Quantum Computation
Construction of the Bell's four states

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A quantum computer is a computation device that makes direct use of quantum-mechanical phenomena, such as superposition and entanglement, to perform operations on data. In classical computing, we use the binary digits – bit - to store information. Each bit has the value 0, or 1. Individual bits are strung together to represent larger binary numbers, such as 00100111000…Each binary number represents an actual number.

In quantum information processing, information is stored in quantum bits or qubits. A qubit is a quantum system with two states $|0\rangle$ and $|1\rangle$. Any two-state quantum system can be used as a qubit. The key difference between bits and qubits is that qubits can exist in a superposition state. This is an stark contrast to a classical bit.

Here we consider a series of quantum circuits that create the four orthonormal entangles Bell's states from the un-entangled computational-base states $|0\rangle$ and $|1\rangle$. We also discuss the transformation of the Bell's state to the computational-base states $|0\rangle$ and $|1\rangle$.

1. Qubits, XOR operation, and Kronecker product

Qubits

A classical bit of information is represented by a system that can be in either of two states, 0, 1. At the quantum mechanical level, the most natural candidate for replacing a classical bit is the state of a two-level system, whose basic components may be written as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is the so-called quantum bit of information, or, in short, a *qubit*.

XOR operation: $\oplus$

$$m \oplus n = m + n \text{ (mudulo 2)}$$

where

$$0 \oplus 0 = 0$$
$$0 \oplus 1 = 1$$
$$1 \oplus 0 = 1$$
$$1 \oplus 1 = 0$$

Kronecker product ⊗

The Kronecker product can be used for the calculation of Mathematica as follows.

((**Mathematica**))

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle , \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \text{KroneckerProduct}[\psi_1,\ \psi_2]$$

In general

$$\left(\hat{A}_1 \otimes \hat{A}_2 \otimes ... \otimes \hat{A}_n\right)\left(|\alpha_1\rangle \otimes |\alpha_1\rangle \otimes ... \otimes |\alpha_n\rangle\right) = \hat{A}_1|\alpha_1\rangle \otimes \hat{A}_2|\alpha_2\rangle \otimes ...\hat{A}_n|\alpha_n\rangle$$

which denotes the action of a tensor product of linear maps on a basis vector of the tensor product space.

((**Note**)) The list of classical gates is given in the APPENDIX.

2. Four Bell's states

The four Bell's states are defined by

$$|\psi_{12}{}^{(+)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$|\psi_{12}{}^{(-)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$|\Phi_{12}{}^{(+)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ S1 \end{pmatrix}$$

$$\left|\Phi_{12}{}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle \otimes \left|0\right\rangle - \left|1\right\rangle \otimes \left|1\right\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The four Bell's states can be derived as follows, using the Mathematica

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
```

```
ψB1 =
 1/Sqrt[2] (KroneckerProduct[ψ1, ψ2] -
    KroneckerProduct[ψ2, ψ1]) // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
ψB2 =
 1/Sqrt[2] (KroneckerProduct[ψ1, ψ2] +
    KroneckerProduct[ψ2, ψ1]) // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
ψB3 =
 1/Sqrt[2] (KroneckerProduct[ψ1, ψ1] +
    KroneckerProduct[ψ2, ψ2]) // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψB4 =
 1/√2 (KroneckerProduct[ψ1, ψ1] -
     KroneckerProduct[ψ2, ψ2]) // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

3. NOT operation (unitary operator)

The NOT gate changes

$$|0\rangle \to |1\rangle, \qquad |1\rangle \to |0\rangle$$

or

$$\hat{U}_{NOT}|0\rangle = |1\rangle, \qquad \hat{U}_{NOT}|1\rangle = |0\rangle$$

where $\hat{U}_{NOT}$ is the NOT operator. The NOT gate is a linear operator, so it also changes a superposition, such that

$$a|0\rangle + b|1\rangle \to a|1\rangle + b|0\rangle = b|0\rangle + a|1\rangle$$

The NOT operator is defined by

$$\begin{aligned}\hat{U}_{NOT} &= \hat{U}_{NOT}(|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= \hat{U}_{NOT}|0\rangle\langle 0| + \hat{U}_{NOT}|1\rangle\langle 1| \qquad \text{(quantum NOT gate transformation)} \\ &= |1\rangle\langle 0| + |0\rangle\langle 1|\end{aligned}$$

where

$$\hat{U}_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{\sigma}_x$$

$$\hat{U}_{NOT}|0\rangle = |1\rangle, \qquad \hat{U}_{NOT}|1\rangle = |0\rangle$$

or

$$\hat{U}_{NOT}|k\rangle = |k \oplus 1\rangle$$

Note that

$$\hat{U}_{NOT}|0\rangle = |0 \oplus 1\rangle = |1\rangle, \qquad \hat{U}_{NOT}|1\rangle = |1 \oplus 1\rangle = |0\rangle$$

Under the basis $\{|0\rangle, |1\rangle\}$, $\hat{U}_{NOT}$ coincides with the Pauli matrix $\hat{\sigma}_x$. This is not a coincidence. All unitary operators for a spin 1/2 system can be expressed as a linear combination of four operators consisting of $\hat{1}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$.

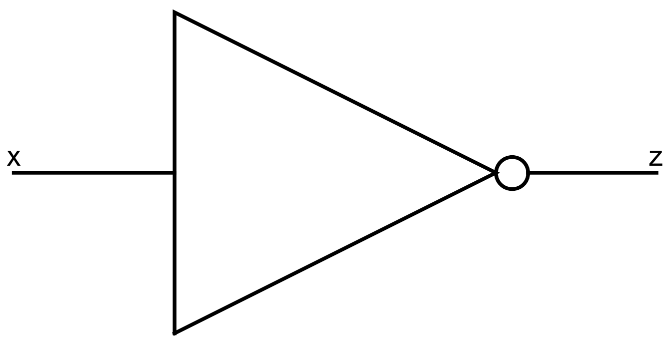


Fig. Classical NOT gate. $x = 0$ ($z = 1$). $x = 1$ ($z = 0$)

4. $\sqrt{NOT}$ gate

The $\sqrt{NOT}$ gate is defined by

$$\hat{U}_{\sqrt{NOT}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

Then we have

$$\hat{U}_{\sqrt{NOT}}|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

$$\hat{U}_{\sqrt{NOT}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Note that

$$\hat{U}_{\sqrt{NOT}}\hat{U}_{\sqrt{NOT}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\hat{\sigma}_x .$$

5. (Walsh)-Hadamard transform $\hat{U}_H$

This gate is one of the most significant quantum logic gate, because it can be used to enable the qubit interference vital to quantum computation.

$$\hat{U}_H|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k|1\rangle) ,$$

with $k = 0, 1$. The Hadamard gate transforms basis states into superposition states.

$$\hat{U}_H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\hat{U}_H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\hat{U}_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \hat{U}_H^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \qquad \text{(Hadamard gate)}$$

The Hadamard transformation is expressed by

$$\begin{aligned} \hat{U}_H &= \hat{U}_H(|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| \\ &= \frac{1}{\sqrt{2}}[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \end{aligned}$$

Note

$$\hat{U}_H^{+}\hat{U}_H = \hat{U}_H^{2} = \hat{1} .$$

and

$$\hat{U}_H{}^{+}|0\rangle = \hat{U}_H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\hat{U}_H{}^{+}|1\rangle = \hat{U}_H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Then we have

$$\begin{aligned} |k\rangle &= \frac{1}{\sqrt{2}}(\hat{U}_H{}^{+}|0\rangle + (-1)^k \hat{U}_H{}^{+}|1\rangle) \\ &= \frac{1}{\sqrt{2}}(\hat{U}_H|0\rangle + (-1)^k \hat{U}_H|1\rangle) \end{aligned}$$

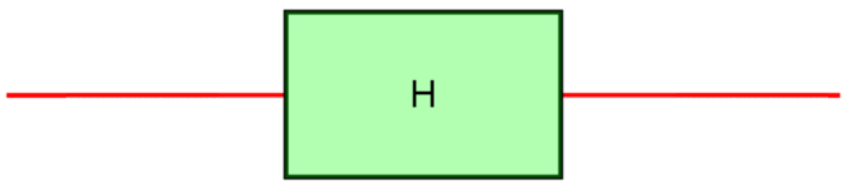


Fig. Circuit representation of Hadamard gate.

6. Quantum *Z* gate (Pauli gate)

We consider the quantum *Z* gate is a π rotation around the *z* axis, with a transformation matrix as

$$\hat{U}_z = \hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$\hat{U}_z|0\rangle = |0\rangle, \qquad \hat{U}_z|1\rangle = -|1\rangle.$$

We show that

$$\hat{U}_H\hat{U}_z\hat{U}_H|0\rangle = |1\rangle, \qquad \hat{U}_H\hat{U}_z\hat{U}_H|1\rangle = |0\rangle$$

((**Proof**))

$$\hat{U}_H\hat{U}_z\hat{U}_H|j\rangle = \frac{1}{\sqrt{2}}\hat{U}_H\hat{U}_z(|0\rangle + (-1)^j|1\rangle)$$
$$= \frac{1}{\sqrt{2}}\hat{U}_H(|0\rangle + (-1)^{j+1}|1\rangle)$$
$$= |\bar{j}\rangle$$

where $\bar{j} = j + 1$. In other words,

$$\hat{U}_H\hat{U}_z\hat{U}_H = \hat{U}_{NOT}$$

7. Properties of the Hamadard gate:

On an intial two-qubit state the Hadamard transform has the form

$$(\hat{U}_H \otimes \hat{U}_H)(|0\rangle \otimes |0\rangle) = \hat{U}_H|0\rangle \otimes \hat{U}_H|0\rangle$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

In the vase of three qubits we have

$$(\hat{U}_H \otimes \hat{U}_H \otimes \hat{U}_H)(|0\rangle \otimes |0\rangle \otimes |0\rangle) = \hat{U}_H|0\rangle \otimes \hat{U}_H|0\rangle \otimes \hat{U}_H|0\rangle$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$= \left(\frac{1}{\sqrt{2}}\right)^3(|000\rangle + |001\rangle + |010\rangle + |011\rangle$$
$$+ |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

8. Controlled NOT operation (unitary operator)

The controlled NOT operation acts on the state of two qubits, known as the control qubit, *C* and the target qubit *T*.

$$\hat{U}_{CNOT} = |0\rangle\langle 0| \otimes \hat{I}_2 + |1\rangle\langle 1| \otimes \hat{U}_{NOT} = |0\rangle\langle 0| \otimes \hat{I}_2 + |1\rangle\langle 1| \otimes \hat{\sigma}_x$$

$$\hat{U}_{CNOT}|0\rangle \otimes |0\rangle = |0\rangle \otimes |0 \oplus 0\rangle = |0\rangle \otimes |0\rangle$$
$$\hat{U}_{CNOT}|1\rangle \otimes |0\rangle = |1\rangle \otimes |1 \oplus 0\rangle = |1\rangle \otimes |1\rangle$$
$$\hat{U}_{CNOT}|0\rangle \otimes |1\rangle = |0\rangle \otimes |0 \oplus 1\rangle = |0\rangle \otimes |1\rangle$$
$$\hat{U}_{CNOT}|1\rangle \otimes |1\rangle = |1\rangle \otimes |1 \oplus 1\rangle = |1\rangle \otimes |0\rangle$$

with

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{U}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

or simplily

$$\hat{U}_{CNOT}|j\rangle \otimes |k\rangle = |j\rangle \otimes |j \oplus k\rangle = |j, j \oplus k\rangle$$

where the notation denotes a mod (2) sum. Note that

$$\hat{U}_{CNOT}{}^{2} = \hat{1}$$

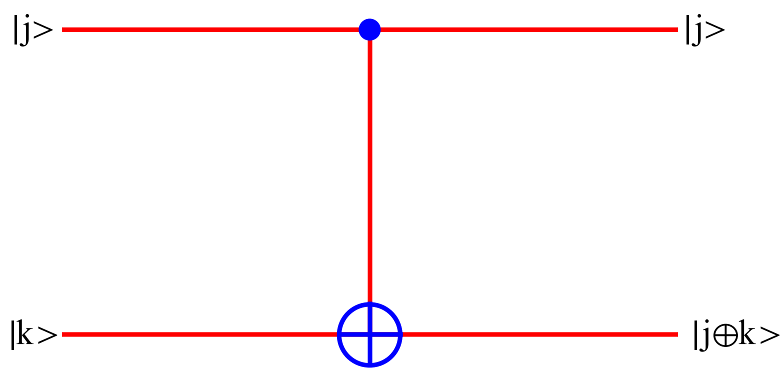


Fig. $\hat{U}_{CNOT}$. The controlled NOT gate is a transformation involving two qubits. The value of the controlled qubit (the upper on in Fig) influences the lower one, whose value is flipped if the upper qubit carries "1". and not flipped if the upper qubit carries "0". This is equivalent to addition mudulo 2. Control input ($|j\rangle$). Target input ($|k\rangle$)

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :→ Complex[re, -im]};

ψ1 = (1
      0); ψ2 = (0
                1); I2 = IdentityMatrix[2];
σx = (0 1
      1 0); ψ11 = ψ1.Transpose[ψ1*];
ψ22 = ψ2.Transpose[ψ2*];

UCNOT = KroneckerProduct[ψ11, I2] +
   KroneckerProduct[ ψ22, σx];
```

```
UCNOT // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
H1 = UCNOT.UCNOT // Simplify;
```

```
H1 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Deutsch called the CNOT gate the measurement gate, because, if the target qubit is prepared in the 0 state, it can always learn about the state of the control qubit.

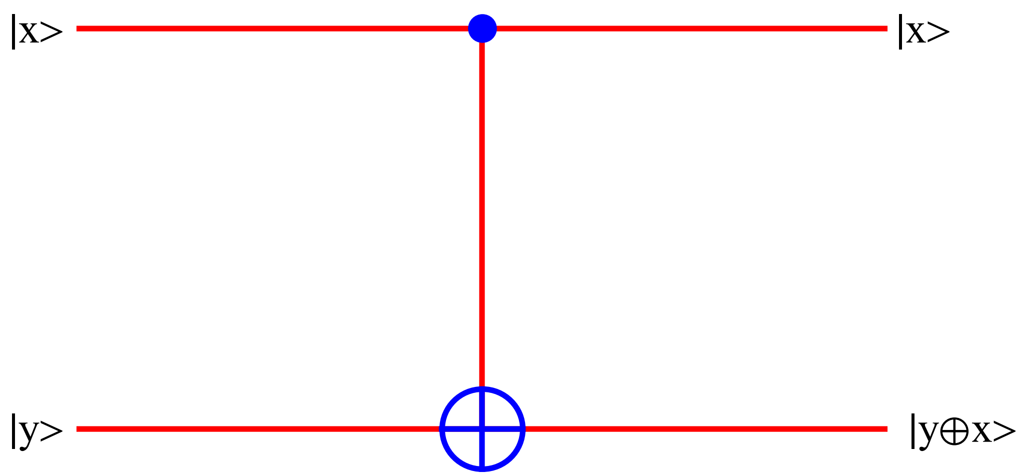
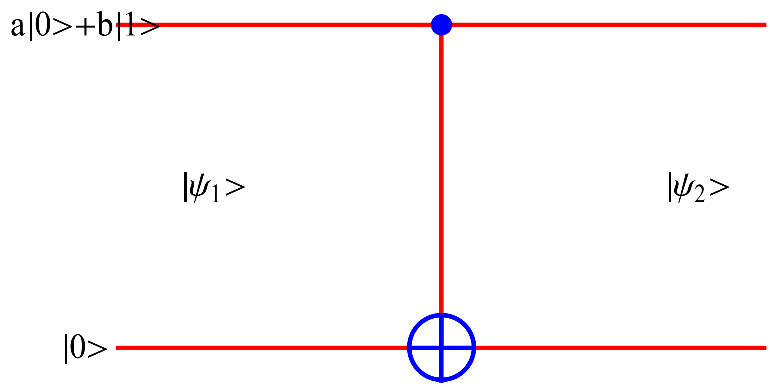


Fig. Representation of the two qubit CNOT gate. The top line represents the control qubit. The bottom line is the target qubit. Note that $|x \oplus y\rangle = |y \oplus x\rangle$.

$$\hat{U}_{CNOT}|00\rangle = |0,0 \oplus 0\rangle = |0,0\rangle$$
$$\hat{U}_{CNOT}|01\rangle = |0,0 \oplus 1\rangle = |0,1\rangle$$
$$\hat{U}_{CNOT}|10\rangle = |1,0 \oplus 1\rangle = |1,1\rangle$$
$$\hat{U}_{CNOT}|11\rangle = |1,1 \oplus 1\rangle = |1,0\rangle$$

We note that

$$\hat{U}_{CNOT}(c_0|0\rangle + c_1|1\rangle)\otimes|0\rangle = c_0|00\rangle + c_1|11\rangle$$
$$\hat{U}_{CNOT}(c_0|00\rangle + c_1|11\rangle) = (c_0|0\rangle + c_1|1\rangle)\otimes|0\rangle$$



$$|\psi_1\rangle = (a|0\rangle + b|1\rangle)_C \otimes |0\rangle_T = a|00\rangle + b|10\rangle$$

$$|\psi_2\rangle = \hat{U}_{CNOT}|\psi_1\rangle = \hat{U}_{CNOT}[a|00\rangle + b|10\rangle] = a|01\rangle + b|10\rangle$$

((**Note**))

$\hat{U}_{CNOT}$ cannot be decomposed into a tensor product of two single-qubit transformation. The importance of $\hat{U}_{CNOT}$ stems from its ability to change the un-entangled two qubits into the entangled state. For example,

$$\hat{U}_{CNOT}\frac{1}{\sqrt{2}}[|00\rangle + |10\rangle] = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

Similarly, since it is its own inverse, it can take an entangled state to an un-entangled one.

$$\hat{U}_{CNOT}{}^2\frac{1}{\sqrt{2}}[|00\rangle + |10\rangle] = \frac{1}{\sqrt{2}}\hat{U}_{CNOT}[|00\rangle + |11\rangle]$$

or

$$\frac{1}{\sqrt{2}}\hat{U}_{CNOT}[|00\rangle + |11\rangle] = \frac{1}{\sqrt{2}}[|00\rangle + |10\rangle]$$

Here $\left|\Phi_{12}{}^{(+)}\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)$ is the Bell's state.

9. Example

Show that

$$(\hat{U}_H \otimes \hat{U}_H)\hat{U}_{CNOT}(\hat{U}_H \otimes \hat{U}_H)\left| j,k\right\rangle = \left| j \oplus k, k\right\rangle .$$

((**Proof**))

$$\begin{aligned}
(\hat{U}_H \otimes \hat{U}_H)\hat{U}_{CNOT}(\hat{U}_H \otimes \hat{U}_H)\left| j,k\right\rangle &= (\hat{U}_H \otimes \hat{U}_H)\hat{U}_{CNOT}(\hat{U}_H\left| j\right\rangle \otimes \hat{U}_H\left| k\right\rangle) \\
&= \frac{1}{2}(\hat{U}_H \otimes \hat{U}_H)\hat{U}_{CNOT}(\left|0\right\rangle + (-1)^j\left|1\right\rangle) \otimes (\left|0\right\rangle + (-1)^k\left|1\right\rangle) \\
&= \frac{1}{2}(\hat{U}_H \otimes \hat{U}_H)(\left|00\right\rangle + (-1)^k\left|01\right\rangle + (-1)^j\left|11\right\rangle) + (-1)^{j+k}\left|10\right\rangle) \\
&= \frac{1}{2}(\hat{U}_H \otimes \hat{U}_H)[(\left|0\right\rangle \otimes (\left|0\right\rangle + (-1)^k\left|1\right\rangle) + (-1)^{j+k}\left|1\right\rangle \otimes (\left|0\right\rangle + (-1)^k\left|1\right\rangle))] \\
&= \frac{1}{2}(\hat{U}_H \otimes \hat{U}_H)[(\left|0\right\rangle + (-1)^{j+k}\left|1\right\rangle) \otimes (\left|0\right\rangle + (-1)^k\left|1\right\rangle)] \\
&= \left| j+k\right\rangle \otimes \left| k\right\rangle = \left| j \oplus k\right\rangle \otimes \left| k\right\rangle
\end{aligned}$$

((**Mathematica**)) We give a proof by this using Mathematica

```
Clear["Global`*"];
exp_ * :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
U = 1/Sqrt[2] {{1, 1}, {1, -1}};
UH = Transpose[U*];

ψ1 = {1, 0}; ψ2 = {0, 1}; ψ11 = {{1}, {0}}; ψ21 = {{0}, {1}};
I2 = IdentityMatrix[2];

χ11 = KroneckerProduct[ψ11, ψ11] // Transpose //
  Flatten ;
χ12 = KroneckerProduct[ψ11, ψ21] // Transpose //
  Flatten;
χ21 = KroneckerProduct[ψ21, ψ11] // Transpose //
  Flatten;
χ22 = KroneckerProduct[ψ21, ψ21] // Transpose //
  Flatten;

A1 = Outer[Times, ψ1, ψ1*] - Outer[Times, ψ2, ψ2*];
U.A1.U.ψ1

{0, 1}

U.A1.U.ψ2

{1, 0}

UNOT = Outer[Times, ψ1, ψ2*] + Outer[Times, ψ2, ψ1*];

UCNOT = KroneckerProduct[Outer[Times, ψ1, ψ1*], I2] +
   KroneckerProduct[Outer[Times, ψ2, ψ2*], UNOT];

UHH = KroneckerProduct[U, U];

K1 = UHH.UCNOT.UHH;

{K1.χ11, K1.χ12, K1.χ21, K1.χ22}

{{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}}
```

10 Controlled-*U* gate

The controlled-U gate is defined by

$$\hat{U} = |0\rangle\langle 0| \otimes \hat{I}_2 + |1\rangle\langle 1| \otimes \hat{u}$$

where $\hat{u}$ is a 2x2 matrix and is given by

$$\hat{u} = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}$$

Then we have

$$\hat{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_{00} & x_{01} \\ 0 & 0 & x_{10} & x_{11} \end{pmatrix}$$

When $\hat{u}$ is one of the Pauli matrices, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$, the respective terms controlled-*X*, controlled-*Y*, or controlled-*Z* are sometimes used.

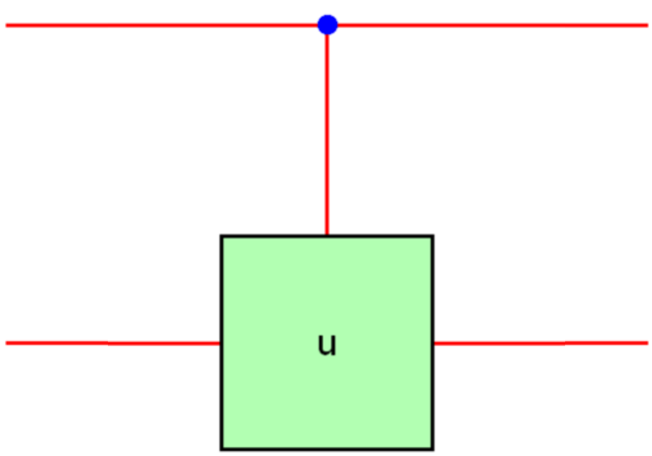


Fig. Circuit representation of controlled-*U* gate.

11. Alternative CNOT

$$\hat{U}_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\hat{U}_A|00\rangle = |01\rangle$$
$$\hat{U}_A|01\rangle = |00\rangle$$
$$\hat{U}_A|10\rangle = |10\rangle$$
$$\hat{U}_A|11\rangle = |11\rangle$$

or

$$\hat{U}_A|00\rangle = |01\rangle$$
$$\hat{U}_A|01\rangle = |00\rangle$$
$$\hat{U}_A|10\rangle = |10\rangle$$
$$\hat{U}_A|11\rangle = |11\rangle$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :→ Complex[re, -im]};
UA = (0 1 0 0
      1 0 0 0
      0 0 1 0
      0 0 0 1); ψ1 = (1
                     0
                     0
                     0); ψ2 = (0
                               1
                               0
                               0); ψ3 = (0
                                         0
                                         1
                                         0);
ψ4 = (0
      0
      0
      1);

UA.ψ1 // MatrixForm
(0
 1
 0
 0)

UA.ψ2 // MatrixForm
(1
 0
 0
 0)
```

```
UA.ψ3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

```
UA.ψ4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

12. Mach-Zender interference

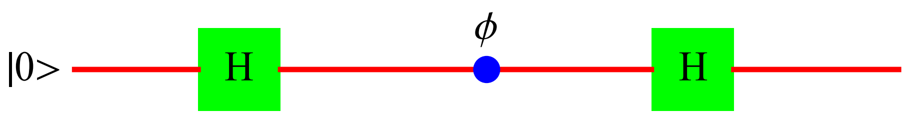


Fig. Quantum computation device as a equivalent of a Mach-Zender interferrometer. The initial state is $|0\rangle$. The final state is $e^{i\frac{\phi}{2}}[\cos\frac{\phi}{2}|0\rangle - i\sin\frac{\phi}{2}|1\rangle]$.

Phase shifter ($\hat{U}_\phi$):phase rotation

$$\hat{U}_\phi|0\rangle = |0\rangle$$
$$\hat{U}_\phi|1\rangle = e^{i\phi}|1\rangle)$$

or

$$\hat{U}_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

((**Phase-flip gate**))

$$\hat{U}_Z = \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{\sigma}_z$$

((*π*/2 phase gate))

$$\hat{U}_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

((π/8 phase gate))

$$\hat{U}_{\pi/4} = e^{i\pi/8}\begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$$

The Hadamard gate ($\hat{U}_H$):

$$\hat{U}_H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad \hat{U}_H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The series of the Hadamard gate and the phase shifter;

$$\hat{U}_\phi \hat{U}_H|0\rangle = \frac{1}{\sqrt{2}}(\hat{U}_\phi|0\rangle + \hat{U}_\phi|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$$

The series of the Hadmard gate-phase shifter - Hadamard gate:

$$\begin{aligned}
\hat{U}_H\hat{U}_\phi\hat{U}_H|0\rangle &= \frac{1}{\sqrt{2}}\hat{U}_H(|0\rangle + e^{i\phi}|1\rangle) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}e^{i\phi}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&= \frac{1}{2}(1+e^{i\phi})|0\rangle + \frac{1}{2}(1-e^{i\phi})|1\rangle \\
&= e^{i\frac{\phi}{2}}[\cos\frac{\phi}{2}|0\rangle - i\sin\frac{\phi}{2}|1\rangle]
\end{aligned}$$

$$\begin{aligned}
\hat{U}_H\hat{U}_\phi\hat{U}_H|1\rangle &= \frac{1}{\sqrt{2}}\hat{U}_H(|0\rangle - e^{i\phi}|1\rangle) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}e^{i\phi}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&= \frac{1}{2}(1-e^{i\phi})|0\rangle + \frac{1}{2}(1+e^{i\phi})|1\rangle \\
&= e^{i\frac{\phi}{2}}[-i\sin\frac{\phi}{2}|0\rangle + i\cos\frac{\phi}{2}|1\rangle]
\end{aligned}$$

Note that

$$\hat{U}_H \hat{U}_\phi \hat{U}_H = e^{i\phi/2} \begin{pmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :→ Complex[re, -im]};

ψ0 = (1
      0); ψ1 = (0
                1); I2 = IdentityMatrix[2];

UH = 1/√2 (1  1
           1 -1); Uϕ = (1  0
                        0  e^(i ϕ)); A1 = UH.Uϕ.UH;

U1 = UH.Uϕ.UH // FullSimplify;
U11 = U1 e^(-i ϕ/2) // FullSimplify;
U12 = U11 e^(i ϕ/2)

{{e^(i ϕ/2) Cos[ϕ/2], -i e^(i ϕ/2) Sin[ϕ/2]},
 {-i e^(i ϕ/2) Sin[ϕ/2], e^(i ϕ/2) Cos[ϕ/2]}}

U12.ψ0 // MatrixForm

( e^(i ϕ/2) Cos[ϕ/2]
  -i e^(i ϕ/2) Sin[ϕ/2] )

U12.ψ1 // MatrixForm

( -i e^(i ϕ/2) Sin[ϕ/2]
  e^(i ϕ/2) Cos[ϕ/2] )
```

13. Pauli gates

Pauli gates are defined by the Pauli matrices

$$\hat{X} = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{Y} = \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\hat{Z} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\sigma}_x \otimes \hat{1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \hat{\sigma}_y \otimes \hat{1} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad \hat{\sigma}_z \otimes \hat{1} = \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\hat{R}_x = \exp(-i\alpha\hat{\sigma}_x \otimes \hat{1}) = \begin{pmatrix} \cos(\alpha) & 0 & -i\sin(\alpha) & 0 \\ 0 & \cos(\alpha) & 0 & -i\sin(\alpha) \\ -i\sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & -i\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}$$

$$\hat{R}_x = \exp(-i\alpha\hat{\sigma}_x \otimes \hat{1}) = \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) & 0 \\ 0 & \cos(\alpha) & 0 & -\sin(\alpha) \\ \sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}$$

$$\hat{R}_z = \exp(-i\alpha\hat{\sigma}_x \otimes \hat{1}) = \begin{pmatrix} e^{-i\alpha} & 0 & 0 & 0 \\ 0 & e^{-i\alpha} & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :> Complex[re, -im]};

σx = {{0, 1}, {1, 0}}; σy = {{0, -ⅈ}, {ⅈ, 0}}; σz = {{1, 0}, {0, -1}};
I2 = IdentityMatrix[2];
Ax = KroneckerProduct[σx, I2];
Ay = KroneckerProduct[σy, I2];
Az = KroneckerProduct[σz, I2];
A1 = KroneckerProduct[σx + σz, I2];

Rx[α_] := MatrixExp[-ⅈ α Ax];
Ry[α_] := MatrixExp[-ⅈ α Ay];
Rz[α_] := MatrixExp[-ⅈ α Az];

Rx[π/2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

```
Ry[π/4] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
K1 = Rx[π/2].Ry[π/4] // Simplify;
```

$$\left\{\left\{-\frac{i}{\sqrt{2}}, 0, -\frac{i}{\sqrt{2}}, 0\right\}, \left\{0, -\frac{i}{\sqrt{2}}, 0, -\frac{i}{\sqrt{2}}\right\}, \left\{-\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}, 0\right\}, \left\{0, -\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}\right\}\right\}$$

```
K2 = A1/√2 (-i) // Simplify;
```

$$\left\{\left\{-\frac{i}{\sqrt{2}}, 0, -\frac{i}{\sqrt{2}}, 0\right\}, \left\{0, -\frac{i}{\sqrt{2}}, 0, -\frac{i}{\sqrt{2}}\right\}, \left\{-\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}, 0\right\}, \left\{0, -\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}\right\}\right\}$$

```
K1 - K2 // Simplify
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}}
```

14. Producing entanglement; Bell's states

Using the simple device shown below, it is possible to produce any of the Bell states

(i) The Bell's state $\left|\Phi_{12}^{(+)}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ form $|0\rangle \otimes |0\rangle$

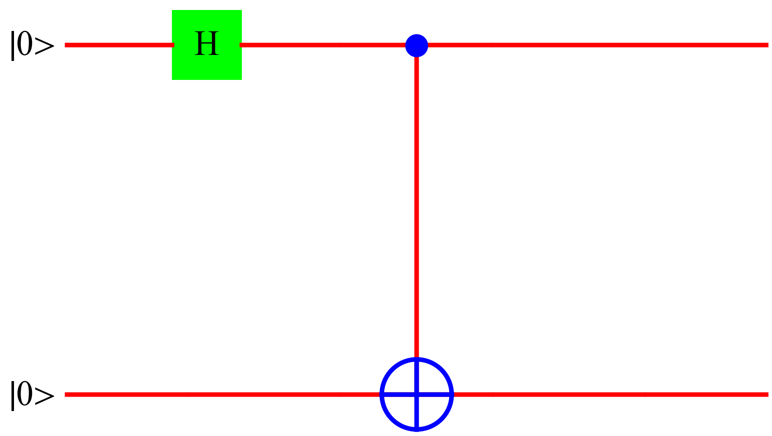
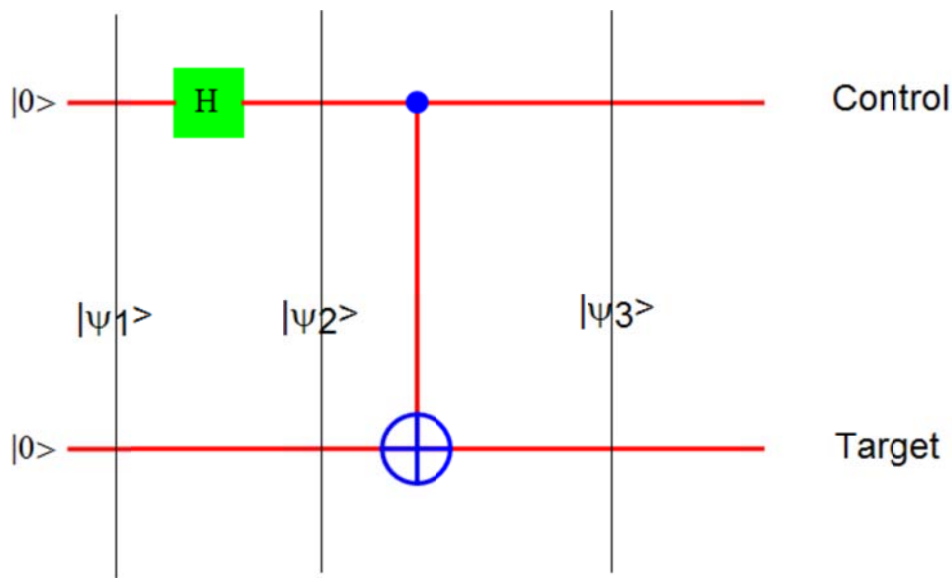


Fig. Generation of the Bell state using a Hamadard gate followed by a CNOT gate. $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



The initial state:

$$|\psi_1\rangle = |0\rangle_C \otimes |0\rangle_T = |0,0\rangle$$

The intermediate state:

$$
\begin{aligned}
|\psi_2\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_1\rangle \\
&= (\hat{U}_H \otimes \hat{1})(|0\rangle_C \otimes |0\rangle_T) \\
&= \hat{U}_H |0\rangle_C \otimes |0\rangle_T \\
&= \frac{1}{\sqrt{2}}(|0\rangle_C + |1\rangle_C) \otimes |0\rangle_T \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|\psi_3\rangle &= \hat{U}_{CNOT}|\psi_2\rangle \\
&= \frac{1}{\sqrt{2}}\hat{U}_{CNOT}(|00\rangle + |10\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

(ii) The Bell's state $\left|\psi_{12}{}^{(+)}\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ form $|0\rangle \otimes |1\rangle$

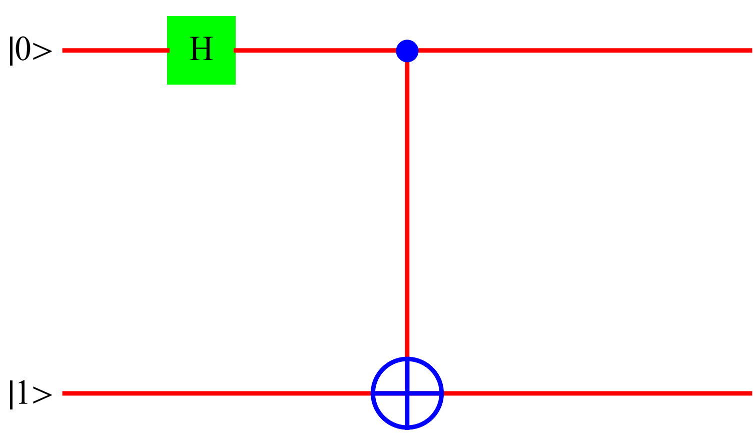


Fig. Generation of the Bell state using a Hadamard gate H followed by a CNOT gate. $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

The initial state:

$$|\psi_1\rangle = |0\rangle \otimes |1\rangle$$

$$\begin{aligned}|\psi_2\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_1\rangle \\ &= (\hat{U}_H \otimes \hat{1})(|0\rangle \otimes |1\rangle) \\ &= \hat{U}_H|0\rangle \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)\end{aligned}$$

$$\begin{aligned}|\psi_3\rangle &= \hat{U}_{CNOT}|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}\hat{U}_{CNOT}(|01\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\end{aligned}$$

(iii) The Bell's state $\left|\Phi_{12}{}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ form $|1\rangle \otimes |0\rangle$

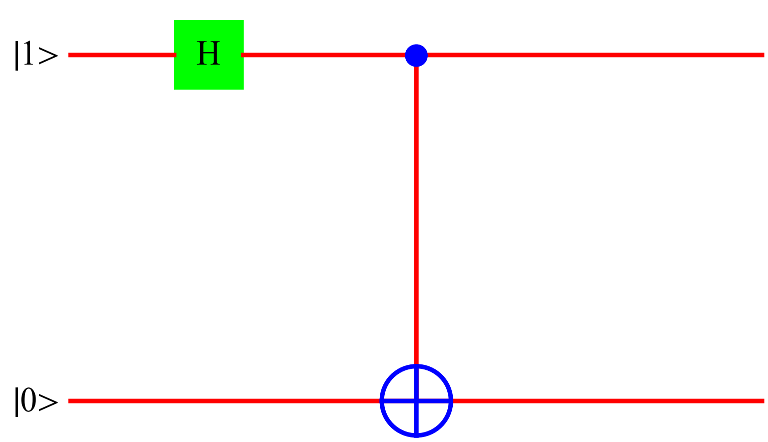


Fig. Generation of the Bell state using a Hadamard gate H followed by a CNOT gate. $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

The initial state:

$$|\psi_1\rangle = |1\rangle \otimes |0\rangle$$

$$\begin{aligned}|\psi_2\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_1\rangle \\ &= \hat{U}_H|1\rangle \otimes |0\rangle \\ &= \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)\end{aligned}$$

$$\begin{aligned}|\psi_3\rangle &= \hat{U}_{CNOT}|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}\hat{U}_{CNOT}(|00\rangle - |10\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\end{aligned}$$

(iv) The Bell's state $|\psi_{12}{}^{(-)}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ form $|1\rangle \otimes |1\rangle$

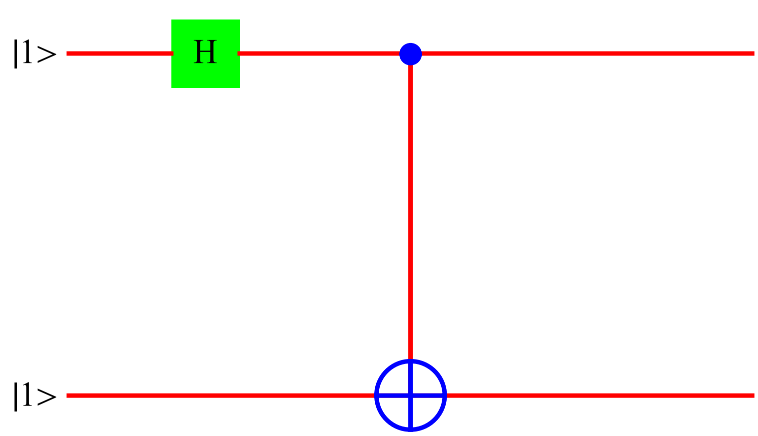


Fig. Generation of the Bell state using a Hadamard gate followed by a CNOT gate. $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

The initial state:

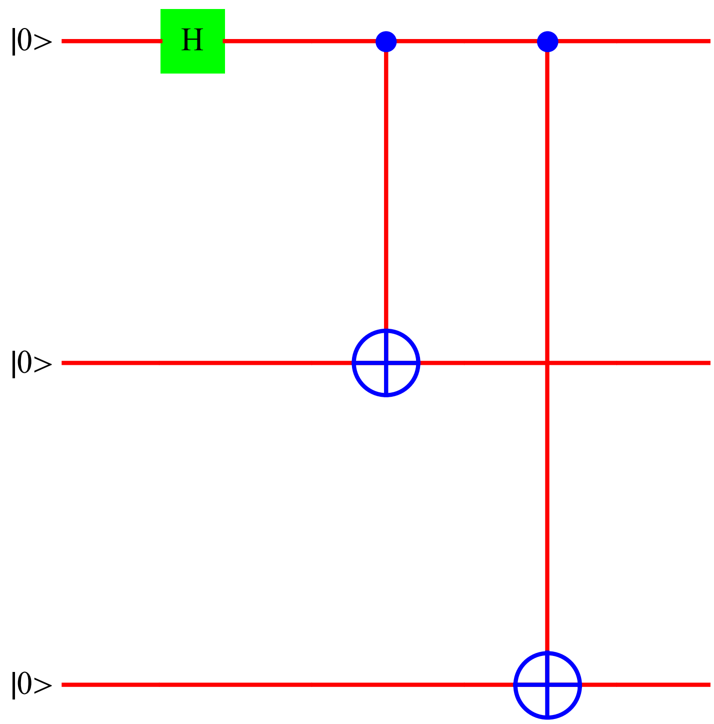
$$|\psi_1\rangle = |1\rangle \otimes |1\rangle$$

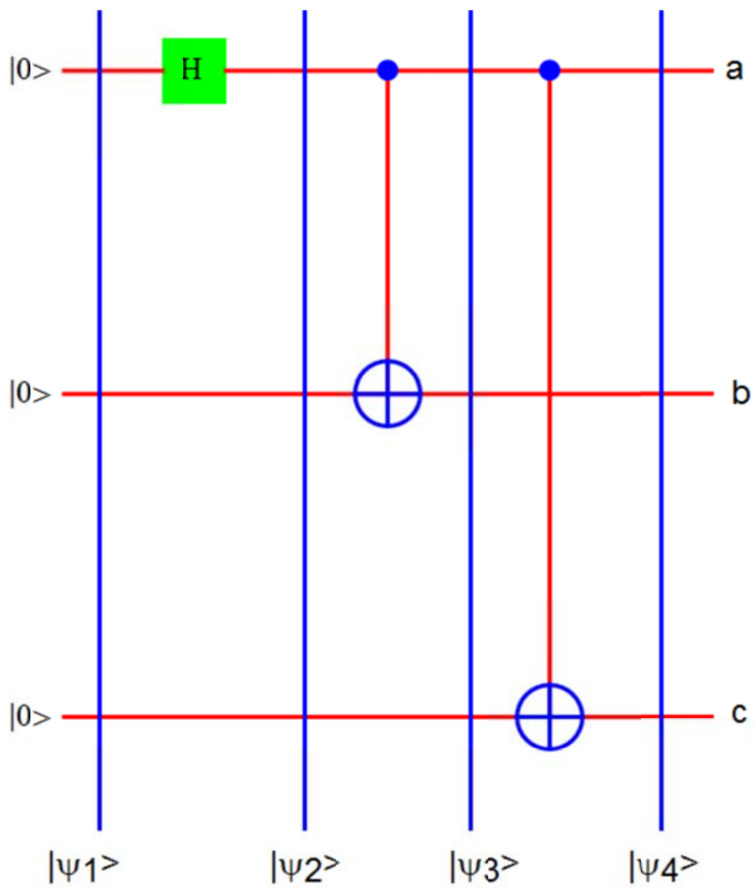
$$\begin{aligned}|\psi_2\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_1\rangle \\ &= \hat{U}_H|1\rangle \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)\end{aligned}$$

$$\begin{aligned}|\psi_3\rangle &= \hat{U}_{CNOT}|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}\hat{U}_{CNOT}(|01\rangle - |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ 1 \\ -1 \\ 0\end{pmatrix}\end{aligned}$$

15. Preparation of a GHZ state

A **Greenberger–Horne–Zeilinger state** (GHZ) is a certain type of entangled quantum state which involves at least three subsystems (particles). It was first studied by D. Greenberger, M.A. Horne and Anton Zeilinger in 1989. They have noticed the extremely non-classical properties of the state.


|0>
H
|0>
|0>


|0>
H
a
|0>
b
|0>
c
|ψ1>
|ψ2>
|ψ3>
|ψ4>

$$|\psi_1\rangle = |0\rangle_a |0\rangle_b |0\rangle_c$$

$$|\psi_2\rangle = (\hat{U}_H |0\rangle_a) \otimes |0\rangle_b |0\rangle_c = \frac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)|0\rangle_b |0\rangle_c$$

$$|\psi_3\rangle = (\hat{U}_{CNOT})_{ab} \frac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)|0\rangle_b \otimes |0\rangle_c$$
$$= \frac{1}{\sqrt{2}}(|0\rangle_a |0\rangle_b + |1\rangle_a)|1\rangle_b \otimes |0\rangle_c$$

$$|\psi_4\rangle = (\hat{U}_{CNOT})_{ac} \frac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)|0\rangle_b \otimes |0\rangle_c$$
$$= \frac{1}{\sqrt{2}}(|0\rangle_a |0\rangle_b |0\rangle_c + |1\rangle_a)|1\rangle_b \otimes |1\rangle_c$$
$$= \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

16. State Swapping

We can represent the controlled NOT operaqtion by the transformation

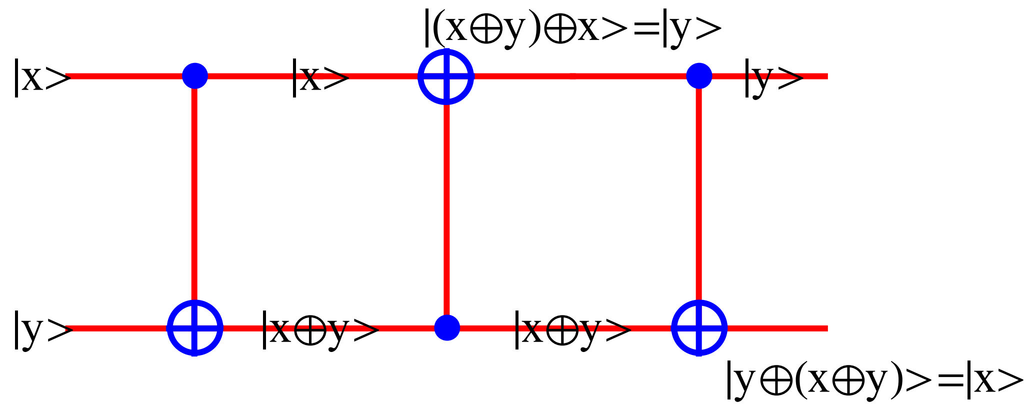
$$U_{CNOT}|x, y\rangle = |x, x \oplus y\rangle \text{, or } \quad |x, y\rangle \Rightarrow |x, x \oplus y\rangle$$

Using this formula, we can see that the gate given below swaps the states

$$|x, y\rangle \Rightarrow |x, x \oplus y\rangle$$
$$\Rightarrow |(x \oplus y) \oplus x, x \oplus y\rangle = |y, x \oplus y\rangle$$
$$\Rightarrow |y, (x \oplus y) \oplus y\rangle = |y, x\rangle$$

where

$x \oplus x = 0$, $y \oplus y = 0$



We use an equivalent schematic symbol notation for the common useful circuit.

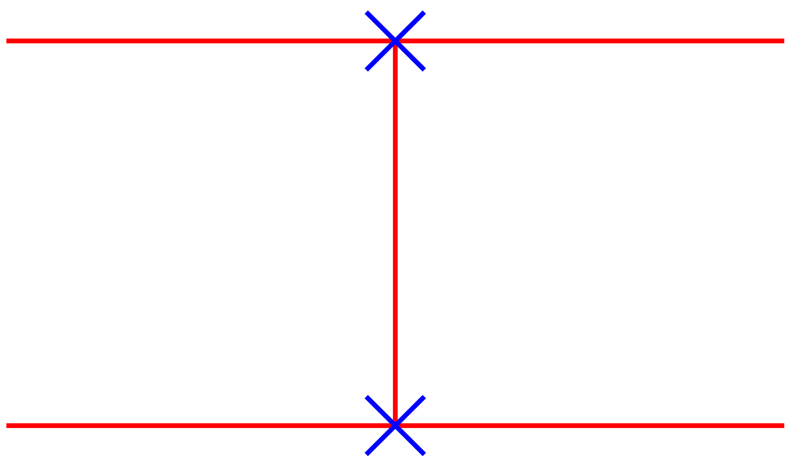


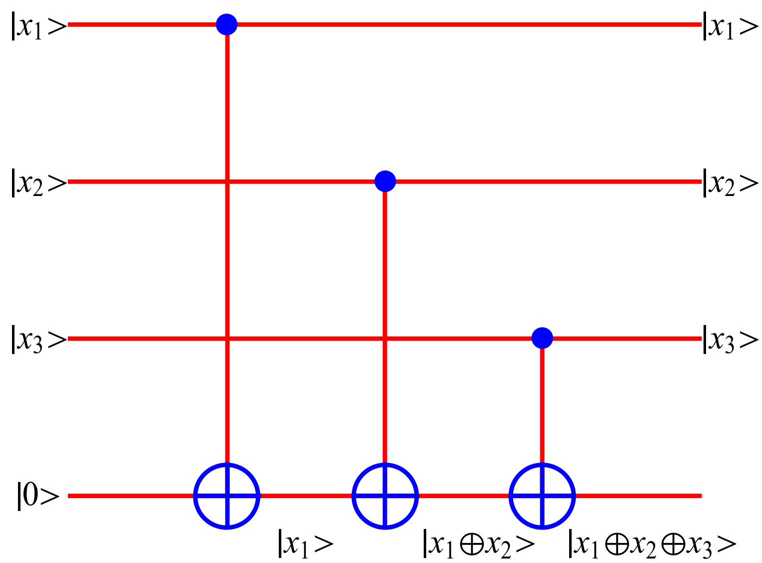
Fig. SWAP gate

The unitary operator for the swapping:

$$\hat{U}_{SW}|00\rangle = |00\rangle$$
$$\hat{U}_{SW}|01\rangle = |10\rangle$$
$$\hat{U}_{SW}|10\rangle = |01\rangle$$
$$\hat{U}_{SW}|11\rangle = |11\rangle$$

with

$$\hat{U}_{SW} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$



17. Transformation of a Bell state to the computational basis.

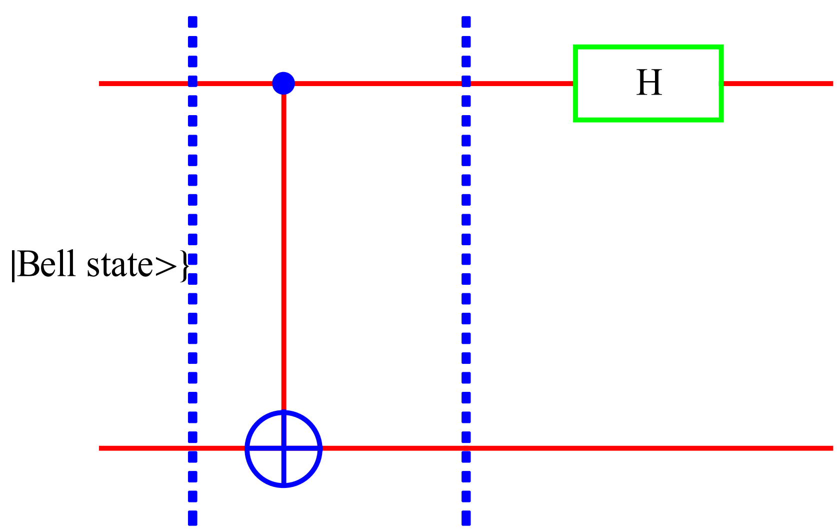


Fig. Transformation of an entangled Bell state to the computational basis ($|1\rangle$, $|0\rangle$)with a CNOT and a Hadamard gate.

(a) From $\frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$ to $|1\rangle \otimes |1\rangle$

Suppose that the initial state is the Bell state,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle],$$

$$|\psi_2\rangle = \hat{U}_{CNOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}[|01\rangle - |11\rangle] = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\otimes|1\rangle,$$

$$\begin{aligned}|\psi_3\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}(\hat{U}_H \otimes \hat{1})(|0\rangle - |1\rangle)\otimes|1\rangle \\ &= \frac{1}{2}[(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)]\otimes|1\rangle \\ &= |1\rangle\otimes|1\rangle\end{aligned}$$

(b) From $\frac{1}{\sqrt{2}}[|01\rangle + |10\rangle]$ to $|0\rangle\otimes|1\rangle$

Suppose that the initial state is the Bell state,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|01\rangle + |10\rangle],$$

$$|\psi_2\rangle = \hat{U}_{CNOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}[|01\rangle + |11\rangle] = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\otimes|1\rangle,$$

$$\begin{aligned}|\psi_3\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}(\hat{U}_H \otimes \hat{1})(|0\rangle + |1\rangle)\otimes|1\rangle \\ &= \frac{1}{2}[(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)]\otimes|1\rangle \\ &= |0\rangle\otimes|1\rangle\end{aligned}$$

(c) From $\frac{1}{\sqrt{2}}[|00\rangle - |11\rangle]$ to $|1\rangle\otimes|0\rangle$

Suppose that the initial state is the Bell state,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle],$$

$$|\psi_2\rangle = \hat{U}_{CNOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}[|00\rangle - |10\rangle] = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle,$$

Then we have

$$\begin{aligned} |\psi_3\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}(\hat{U}_H \otimes \hat{1})(|0\rangle - |1\rangle) \otimes |0\rangle \\ &= \frac{1}{2}[(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)] \otimes |0\rangle \\ &= |1\rangle \otimes |0\rangle \end{aligned}$$

(d) From $\frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$ to $|0\rangle \otimes |0\rangle$

Suppose that the initial state is the Bell state,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle],$$

$$|\psi_2\rangle = \hat{U}_{CNOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |10\rangle] = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle,$$

Then we have

$$\begin{aligned} |\psi_3\rangle &= (\hat{U}_H \otimes \hat{1})|\psi_2\rangle \\ &= \frac{1}{\sqrt{2}}(\hat{U}_H \otimes \hat{1})(|0\rangle + |1\rangle) \otimes |0\rangle \\ &= \frac{1}{2}[(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)] \otimes |0\rangle \\ &= |0\rangle \otimes |0\rangle \end{aligned}$$

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APPENDIX-I Quantum gates

http://en.wikipedia.org/wiki/Quantum_gate

Hadamard gate

The Hadamard gate acts on a single qubit. It maps the basis state $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ and represents a rotation of π about the axis $\frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$. It is represented by the Hadamard matrix:

$$\hat{H}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Pauli-X gate

The Pauli-X gate acts on a single qubit. It is the quantum equivalent of a NOT gate. It equates to a rotation of the Bloch Sphere around the X-axis by π radians. It maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$. It is represented by the Pauli matrix:

$$\hat{X}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Pauli-Y gate

The Pauli-Y gate acts on a single qubit. It equates to a rotation around the *Y*-axis of the Bloch Sphere by π radians. It maps $|0\rangle$ to $i|1\rangle$ and $|1\rangle$ to $-i|0\rangle$ .It is represented by the Pauli Y matrix:

$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli-Z gate

The Pauli-Z gate acts on a single qubit. It equates to a rotation around the Z-axis of the Bloch Sphere by π radians. Thus, it is a special case of a phase shift gate (next) with θ=π. It leaves the basis state $|0\rangle$ unchanged and maps $|1\rangle$ to $-|1\rangle$. It is represented by the Pauli Z matrix:

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Phase shift gate

This is a family of single-qubit gates that leave the basis state $|0\rangle$ unchanged and map $|1\rangle$ to $e^{i\theta}|1\rangle$. The probability of measuring a $|0\rangle$ or $|1\rangle$ is unchanged after applying this gate, however it modifies the phase of the quantum state. This is equivalent to tracing a horizontal circle (a line of latitude) on the Bloch Sphere by θ radians.

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

where θ is the *phase shift*. Some common examples are the $\pi/8$ gate where $\theta = \pi/4$, θ = , the phase gate where θ = π/2, and the Pauli-Z gate where θ = π.

Controlled-NOT

In computing science, the **controlled NOT gate** (also **C-NOT** or **CNOT**) is a quantum gate that is an essential component in the construction of a quantum computer. It can be used to entangle and disentangle EPR states. Specifically, any quantum circuit can be simulated to an arbitrary degree of accuracy using a combination of CNOT gates and single qubit rotations.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Swap gate

The swap gate swaps two qubits. It is represented by the matrix given by

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Controlled-X
Controlled-Y
Controlled-Z

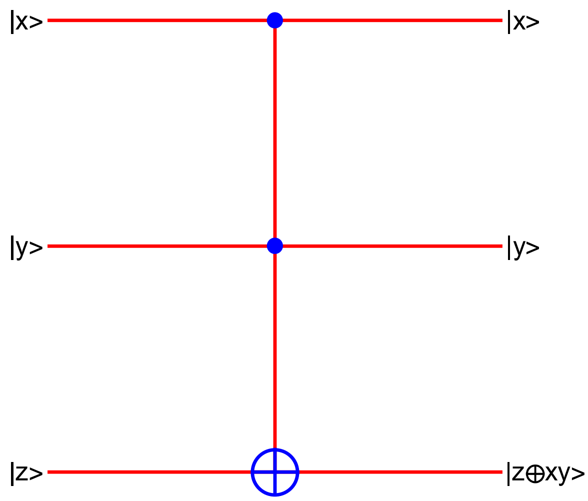
The matrix representing the controlled *U* is

$$C(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_{00} & x_{01} \\ 0 & 0 & x_{10} & x_{11} \end{pmatrix}$$

When $U = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}$ is one of the Pauli matrices, σ_x, σ_y, or σ_z, the respective terms "controlled-*X*", "controlled-*Y*", or "controlled-*Z*" are sometimes used.

Toffoli gate

In computer science, the **Toffoli gate** (also **CCNOT gate**), invented by Tommaso Toffoli, is a universal reversible logic gate, which means that any reversible circuit can be constructed from Toffoli gates. It is also known as the "controlled-controlled-not" gate, which describes its action. It has 3-bit inputs and outputs; if the first two bits are set, it inverts the third bit, otherwise all bits stay the same.



Truth table

INPUT			OUTPUT		
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fredkin (controlled-swap)

The Fredkin gate (also CSWAP gate) is a 3-bit gate that performs a controlled swap. It is universal for classical computation. It has the useful property that the numbers of 0s and 1s are conserved throughout, which in the billiard ball model means the same number of balls are output as input.

qubit

In quantum computing, a **qubit** or **quantum bit** is a unit of quantum information—the quantum analogue of the classical bit. A qubit is a two-state quantum-mechanical system, such as the polarization of a single photon: here the two states are vertical polarization and horizontal polarization. In a classical system, a bit would have to be in one state or the other, but quantum mechanics allows the qubit to be in a superposition of both states at the same time, a property which is fundamental to quantum computing.

Quantum decoherence

Quantum decoherence is the loss of coherence or ordering of the phase angles between the components of a system in a quantum superposition. One consequence of this dephasing is classical or probabilistically additive behavior. Quantum decoherence gives the appearance of wave function collapse (the reduction of the physical possibilities into a single possibility as seen by an observer) and justifies the framework and intuition of classical physics as an acceptable approximation: decoherence is the mechanism by which the classical limit emerges from a quantum starting point and it determines the location of the quantum-classical boundary. Decoherence occurs when a system interacts with its environment in a thermodynamically irreversible way. This prevents different elements in the quantum superposition of the total system's wavefunction from interfering with each other.

http://en.wikipedia.org/wiki/Quantum_decoherence

Quantum parallelism

Quantum parallelism is the method in which a quantum computer is able to perform two computations simultaneously. The term was coined by physicist David Deutsch, so as to distinguish it from classical parallel computation in standard computers. In classical computers, parallel computing is performed by having several processors linked together, so that each processor performs one computation while the other processors are performing other computations. In a quantum computer, a single quantum processor is able to perform multiple computations on its own by utilizing the fact that the qubit (or quantum bit of information) exists in multiple states simultaneously (a key feature of quantum physics is the ability of the quantum wavefunction to exist in multiple states at the same time). This gives a quantum computer much greater raw computation ability than a traditional computer.
http://physics.about.com/od/physicsqtot/g/quantumparallel.htm

Quantum error correction

Quantum error correction is used in quantum computing to protect quantum information from errors due to decoherence and other quantum noise. Quantum error correction is essential if one is to achieve fault-tolerant quantum computation that can deal not only with noise on stored quantum information, but also with faulty quantum gates, faulty quantum preparation, and faulty measurements.

Quantum Correlation

One of the most counterintuitive features of quantum mechanics is its non-local nature, which makes a fundamental departure from classical physics. Quantum mechanics allows correlations between values of measurements performed at spatially separated locations that can never occur according to classical physics. These correlations are manifestations of the phenomenon Einstein coined as the spooky action at a distance. The inequalities invented by John Bell enable to put into a testable form the non-local nature of quantum mechanics

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APPENDIX-II

The method of calculation for $|\psi_1\rangle\langle\psi_2|$ by using the Mathematica

There are two methods for the calculation.

- `Outer[Times, list1, list2]` gives an outer product.

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :→ Complex[re, -im]};
```

method - 1

```
ψ1 = {1, 0}; ψ2 = {0, 1};
A1 = Outer[Times, ψ1, ψ2*]
```

```
{{0, 1}, {0, 0}}
```

method - 2

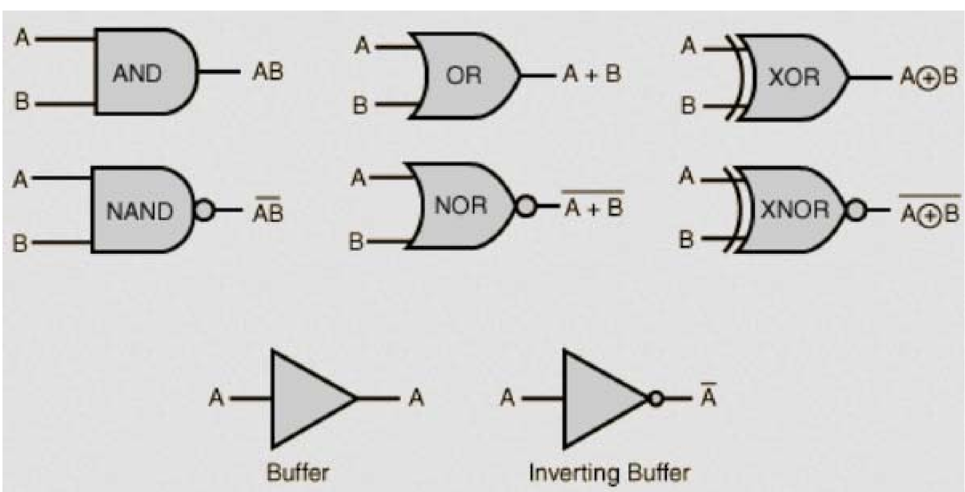
$$\phi 1 = \begin{pmatrix}1\\0\end{pmatrix};\ \phi 2 = \begin{pmatrix}0\\1\end{pmatrix};\ \phi 2H = \text{Transpose}[\phi 2^*];$$

```
A2 = ϕ1.ϕ2H
```

```
{{0, 1}, {0, 0}}
```

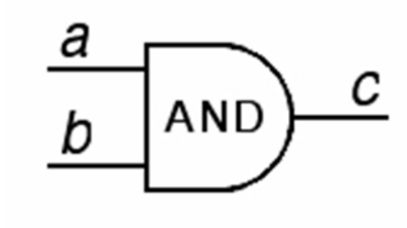
APPENDIX-III Logic gate

In electronics, a **logic gate** is an idealized or physical device implementing a Boolean function; that is, it performs a logical operation on one or more logical inputs, and produces a single logical output.).



AND gate

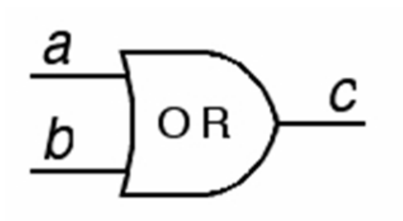
ab or $a \cdot b$



INPUT		OUTPUT
A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

OR gate

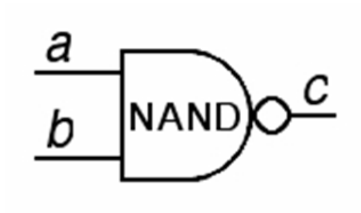
a+b



INPUT		OUTPUT
A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

NAND gate

$\overline{a.b}$

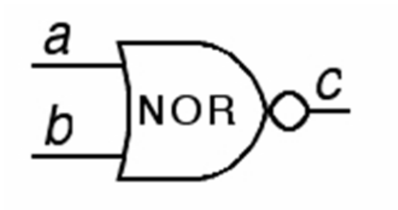


INPUT		OUTPUT
A	B	A NAND B
0	0	1
0	1	1
1	0	1
1	1	0

NOR gate

Exclusive or

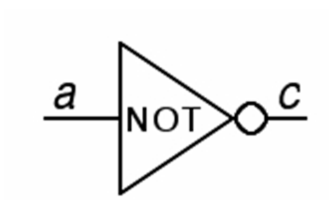
$\overline{a+b}$



INPUT		OUTPUT
A	B	A NOR B
0	0	1
0	1	0
1	0	0
1	1	0

<u>NOT gate</u>

$\overline{a}$

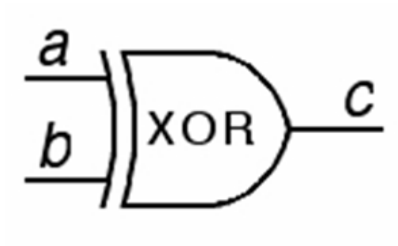


INPUT	OUTPUT
A	NOT A
0	1
1	0

<u>XOR gate</u>

exclusive OR gate.

$a \oplus b$

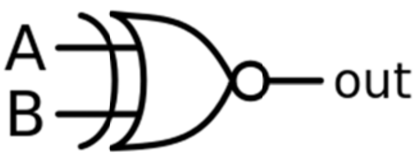


INPUT		OUTPUT
A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

XNOR gate

Inverse of exclusive OR gate.

$\overline{a \oplus b}$



INPUT		OUTPUT
A	B	A XNOR B
0	0	1
0	1	0
1	0	0
1	1	1