Radiation field

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: February 06, 2017)

We discuss phenomena on the interaction between atoms and electromagnetic field, in terms of the quantum mechanics. The electromagnetic field is classically treated, while the state of atoms is quantum mechanically treated. Such a method is called *semi-classical treatment*.

1. Lagrangian (Goldstein, Classical Mechanics)

We start with the Lorentz force given by

$$\boldsymbol{F} = q(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}).$$

We use the vector potential $\boldsymbol{A}$ and scalar potential ϕ as

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}.$$

The Newton's second law:

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} = q(-\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}) + \frac{q}{c}\boldsymbol{v}\times(\nabla\times\boldsymbol{A})$$

Here we note that

$$\begin{aligned}
[\boldsymbol{v}\times(\nabla\times\boldsymbol{A})]_x &= v_y(-\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}) + v_z(-\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x}) \\
&= (v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}) - (v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}) \\
&= (v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}) - (v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z})
\end{aligned}$$

We also note that

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + (v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}),$$

or

$$v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} = \frac{dA_x}{dt} - \frac{\partial A_x}{\partial t}.$$

Then we get

$$[\boldsymbol{v}\times(\nabla\times\boldsymbol{A})]_x = (v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}.$$

In considering the Lagrange equation, it is supposed that there are two kinds of independent variables, ***r*** and ***v***. In this case

$$(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}) = \frac{\partial}{\partial x}(\boldsymbol{v}\cdot\boldsymbol{A}).$$

Then we have

$$[\boldsymbol{v}\times(\nabla\times\boldsymbol{A})]_x = \frac{\partial}{\partial x}(\boldsymbol{v}\cdot\boldsymbol{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t},$$

or

$$[\boldsymbol{v}\times(\nabla\times\boldsymbol{A})] = \nabla(\boldsymbol{v}\cdot\boldsymbol{A}) - \frac{d\boldsymbol{A}}{dt} + \frac{\partial\boldsymbol{A}}{\partial t},$$

Using this formula, we get

$$\begin{aligned} m\frac{d\boldsymbol{v}}{dt} &= \boldsymbol{F} \\ &= q(-\nabla\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}) + \frac{q}{c}[\nabla(\boldsymbol{v}\cdot\boldsymbol{A}) - \frac{d\boldsymbol{A}}{dt} + \frac{\partial\boldsymbol{A}}{\partial t}] \\ &= -q\nabla(\phi - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}) - \frac{q}{c}\frac{d\boldsymbol{A}}{dt} \end{aligned}$$

Then we have

$$\frac{d}{dt}(m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}) = -q\nabla(\phi - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}).$$

This equation takes the form of Newton’s second law. The rate of change of a quantity that looks like momentum is equal to the gradient of a quantity that looks like potential energy. It therefore motivates the definition of the canonical momentum

$$\boldsymbol{p} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A},$$

and an effective potential energy experienced by the charged particle,

$$q(\phi - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}),$$

which is velocity-dependent. The force on the charge can be derived from the velocity-dependent potential energy

$$U = q(\phi - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}).$$

So the Lagrangian L is defined by

$$L = T - U = \frac{1}{2}m\boldsymbol{v}^2 - q(\phi - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A})$$

The canonical momentum is defined as

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}.$$

Then the mechanical momentum (the measurable quantity) is given by

$$\boldsymbol{\pi} = m\boldsymbol{v} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A}.$$

The Hamiltonian H is given by

$$\begin{aligned} H &= \boldsymbol{p}\cdot\boldsymbol{v} - L \\ &= (m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A})\cdot\boldsymbol{v} - L \\ &= \frac{1}{2}m\boldsymbol{v}^2 + q\phi \\ &= \frac{1}{2m}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 + q\phi \end{aligned}.$$

The Hamiltonian formalism uses $\boldsymbol{A}$ and ϕ, and not $\boldsymbol{E}$ and $\boldsymbol{B}$, directly. The result is that the description of the particle depends on the gauge chosen.

In conclusion we have two kinds of momentum.

Mechanical momentum	$\boldsymbol{\pi} = m\boldsymbol{v}$
Canonical momentum:	$\boldsymbol{p} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}.$

or

$$\boldsymbol{\pi} = m\boldsymbol{v} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A}$$

2. Hamiltonian

The Hamiltonian of the classical radiation field ($\hat{\boldsymbol{p}}$: momentum operator of the system, Quantum mechanical operator) is given by

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}\right)^2 \\ &= \frac{1}{2m}\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}\right)\cdot\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}\right) \\ &= \frac{1}{2m}[\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{e}{c}\left(\boldsymbol{A}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\boldsymbol{A}\right)]\end{aligned}$$

where $q = -e$ is the charge of electron (e>0) and $\phi = 0$.

$$\begin{aligned}\left(\boldsymbol{A}\cdot\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\cdot\boldsymbol{A}\right)\psi(\boldsymbol{r}) &= \boldsymbol{A}\cdot\frac{\hbar}{i}\nabla\psi(\boldsymbol{r}) + \frac{\hbar}{i}\nabla\cdot\left(\boldsymbol{A}\psi(\boldsymbol{r})\right) \\ &= \boldsymbol{A}\cdot\frac{\hbar}{i}\nabla\psi(\boldsymbol{r}) + \frac{\hbar}{i}\left(\nabla\psi(\boldsymbol{r})\cdot\boldsymbol{A} + \psi(\boldsymbol{r})\nabla\cdot\boldsymbol{A}\right) \\ &= \frac{2\hbar}{i}\boldsymbol{A}\cdot\nabla\psi(\boldsymbol{r}) + \frac{\hbar}{i}\psi(\boldsymbol{r})\left(\nabla\cdot\boldsymbol{A}\right)\end{aligned}$$

Thus

$$\hat{H} = \frac{1}{2m}\left[\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{2e}{c}\boldsymbol{A}\cdot\hat{\boldsymbol{p}} + \frac{e\hbar}{ic}\left(\nabla\cdot\boldsymbol{A}\right)\right].$$

We use the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$. Then we have the perturbations such that

$$\begin{cases}\hat{H}' = \dfrac{e}{mc}\boldsymbol{A}\cdot\hat{\boldsymbol{p}} \\ \hat{H}'' = \dfrac{e^2}{2mc^2}\boldsymbol{A}^2\end{cases}.$$

where we use the vector potential $\boldsymbol{A}$ for the classical case.

3. Classical radiation field

Maxwell's equation:

$$\begin{cases} \nabla \cdot \boldsymbol{E} = 4\pi\rho \\ \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \end{cases}$$

The equation of the continuity

$$\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0,$$

$$\begin{cases} \boldsymbol{B} = \nabla \times \boldsymbol{A} \\ \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi \end{cases},$$

where $\boldsymbol{A}$ is the vector potential and ϕ is the scalar potential.

$$\begin{cases} \nabla^2 \phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -4\pi\rho \\ \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = -\frac{4\pi \boldsymbol{j}}{c} \end{cases}$$

Gauge transformation:

$$\begin{cases} \boldsymbol{A}' = \boldsymbol{A} - \nabla \chi \\ \phi' = \phi + \frac{1}{c}\frac{\partial \chi}{\partial t} \end{cases}$$

4. Coulomb gauge

We start any pair of $\boldsymbol{A}$ and ϕ. Using the Gauge transformation we have a pair of $\boldsymbol{A}$'and ϕ', where

$$\nabla \cdot \boldsymbol{A}' = 0$$

or

$$\nabla \cdot (\boldsymbol{A} - \nabla \chi) = 0$$

or

$$\nabla^2 \chi = \nabla \cdot \boldsymbol{A}$$

This is a Poisson equation with known value of $\nabla\cdot A$. The solution of χ is uniquely determined. Therefore we can always choose the Coulomb gauge with $\nabla\cdot\boldsymbol{A}'=0$.

5. Vector potential *A* in the Coulomb gauge

Here we assume that

$$\nabla\cdot\boldsymbol{A}=0 \qquad \text{(Coulomb gauge)}$$

In the vacuum, we have

$$\rho=0,\ \boldsymbol{j}=0$$

$$\begin{cases}\nabla^2\phi=0\\ \nabla^2\boldsymbol{A}-\dfrac{1}{c^2}\dfrac{\partial^2\boldsymbol{A}}{\partial t^2}-\nabla\left(\dfrac{1}{c}\dfrac{\partial\phi}{\partial t}\right)=0\end{cases}$$

From the first equation, we have $\phi=0$

or

$$(\nabla\cdot\boldsymbol{A}=0,\ \phi=0)$$

Then we have

$$\nabla^2\boldsymbol{A}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}=0,\ \text{with}\ \nabla\cdot\boldsymbol{A}=0$$

The plane wave monochromatic solution for the wave equation is

$$\boldsymbol{A}=2\boldsymbol{A}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t),$$

where

$$\omega^2=k^2c^2 \qquad \text{or} \qquad \omega=ck \qquad \text{(Dispersion relation)}$$

Note that

$$\nabla\cdot\boldsymbol{A}=-2(\boldsymbol{k}\cdot\boldsymbol{A}_0)\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)=0$$

since

$$\begin{aligned}\nabla\cdot(\boldsymbol{A}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t))&=[\nabla\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\cdot\boldsymbol{A}_0+\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\nabla\cdot\boldsymbol{A}_0\\&=-\boldsymbol{k}\cdot\boldsymbol{A}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\end{aligned}.$$

Then we have

$$\boldsymbol{k} \cdot \boldsymbol{A}_0 = 0$$

In other words, $\boldsymbol{A}_0$ is perpendicular to the wavevector $\boldsymbol{k}$.

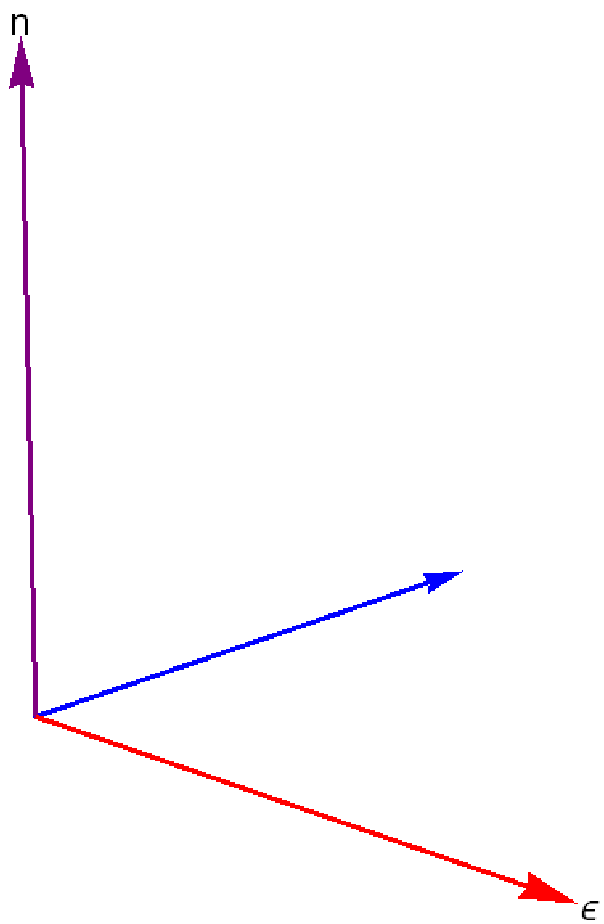


Fig. $\boldsymbol{n}$; propagation vector of the light. $\boldsymbol{\varepsilon}$ is the polarization vector. The vector potential $\boldsymbol{A}$ is parallel to the polarization vector.

$\boldsymbol{A}$ must lie in a plane perpendicular to the direction of the propagation vector ($\boldsymbol{n}$).

$$\boldsymbol{A} \equiv 2|\boldsymbol{A}_0|\boldsymbol{\varepsilon}\cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) = |\boldsymbol{A}_0|\boldsymbol{\varepsilon}\left[e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} + e^{-i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}\right] = \mathrm{Re}[2\boldsymbol{\varepsilon}|\boldsymbol{A}_0|e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}],$$

$$
\begin{aligned}
\boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \\
&= -\frac{2|\boldsymbol{A}_0|}{c}\boldsymbol{\varepsilon}(-\frac{1}{c})(-\omega)\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\
&= -2|\boldsymbol{A}_0|\frac{\omega}{c}\boldsymbol{\varepsilon}\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\
&= -2k|\boldsymbol{A}_0|\boldsymbol{\varepsilon}\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\
&= \mathrm{Re}[2ik|\boldsymbol{A}_0|\boldsymbol{\varepsilon}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{B} &= \nabla\times\boldsymbol{A} \\
&= -2|\boldsymbol{A}_0|(\boldsymbol{k}\times\boldsymbol{\varepsilon})\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\
&= -2|\boldsymbol{A}_0|\frac{\omega}{c}(\boldsymbol{n}\times\boldsymbol{\varepsilon})\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\text{'} \\
&= \mathrm{Re}[2i|\boldsymbol{A}_0|(k\boldsymbol{n}\times\boldsymbol{\varepsilon})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]
\end{aligned}
$$

where $\boldsymbol{n}$ is the unit vector defined by $\boldsymbol{n}=\boldsymbol{k}/|\boldsymbol{k}|$ and $\boldsymbol{\varepsilon}$ is the polarization vector (unit vector). The direction of the magnetic field and electric field is perpendicular to the propagation direction, forming the transverse wave.

((**Note**)) $\nabla\times(f\boldsymbol{F}) = \nabla f\times\boldsymbol{F} + f(\nabla\times\boldsymbol{F})$

6. Poynting vector

The electromagnetic energy is given by

$$
\begin{aligned}
\frac{1}{8\pi}(\boldsymbol{E}^2+\boldsymbol{B}^2) &= \frac{1}{8\pi}\left(4|\boldsymbol{A}_0|^2\frac{1}{c^2}\omega^2+4|\boldsymbol{A}_0|^2\frac{1}{c^2}\omega^2\right)\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\
&= \frac{\omega^2|\boldsymbol{A}_0|^2}{2\pi c^2}[1-\cos(2\boldsymbol{k}\cdot\boldsymbol{r}-2\omega t)]
\end{aligned}
$$

The time average is

$$
\frac{1}{T}\int_0^T\frac{1}{8\pi}(\boldsymbol{E}^2+\boldsymbol{B}^2)dt = \frac{\omega^2|\boldsymbol{A}_0|^2}{2\pi c^2} = u\ (\mathrm{erg/cm^3}),
$$

where u is the energy density. The Poynting vector is given by

$$
\boldsymbol{S} = \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B}).
$$

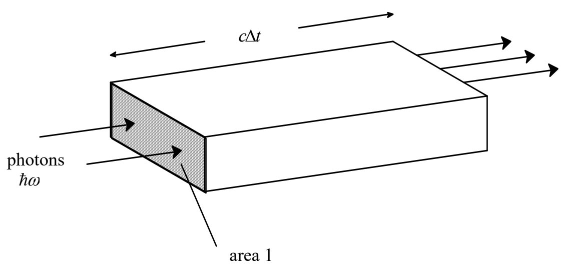
Since

$$\boldsymbol{E}\times\boldsymbol{B} = 2|\boldsymbol{A}_0|\frac{\omega}{c}\hat{\boldsymbol{\varepsilon}}\times\left[2|\boldsymbol{A}_0|\frac{\omega}{c}(\boldsymbol{n}\times\hat{\boldsymbol{\varepsilon}})\right]\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)$$

$$= 4|\boldsymbol{A}_0|^2\frac{\omega^2}{c^2}\boldsymbol{n}\frac{1}{2}[1-\cos(2\boldsymbol{k}\cdot\boldsymbol{r}-2\omega t)]$$

the time average of the Poynting vector is obtained as

$$\frac{1}{T}\int_0^T \boldsymbol{S}dt = \frac{|\boldsymbol{A}_0|^2\omega^2}{2\pi c}\boldsymbol{n} = \boldsymbol{n}cu\,. \qquad (\text{erg/s}\cdot\text{cm}^2)$$



The intensity *s*; the energy flow per unit area per unit time.

$$s = cu = \frac{\omega^2|\boldsymbol{A}_0|^2}{2\pi c} \qquad (\text{erg/s cm}^2).$$

The flux of photons (the number of photons per unit area per unit time)

$$f = \frac{s}{\hbar\omega} = \frac{\omega|\mathbf{A}_0|^2}{2\pi\hbar c}\,. \qquad (1/\text{s cm}^2).$$

7. Application —interaction with the classical radiation field

We consider the absorption and emission of light which is caused through the interaction between atoms and electromagnetic fields. The light is the electromagnetic field which periodically varies with time. Here we discuss the absorption and stimulated emission, where the electromagnetic field is semi-classically treated and the atoms are quantum-mechanically treated. There is another emission, so-called the spontaneous emission, where the electromagnetic field should be quantum-mechanically treated.

((**Classical radiation field**))

The electric or magnetic field is derivable from a classical radiation field as opposed to quantized field,

$$\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + e\phi(\hat{\boldsymbol{r}}) + \frac{e}{mc}\boldsymbol{A}\cdot\hat{\boldsymbol{p}}$$

which is justified if

$$\nabla\cdot\boldsymbol{A} = 0. \qquad \text{(Coulomb gauge)}$$

We work with a monochromatic field of the plane wave

$$\boldsymbol{A} = 2|\boldsymbol{A}_0|\boldsymbol{\varepsilon}\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

$$\boldsymbol{k} = \frac{\omega}{c}\boldsymbol{n}, \qquad \boldsymbol{\varepsilon}\cdot\boldsymbol{k} = 0$$

($\boldsymbol{\varepsilon}$ and $\boldsymbol{n}$ are the (linear) polarization and propagation directions.)

or

$$\boldsymbol{A} = |\boldsymbol{A}_0|\boldsymbol{\varepsilon}\left[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]$$

The Hamiltonian is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_1,$$

where $\hat{H}_1$ is the time dependent perturbation

$$\begin{aligned}\hat{H}_1 &= \frac{e}{mc}|\boldsymbol{A}_0|[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}](\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) \\ &= \hat{H}_1^+ e^{-i\omega t} + \hat{H}_1 e^{i\omega t}\end{aligned}$$

where

$$\hat{H}_1^+ = \frac{e}{mc}|\boldsymbol{A}_0|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}), \qquad \hat{H}_1 = \frac{e}{mc}|\boldsymbol{A}_0|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}})$$

(i) Absorption

The first term: responsible for absorption

$$\left(\hat{H}_1^+\right)_{fi} = \frac{e|\boldsymbol{A}_0|}{mc}\langle\phi_f|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}|\phi_i\rangle$$

and

$$W_{i\to f}=\frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|\boldsymbol{A}_0|^2\left|\langle\phi_f\left|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}\right|\phi_i\rangle\right|^2\delta(E_f-E_i-\hbar\omega)$$

((**Fermi's golden rule**))

where the energy is conserved during the process; $E_f-E_i=\hbar\omega$

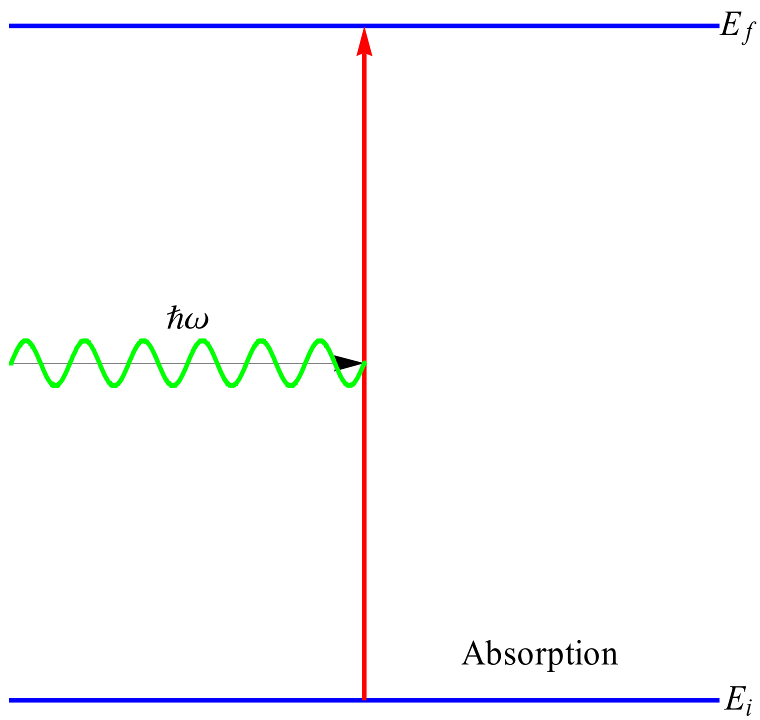


Fig. Absorption process. $E_f=E_i+\hbar\omega$.

(ii) **Stimulated emission**

The second term: responsible for the stimulated emission

$$(\hat{H}_1)_{fi}=\frac{e|\boldsymbol{A}_0|}{mc}\langle\phi_f\left|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}\right|\phi_i\rangle$$

and

$$W_{i\to f}=\frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|\boldsymbol{A}_0|^2\left|\langle\phi_f\left|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}\right|\phi_i\rangle\right|^2\delta(E_f-E_i+\hbar\omega)$$

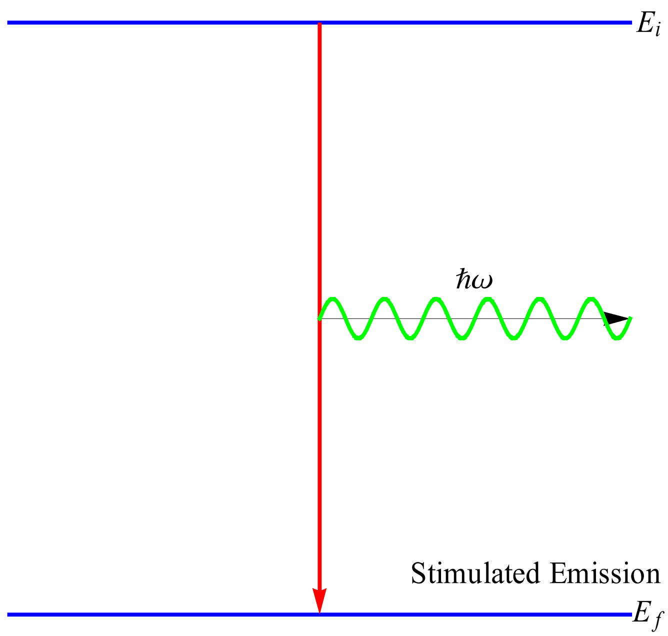


Fig. Stimulated emission process. $E_f = E_i - \hbar\omega$.

8. The semi-classical form of $|A_0|^2$

As is derived above, the transition probability rate is given by

$$W_{i\to f} = \frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|A_0|^2 \left|\langle\phi_f|e^{\mp i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}|\phi_i\rangle\right|^2 \delta(E_f - E_i \pm \hbar\omega)$$

The semiclassical form of $|A_0|^2$ can be obtained using the expressions of the energy density u, and the magnitude of the Poynting vector, s,

$$u = \frac{\omega^2}{2\pi c^2}|A_0|^2 \left(\frac{\text{erg}}{\text{cm}^3}\right) \to \overline{W}(\omega)d\omega \left(\text{erg}\frac{\text{s}}{\text{cm}^3}\frac{1}{\text{s}}\right)$$

$$s = cu\left(\text{erg}\frac{1}{\text{s cm}^2}\right) \to I(\omega)d\omega \ \left(\text{erg}\frac{1}{\text{cm}^2}\frac{1}{\text{s}}\right)$$

Note that the units of $|A_0|$ is G.cm since erg =G^2 cm^3, $I(\omega)d\omega$ is the intensity between ω and ω + dω, and $\overline{W}(\omega)d\omega$ is the energy density between ω and ω + dω,

$$I(\omega) = c\overline{W}(\omega), \qquad \overline{W}(\omega) = \frac{1}{c}I(\omega)$$

We assume that these quantities are dependent on the angular frequency ω. We note that the units of $\overline{W}(\omega)$ and $I(\omega)$ are given by

$$\left[\overline{W}(\omega)\right]=[\frac{\mathrm{erg}\cdot\mathrm{s}}{\mathrm{cm}^3}], \qquad \left[\overline{W}(\omega)d\omega\right]=[\frac{\mathrm{erg}}{\mathrm{cm}^3}],$$

$$\left[I(\omega)\right]=\left[\frac{\mathrm{erg}\cdot\mathrm{s}}{\mathrm{cm}^3}\cdot\frac{\mathrm{cm}}{\mathrm{s}}\right]=\left[\frac{\mathrm{erg}}{\mathrm{cm}^2}\right], \qquad \left[I(\omega)d\omega\right]=\left[\frac{\mathrm{erg}}{\mathrm{cm}^2 s}\right].$$

Using the relation

$$\left|\boldsymbol{A}_0\right|^2=\frac{2\pi c}{\omega^2}I(\omega)d\omega=\frac{2\pi c^2}{\omega^2}\overline{W}(\omega)d\omega,$$

we get the transition probability as

$$W_{i\to f}=\frac{2\pi}{\hbar}\int\frac{e^2}{m^2c^2}\left[\frac{2\pi c^2}{\omega^2}\overline{W}(\omega)\right]d\omega\left|\left\langle\phi_f\left|e^{\mp i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}\right|\phi_i\right\rangle\right|^2\delta\left(E_f-E_i\pm\hbar\omega\right),$$

where $\delta\left(E_f-E_i\pm\hbar\omega\right)=\frac{1}{\hbar}\delta\left(\omega_0\pm\omega\right)$ and $E_f-E_i=\hbar\omega_0$.

(i) Absorption:

$$W_{i\to f}^{(a)}=\frac{4\pi^2e^2}{\hbar^2m^2\omega_0^2}\overline{W}(\omega_0)\left|\left\langle\phi_f\left|e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}\right|\phi_i\right\rangle\right|^2.$$

(ii) Stimulated emission:

$$W_{i\to f}^{(e)}=\frac{4\pi^2e^2}{\hbar^2m^2\omega_0^2}\overline{W}(\omega_0)\left|\left\langle\phi_f\left|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}\right|\phi_i\right\rangle\right|^2$$

9. Electric dipole approximation

The vector potential periodically change over the order of the distance (wavelength, 600 nm = 6000 Å). The radius of electron in atoms is much smaller than the wavelength. In this case, we can use the approximation,

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}}=1+i\boldsymbol{k}\cdot\boldsymbol{r}+\cdots\cong 1.$$

Here

$$\frac{\omega}{c} = k = \frac{2\pi}{\lambda},$$

$$\boldsymbol{k}\cdot\boldsymbol{r} = \frac{\omega}{c}\boldsymbol{n}\cdot\boldsymbol{r} \cong \frac{2\pi}{\lambda}(\boldsymbol{n}\cdot\boldsymbol{r}) = \frac{\boldsymbol{n}\cdot\boldsymbol{r}}{\lambda\!\!\!^{-}} \cong \frac{r_{\text{atom}}}{\lambda\!\!\!^{-}} \ll 1.$$

This approximation is valid for $\lambda \gg r_{\text{atom}}$ (atomic dimension).

((**Note**))

$$\hbar\omega \approx \frac{Ze^2}{a_0 / Z} \cong \frac{Ze^2}{r_{\text{atom}}},$$ a_0/Z: atomic level spacing

$$\frac{c}{\omega} = \lambda\!\!\!^{-} = \frac{\lambda}{2\pi} \approx \frac{c\hbar r_{\text{atom}}}{Ze^2} \cong \frac{137}{Z} r_{\text{atom}}$$

where

$$(\alpha = \frac{e^2}{\hbar c} \cong \frac{1}{137})$$

The velocity v_n and radius r_n are

$$v_n = \frac{Ze^2}{n\hbar} \qquad \text{and} \qquad \frac{v_n}{c} = \frac{Ze^2}{n\hbar c} = \frac{Z\alpha}{n} = \frac{Z}{137}\frac{1}{n},$$

The energy level is

$$E_n = -\frac{Ze^2}{2r_n} = -\frac{Z^2\mu e^4}{2\hbar^2 n^2} = -\frac{Z^2 e^2}{2n^2 a},$$

In other words

$$\frac{1}{\lambda\!\!\!^{-}} r_{\text{atom}} \cong \frac{Z}{137} \ll 1$$

for the light atoms (small Z). In this approximation, we get

$$W_{i\to f}^{(a)} \approx \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2} \overline{W}(\omega_0) \left| \langle \phi_f | \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} | \phi_i \rangle \right|^2.$$

(ii) Stimulated emission

$$W_{i\to f}^{(e)} = \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2}\overline{W}(\omega_0)\left|\langle\phi_f|\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}|\phi_i\rangle\right|^2.$$

Next we need to calculate the matrix element $\langle\phi_f|\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}|\phi_i\rangle$. For simplicity we take

$$\boldsymbol{\varepsilon} = \boldsymbol{e}_x \qquad (\boldsymbol{n} = \boldsymbol{e}_z)$$

Then we get the matrix element as

$$\langle\varphi_f|\hat{p}_x|\phi_i\rangle.$$

Suppose that the Hamiltonian is given by

$$\hat{H}_0 = \frac{1}{2m}(\hat{p}_x{}^2 + \hat{p}_y{}^2 + \hat{p}_z{}^2) + V(\hat{x}, \hat{y}, \hat{z}).$$

Then we have

$$[\hat{x}, \hat{H}_0] = \frac{1}{2m}[\hat{x}, \hat{p}_x{}^2] = \frac{i\hbar}{m}\hat{p}_x,$$

where $|\phi_i\rangle$ and $|\phi_i\rangle$ are the eigenkets of $\hat{H}_0$,

$$\hat{H}_0|\phi_i\rangle = E_i|\phi_i\rangle, \qquad \hat{H}_0|\phi_f\rangle = E_f|\varphi_f\rangle.$$

We have

$$\langle\phi_f|[\hat{x}, \hat{H}_0]|\phi_i\rangle = \langle\phi_f|\hat{x}\hat{H}_0 - \hat{H}_0\hat{x}]|\phi_i\rangle = (E_i - E_f)\langle\phi_f|\hat{x}|\phi_i\rangle = -\hbar\omega_{fi}\langle\phi_f|\hat{x}|\phi_i\rangle$$

or

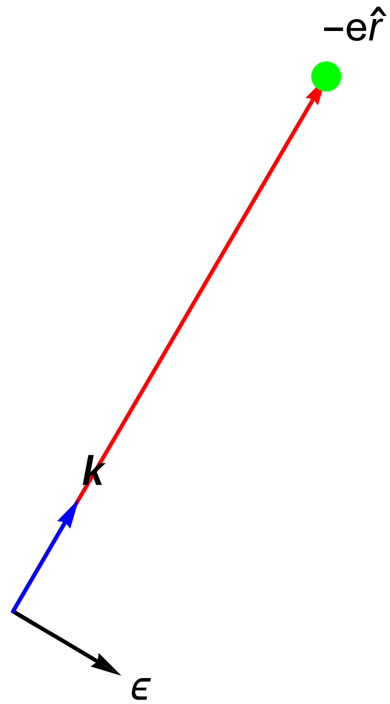
$$\langle\phi_f|\frac{i\hbar}{m}\hat{p}_x|\phi_i\rangle = -\hbar\omega_{fi}\langle\phi_f|\hat{x}|\phi_i\rangle$$

or

$$\langle\phi_f|\hat{p}_x|\phi_i\rangle = im\omega_{fi}\langle\phi_f|\hat{x}|\phi_i\rangle$$

In the direction of the electric polarization vector, we have

$$\langle \phi_f | \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{p}} | \phi_i \rangle \approx m\omega_0 \langle \phi_f | \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{r}} | \phi_i \rangle = m\omega_0 \langle \phi_f | \hat{r}_\varepsilon | \phi_i \rangle$$



In other words, the transition matrix element is expressed by

$$\langle f | \boldsymbol{\varepsilon} \cdot (-e\hat{\boldsymbol{r}}) | i \rangle .$$

where $(-e\hat{\boldsymbol{r}})$ is the electric dipole moment of electron and $\boldsymbol{\varepsilon}$ is the polarization vector of the electromagnetic wave,

Using this expression, we get the final form

$$\begin{aligned} W_{i\to f} &= \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2} \overline{W}(\omega_0) \left| \langle \phi_f | \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{p}} | \phi_i \rangle \right|^2 \\ &\approx \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2} \overline{W}(\omega_0) m^2 \omega_0^2 \left| \langle \phi_f | \hat{r}_\varepsilon | \phi_i \rangle \right|^2 \\ &= \frac{4\pi^2 e^2}{\hbar^2} \overline{W}(\omega_0) \left| \langle \varphi_f | \hat{r}_\varepsilon | \varphi_i \rangle \right|^2 \\ &= B_{12} \overline{W}(\omega_0) \end{aligned}$$

or

$$W_{i\to f} = \frac{4\pi^2 e^2}{\hbar^2 c} I(\omega_0) \left| \langle \varphi_f | \hat{r}_\varepsilon | \varphi_i \rangle \right|^2$$

where we use the electric dipole approximation

$$\overline{W}(\omega_0) = \frac{1}{c} I(\omega_0)$$

B_{12} and B_{21} are called the Einstein B-coefficient. We have

$$\begin{aligned} B_{12} &= B_{21} \\ &= \frac{4\pi^2 e^2}{\hbar^2} \left| \langle \phi_f | \hat{r}_\varepsilon | \phi_i \rangle \right|^2 \qquad \text{(Average)} \\ &\cong \frac{4\pi^2 e^2}{3\hbar^2} \left| \langle \phi_f | \hat{\boldsymbol{r}} | \phi_i \rangle \right|^2 \end{aligned}$$

The factor 1/3 arises from the random distribution of $\boldsymbol{\varepsilon}$, since the radiation is isotropic.
((**Note**))

$$\langle \phi_f | \hat{r}_\varepsilon | \phi_i \rangle = \boldsymbol{\varepsilon} \cdot \langle \phi_f | \hat{\boldsymbol{r}} | \phi_i \rangle$$

where θ is the angle between $\boldsymbol{\varepsilon}$ and $\langle \phi_f | \hat{\boldsymbol{r}} | \phi_i \rangle$, and $|\boldsymbol{\varepsilon}| = 1$. We need to take an average over the random orientations of the electric dipole moment $-e\left| \langle \phi_f | \hat{\boldsymbol{r}} | \phi_i \rangle \right|^2$.

$$\left| \langle \phi_f | \hat{r}_\varepsilon | \phi_i \rangle \right|^2 = \left| \langle \phi_f | \hat{\boldsymbol{r}} | \phi_i \rangle \right|^2 \cos^2\theta = |\boldsymbol{r}_0|^2 \cos^2\theta \to \frac{1}{3} |\boldsymbol{r}_0|^2$$

Here

$$\langle \cos^2\theta \rangle = \frac{1}{4\pi} \int_0^\pi \cos^2\theta (2\pi \sin\theta) d\theta = \frac{1}{2} \int_0^\pi \cos^2\theta \sin\theta d\theta = \frac{1}{3}.$$

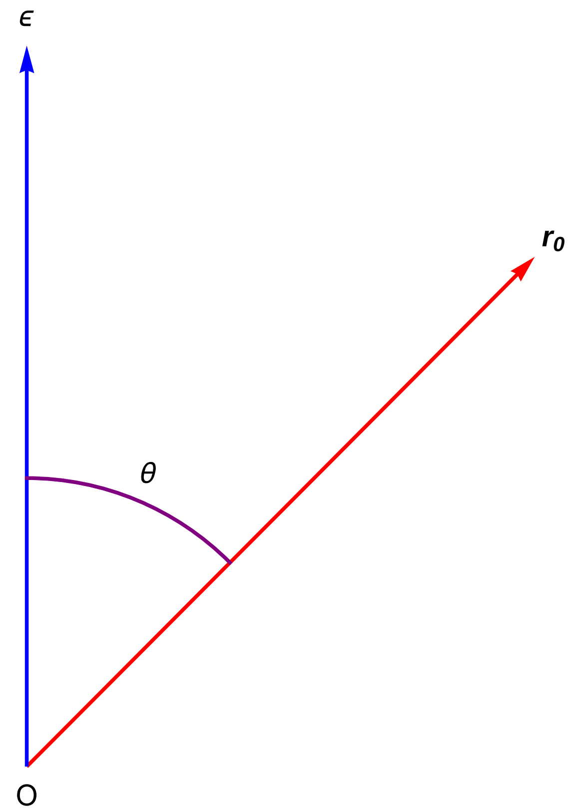


Fig. $\boldsymbol{r}_0 = \langle f|\hat{\boldsymbol{r}}|i\rangle$. $\boldsymbol{\varepsilon}\cdot\boldsymbol{r}_0 = \cos\theta|\boldsymbol{\varepsilon}||\boldsymbol{r}_0| = \cos\theta|\boldsymbol{r}_0|$ with $|\boldsymbol{\varepsilon}| = 1$. The angle θ is variable, since the orientation of the $\boldsymbol{r}_0$ is not fixed.

12. Relation between Einstein's A & B co-efficients

The energy density in thermal equilibrium between ω and $\omega + d\omega$ is given by $\overline{W}_T(\omega)d\omega$. We know that the Planck's law for the radiative energy density is given by

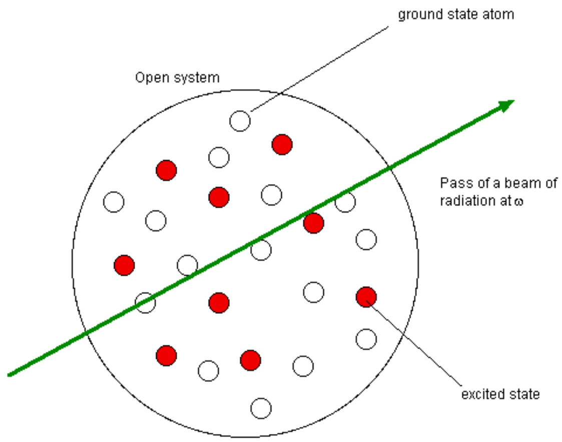
$$\overline{W}_T(\omega) = \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}.$$

from the Black-body problem (see the Black body problem in the APPENDIX)

Suppose that a gas of N identical atoms is placed in the interior of the cavity:

$$\hbar\omega = E_2 - E_1.$$

Two atomic levels are not degenerate. N_1, N_2 are the level population.



We assume that

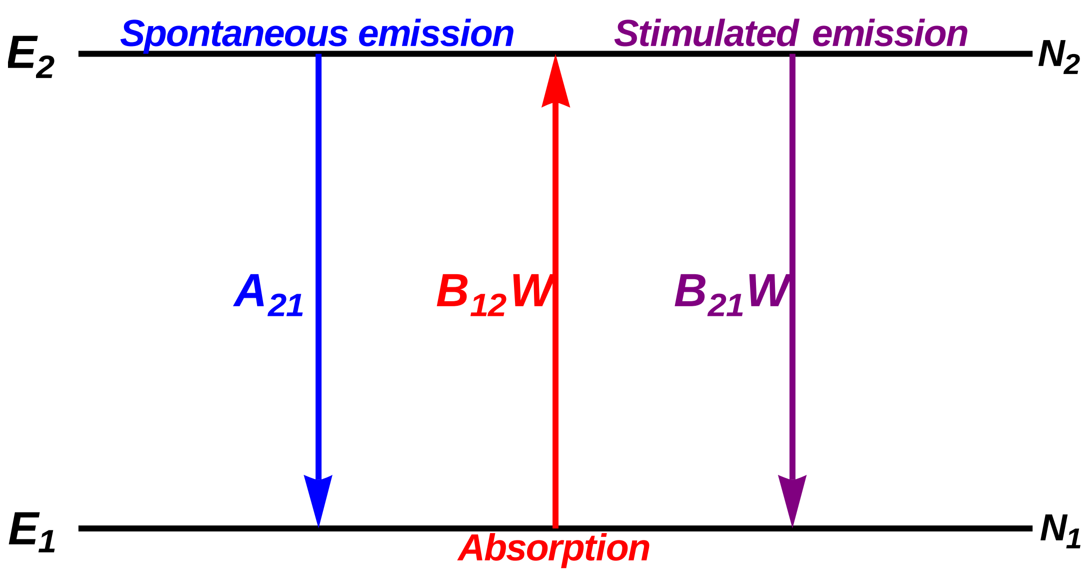
$$\overline{W}(\omega) = \overline{W_T}(\omega) + \overline{W_E}(\omega)$$

where

$\overline{W}(\omega)$: cycle-average energy density of radiation at ω

$\overline{W_T}(\omega)$: thermal part

$\overline{W_E}(\omega)$: contribution from some external source of electromagnetic radiation



We set up the rate equations for N_1 and N_2

$$\begin{cases} \dfrac{dN_1}{dt} = A_{21}N_2 - N_1B_{12}\overline{W}(\omega) + N_2B_{21}\overline{W}(\omega) \\ \dfrac{dN_2}{dt} = -A_{21}N_2 + N_1B_{12}\overline{W}(\omega) - N_2B_{21}\overline{W}(\omega) \end{cases}$$

Note that the spontaneous emission is independent of $\overline{W}(\omega)$. In the case of thermal equilibrium, we have

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0,$$

or

$$N_2A_{21} - N_1B_{12}\overline{W}(\omega) + N_2B_{21}\overline{W}(\omega) = 0.$$

For thermal equilibrium with no external radiation introduced into the cavity

$$\overline{W}(\omega) = \overline{W_T}(\omega)$$

with

$$\overline{W_T}(\omega) = \frac{A_{21}}{\left(\dfrac{N_1}{N_2}B_{12} - B_{21}\right)}$$

The level populations N_1 and N_2 are related in thermal equilibrium by Boltzman's law

$$\frac{N_1}{N_2} = \frac{e^{-\beta E_1}}{e^{-\beta E_2}} = \exp(\beta\hbar\omega), \ (\beta = 1/k_BT)$$

Then

$$\overline{W}_T(\omega) = \frac{A_{21}}{B_{12}e^{\beta\hbar\omega} - B_{21}} = \frac{\dfrac{A_{21}}{B_{21}}}{e^{\beta\hbar\omega} - \dfrac{B_{21}}{B_{12}}},$$

which is compared with the Planck's law,

$$\overline{W}_T(\omega) = \frac{\left(\dfrac{\hbar\omega^3}{\pi^2c^3}\right)}{e^{\beta\hbar\omega} - 1}$$

with

$$\begin{cases} B_{12} = B_{21} \\ \dfrac{A_{21}}{B_{12}} = \dfrac{\hbar\omega^3}{\pi^2 c^3} \end{cases}$$

$$\overline{W}_T(\omega) = \frac{A_{21}}{B_{12}}\overline{n},$$

where

$$\overline{n} = \frac{1}{e^{\beta\hbar\omega} - 1}$$

or

$$\frac{A_{21}}{B_{21}\overline{W}_T(\omega)} = e^{\beta\hbar\omega} - 1$$

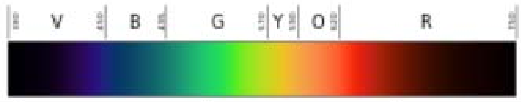
((Example)) $\hbar\omega_T = k_B T$ ($\nu = \nu_T$, $\omega_T = 2\pi\nu_T$)

For T = 300 K, $\nu_T = 6.25 \times 10^{12}$ Hz = 6.25 THz

For $\hbar\omega << k_B T$, $A_{21} << B_{21}\overline{W}_T(\omega)$ ($\nu << \nu_T$)
For $\hbar\omega >> k_B T$, $A_{21} >> B_{21}\overline{W}_T(\omega)$ ($\nu >> \nu_T$)

For optical experiments that use electromagnetic radiation, $\nu >> 5$ THz.

or $A_{21} >> B_{21}\overline{W}(\omega)$



Color	Wavelength	Frequency	Photon energy
violet	380–450 nm	668–789 THz	2.75–3.26 eV
blue	450–495 nm	606–668 THz	2.50–2.75 eV
green	495–570 nm	526–606 THz	2.17–2.50 eV
yellow	570–590 nm	508–526 THz	2.10–2.17 eV
orange	590–620 nm	484–508 THz	2.00–2.10 eV
red	620–750 nm	400–484 THz	1.65–2.00 eV

https://en.wikipedia.org/wiki/Visible_spectrum

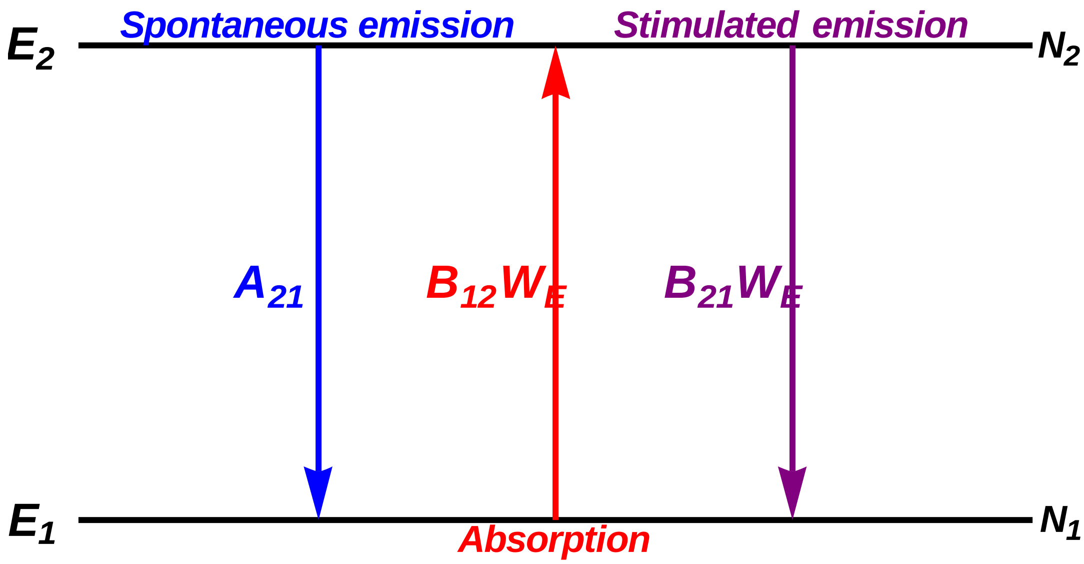
So we have

(i) $A_{21} >> B_{21}\overline{W}_T(\omega)$

A_{21}: spontaneous emission rate
$B_{21}\overline{W}_T(\omega)$: rate of thermally stimulated emission

(ii) $\overline{W}(\omega) = \overline{W}_T(\omega) + \overline{W}_E(\omega) \cong \overline{W}_E(\omega)$

Therefore the radioactive process of interest involve the absorption and stimulated emission associated with the external source.



$$\begin{cases} \dfrac{dN_1}{dt} = A_{21}N_2 - N_1B_{12}\overline{W}_E(\omega) + N_2B_{21}\overline{W}_E(\omega)) \\ \dfrac{dN_2}{dt} = -A_{21}N_2 + N_1B_{12}\overline{W}_E(\omega) - N_2B_{21}\overline{W}_E(\omega) \end{cases}$$

13. Spontaneous emission (quantum mechanics)

From the above discussion, we get

$$\begin{cases} B_{12} = B_{21} \\ \dfrac{A_{21}}{B_{12}} = \dfrac{\hbar\omega^3}{\pi^2c^3} \end{cases}$$

We note that

$$B_{12} = B_{21} = \frac{4\pi^2e^2}{\hbar^2}\left|\langle\varphi_f|\hat{r}_\varepsilon|\varphi_i\rangle\right|^2 \cong \frac{4\pi^2e^2}{3\hbar^2}\left|\langle\varphi_f|\hat{\boldsymbol{r}}|\varphi_i\rangle\right|^2$$

Thus we have

$$\begin{aligned} A_{21} &= \frac{\hbar\omega^3}{\pi^2c^3}B_{21} \\ &= \frac{\hbar\omega^3}{\pi^2c^3}\frac{4\pi^2e^2}{3\hbar^2}\left|\langle\varphi_f|\hat{\boldsymbol{r}}|\varphi_i\rangle\right|^2 \\ &= \frac{4\omega^3e^2}{3\hbar c^3}\left|\langle\varphi_f|\hat{\boldsymbol{r}}|\varphi_i\rangle\right|^2 \end{aligned}$$

When $\overline{W}_E(\omega) = 0$

$$\frac{dN_2}{dt} = -A_{21}N_2 = -\frac{1}{\tau}N_2$$

or

$$N_2 = N_2^0 \exp(-\frac{1}{\tau}t)$$

The radiative lifetime is given by

$$\tau = \frac{1}{A_{21}}$$

or

$$\frac{1}{\tau} = A_{21} = \frac{4\omega^3 e^2}{3\hbar c^3}\left|\left\langle \varphi_f \left| \hat{r} \right| \varphi_i \right\rangle\right|^2 \approx \frac{4\omega^3 e^2}{3\hbar c^3} < x >^2 = \frac{4}{3}c\frac{e^2}{\hbar c}\frac{\omega^3}{c^3} < x >^2 = \frac{4}{3}c\alpha\left(\frac{2\pi}{\lambda}\right)^3 < x >^2 \tag{1}$$

or

$$\frac{1}{\tau} = \frac{7.2354\times 10^{18}}{[\lambda(A)]^3}[x[A]]^2 \qquad \text{s}^{-1}.$$

with

$$\frac{\omega}{c} = \frac{2\pi}{\lambda}, \qquad \alpha = \frac{e^2}{\hbar c}.$$

Setting x = 1 Å and $\lambda = 400$ nm = 4000 Å (typical value), we get

$$\frac{1}{\tau} \approx \frac{7.2354\times 10^{18}}{4000^3}1^2 = 1.13 \text{ x } 10^8 \text{ s}^{-1}.$$

((**Note**))

$$\frac{1}{\tau} \approx 6\times 10^8 \text{ s}^{-1}$$

for the transition 2P to 1S in atomic hydrogen.

It is interesting to compare Eq.(1) with the result obtained from the classical radiation theory (see later). The power radiated by accelerated particle of the charge (-e) is given by the Larmor formula

$$P = \frac{2e^2(\dot{v})^2}{3c^3},$$

where $\dot{v}$ is the acceleration. If we assume that the particle undergoes a circular motion of radius r, with uniform angular velocity. The centripetal acceleration is

$$\dot{v} = \frac{v^2}{r} = \omega^2 r .$$

We can argue that the time requires for the classical system to radiate energy $\hbar\omega/2$ is equivalent to the lifetime τ. Thus

$$\frac{1}{\tau} \to \frac{2P}{\hbar\omega} = \frac{4e^2(\dot{v})^2}{3c^3\hbar\omega} = \frac{4e^2\omega^4 r^2}{3c^3\hbar\omega} = \frac{4e^2\omega^3 r^2}{3c^3\hbar} \qquad \text{(classical)}$$

Note that the qualitative agreement between the classical and quantum mechanical results is a manifestation of the correspondence principle. However, the mechanism for the emission of the radiation is completely different in the two cases, and the classical argument can never produce the discrete spectrum of the radiation.

14. Larmor's formula (classical)

The classical electrodynamics tells us that an accelerating charge radiates an electromagnetic field with far-field electric and magnetic field values. The instantaneous electromagnetic energy flow is given by the Poynting vector

$$\begin{aligned}
\boldsymbol{S} &= \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} \\
&= \frac{c}{4\pi}\boldsymbol{E}\times(\boldsymbol{n}\times\boldsymbol{E}) \\
&= \frac{c}{4\pi}[\boldsymbol{E}^2\boldsymbol{n} - (\boldsymbol{E}\cdot\boldsymbol{n})\boldsymbol{E}] \\
&= \frac{c}{4\pi}\boldsymbol{E}^2\boldsymbol{n}
\end{aligned}$$

The electric field is given by

$$\boldsymbol{E} = \frac{e}{c^2}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{v}})}{R}\right]_{ret}$$

$\dot{\boldsymbol{v}}$ is evaluated at the retarded time $t_{ret} = t - R/c$. This radiation has the characteristic dipole pattern and causes the electric dipole to lose energy, that is, to be damped. The pointing vector is then obtained as

$$\boldsymbol{S} = \frac{e^2}{4\pi c^3}\frac{1}{R^2}[\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{v}})]^2\boldsymbol{n} = \frac{e^2}{4\pi c^3}\frac{1}{R^2}(\boldsymbol{n}\times\dot{\boldsymbol{v}})^2\boldsymbol{n} = \frac{e^2}{4\pi c^3}\frac{1}{R^2}|\dot{\boldsymbol{v}}|^2(\sin^2\theta)\boldsymbol{n},$$

where θ is the angle between $\boldsymbol{n}$ and $\dot{\boldsymbol{v}}$.

$$[\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{v}})]^2 = (\boldsymbol{n}\times\dot{\boldsymbol{v}})^2 = |\dot{\boldsymbol{v}}|^2\sin^2\theta.$$

The total power radiated is given by the integration of $\boldsymbol{S}$ over a sphere surrounding the charge,

$$\begin{aligned}\int \boldsymbol{S}\cdot d\boldsymbol{a} &= \frac{e^2}{4\pi c^3}|\dot{\boldsymbol{v}}|^2\int\frac{1}{R^2}\sin^2\theta\boldsymbol{n}\cdot d\boldsymbol{a}\\ &= \frac{e^2}{4\pi c^3}|\dot{\boldsymbol{v}}|^2\int\frac{1}{R^2}\sin^3\theta(2\pi R^2)d\theta\\ &= \frac{e^2}{2c^3}|\dot{\boldsymbol{v}}|^2\int_0^\pi\sin^3\theta d\theta\\ &= \frac{2e^2}{3c^3}|\dot{\boldsymbol{v}}|^2\end{aligned}$$

where

$$\int_0^\pi \sin^3\theta d\theta = \frac{4}{3}.$$

15. Larmor formula ((Griffiths))

Suppose that a charge $q = -e$ is attached to a spring and constrained to oscillate along the x axis. It starts out in the state $|n\rangle$, and decays by spontaneous emission to the state $|n\rangle$.

The electric dipole moment is given by

$$p = -e\langle n'|\hat{x}|n\rangle\boldsymbol{e}_x$$

where

$$\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^+ + \hat{a})|n\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle$$

and

$$\langle n'|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\langle n'|n+1\rangle + \sqrt{n}\langle n'|n-1\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$$

A photon is emitted with the angular frequency given by

$$\hbar\omega_{n',n} = E_{n'} - E_n = \hbar\omega(n'+\frac{1}{2}) - \hbar\omega(n+\frac{1}{2}) = \hbar\omega(n'-n),$$

during the transition from the state $|n\rangle$ to the state $|n'\rangle$. When $n'=n+1$, we have

$$p^2 = \frac{\hbar e^2}{2m\omega}(n+1)$$

The energy of the n-th state is

$$E = \hbar\omega(n+\frac{1}{2})$$

The transition rate is

$$A = \frac{4\omega^3}{3\hbar c^3}p^2 = \frac{4\omega^3}{3\hbar c^3}\frac{\hbar e^2}{2m\omega}(n+1) \approx \frac{2\omega^2 e^2}{3mc^3}n$$

The life time of the n-th state is

$$\tau = \frac{1}{A} \approx \frac{3mc^3}{2\omega^2 e^2 n}.$$

Each radiated photon carries an energy $\hbar\omega$. The power radiated is evaluated as

$$P = \hbar\omega A = \frac{2\omega^2 e^2}{3mc^3}n\hbar\omega = \frac{2\omega^2 e^2}{3mc^3}(E - \frac{1}{2}\hbar\omega)$$

((**Classical theory**))

In the classical electrodynamics,

$$P = \frac{2a^2e^2}{3c^3}$$

where a is the acceleration

$$a = -\omega^2 x_0 \cos(\omega t)$$

Then we have

$$P = \frac{2e^2\omega^4 {x_0}^2}{3c^3}\left\langle \cos^2(\omega t)\right\rangle = \frac{e^2 {x_0}^2\omega^4}{3c^3}$$

where the time dependence is estimated by the average value as

$$\left\langle \cos^2(\omega t)\right\rangle = \frac{1}{2}\left\langle 1-\cos(2\omega t)\right\rangle = \frac{1}{2}$$

Noting that

$$E = \frac{1}{2}m\omega^2 {x_0}^2 \qquad \text{(the total energy)}$$

we have

$$P = \frac{e^2\omega^4}{3c^3}\frac{2E}{m\omega^2} = \frac{2\omega^2 e^2}{3mc^3}E$$

which is the same as that based on the quantum mechanics.

16. Absorption cross section

The absorption cross section is defined as

σ_{abs} = absorption cross section

$$= \frac{\text{(energy/unit time) absorpted by the atom } (i \to f)}{\text{Energy flux of the radiation field (erg/cm}^2\text{s)}}$$

$$== \frac{\hbar\omega W_{i\to f}\,\text{(erg/s)}}{\frac{1}{2\pi}\frac{\omega^2}{c}\left|\boldsymbol{A}_0\right|^2 \text{(ergs/cm}^2)}[\text{cm}^2]$$

$$\sigma_{\text{abs}} = \frac{\hbar\omega \frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|\boldsymbol{A}_0|^2 \left|\left\langle \varphi_f \left| e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} \right| \varphi_i \right\rangle\right|^2 \delta\left(E_f - E_i - \hbar\omega\right)}{\frac{1}{2\pi}\frac{\omega^2}{c}|\boldsymbol{A}_0|^2}$$

$$= \frac{4\pi^2\hbar}{m^2\omega}\left(\frac{e^2}{\hbar c}\right)\left|\left\langle \varphi_f \left| e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} \right| \varphi_i \right\rangle\right|^2 \delta\left(E_f - E_i - \hbar\omega\right)$$

In the electric dipole approximation,

$$\sigma_{abs} = \frac{4\pi^2}{m^2\omega}\frac{e^2}{\hbar c}\left|\left\langle \varphi_f \left| e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} \right| \varphi_i \right\rangle\right|^2 \delta\left(\omega_{fi} - \omega\right)$$

$$= \frac{4\pi^2}{m^2\omega}\left(\frac{e^2}{\hbar c}\right)\left|\left\langle \varphi_f \left| \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} \right| \phi_i \right\rangle\right|^2 \delta\left(\omega_{fi} - \omega\right)$$

$$= \frac{4\pi^2}{m^2\omega}\alpha\left|\left\langle \varphi_f \left| \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} \right| \phi_i \right\rangle\right|^2 \delta\left(\omega_{fi} - \omega\right)$$

When $\boldsymbol{\varepsilon} = \boldsymbol{e}_x$, we have

$$\left\langle \varphi_f \left| \hat{p}_x \right| \phi_i \right\rangle = im\omega_{fi}\left\langle \varphi_f \left| \hat{x} \right| \phi_i \right\rangle$$

Then

$$\sigma_{abs} = \frac{4\pi^2}{m^2\omega}\alpha(m^2\omega_{fi}^2)\left|\left\langle \varphi_f \left| \hat{x} \right| \phi_i \right\rangle\right|^2 \delta\left(\omega_{fi} - \omega\right)$$

$$= 4\pi^2\alpha\omega_{fi}\left|\left\langle \varphi_f \left| \hat{x} \right| \phi_i \right\rangle\right|^2 \delta\left(\omega_{fi} - \omega\right)$$

where

$$\alpha = \frac{e^2}{\hbar c}. \qquad \text{(fine structure constant).}$$

In atomic physics, we define oscillator strength f_{fi}

$$f_{fi} \equiv \frac{2m\omega_{fi}}{\hbar}\left|\left\langle \varphi_f \left| \hat{x} \right| \phi_i \right\rangle\right|^2$$

Thomas-Reiche-Kuhn sum rule indicates that

$$\sum_f f_{fi} = 1.$$

Using this rule

$$\begin{aligned}
\int \sigma_{abs}(\omega) d\omega &= 4\pi^2 \alpha \sum_f \omega_{fi} \left| \langle \varphi_f | \hat{x} | \phi_i \rangle \right|^2 \\
&= 4\pi^2 \alpha \frac{\hbar}{2m} \sum_f \frac{2m\omega_{fi}}{\hbar} \left| \langle \varphi_f | \hat{x} | \phi_i \rangle \right|^2 \\
&= 4\pi^2 \alpha \frac{\hbar}{2m} \sum_f f_{fi} \\
&= 4\pi^2 \alpha \frac{\hbar}{2m} \\
&= 2\pi^2 c \left(\frac{e^2}{mc^2} \right)
\end{aligned}$$

17. Thomas-Reiche-Kuhn sum rule

We consider a particle in one dimension whose Hamiltonian is given by

$$\hat{H} = \frac{1}{2m} \hat{p}_x{}^2 + V(\hat{x})$$

We have the Thomas-Reiche-Kuhn rule that

$$\sum_n \left| \langle 0 | \hat{x} | n \rangle \right|^2 (E_n - E_0) = \frac{\hbar^2}{2m}$$

where

$$\hat{H} | n \rangle = E_n | n \rangle$$

$$
\begin{aligned}
[[\hat{H},\hat{x}],\hat{x}] &= [[\frac{\hat{p}^2}{2m},\hat{x}],\hat{x}] + [[V(\hat{x}),\hat{x}],\hat{x}] \\
&= [[\frac{\hat{p}^2}{2m},\hat{x}],\hat{x}] \\
&= [-i\hbar\frac{\partial}{\partial\hat{p}}\left(\frac{\hat{p}^2}{2m}\right),\hat{x}] \\
&= [-i\hbar\frac{\hat{p}}{m},\hat{x}] \\
&= i\hbar\frac{1}{m}[\hat{x},\hat{p}] \\
&= (i\hbar)^2\frac{1}{m} \\
&= -\frac{\hbar^2}{m}
\end{aligned}
$$

Then

$$
\langle 0|[[\hat{H},\hat{x}],\hat{x}]|n\rangle = -\frac{\hbar^2}{m} \qquad (1)
$$

On the other hand

$$
\begin{aligned}
\langle 0|[[\hat{H},\hat{x}],\hat{x}]|0\rangle &= \langle 0|[\hat{H},\hat{x}]\hat{x}|0\rangle - \langle 0|\hat{x}[\hat{H},\hat{x}]|0\rangle \\
&= \sum_n \{\langle 0|[\hat{H},\hat{x}]|n\rangle\langle n|\hat{x}|0\rangle - \langle 0|\hat{x}|n\rangle\langle n|[\hat{H},\hat{x}]|0\rangle\} \\
&= \sum_n \{(E_0-E_n)\langle 0|\hat{x}|n\rangle\langle n|\hat{x}|0\rangle - (E_n-E_0)\langle 0|\hat{x}|n\rangle\langle n|\hat{x}|0\rangle\} \\
&= -2\sum_n (E_n-E_0)|\langle 0|x|n\rangle|^2
\end{aligned}
\qquad (2)
$$

Combining Eqs. (1) and (2), we obtain

$$
\sum_n (E_n-E_0)|\langle 0|\hat{x}|n\rangle|^2 = \frac{\hbar^2}{2m}.
$$

or

$$
\sum_n \omega_{n0}|\langle 0|\hat{x}|n\rangle|^2 = \frac{\hbar}{2m}
$$

where

$$E_n - E_0 = \hbar\omega_{n0}$$

18. Electric dipole transition selection rule: hydrogen atom

$$I = \left|\langle\varphi_f|\hat{\boldsymbol{r}}|\phi_i\rangle\right|^2$$

$$\langle\varphi_f|\hat{\boldsymbol{r}}|\phi_i\rangle = \langle\varphi_f|\hat{x}|\phi_i\rangle\boldsymbol{e}_x + \langle\varphi_f|\hat{y}|\phi_i\rangle\boldsymbol{e}_y + \langle\varphi_f|\hat{z}|\phi_i\rangle\boldsymbol{e}_z$$

is a vector. Then we have

$$I = \left|\langle\varphi_f|\hat{\boldsymbol{r}}|\phi_i\rangle\right|^2 = \left|\langle\varphi_f|\hat{x}|\phi_i\rangle\right|^2 + \left|\langle\varphi_f|\hat{y}|\phi_i\rangle\right|^2 + \left|\langle\varphi_f|\hat{z}|\phi_i\rangle\right|^2$$

Here we note that

$$\left|\langle\varphi_f|\hat{x}+i\hat{y}|\phi_i\rangle\right|^2 = \langle\varphi_f|\hat{x}+i\hat{y}|\phi_i\rangle^*\langle\varphi_f|\hat{x}+i\hat{y}|\phi_i\rangle = \langle\varphi_i|\hat{x}-i\hat{y}|\phi_f\rangle\langle\varphi_f|\hat{x}+i\hat{y}|\phi_i\rangle$$

or

$$\left|\langle\varphi_f|\hat{x}+i\hat{y}|\phi_i\rangle\right|^2 = \left|\langle\varphi_i|\hat{x}|\phi_f\rangle\right|^2 + \left|\langle\varphi_f|\hat{y}|\phi_i\rangle\right|^2$$

Similarly we have

$$\left|\langle\varphi_f|\hat{x}-i\hat{y}|\phi_i\rangle\right|^2 = \left|\langle\varphi_i|\hat{x}|\phi_f\rangle\right|^2 + \left|\langle\varphi_f|\hat{y}|\phi_i\rangle\right|^2$$

Then we have

$$I = \left|\langle\varphi_f|\hat{\boldsymbol{r}}|\phi_i\rangle\right|^2 = \frac{1}{2}\left|\langle\varphi_f|\hat{x}+i\hat{y}|\phi_i\rangle\right|^2 + \frac{1}{2}\left|\langle\varphi_f|\hat{x}-i\hat{y}|\phi_i\rangle\right|^2 + \left|\langle\varphi_f|\hat{z}|\phi_i\rangle\right|^2$$

Spherical tensor of rank 1

$$T_1^{(1)} = -\left(\frac{\hat{x}+i\hat{y}}{\sqrt{2}}\right)$$

$$T_0^{(1)} = \hat{z}$$

$$T_{-1}^{(1)} = \left(\frac{\hat{x} - i\hat{y}}{\sqrt{2}} \right)$$

From the Wigner-Eckart theorem,

$$\langle n', l', m' | \hat{T}_q^{(1)} | n, l, m \rangle \neq 0 \text{ for } m' = m + q \text{ and for } l' = l+1, l, l-1$$

$\hat{T}_q^{(1)}$ is the odd parity operator,

$$\hat{\pi} \hat{T}_q^{(1)} \hat{\pi} = -\hat{T}_q^{(1)}$$

and

$$\hat{\pi} | n, l, m \rangle = (-1)^l | n, l, m \rangle$$

Then the matrix element $\langle n', l', m' | \hat{T}_q^{(1)} | n, l, m \rangle$ is equal to zero for $l' = l$.

Then we have

(i) $\langle n', l', m' | \hat{T}_1^{(1)} | n, l, m \rangle = -\frac{1}{\sqrt{2}} \langle n', l', m' | \hat{x} + i\hat{y} | n, l, m \rangle \neq 0$

for $m' = m + 1$ and for $l' = l \pm 1$

(ii) $\langle n', l', m' | \hat{T}_{-1}^{(1)} | n, l, m \rangle = \frac{1}{\sqrt{2}} \langle n', l', m' | \hat{x} - i\hat{y} | n, l, m \rangle \neq 0$

for $m' = m - 1$ and for $l' = l \pm 1$.

(iii) $\langle n', l', m' | \hat{T}_0^{(1)} | n, l, m \rangle = \langle n', l', m' | \hat{z} | n, l, m \rangle = \neq 0$

for $m' = m$ and for $l' = l \pm 1$.

19. Radiation due to Electric quadrupole and magnetic dipole

A. Radiation due to electric dipole moment (review)

$$\boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{p}} = i \frac{m}{\hbar} \boldsymbol{\varepsilon} \cdot [\hat{H}_0, \hat{\boldsymbol{r}}]$$

since

$$\hat{\boldsymbol{p}} = i\frac{m}{\hbar}[\hat{H}_0, \hat{\boldsymbol{r}}].$$

The matrix element:

$$\begin{aligned}
\langle \phi_n | \boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}} | \phi_m \rangle &= i\frac{m}{\hbar}\boldsymbol{\varepsilon}\cdot\langle \phi_n | [\hat{H}_0, \hat{\boldsymbol{r}}] | \phi_m \rangle \\
&= i\frac{m}{\hbar}\boldsymbol{\varepsilon}\cdot(E_n - E_m)\langle \phi_n | \hat{\boldsymbol{r}} | \phi_m \rangle \\
&= im\omega_{nm}\boldsymbol{\varepsilon}\cdot\langle \phi_n | \hat{\boldsymbol{r}} | \phi_m \rangle \\
&= i\frac{m\omega_{nm}}{-e}\boldsymbol{\varepsilon}\cdot\langle \phi_n | -e\hat{\boldsymbol{r}} | \phi_m \rangle \\
&= -i\frac{m\omega_{nm}}{e}\boldsymbol{\varepsilon}\cdot\langle \phi_n | \hat{\boldsymbol{D}} | \phi_m \rangle
\end{aligned}$$

The electric dipole moment:

$$\hat{\boldsymbol{D}} = q\hat{\boldsymbol{r}} = -e\hat{\boldsymbol{r}}$$

B. Radiation due to the magnetic dipole moment

We can manipulate the operator $(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}})$ into a form that consists of two terms, one representing a magnetic moment, the other an electric quadrupole moment

$$(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) = \frac{1}{2}[(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) - (\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}}) + (\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) + (\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})]$$

We consider the first term,

$$\begin{aligned}
\langle \boldsymbol{r} | (\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) - (\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}}) | \psi \rangle &= \frac{\hbar}{i}(\boldsymbol{k}\cdot\boldsymbol{r})[\boldsymbol{\varepsilon}\cdot\nabla\psi(\boldsymbol{r})] - \frac{\hbar}{i}(\boldsymbol{\varepsilon}\cdot\boldsymbol{r})[\boldsymbol{k}\cdot\nabla\psi(\boldsymbol{r})] \\
&= \frac{\hbar}{i}\{(\boldsymbol{k}\cdot\boldsymbol{r})[\boldsymbol{\varepsilon}\cdot\nabla\psi(\boldsymbol{r})] - [\boldsymbol{k}\cdot\nabla\psi(\boldsymbol{r})](\boldsymbol{\varepsilon}\cdot\boldsymbol{r})\} \\
&= \frac{\hbar}{i}(\boldsymbol{k}\times\boldsymbol{\varepsilon})[\boldsymbol{r}\times\nabla\psi(\boldsymbol{r})] \\
&= (\boldsymbol{k}\times\boldsymbol{\varepsilon})\langle \boldsymbol{r} | \hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}} | \psi \rangle \\
&= (\boldsymbol{k}\times\boldsymbol{\varepsilon})\langle \boldsymbol{r} | \hat{\boldsymbol{L}} | \psi \rangle
\end{aligned}$$

where $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}}$ is the orbital angular momentum. We can identify $(\boldsymbol{k}\times\boldsymbol{\varepsilon})$ with the direction of the magnetic field The orbital magnetic moment is given by

$$\boldsymbol{\mu} = -\frac{\mu_B}{\hbar}\boldsymbol{L} = -\frac{e}{2mc}\boldsymbol{L}$$

So that this corresponds to the transition due to the interaction of the magnetic dipole moment of the charged particle with the magnetic field associated with the photon. This is in close analogy with the electric dipole transition, except that the magnetic dipole transition probability is much smaller.

Next we consider the second term.

$$(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) + (\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}}) \tag{1}$$

Here we show that

$$(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}}) = (\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})$$

((**Proof**))

$$\begin{aligned}
\langle \boldsymbol{r}|(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})|\psi\rangle &= \frac{\hbar}{i}(\boldsymbol{k}\cdot\nabla)(\boldsymbol{\varepsilon}\cdot\boldsymbol{r})\psi(\boldsymbol{r}) \\
&= \frac{\hbar}{i}\sum_{i,j} k_i \frac{\partial}{\partial x_i}[\varepsilon_j x_j \psi(\boldsymbol{r})] \\
&= \frac{\hbar}{i}\sum_{i,j} k_i\varepsilon_j[\delta_{i,j}\psi(\boldsymbol{r}) + x_j\frac{\partial}{\partial x_i}\psi(\boldsymbol{r})] \\
&= \frac{\hbar}{i}\sum_{i,j} k_i\varepsilon_j\delta_{i,j}\psi(\boldsymbol{r}) + \frac{\hbar}{i}\sum_{i,j}\varepsilon_j x_j k_i\frac{\partial}{\partial x_i}\psi(\boldsymbol{r}) \\
&= \frac{\hbar}{i}(\boldsymbol{k}\cdot\boldsymbol{\varepsilon})\psi(\boldsymbol{r}) + \frac{\hbar}{i}(\boldsymbol{\varepsilon}\cdot\boldsymbol{r})(\boldsymbol{k}\cdot\nabla\psi(\boldsymbol{r})) \\
&= \frac{\hbar}{i}(\boldsymbol{k}\cdot\boldsymbol{\varepsilon})\langle \boldsymbol{r}|\psi\rangle + \langle \boldsymbol{r}|(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\boldsymbol{p})|\psi\rangle
\end{aligned}$$

Since $\boldsymbol{k}\cdot\boldsymbol{\varepsilon} = 0$, we have

$$\langle \boldsymbol{r}|(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})|\psi\rangle = \langle \boldsymbol{r}|(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\boldsymbol{p})|\psi\rangle$$

or

$$(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\boldsymbol{p}) = (\boldsymbol{k}\cdot\hat{\boldsymbol{p}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}}).$$

Here we use the relation

$$\hat{\boldsymbol{p}} = -\frac{1}{2i\hbar}[2m\hat{H}_0, \hat{\boldsymbol{r}}] = i\frac{m}{\hbar}[\hat{H}_0, \hat{\boldsymbol{r}}],$$

since

$$[\hat{x}, \hat{p}_x{}^2] = 2i\hbar\hat{p}_x, \qquad [\hat{y}, \hat{p}_y{}^2] = 2i\hbar\hat{p}_y, \qquad [\hat{z}, \hat{p}_z{}^2] = 2i\hbar\hat{p}_z.$$

Then we get

$$\begin{aligned}(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\boldsymbol{p}) &= (\boldsymbol{k}\cdot\hat{\boldsymbol{p}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}}) \\ &= i\frac{m}{\hbar}(\boldsymbol{k}\cdot[\hat{H}_0, \hat{\boldsymbol{r}}])(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})\end{aligned} \tag{2}$$

We must add the relations Eqs(1) and (2). As we do this, we keep ***k*** to the left,

$$\begin{aligned}(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}}) + (\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}}) &= i\frac{m}{\hbar}[(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot[\hat{H}_0, \hat{\boldsymbol{r}}]) + (\boldsymbol{k}\cdot[\hat{H}_0, \hat{\boldsymbol{r}}])(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})] \\ &= i\frac{m}{\hbar}\sum_{i,j}\{k_j\hat{x}_j\varepsilon_i[\hat{H}_0, \hat{x}_i] + k_j[\hat{H}_0, \hat{x}_j]\varepsilon_i\hat{x}_j\} \\ &= i\frac{m}{\hbar}\sum_{i,j}\varepsilon_i k_j\{\hat{x}_j[\hat{H}_0, \hat{x}_i] + [\hat{H}_0, \hat{x}_j]\hat{x}_i\} \\ &= i\frac{m}{\hbar}\sum_{i,j}\varepsilon_i k_j[\hat{H}_0, \hat{x}_i\hat{x}_i]\end{aligned}$$

since

$$[\hat{H}_0, \hat{x}_i\hat{x}_j] = \hat{x}_j[\hat{H}_0, \hat{x}_i] + [\hat{H}_0, \hat{x}_j]\hat{x}_i.$$

The matrix element can be calculated as

$$\langle\phi_n|[\hat{H}_0, \hat{x}_i\hat{x}_j]|\phi_m\rangle = (E_n - E_m)\langle\phi_n|\hat{x}_i\hat{x}_j|\phi_m\rangle = \hbar\omega_{nm}\langle\phi_n|\hat{x}_i\hat{x}_j|\phi_m\rangle,$$

where

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m).$$

We note that the electric quadrupole moment is defined as

$$\hat{Q}_{ij} = q(\hat{x}_i\hat{x}_j - \frac{1}{3}|\hat{\boldsymbol{r}}|^2\delta_{i,j}), \tag{3}$$

where q = -e (e>0) for the electron.

The extra term proportional to $\delta_{i,j}$ in Eq.(3) does not matter because it gets multiplied by $\varepsilon_i k_j$ giving $\boldsymbol{\varepsilon}\cdot\boldsymbol{k}$ which is zero. We interpret this as an electric quadrupole transition. Its transition probability is of the same order of magnitude as the one from the magnetic dipole moment and much smaller than the transition probability from the electric dipole moment.

((**Note**))

$$\begin{aligned}\sum_{i,j}\varepsilon_i k_j\langle\phi_n|\hat{Q}_{ij}|\phi_m\rangle &= q\sum_{i,j}\varepsilon_i k_j\langle\phi_n|\hat{x}_i\hat{x}_j-\frac{1}{3}|\hat{\boldsymbol{r}}|^2\delta_{i,j}|\phi_m\rangle\\ &= q\sum_{i,j}\varepsilon_i k_j\langle\phi_n|\hat{x}_i\hat{x}_j|\phi_m\rangle-\frac{q}{3}\sum_{i,j}\varepsilon_i k_j\delta_{i,j}\langle\phi_n||\hat{\boldsymbol{r}}|^2|\phi_m\rangle\\ &= q\sum_{i,j}\varepsilon_i k_j\langle\phi_n|\hat{x}_i\hat{x}_j|\phi_m\rangle-\frac{q}{3}\langle\phi_n||\hat{\boldsymbol{r}}|^2|\phi_m\rangle(\boldsymbol{\varepsilon}\cdot\boldsymbol{k})\\ &= q\sum_{i,j}\varepsilon_i k_j\langle\phi_n|\hat{x}_i\hat{x}_j|\phi_m\rangle\end{aligned}$$

since $\boldsymbol{\varepsilon}\cdot\boldsymbol{k}=0$.

C. M1 transition due to the magnetic moment: 21 cm H1 [hydrogen (1)] atom

Here we introduce a full magnetic dipole operator which is defined as

$$-\frac{\mu_B}{\hbar}(\hat{\boldsymbol{L}}+2\hat{\boldsymbol{S}})=-\frac{\mu_B}{\hbar}(\hat{\boldsymbol{J}}+\hat{\boldsymbol{S}})$$

The hyper transition in an atomic hydrogen from F = 1 to F = 0 state at 1420 MHz transition is an M1 transition. The M1 transition involves a change in a spin (a) from the state $|F=1,m_f=1,\rangle$ to $|F=0,m_f=0\rangle$, and (b) from the state $|F=1,m_f=0,\rangle$ to $|F=,m_f=0\rangle$.

$$\hat{O}_{M1}=\frac{i}{2}(\boldsymbol{k}\times\boldsymbol{\varepsilon})\cdot(\hat{\boldsymbol{L}}+2\hat{\boldsymbol{S}})$$

The matrix element is given by

$$\begin{aligned}\langle 1,m|\hat{O}_{M1}|0,0\rangle &= \frac{i}{2}(\boldsymbol{k}\times\boldsymbol{\varepsilon})\cdot\langle 1,m|(\hat{\boldsymbol{L}}+2\hat{\boldsymbol{S}})|0,0\rangle\\ &\approx i(\boldsymbol{k}\times\boldsymbol{\varepsilon})\cdot\langle 1,m|\hat{\boldsymbol{S}}|0,0\rangle\end{aligned}$$

Note that the magnetic moment of the proton is much smaller than that of the magnetic moment. Then $\hat{\boldsymbol{S}}$ is actually equal to the spin of electrons. Using the kronecker product, the spin operator of electron are given by

$$\hat{S}_{xe} = \frac{\hbar}{2}(\hat{\sigma}_x \otimes \hat{1}_2) = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{S}_{ye} = \frac{\hbar}{2}(\hat{\sigma}_y \otimes \hat{1}_2) = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\hat{S}_{ze} = \frac{\hbar}{2}(\hat{\sigma}_z \otimes \hat{1}_2) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The matrix elements can be evaluated as

$$\langle 1,1|\hat{S}_{ze}|0,0\rangle = 0, \qquad \langle 1,0|\hat{S}_{ze}|0,0\rangle = \frac{\hbar}{2}, \qquad \langle 1,-1|\hat{S}_{ze}|0,0\rangle = 0$$

$$\langle 1,1|\hat{S}_{xe}|0,0\rangle = -\frac{\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{xe}|0,0\rangle = 0, \qquad \langle 1,-1|\hat{S}_{xe}|0,0\rangle = \frac{\hbar}{2\sqrt{2}}$$

$$\langle 1,1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{ye}|0,0\rangle = 0 \qquad \langle 1,-1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}$$

Then we have the transitions

S_{ze} component:	$\|1,0\rangle \to \|0,0\rangle$	allowed	(photon: linear polarization)
S_{xe}, S_{ye} component:	$\|1,1\rangle \to \|0,0\rangle$	allowed	(photon $\|R\rangle$ polarization wit $\hbar$)
	$\|1,-1\rangle \to \|0,0\rangle$	allowed	(photon $\|L\rangle$ polarization wit $-\hbar$)

((**Note**))
(a)

$$\langle 1,1|\hat{S}_{ze}|0,0\rangle = 0, \qquad \langle 1,0|\hat{S}_{ze}|0,0\rangle = \frac{\hbar}{2}, \qquad \langle 1,-1|\hat{S}_{ze}|0,0\rangle = 0$$

((**Proof**))

$$\hat{S}_{ze}\left|1,1\right\rangle = \frac{\hbar}{2}\hat{\sigma}_{ze}\left|+_e,+_p\right\rangle = \frac{\hbar}{2}\left|+_e,+_p\right\rangle = \frac{\hbar}{2}\left|1,1\right\rangle,$$

$$\begin{aligned}\hat{S}_{ze}\left|1,0\right\rangle &= \frac{\hbar}{2}\hat{\sigma}_{ze}\frac{\left|+_e,-_p\right\rangle + \left|-_e,+_p\right\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{\left|+_e,-_p\right\rangle - \left|-_e,+_p\right\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\left|0,0\right\rangle\end{aligned}$$

$$\hat{S}_{ze}\left|1,-1\right\rangle = \frac{\hbar}{2}\hat{\sigma}_{ze}\left|-_e,-_p\right\rangle = -\frac{\hbar}{2}\left|-_e,-_p\right\rangle = -\frac{\hbar}{2}\left|1,-1\right\rangle$$

(b)

$$\left\langle 1,1\right|\hat{S}_{xe}\left|0,0\right\rangle = -\frac{\hbar}{2\sqrt{2}}, \qquad \left\langle 1,0\right|\hat{S}_{xe}\left|0,0\right\rangle = 0, \qquad \left\langle 1,-1\right|\hat{S}_{xe}\left|0,0\right\rangle = \frac{\hbar}{2\sqrt{2}}$$

((**Proof**))

$$\hat{S}_{xe}\left|1,1\right\rangle = \frac{\hbar}{2}\hat{\sigma}_{xe}\left|+_e,+_p\right\rangle = \frac{\hbar}{2}\left|-_e,+_p\right\rangle = \frac{\hbar}{2}\frac{\left|1,0\right\rangle - \left|0,0\right\rangle}{\sqrt{2}},$$

$$\begin{aligned}\hat{S}_{xe}\left|1,0\right\rangle &= \frac{\hbar}{2}\hat{\sigma}_{xe}\frac{\left|+_e,-_p\right\rangle + \left|-_e,+_p\right\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{\left|-_e,-_p\right\rangle + \left|+_e,+_p\right\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{\left|1,1\right\rangle + \left|1,-1\right\rangle}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\hat{S}_{xe}\left|1,-1\right\rangle &= \frac{\hbar}{2}\hat{\sigma}_{xe}\left|-_e,-_p\right\rangle \\ &= \frac{\hbar}{2}\left|+_e,-_p\right\rangle \\ &= \frac{\hbar}{2}\frac{\left|1,0\right\rangle + \left|0,0\right\rangle}{\sqrt{2}}\end{aligned}$$

(c)

$$\langle 1,1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{ye}|0,0\rangle = 0 \qquad \langle 1,-1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}$$

((**Proof**))

$$\hat{S}_{ye}|1,1\rangle = \frac{\hbar}{2}\hat{\sigma}_{ye}|+_e,+_p\rangle = i\frac{\hbar}{2}|-_e,+_p\rangle = i\frac{\hbar}{2}\frac{|1,0\rangle - |0,0\rangle}{\sqrt{2}},$$

$$\begin{aligned}\hat{S}_{ye}|1,0\rangle &= \frac{\hbar}{2}\hat{\sigma}_{ye}\frac{|+_e,-_p\rangle + |-_e,+_p\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{i|-_e,-_p\rangle - i|+_e,+_p\rangle}{\sqrt{2}} \\ &= -i\frac{\hbar}{2}\frac{|1,1\rangle - |1,-1\rangle}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\hat{S}_{ye}|1,-1\rangle &= \frac{\hbar}{2}\hat{\sigma}_{ye}|-_e,-_p\rangle \\ &= -i\frac{\hbar}{2}|+_e,-_p\rangle \\ &= -i\frac{\hbar}{2}\frac{|1,0\rangle + |0,0\rangle}{\sqrt{2}}\end{aligned}$$

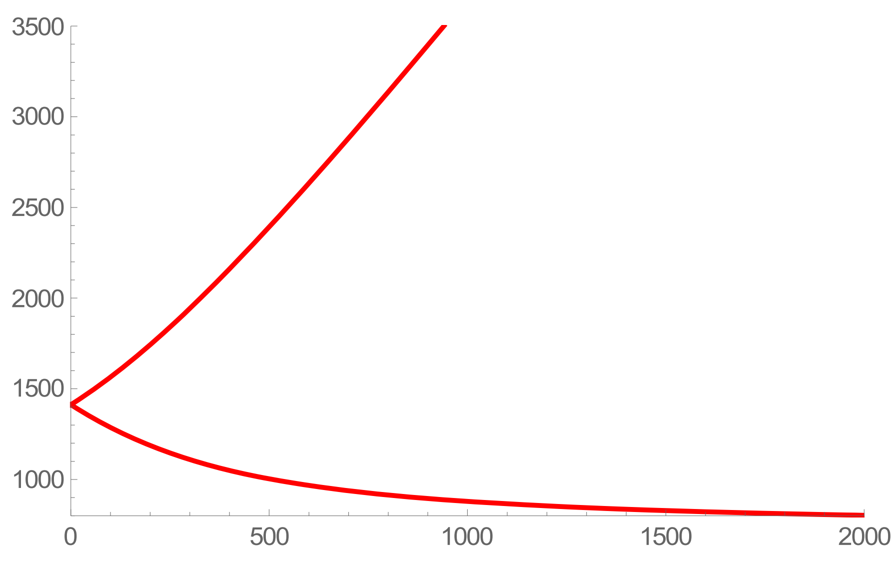
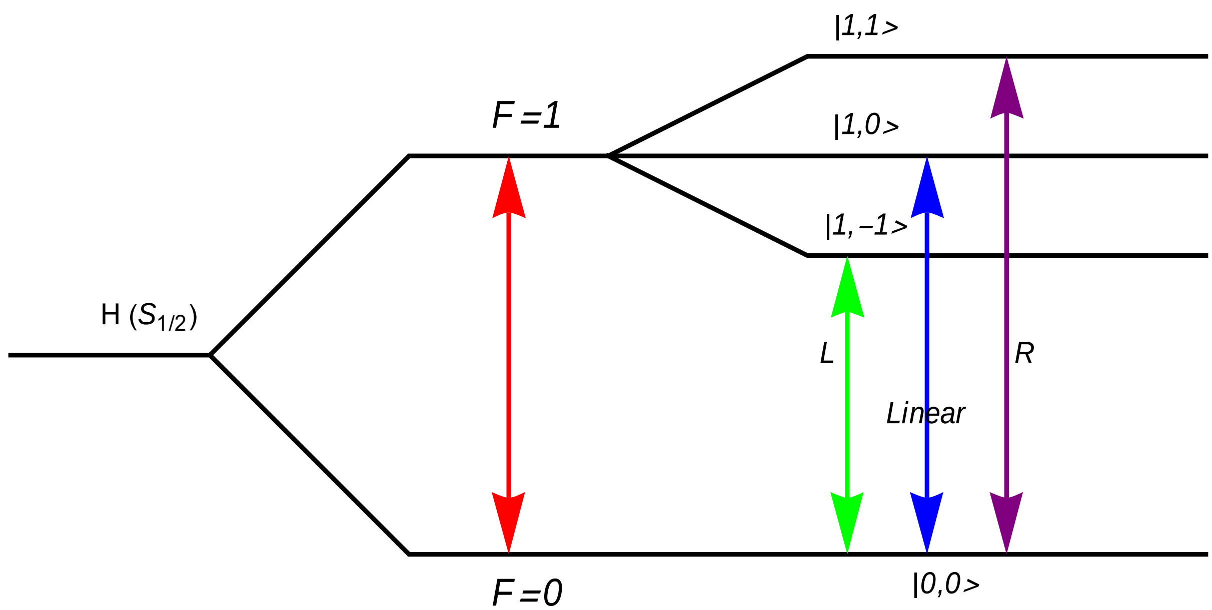


Fig. Plot of the frequencies for the transitions $|1{,}1\rangle \to |0{,}0\rangle$ and $|1,-1\rangle \to |0{,}0\rangle$. The *x*-axis: the magnetic field *B*(Oe). The *y* axis is the frequency (in units of MHz).



The enrgy level of the ground state 1 $S_{1/2}$ state in hydrogen (H) atom. The levels are depicted wuth the account for the proton-electron spin system (the total momentum F) and the Zeeman splitting. The energy levels resulting from the Zeeman effect are denoted by the states $|F=1, m_F=1,0,-1\rangle$ and $|F=0, m_F=0\rangle$. The photons with right hand circular polarization are generated during the transition from $|1{,}1\rangle$ to $|0{,}0\rangle$, with the angular momentum $\hbar$. The photons with left hand circular polarization are generated during the transition from $|1,-1\rangle$ to $|0{,}0\rangle$, with the angular momentum $-\hbar$. The photons with the ;linear polarization generated during the transition from $|1{,}0\rangle$ to $|0{,}0\rangle$, with the angular momentum 0.

20. The coefficient A for the spontaneous emission

The constant *A* for the spontaneous emission is given by

$$\frac{4\pi^2 e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3 c^3\hbar}d\Omega\sum_s\left|\langle f|[(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}}+i(\boldsymbol{k}\cdot\boldsymbol{r})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})]|i\rangle\right|^2$$

$$=\frac{4\pi^2 e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3 c^3\hbar}d\Omega$$

$$\times\sum_s\left|\langle f|(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}}+i\frac{(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})-(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{r}})(i\boldsymbol{k}\cdot\hat{\boldsymbol{p}})}{2}+i\frac{(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})+(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{r}})(i\boldsymbol{k}\cdot\hat{\boldsymbol{p}})}{2}|i\rangle\right|^2$$

(a) Electric dipole contribution

$$
\begin{aligned}
A_{ed} &= \frac{4\pi^2 e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3c^3\hbar}d\Omega\sum_s\left|\langle f\left|(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}}\right|i\rangle\right|^2 \\
&= \frac{4\pi^2 e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3c^3\hbar}\frac{8\pi}{3}\left|\langle f\left|\hat{\boldsymbol{p}}\right|i\rangle\right|^2 \\
&= \frac{4\pi^2 e^2}{Vm^2\omega_0}\frac{V\omega_0{}^2}{(2\pi)^3c^3\hbar}\frac{8\pi}{3}\left|\langle f\left|\hat{\boldsymbol{p}}\right|i\rangle\right|^2 \\
&= \frac{4e^2\omega_0}{3m^2c^3\hbar}\left|\langle f\left|\hat{\boldsymbol{p}}\right|i\rangle\right|^2 \\
&= \frac{4e^2\omega_0{}^3}{3\hbar c^3}\left|\langle f\left|\hat{\boldsymbol{r}}\right|i\rangle\right|^2 \\
&= \frac{4\omega_0{}^3}{3\hbar c^3}\left|\langle f\left|e\hat{\boldsymbol{r}}\right|i\rangle\right|^2
\end{aligned}
$$

(b) Magnetic dipole contribution

$$
A_{mag} = \frac{4\pi^2 e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3c^3\hbar}d\Omega\sum_s\left|\langle f\left|(i\frac{(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})-(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{r}})(i\boldsymbol{k}\cdot\hat{\boldsymbol{p}})}{2}\right|i\rangle\right|^2
$$

$$
\begin{aligned}
\langle \boldsymbol{r}\left|(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})-(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})\right|\psi\rangle &= \frac{\hbar}{i}(\boldsymbol{k}\cdot\boldsymbol{r})[\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\nabla\psi(\boldsymbol{r})]-\frac{\hbar}{i}(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{r})[\boldsymbol{k}\cdot\nabla\psi(\boldsymbol{r})] \\
&= \frac{\hbar}{i}\{(\boldsymbol{k}\cdot\boldsymbol{r})[\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\nabla\psi(\boldsymbol{r})]-[\boldsymbol{k}\cdot\nabla\psi(\boldsymbol{r})](\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{r})\} \\
&= \frac{\hbar}{i}(\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))[\boldsymbol{r}\times\nabla\psi(\boldsymbol{r})] \\
&= (\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))\langle\boldsymbol{r}\left|\hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}}\right|\psi\rangle \\
&= (\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))\langle\boldsymbol{r}\left|\hat{\boldsymbol{L}}\right|\psi\rangle
\end{aligned}
$$

The orbital magnetic moment is defined by

$$
\boldsymbol{\mu}_L = -\frac{\mu_B}{\hbar}\boldsymbol{L} = -\frac{e}{2mc}\boldsymbol{L}
$$

leading to

$$
\langle \boldsymbol{r}\left|(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})-(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})\right|\psi\rangle = (\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))(-\frac{2mc}{e})\langle\boldsymbol{r}\left|\hat{\boldsymbol{\mu}}\right|\psi\rangle
$$

or

$$(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{p}})-(\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})=-\frac{2mc}{e}(\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))\hat{\boldsymbol{\mu}}\ ,$$

Thus we have

$$\begin{aligned}
A_{mag} &= \frac{4\pi^2e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3c^3\hbar}d\Omega\sum_s\left|\langle f|-\frac{imc}{e}(\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))\hat{\boldsymbol{\mu}}|i\rangle\right|^2 \\
&= \frac{4\pi^2e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3c^3\hbar}\frac{m^2c^2}{e^2}\frac{\omega^2}{c^2}d\Omega\sum_s\left|\langle f|(\boldsymbol{e}_z\times\boldsymbol{\varepsilon}(\boldsymbol{k},s))\hat{\boldsymbol{\mu}}|i\rangle\right|^2 \\
&= \frac{4\pi^2e^2}{Vm^2\omega_0}\frac{V{\omega_0}^2}{(2\pi)^3c^3\hbar}\frac{m^2c^2}{e^2}\frac{{\omega_0}^3}{c^2}\frac{8\pi}{3}\left|\langle f|\hat{\boldsymbol{\mu}}|i\rangle\right|^2 \\
&= \frac{4{\omega_0}^3}{3\hbar c^3}\left|\langle f|\hat{\boldsymbol{\mu}}|i\rangle\right|^2
\end{aligned}$$

(c) Electric quadrupole contribution

$$A_{eq}=\frac{4\pi^2e^2}{Vm^2\omega_0}\int d\omega\delta(\omega_0-\omega)\frac{V\omega^2}{(2\pi)^3c^3\hbar}d\Omega\sum_s\left|\langle f|(i\frac{(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}})+(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(i\boldsymbol{k}\cdot\hat{\boldsymbol{p}})}{2}|i\rangle\right|^2$$

Note that

$$\begin{aligned}
\langle f|(\boldsymbol{k}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{p}})+(\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{r}})(\boldsymbol{k}\cdot\hat{\boldsymbol{p}})|i\rangle &= i\frac{m}{\hbar}\sum_{i,j}\varepsilon_ik_j\langle f|[\hat{H}_0,\hat{x}_i\hat{x}_i]|i\rangle \\
&= im\sum_{i,j}\varepsilon_ik_j\omega_{fi}\langle f|\hat{x}_i\hat{x}_j|i\rangle
\end{aligned}$$

where

$$[\hat{H}_0,\hat{x}_i\hat{x}_j]=\hat{x}_j[\hat{H}_0,\hat{x}_i]+[\hat{H}_0,\hat{x}_j]\hat{x}_i\ .$$

and

$$\langle f|[\hat{H}_0,\hat{x}_i\hat{x}_j]|i\rangle=(E_f-E_i)\langle f|\hat{x}_i\hat{x}_j|i\rangle=\hbar\omega_{fi}\langle f|\hat{x}_i\hat{x}_j|i\rangle\ ,$$

We note that the electric quadrupole moment is defined as

$$\hat{Q}_{ij}=q(\hat{x}_i\hat{x}_j-\frac{1}{3}|\hat{\boldsymbol{r}}|^2\delta_{i,j})\ , \qquad (1)$$

where q = -e (e>0) for the electron.

REFERENCES

J.J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, second edition (Addison-Wesley, New York, 2011).

E. Merzbacher, *Quantum Mechanics*, third edition (John Wiley & Sons, New York, 1998).

L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc, New York, 1955).

G. Baym, Lectures on Quantum Mechanics (Westview Press, 1990).

D. Bohm, Quantum Theory (Prentice Hall, New York, 1951).

H. Lüth, Quantum Physics in the Nanoworld (Springer, 2013).

D. Park, Introduction to the Quantum Theory, 2nd-edition (McGraw-Hill, New York, 1974).

E. Hecht, Optics, 4-th edition (Addison Wesley, 2002).

J.D.Jackson, Classical Electrodynamics (John Wiley & Sons, 1962)

A. Das and A.C. Melissinos, Quantum Mechanics A Modern Introduction (Gordon and Breach Science Publishers, New York, 1986).

A. Das, Lectures on Quantum Mechanics, 2nd edition (World Scientific Publishing Co.. Ltd, 2012).

H.A.Bethe and R. Jackiw, Intermediate Quantum Mechanics (Westview, 1997).

APPENDIX Black body problem

A.1 Maxwell's equation

We start with the Maxwell's equation (in cgs units)

$$\nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

We assume that

$$\boldsymbol{E} = \mathrm{Re}[\widetilde{\boldsymbol{E}}_0 e^{-i\omega t}]$$

$$\boldsymbol{B} = \mathrm{Re}[\widetilde{\boldsymbol{B}}_0 e^{-i\omega t)}]$$

Then we have

$$\nabla \cdot \widetilde{\boldsymbol{E}}_0 = 0$$

$$\nabla \cdot \widetilde{\boldsymbol{B}}_0 = 0$$

$$\nabla \times \widetilde{\boldsymbol{E}}_0 = i\frac{\omega}{c}\widetilde{\boldsymbol{B}}_0$$

$$\nabla \times \tilde{\boldsymbol{B}}_0 = -i\frac{\omega}{c}\tilde{\boldsymbol{E}}_0$$

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}$$

or

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}$$

or

$$\nabla^2 \tilde{\boldsymbol{E}}_0 + k^2 \tilde{\boldsymbol{E}}_0 = 0$$

with $\omega = ck$. Similarly, we have

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla^2 \tilde{\boldsymbol{B}}_0 + \frac{\omega}{c^2}^2 \tilde{\boldsymbol{B}}_0 = 0$$

or

$$\nabla^2 \tilde{\boldsymbol{B}}_0 + k^2 \tilde{\boldsymbol{B}}_0 = 0$$

We now consider an electromagnetic wave in the closed cube with side *L*.

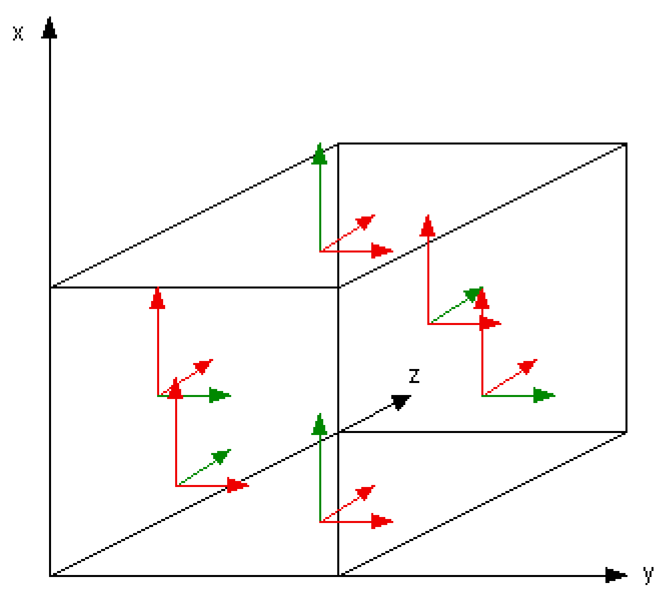


Fig. Boundary condition for the electric field (red) (tangential component continuous) and the magnetic field (green) (normal component continuous).

From the boundary conditions we have

$$E_x = E_1 \begin{pmatrix} \sin(k_1 x) \\ \cos(k_1 x) \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \end{pmatrix}$$

$$E_y = E_2 \begin{pmatrix} \sin(k_1 x) \\ \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \cos(k_2 y) \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \end{pmatrix}$$

$$E_z = E_3 \begin{pmatrix} \sin(k_1 x) \\ \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \cos(k_3 y) \end{pmatrix}$$

where

$$k_1 = \frac{\pi}{L} n_x , \qquad k_2 = \frac{\pi}{L} n_y , \qquad k_3 = \frac{\pi}{L} n_z$$

$(n_{\mathrm{x}}, n_{\mathrm{y}}, n_{\mathrm{z}} = 1, 2, 3, \ldots)$

$$\begin{pmatrix} \sin(k_1 x) \\ \cos(k_1 x) \end{pmatrix} \rightarrow A \sin(k_1 x) + B \cos(k_1 x) .$$

Note that

$E_{\mathrm{x}} = 0$	for $y = 0$ and $y = L$ planes and $z = 0$ and $z = L$ planes.
$E_{\mathrm{y}} = 0$	for $z = 0$ and $z = L$ planes and $x = 0$ and $x = L$ planes.
$E_{\mathrm{z}} = 0$	for $x = 0$ and $x = L$ planes and $y = 0$ and $y = L$ planes.

From the condition

$$\nabla \cdot \tilde{\boldsymbol{E}} = 0$$

we have

$$E_x = E_1 \cos(k_1 x) \sin(k_2 y) \sin(k_3 z) ,$$

$$E_y = E_2 \sin(k_1 x) \cos(k_2 y) \sin(k_3 z) ,$$

$$E_z = E_3 \sin(k_1 x) \sin(k_2 y) \cos(k_3 y)$$

From the condition

$$\nabla \times \tilde{\boldsymbol{E}}_0 = i\frac{\omega}{c}\tilde{\boldsymbol{B}}_0 ,$$

we have

$$B_x = B_1 \sin(k_1 x)\cos(k_2 y)\cos(k_3 z) ,$$

$$B_y = B_2 \cos(k_1 x)\sin(k_2 y)\cos(k_3 z) ,$$

$$B_z = B_3 \cos(k_1 x)\cos(k_2 y)\sin(k_3 z)$$

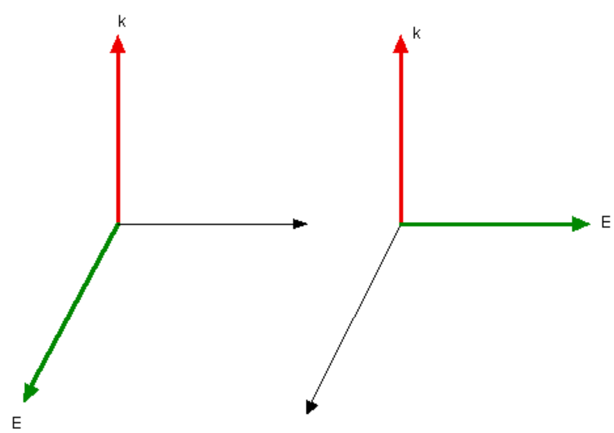
where

$B_x = 0$	for $x = 0$ and $x = L$ planes
$B_y = 0$	for $y = 0$ and $y = L$ planes.
$B_z = 0$	for $z = 0$ and $z = L$ planes.

We note that

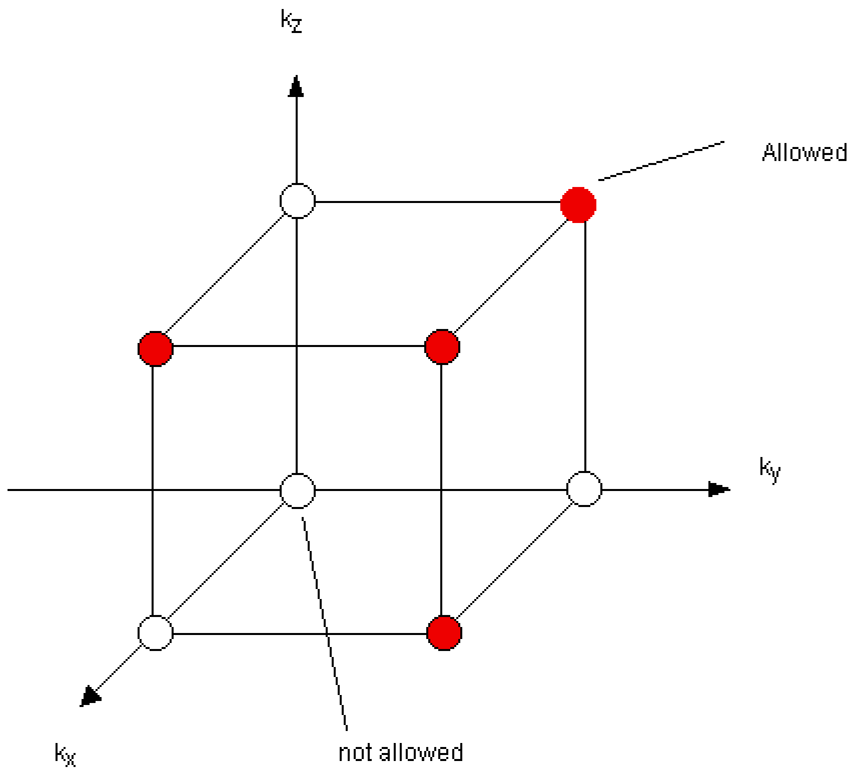
$$\nabla \cdot \boldsymbol{E} = (E_1 k_1 + E_2 k_2 + E_3 k_3)\sin(k_1 x)\sin(k_2 y)\sin(k_3 z) = 0$$

This means that the vector (E_1, E_2, E_3) is perpendicular to the wave vector $\boldsymbol{k} = (k_1, k_2, k_3)$.
For each $\boldsymbol{k}$, there are two independent directions for (E_1, E_2, E_3); polarization.



A.2. Density of states for the modes

Since $E_1 k_1 + E_2 k_2 + E_3 k_3 = 0$, only one of k_1, k_2, k_3 can be zero at a time. Since if two or three are zero, $E_1 = E_2 = E_3 = 0$. There is no electromagnetic field in the cavity. Each set of integers (n_x, n_y, n_z) defines a mode of the radiation field and corresponds to two degrees of freedom of the field when two polarization directions are taken into account.



There are 2 states per $\left(\frac{\pi}{L}\right)^3$.

$$\omega = ck = c\sqrt{k_x^{\ 2} + k_y^{\ 2} + k_z^{\ 2}}$$

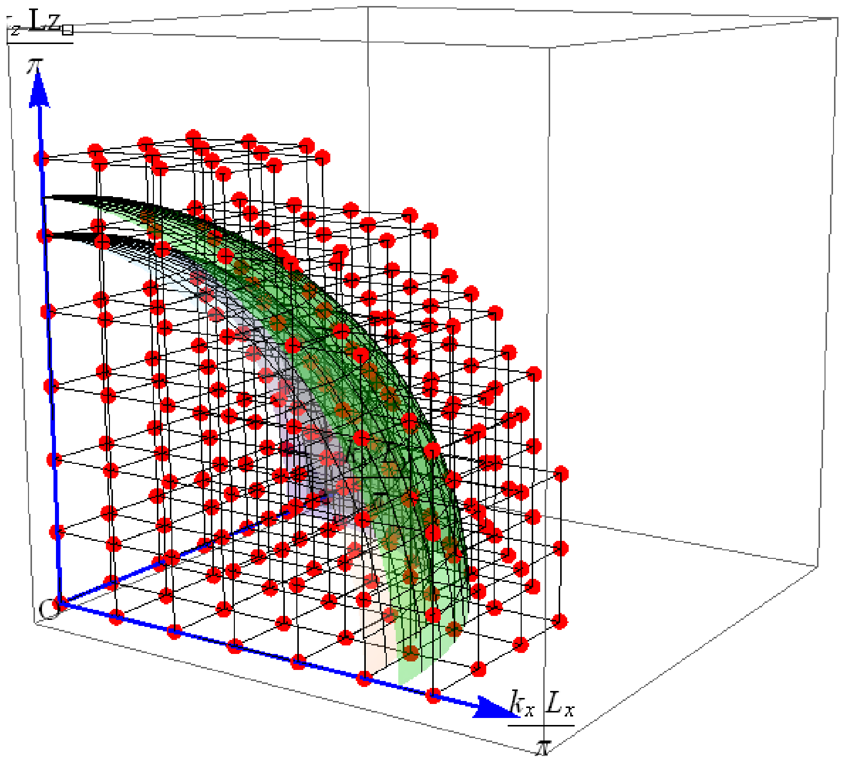
or

$$\frac{\omega^2}{c^2} = k_x^{\ 2} + k_y^{\ 2} + k_z^{\ 2}$$

The density of states (k to k +dk)

$$\rho_k dk = \frac{1}{8}\frac{4\pi k^2 dk}{\left(\frac{\pi}{L}\right)^3}2 = \frac{Vk^2 dk}{\pi^2}.$$

where $V = L^3$.



Since $\omega = ck$,

$$\rho_\omega d\omega = \frac{V\left(\frac{\omega}{c}\right)^2 d\frac{\omega}{c}}{\pi^2} = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

of modes having their frequencies between ω and ω+dω,

$$\rho_\omega = \frac{V\omega^2}{\pi^2 c^3} = VD(\omega) \quad \text{(density of modes)}$$

where c is the velocity of light and

$$D(\omega) = \frac{\omega^2}{\pi^2 c^3} .$$

We have the following formula;

$$\sum_k \rightarrow \int \rho_k dk ,$$

or

$$\sum_k \rightarrow \int \rho_\omega d\omega = V\int D(\omega)d\omega .$$

For single mode $|\boldsymbol{k}\rangle$, the energy is given by

$$E_{n.\boldsymbol{k}} = (n_k + \frac{1}{2})\hbar\omega_{\boldsymbol{k}} .$$

We use the Planck distribution. The total energy is given by

$$E_{tot} = \sum_{\boldsymbol{k}} n_{\boldsymbol{k}}\hbar\omega_{\boldsymbol{k}} = \int \frac{V\omega^2}{\pi^2 c^3} d\omega n_\omega \hbar\omega = V\int u(\omega)d\omega ,$$

or the energy density by

$$\frac{E_{tot}}{V} = \int_0^\infty u(\omega)d\omega = \int_0^\infty u(\lambda)d\lambda ,$$

where

$$u(\omega) = \overline{W_T}(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_B T})-1} = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(x)-1} = \frac{k_B{}^3 T^3}{\pi^2\hbar^2 c^3}\frac{x^3}{\exp(x)-1} .$$

(Planck's law for the radiation energy density). It is clear that

$$\frac{u(\omega)}{\frac{k_B{}^3 T^3}{\pi^2\hbar^2 c^3}} = f(x) = \frac{x^3}{\exp(x)-1} ,$$

is dependent on a variable *x* given by

$$x = \frac{\hbar\omega}{k_B T} .$$

(the scaling relation). The experimentally observed spectral distribution of the black body radiation is very well fitted by the formula discovered by Planck.

(1) Region of Wien ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_W(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^3 e^{-x}.$$

(2) Region of Rayleigh-Jeans ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_{RJ}(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1} \approx \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^2.$$

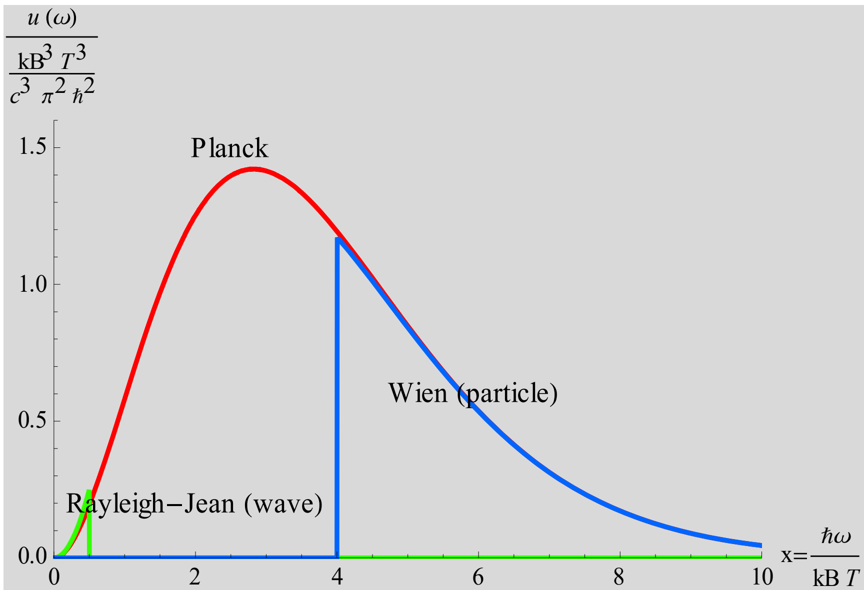


Fig. Scaling plot of $f(x)$ vs x for the Planck's law for the energy density of electromagnetic radiation at angular frequency ω and temperature T. Planck (red). Wien (blue, particle-like). Rayleigh-Jean (green, wave-like).

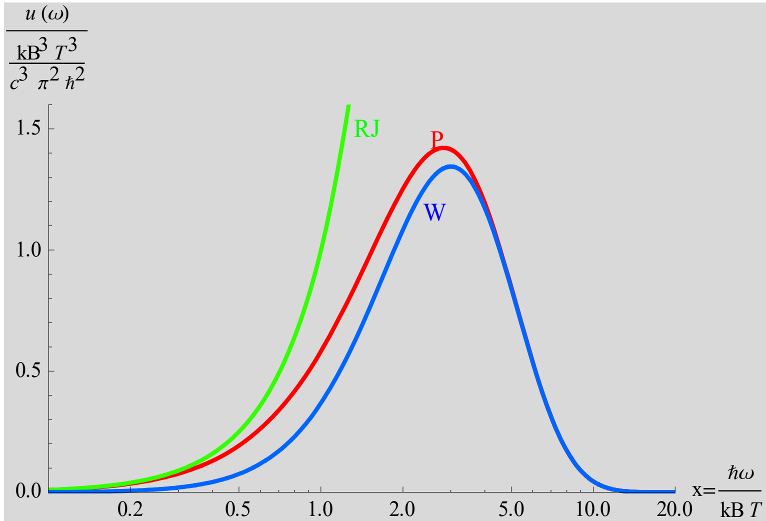


Fig. Scaling plot of Planck's law. Wien's law, and Rayleigh-Jean's law.

A.3 Deivation of $u(\lambda, T)$

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_B T})-1}d\omega$$

Since $\omega = \frac{2\pi c}{\lambda}$, $\qquad d\omega = -2\pi c\frac{d\lambda}{\lambda^2}$

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\frac{\hbar\omega}{k_B T})-1}d\omega$$

$$= \int_0^\infty \frac{\hbar\left(\frac{2\pi c}{\lambda}\right)^3}{\pi^2 c^3}\frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T})-1}2\pi c\frac{d\lambda}{\lambda^2}$$

or

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty u(\lambda)d\lambda = \int_0^\infty 16\pi^2\hbar c\frac{1}{\lambda^5}\frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T})-1}d\lambda$$

Then we have

$$u(\lambda) = \frac{16\pi^2\hbar c}{\lambda^5} \frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T}) - 1}$$

where

$\hbar$ = 1.054571596 x 10^{-27} erg s, k_B = 1.380650324 x 10^{-16} erg/K
c = 2.99792458 x 10^{10} cm/s.
J = 10^7 erg

A.4 Wien's displacement law

$u(\lambda)$ has a maximum at

$$\frac{2\pi\hbar c}{\lambda k_B T} = 4.96511, \qquad \text{(dimensionless)}$$

or

$$\lambda = \frac{0.28977}{T(K)} \qquad (\lambda \text{ in the units of cm})$$

or

$$\lambda = \frac{2.897768551}{T(K)} \times 10^6. \qquad (\lambda \text{ in the units of nm})$$

T is the temperature in the units of K. λ is the wave-length in the unit of nm

T(K)	λ (nm)

1000	2897.77
1500	1931.85
2000	1448.89
2500	1159.11
3000	965.924
3500	827.935
4000	724.443
4500	643.949
5000	579.554
5500	526.867
6000	482.962
6500	445.811
7000	413.967
7500	386.369
8000	362.221
8500	340.914
9000	321.975
9500	305.029
10 000	289.777

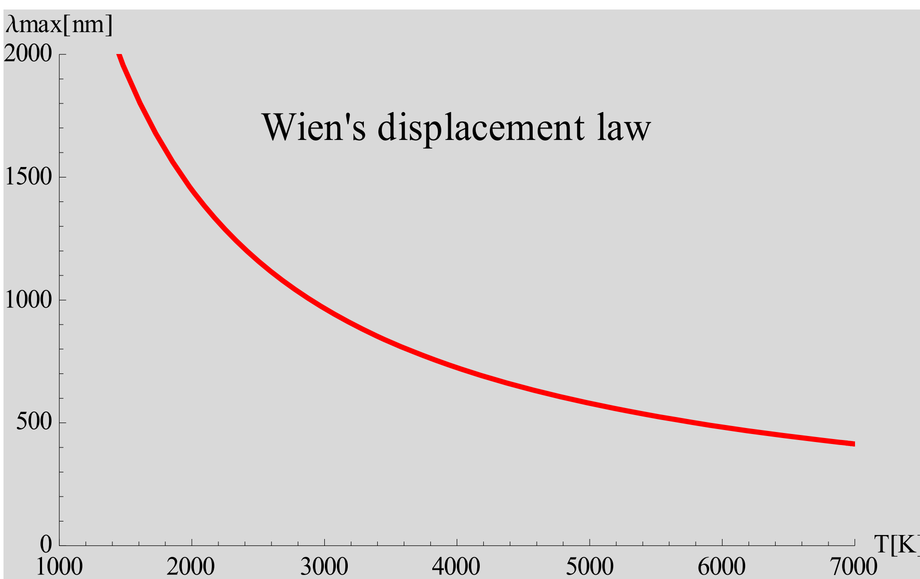


Fig. Wien's displacement law. The peak wavelength vs temperature T(K).

A.5 Rate of the energy flux density

It is assumed that the thermal equilibrium of the electromagnetic waves is not disturbed even when a small hole is bored through the wall of the box. The area of the hole is dS. The energy which passes in unit time through a solid angle dΩ, making an angle θ with the normal to dS is

$$J(\lambda, T, \theta) d\lambda d\Omega dS = cu(\lambda, T) d\lambda \cos\theta \frac{d\Omega}{4\pi} dS,$$

where c is the velocity of light. The right hand side is divided by 4π, because the energy density u comprises all waves propagating along different directions. The emitted energy unit time, per unit area is

$$\begin{aligned}
\iint J(\lambda, T, \theta) d\lambda d\Omega = \int cu(\lambda, T) d\lambda \int \cos\theta \frac{d\Omega}{4\pi} = \int \frac{cu(\lambda, T)}{4} d\lambda \\
\equiv \frac{c}{4} \int u(\lambda, T) d\lambda \\
= \frac{c}{4} \varepsilon
\end{aligned}$$

where

$$\varepsilon = \int u(\lambda, T) d\lambda,$$

$$\begin{aligned}
\int \cos\theta \frac{d\Omega}{4\pi} &= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta \\
&= \frac{1}{4\pi} 2\pi \int_0^{\pi/2} \frac{1}{2} \sin(2\theta) d\theta \\
&= \frac{1}{4} \frac{1}{2} [-\cos(2\theta)] |_0^{\pi/2} = \frac{1}{4}
\end{aligned}$$

(only for the half upper plane).

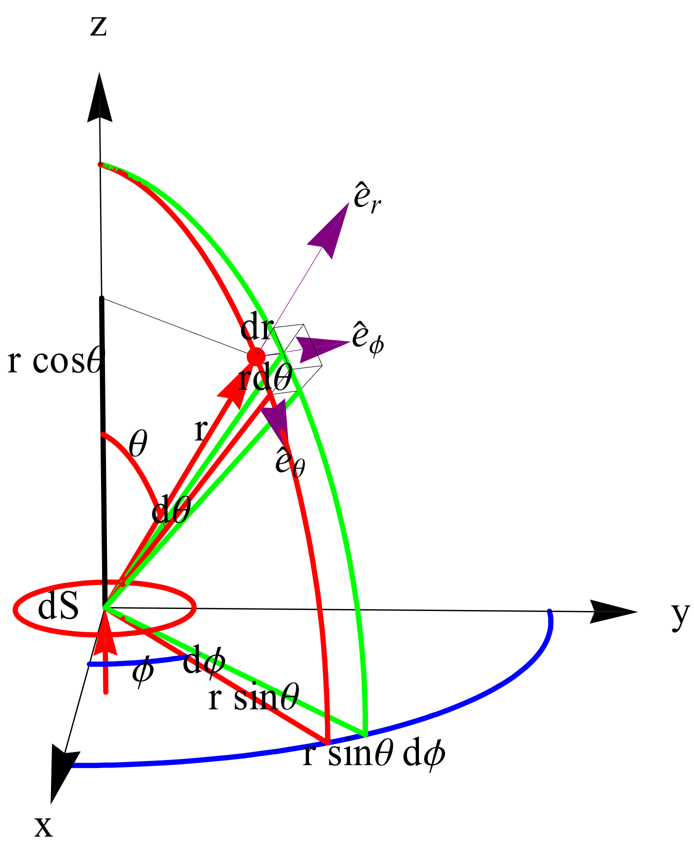


Fig. Radiation intensity is used to describe the variation of radiation energy with direction.

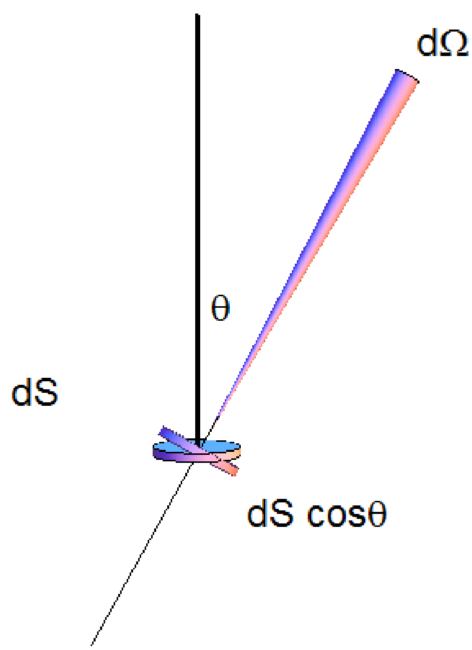


Fig. Geometrical factor. The photons pass from the lower half plane to the upper half plane in a straight way through a pin hole with the effective area ($dS\cos\theta$). Since the area dS is small enough, the form of the wave changes from plane wave to spherical wave. The spherical wave propagates in all directions (the total solid angle 4π) after passing through the pin hole. The fraction of the photons propagating over the solid angle ($d\Omega$) is $d\Omega/4\pi$

In other words, the geometrical factor is equal to 1/4. Then we have a measure for the intensity of radiation (the rate of energy flux density);

$$S(\lambda,T)=\frac{cu(\lambda,T)}{4}=\frac{4\pi^2\hbar c^2}{\lambda^5}\frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T})-1}$$

where

$S(\lambda,T)d\lambda$ = power radiated per unit area in $(\lambda,\lambda+d\lambda)$

Unit

$$[\hbar c^2\frac{1}{\lambda^5}]=\frac{erg.s}{cm^5}\frac{cm^2}{s^2}=\frac{erg}{cm^3}\frac{1}{s}=\frac{10^{-7}J}{(10^{-2}m)^3}\frac{1}{s}=10^{-1}\frac{W}{m^3}=[\frac{W}{m^3}]$$

The energy flux density $S(\lambda,T)$ is defined as the rate of energy emission per unit area.

((Note)) The unit of the poynting vector $<S>$ is [W/m^2]. $\langle S\rangle$ is the energy flux (energy per unit area per unit time).

(1) Rayleigh-Jeans law (in the long-wavelength limit)

$$S_{RJ}(\lambda)=\frac{1}{4}cu_{RJ}(\lambda)=4\pi^2\hbar c\frac{1}{\lambda^5}\frac{1}{\frac{2\pi\hbar c}{\lambda k_B T}}=\frac{2\pi c k_B T}{\lambda^4}$$

for

$$\frac{\lambda k_B T}{2\pi\hbar c}>>1$$

(2) Wien's law (in short-wavelength limit)

$$S_W(\lambda) = \frac{1}{4}cu_W(\lambda) = \frac{\pi^2\hbar c}{\lambda^5}\exp(-\frac{2\pi\hbar c}{\lambda k_B T})$$

$$\frac{\lambda k_B T}{2\pi\hbar c} << 1$$

We make a plot of $S(\lambda,T)$ as a function of the wavelength, where $S(\lambda,T)$ is in the units of W/m^3 and the wavelength is in the units of nm.

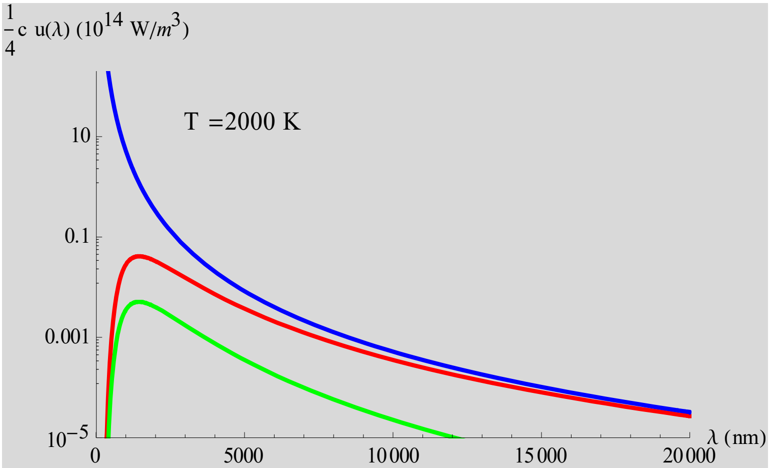
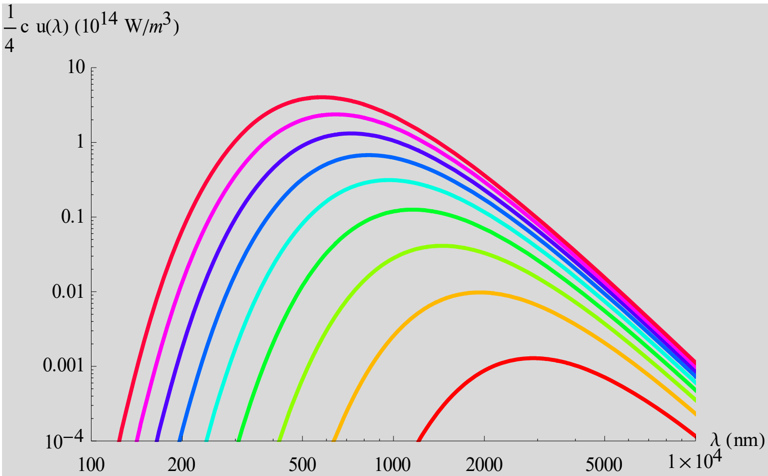


Fig. $cu(\lambda)/4$ (W/m^3) vs λ (nm). T = 2 x 10^3 K. Red [Planck]. Green [Wien]. Blue [Rayleigh-Jean]. Wien's displacement law: The peak appears at λ = 1448.89 nm for T = 2 x 10^3 K. This figure shows the misfit of Wien's law at long wavelength and the failure of the Rayleigh-Jean's law at short wavelangth.



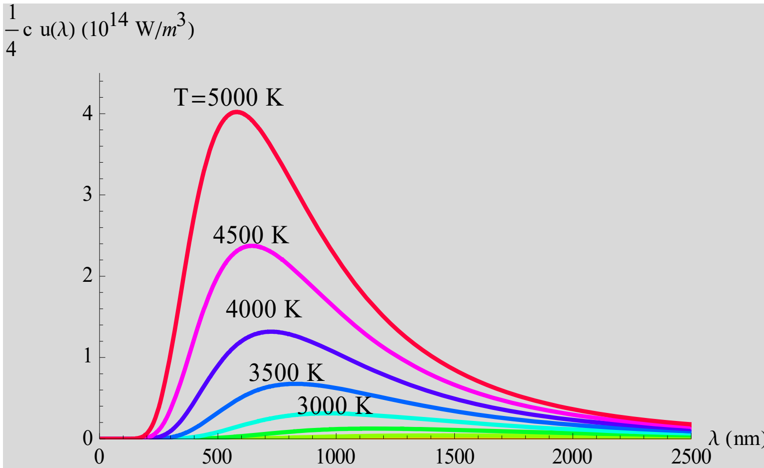


Fig. (a) and (b) $cu(\lambda)/4$ (W/m^3) vs λ (nm) for the Plank's law. T = 1000 K (red), 1500 K, 2000 K, 2500 K, 3000 K (blue), 3500 K, 4000 K (purple), 4500 K, and 5000 K. The peak shifts to the higher wavelength side as T decreases according to the Wien's displacement law.

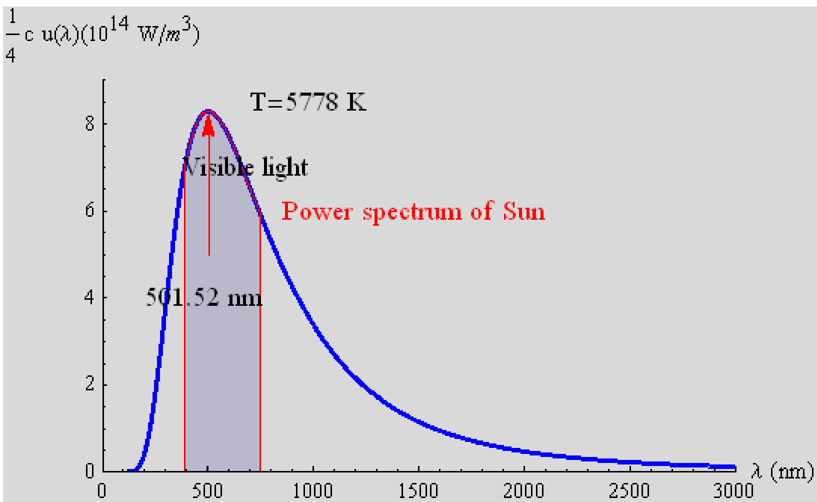


Fig. Power spectrum of sun. $cu(\lambda)/4$ (W/m^3) vs λ (nm). T = 5778 K. The peak wavelength is 501.52 nm according to the Wien's displacement law.

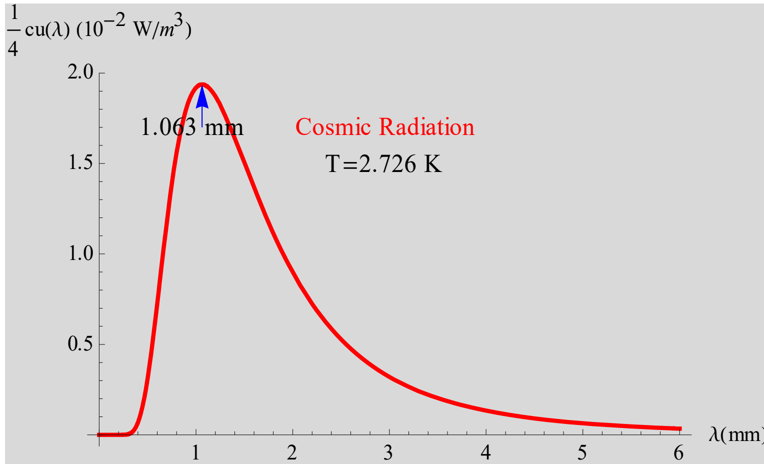


Fig. Power spectrum of cosmic blackbody radiation at T = 2.726 K. The peak wavelength is 1.063 mm (Wien's displacement law.

A.6. Stefan-Boltzmann radiation law for a black body (1879).

Joseph Stefan (24 March 1835 – 7 January 1893) was a physicist, mathematician and poet of Slovene mother tongue and Austrian citizenship.

http://en.wikipedia.org/wiki/Joseph_Stefan

Ludwig Eduard Boltzmann (February 20, 1844 – September 5, 1906) was an Austrian physicist famous for his founding contributions in the fields of statistical mechanics and statistical thermodynamics. He was one of the most important advocates for atomic theory at a time when that scientific model was still highly controversial.

http://en.wikipedia.org/wiki/Ludwig_Boltzmann

The total energy per unit volume is given by

$$\varepsilon = \frac{E_{tot}}{V} = \int u(\omega)d\omega = \int u(\lambda)d\lambda = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3}\int_0^\infty \frac{x^3}{\exp(x)-1} = \frac{\pi^2 (k_B T)^4}{15\hbar^3 c^3}$$

((**Mathematica**))

$$\int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$\frac{\pi^4}{15}$$

A spherical enclosure is in equilibrium at the temperature T with a radiation field that it contains. The power emitted through a hole of unit area in the wall of enclosure is

$$P = \frac{1}{4}c\varepsilon = \frac{\pi^2 k_B{}^4}{60\hbar^3 c^2}T^4 = \sigma T^4$$

where σ is the Stefan-Boltzmann constant

$$\sigma = \frac{\pi^2 k_B{}^4}{60\hbar^3 c^2} = 0.5670400\times 10^{-4}$$ erg/s-cm^2-K^4 = 5.670400 x 10^{-8} W m^{-2} K^{-4}

and the geometrical factor is equal to 1/4. The application of the Stefan-Boltzmann law is discussed in lecture notes of Phys.131 (Chapter 18) (see URL at

http://bingweb.binghamton.edu/~suzuki/GeneralPhysLN.html

A.7 Duality of wave and particle

Region of Rayleigh-Jeans: wave-like nature
Region of Wien: particle-like nature

The mean energy contained in a volume ΔV in the frequency range between ω and $\omega+\Delta\omega$, is given by

$$\overline{E}(\omega) = \langle E(\omega)\rangle = \Delta V \overline{W}_T(\omega)\Delta\omega = \Delta V D(\omega)\hbar\omega\Delta\omega\overline{n} = \Delta V D(\omega)\hbar\omega\Delta\omega\frac{1}{e^{\beta\hbar\omega}-1}$$

where

$$\overline{n} = \frac{1}{e^{\beta\hbar\omega}-1},$$

and

$$D(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

The mean-square of the fluctuation in energy is obtained as

$$\langle[\Delta E(\omega)]^2\rangle = \langle[E(\omega)]^2\rangle - \langle E(\omega)\rangle^2 = k_B T^2 \frac{\partial}{\partial T}\langle E(\omega)\rangle$$

from the general theory of thermodynamics,

or

$$\langle[\Delta E(\omega)]^2\rangle = \Delta V\Delta\omega D(\omega)\hbar\omega k_B T^2 \frac{\partial}{\partial T}\frac{1}{e^{\beta\hbar\omega}-1} = \Delta V\Delta\omega D(\omega)\hbar^2\omega^2[\frac{1}{e^{\beta\hbar\omega}-1} + \frac{1}{\left(e^{\beta\hbar\omega}-1\right)^2}]$$

or

$$\langle[\Delta E(\omega)]^2\rangle = \Delta V\Delta\omega D(\omega)\hbar^2\omega^2[\overline{n}+\overline{n}^2] = \Delta V\Delta\omega D(\omega)\hbar^2\omega^2(\Delta n)^2$$

where

$$\overline{n}+\overline{n}^2 = (\Delta n)^2$$

(See the Appendix for the detail). Note that

$$(\Delta n)^2 = <(n-\overline{n})^2> = <n^2> - \overline{n}^2 \qquad \text{(from the definition).}$$

(i) Rayleigh-Jean (wave-like)

For $\frac{\hbar\omega}{k_B T} = \beta\hbar\omega << 1$, $\qquad \overline{n}^2 >> \overline{n}$

$$(\Delta n)^2 \approx \overline{n}^2, \quad \text{or} \quad (\Delta n) \approx \overline{n} \qquad \text{(wave-like, Rayleigh-Jeans)}$$

$$\left\langle [\Delta E(\omega)]^2 \right\rangle \approx \Delta V \Delta\omega D(\omega)\hbar^2\omega^2\overline{n}^2$$

Then we have

$$\frac{\left\langle [\Delta E(\omega)]^2 \right\rangle}{\left\langle E(\omega) \right\rangle^2} = \frac{\Delta V \Delta\omega D(\omega)\hbar^2\omega^2\overline{n}^2}{(\Delta V \Delta\omega D(\omega)\hbar\omega\overline{n})^2} = \frac{1}{\Delta V \Delta\omega D(\omega)} = \frac{1}{\Delta V \Delta\omega}\frac{c^3}{\omega^2}$$

(ii) Wien (particle-like)

For $\frac{\hbar\omega}{k_B T} = \beta\hbar\omega >> 1$, $\qquad \overline{n}^2 < \overline{n}$

$$(\Delta n)^2 \approx \overline{n} \qquad \text{(particle-like, corpuscle, Wien)}$$

$$\left\langle [\Delta E(\omega)]^2 \right\rangle \approx \Delta V \Delta\omega D(\omega)\hbar^2\omega^2\overline{n} = \hbar\omega(\Delta V \Delta\omega D(\omega)\hbar\omega\overline{n}) = \hbar\omega\left\langle E(\omega) \right\rangle$$

or

$$\frac{\left\langle [\Delta E(\omega)]^2 \right\rangle}{\left\langle E(\omega) \right\rangle} = \hbar\omega$$

(iii) Planck

$$\left\langle [\Delta E(\omega)]^2 \right\rangle = \hbar\omega\left\langle E(\omega) \right\rangle + \frac{1}{\Delta V \Delta\omega}\frac{c^3}{\omega^2}\left\langle E(\omega) \right\rangle^2$$
