\psi\rangle = \hat{P}_j |\psi_i\rangle\langle\psi_i|\psi\rangle = 0.$$

(d) The product of two commuting projection operators \hat{P}_1 and \hat{P}_2 , is also a projection operator, since

$$(\hat{P}_1 \hat{P}_2)^+ = \hat{P}_2^+ \hat{P}_1^+ = \hat{P}_2 \hat{P}_1 = \hat{P}_1 \hat{P}_2,$$

$$(\hat{P}_1 \hat{P}_2)^2 = \hat{P}_1 \hat{P}_2 \hat{P}_1 \hat{P}_2 = \hat{P}_1^2 \hat{P}_2^2 = \hat{P}_1 \hat{P}_2.$$

(e)

$$|\psi\rangle = \sum_{i=1}^n c_i |\phi_i\rangle,$$

$$\sum_{i=1}^n \hat{P}_i |\psi\rangle = \sum_{i=1}^n |\phi_i\rangle\langle\phi_i|\psi\rangle = \sum_{i=1}^n c_i |\phi_i\rangle = |\psi\rangle,$$

where

$$\hat{P}_i = |\phi_i\rangle\langle\phi_i|.$$

Hence

$$\sum_{i=1}^n \hat{P}_i = \hat{1}. \quad (\text{closure relation})$$

(f)

$|a_n\rangle$ is an eigenket of \hat{A} with the eigenvalue a_n ,

$$\hat{A}|a_n\rangle = a_n|a_n\rangle.$$

Then we have

$$\hat{A} = \sum_n \hat{A}|a_n\rangle\langle a_n| = \sum_n a_n|a_n\rangle\langle a_n|.$$

3. Measurement

Suppose we measure the physical quantity A . The eigenstate of the operator \hat{A} is given by

$$\hat{A}|\psi_n\rangle = \lambda_n|\psi_n\rangle.$$

The initial state is given by $|\psi\rangle$. After the measurement we have

$$\hat{P}_n|\psi\rangle = |\psi_n\rangle\langle\psi_n|\psi\rangle,$$

with the probability as

$$|\langle\psi_n|\psi\rangle|^2.$$

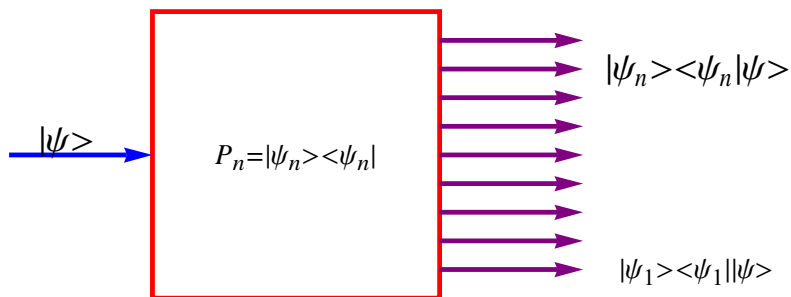


Fig. Projection operator. $|\psi_n\rangle$ is the eigenket of the operator \hat{A} corresponding to the measurement of physical quantity A .

4. Examples of projection operator (2x2 matrix)

$$\hat{P}_+ = |+z\rangle\langle+z|, \quad \hat{P}_- = |-z\rangle\langle-z|,$$

where

$$\hat{P}_+ + \hat{P}_- = \hat{1}. \quad (\text{Closure relation})$$

The matrix representation of \hat{P}_+ ,

$$\hat{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix representation of \hat{P}_- ,

$$\hat{P}_- = 1 - \hat{P}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

5. Property of the projection operator:

$$\hat{P}_+|+z\rangle = |+z\rangle\langle+z|+z\rangle = |+z\rangle, \quad \hat{P}_+|-z\rangle = |+z\rangle\langle+z|-z\rangle = 0.$$

$$\hat{P}_-|+z\rangle = |-z\rangle\langle-z|+z\rangle = 0, \quad \hat{P}_-|-z\rangle = |-z\rangle\langle-z|-z\rangle = |-z\rangle$$

This means that

- (i) $|+z\rangle$ is the eigenket of \hat{P}_+ with the eigenvalue 1, $|-z\rangle$ is the eigenket of \hat{P}_+ with the eigenvalue 0.
- (ii) $|+z\rangle$ is the eigenket of \hat{P}_- with the eigenvalue 0, $|-z\rangle$ is the eigenket of \hat{P}_- with the eigenvalue 1.

$$\hat{P}_+^2 = \hat{P}_+|+z\rangle\langle+z| = |+z\rangle\langle+z| = \hat{P}_+,$$

$$\hat{P}_-^2 = \hat{P}_-|-z\rangle\langle-z| = |-z\rangle\langle-z| = \hat{P}_-,$$

$$\hat{P}_+\hat{P}_- = \hat{P}_+|-z\rangle\langle-z| = 0,$$

$$\hat{P}_-\hat{P}_+ = \hat{P}_-|+z\rangle\langle+z| = 0,$$

6. Eigenvalue problems for the projection operator

((Mathematica))

```
Clear["Global`*"]; P1 =  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ; P2 =  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ;
```

```
eq1 = Eigensystem[P1]
```

```
{{1, 0}, {{1, 0}, {0, 1}}}
```

```
 $\psi_1$  = Normalize[eq1[[2, 1]]]
```

```
{1, 0}
```

```
 $\psi_0$  = Normalize[eq1[[2, 2]]]
```

```
{0, 1}
```

```
eq2 = Eigensystem[P2]
```

```
{{1, 0}, {{0, 1}, {1, 0}}}
```

```
 $\phi_1$  = Normalize[eq2[[2, 1]]]
```

```
{0, 1}
```

```
 $\phi_0$  = Normalize[eq2[[2, 2]]]
```

```
{1, 0}
```

7. Definition of Outer[Times,...] in the Mathematica

We assume that

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_n \end{pmatrix}, \quad \langle\psi| = (a_1^* \ a_2^* \ a_3^* \ \cdot \ \cdot \ \cdot \ a_n^*).$$

Then we have the matrix of the projection operator as

$$\begin{aligned}
|\psi\rangle\langle\psi| &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* & a_3^* & \cdot & \cdot & \cdot & a_n^* \end{pmatrix} \\
&= \begin{pmatrix} a_1 a_1^* & a_1 a_2^* & a_1 a_3^* & \cdot & \cdot & \cdot & a_1 a_n^* \\ a_2 a_1^* & a_2 a_2^* & a_2 a_3^* & \cdot & \cdot & \cdot & a_2 a_n^* \\ a_3 a_1^* & a_3 a_2^* & a_3 a_3^* & \cdot & \cdot & \cdot & a_3 a_n^* \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n a_1^* & a_n a_2^* & a_n a_3^* & \cdot & \cdot & \cdot & a_n a_n^* \end{pmatrix}
\end{aligned}$$

We note that the matrix of the projection operator can be calculated using the Mathematica as
(a)

$$|\psi\rangle\langle\psi| \rightarrow |\psi\rangle \cdot \langle\psi|,$$

where we use the dot mark "." between $|\psi\rangle$ and $\langle\psi|$, we use the standard form of $|\psi\rangle$ and $\langle\psi|$ as

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ a_n \end{pmatrix}, \quad \langle\psi| = (a_1^* \quad a_2^* \quad a_3^* \quad \cdot \quad \cdot \quad \cdot \quad a_n^*).$$

(b)

$$|\psi\rangle\langle\psi| \rightarrow \text{Outer}[\text{Times}, |\psi\rangle, \langle\psi|]$$

where we use

$$|\psi\rangle \rightarrow \{a_1 \quad a_2 \quad a_3 \quad \cdot \quad \cdot \quad \cdot \quad a_n\},$$

$$\langle \psi | \rightarrow \{a_1^* \ a_2^* \ a_3^* \ . \ . \ . \ a_n^*\}.$$

((Mathematica))

```
Clear["Global`*"];
```

```
 $\psi_1 = \{a_1, a_2, a_3, a_4, a_5\}; \psi_2 = \{b_1, b_2, b_3, b_4, b_5\};$ 
```

```
A1 = Outer[Times,  $\psi_1$ ,  $\psi_2$ ] // Simplify // MatrixForm
```

```
( a1 b1  a1 b2  a1 b3  a1 b4  a1 b5 )
( a2 b1  a2 b2  a2 b3  a2 b4  a2 b5 )
( a3 b1  a3 b2  a3 b3  a3 b4  a3 b5 )
( a4 b1  a4 b2  a4 b3  a4 b4  a4 b5 )
( a5 b1  a5 b2  a5 b3  a5 b4  a5 b5 )
```

8. Examples: Projection operators

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-x\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$|+y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The projection operator

(a)

$$|+z\rangle\langle +z| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |-z\rangle\langle -z| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$|+z\rangle\langle +z| + |-z\rangle\langle -z| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b)

$$|+x\rangle\langle+x| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad |-x\rangle\langle-x| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$|+x\rangle\langle+x| + |-x\rangle\langle-x| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c)

$$|+y\rangle\langle+y| = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad |-y\rangle\langle-y| = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

$$|+y\rangle\langle+y| + |-y\rangle\langle-y| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

((Mathematica))

```
Clear["Global`*"];
```

```
expr_* := expr /. Complex[a_, b_] :=> Complex[a, -b];
```

$$\psi_{xp} = \frac{1}{\sqrt{2}} \{1, 1\}; \quad \psi_{xn} = \frac{1}{\sqrt{2}} \{1, -1\}; \quad \psi_{yp} = \frac{1}{\sqrt{2}} \{1, i\};$$

$$\psi_{yn} = \frac{1}{\sqrt{2}} \{1, -i\}; \quad \psi_{zp} = \{1, 0\};$$

$$\psi_{zn} = \{0, 1\};$$

```
Pxp = Outer[Times, \psi_{xp}, \psi_{xp}^*] // Simplify;
```

```
Pxn = Outer[Times, \psi_{xn}, \psi_{xn}^*] // Simplify;
```

```
Pyp = Outer[Times, \psi_{yp}, \psi_{yp}^*] // Simplify;
```

```
Pyn = Outer[Times, \psi_{yn}, \psi_{yn}^*] // Simplify;
```

```
Pzp = Outer[Times, \psi_{zp}, \psi_{zp}^*] // Simplify;
```

```
Pzn = Outer[Times, \psi_{zn}, \psi_{zn}^*] // Simplify;
```

Pzp // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Pzn // MatrixForm

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Pzp + Pzn // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pxp // MatrixForm

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Pxn // MatrixForm

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Pxp + Pxn // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pyp // MatrixForm

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

Pyn // MatrixForm

$$\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

Pyp + Pyn // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

9. Spin operators

$$\begin{aligned} \hat{S}_z &= \hat{S}_z(|+z\rangle\langle+z| + |-z\rangle\langle-z|) \\ &= \frac{\hbar}{2}[|+z\rangle\langle+z| - |-z\rangle\langle-z|] \\ &= \frac{\hbar}{2}(\hat{P}_{+z} - \hat{P}_{-z}) \end{aligned}$$

$$\begin{aligned} \hat{S}_x &= \hat{S}_x(|+x\rangle\langle+x| + |-x\rangle\langle-x|) \\ &= \frac{\hbar}{2}[|+x\rangle\langle+x| - |-x\rangle\langle-x|] \\ &= \frac{\hbar}{2}(\hat{P}_{+x} - \hat{P}_{-x}) \end{aligned}$$

$$\begin{aligned} \hat{S}_y &= \hat{S}_y(|+y\rangle\langle+y| + |-y\rangle\langle-y|) \\ &= \frac{\hbar}{2}[|+y\rangle\langle+y| - |-y\rangle\langle-y|] \\ &= \frac{\hbar}{2}(\hat{P}_{+y} - \hat{P}_{-y}) \end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{S}} \cdot \mathbf{n} &= \frac{\hbar}{2} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}) \\
&= \frac{\hbar}{2} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}) [|+\mathbf{n}\rangle\langle+\mathbf{n}| + |-\mathbf{n}\rangle\langle-\mathbf{n}|] \\
&= \frac{\hbar}{2} [|+\mathbf{n}\rangle\langle+\mathbf{n}| - |-\mathbf{n}\rangle\langle-\mathbf{n}|] \\
&= \frac{\hbar}{2} (\hat{P}_{+\mathbf{n}} - \hat{P}_{-\mathbf{n}})
\end{aligned}$$