

**Matrix of angular momentum and rotation operator with  $j=1$**   
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**(Date: October 24, 2014)**

**Eigenvalue problem for  $S = 1$  (or  $J = 1$ )**  
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**(Date: October 13, 2014)**

We determine the eigenstates of  $\hat{S}_x$  and  $\hat{S}_y$  for a spin-1 particle in terms of the eigenstates  $|j=1, m\rangle$  ( $m = 1, 0, -1$ ) of  $\hat{S}_z$

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

**1. Eigenvalue and eigenkets of  $\hat{S}_x$**

$$\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$|1,1\rangle_x = \hat{U}_x |1,1\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix},$$

$$|1,0\rangle_x = \hat{U}_x |1,0\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix},$$

$$|1,-1\rangle_x = \hat{U}_x |1,-1\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

where  $\hat{U}_x$  is a unitary operator.

Eigenvalue problem

$$\hat{S}_x|\psi\rangle = \lambda\hbar|\psi\rangle$$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix} = \lambda \begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix}$$

or

$$\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For nontrivial solution, the determinant should be zero,

$$\begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = 0$$

or

$$\lambda(\lambda - 1)(\lambda + 1) = 0$$

Note that

$$\hat{S}_x|1,1\rangle_x = \hbar|1,1\rangle_x \quad (\lambda = 1)$$

$$\hat{J}_x|1,0\rangle_x = 0|1,0\rangle_x \quad (\lambda = 0)$$

$$\hat{J}_x|1,-1\rangle_x = -\hbar|1,-1\rangle_x \quad (\lambda = -1)$$

(a)  $\hat{S}_x|1,1\rangle_x = \hbar|1,1\rangle_x$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{11} = \frac{1}{\sqrt{2}}U_{21}, \quad U_{31} = \frac{1}{\sqrt{2}}U_{21}$$

with the normalization condition

$$|U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 = 1$$

So we get  $|U_{21}| = \frac{1}{\sqrt{2}}$ . Here we choose  $U_{21} = \frac{1}{\sqrt{2}}$

$$U_{11} = \frac{1}{2}, \quad U_{31} = \frac{1}{2}$$

Finally we obtain the eigenket  $|1,1\rangle_x$ ,

$$|1,1\rangle_x = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2} (|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle)$$

(b)  $\hat{S}_x|1,0\rangle_x = 0|1,0\rangle_x = 0$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Then we have

$$U_{22} = 0, \quad U_{12} + U_{32} = 0$$

with the normalization condition

$$|U_{12}|^2 + |U_{22}|^2 + |U_{32}|^2 = 1$$

So we have  $U_{12} = \frac{1}{\sqrt{2}}, \quad U_{32} = -\frac{1}{\sqrt{2}}$

In summary we get

$$|1,0\rangle_x = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} = \frac{1}{2} (\sqrt{2}|1,1\rangle - \sqrt{2}|1,-1\rangle)$$

(c)  $\hat{S}_x |1,-1\rangle_x = -\hbar |1,-1\rangle_x$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = - \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{33} + \frac{1}{\sqrt{2}}U_{23} = 0, \quad U_{13} + \frac{1}{\sqrt{2}}U_{23} = 0$$

with the normalization condition

$$|U_{13}|^2 + |U_{23}|^2 + |U_{33}|^2 = 1$$

So we get  $|U_{23}| = \frac{1}{\sqrt{2}}$ . Here we choose  $U_{23} = -\frac{1}{\sqrt{2}}$

$$U_{13} = \frac{1}{2}, \quad U_{33} = \frac{1}{2}$$

$$|1,-1\rangle_x = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2} (|1,1\rangle - \sqrt{2}|1,0\rangle + |1,-1\rangle)$$

The unitary operator is obtained as

$$\hat{U}_x = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\hat{U}_x^+ = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\hat{U}_x^+ \hat{J}_x \hat{U}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We now calculate the rotation operator  $\hat{R}_y(\alpha) = \exp(-\frac{i}{\hbar} \hat{J}_x \alpha)$

$$\begin{aligned} \exp(-\frac{i}{\hbar} \hat{J}_x \alpha) &= \hat{U}_x \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \hat{U}_x^+ \\ &= \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \frac{\alpha}{2} & -\frac{i}{\sqrt{2}} \sin \alpha & -\sin^2 \frac{\alpha}{2} \\ -\frac{i}{\sqrt{2}} \sin \alpha & \cos \alpha & -\frac{i}{\sqrt{2}} \sin \alpha \\ -\sin^2 \frac{\alpha}{2} & -\frac{i}{\sqrt{2}} \sin \alpha & \cos^2 \frac{\alpha}{2} \end{pmatrix} \end{aligned}$$

## 2. Eigenvalues and eigenkets of $\hat{S}_y$

$$\hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$|1,1\rangle_y = \hat{U}_y |1,1\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix},$$

$$|1,0\rangle_y = \hat{U}_y |1,0\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix},$$

$$|1,-1\rangle_y = \hat{U}_y |1,-1\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

where  $\hat{U}_y$  is a unitary operator. Note that

$$\hat{S}_y |1,1\rangle_y = \hbar |1,1\rangle_y,$$

$$\hat{S}_y |1,0\rangle_y = 0 |1,0\rangle_y,$$

$$\hat{S}_y |1,-1\rangle_y = -\hbar |1,-1\rangle_y,$$

(a)  $\hat{S}_y |1,1\rangle_y = \hbar |1,1\rangle_y$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}, \text{ or } \begin{pmatrix} -1 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -1 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -1 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{11} = -\frac{i}{\sqrt{2}} U_{21}, \quad U_{31} = \frac{i}{\sqrt{2}} U_{21}$$

with the normalization condition

$$|U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 = 1$$

So we get  $|U_{21}| = \frac{1}{\sqrt{2}}$ . Here we choose  $U_{21} = \frac{i}{\sqrt{2}}$

$$U_{11} = \frac{1}{2}, \quad U_{31} = -\frac{1}{2}$$

Finally we obtain the eigenket  $|1,1\rangle_y$ ,

$$|1,1\rangle_y = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix} = \frac{1}{2} (|1,1\rangle + i\sqrt{2}|1,0\rangle - |1,-1\rangle)$$

(b)  $\hat{S}_y|1,0\rangle_y = 0|1,0\rangle_y = 0$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Then we have

$$U_{22} = 0, \quad U_{12} = U_{32}$$

with the normalization condition

$$|U_{12}|^2 + |U_{22}|^2 + |U_{32}|^2 = 1$$

So we have  $U_{12} = \frac{1}{\sqrt{2}}$ ,  $U_{32} = \frac{1}{\sqrt{2}}$

In summary we get

$$|1,0\rangle_y = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} = \frac{1}{2} (\sqrt{2}|1,1\rangle + \sqrt{2}|1,-1\rangle)$$

(c)  $S_y|1,-1\rangle_y = -\hbar|1,-1\rangle_y$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = - \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 1 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{13} - \frac{i}{\sqrt{2}}U_{23} = 0, \quad U_{33} + \frac{i}{\sqrt{2}}U_{23} = 0$$

with the normalization condition

$$|U_{13}|^2 + |U_{23}|^2 + |U_{33}|^2 = 1$$

So we get  $|U_{23}| = \frac{1}{\sqrt{2}}$ . Here we choose  $U_{23} = -\frac{i}{\sqrt{2}}$

$$U_{13} = \frac{1}{2}, \quad U_{33} = -\frac{1}{2}$$

$$|1,-1\rangle_y = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix} = \frac{1}{2} (|1,1\rangle - i\sqrt{2}|1,0\rangle - |1,-1\rangle)$$

The unitary operator is obtained as

$$\hat{U}_y = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}$$



$$\hat{U}_y^+ = \frac{1}{2} \begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix}$$

$$\hat{U}_y^+ \hat{J}_y \hat{U}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

### 3. Rotation operator $\hat{R}_y(\theta)$

Here we discuss the matrix of the rotation operator with  $j = 1$ . The rotation operator is defined by

$$\hat{R}_y(\theta) = \exp\left(-\frac{i}{\hbar} \hat{J}_y \theta\right)$$

$$\begin{aligned} \hat{R}_y(\theta) &= \exp\left(-\frac{i}{\hbar} \hat{J}_y \theta\right) (|1,1; y\rangle\langle 1,1; y| + |1,0; y\rangle\langle 1,0; y| + |1,-1; y\rangle\langle 1,-1; y|) \\ &= \exp\left(-\frac{i}{\hbar} \hat{J}_y \theta\right) |1,1; y\rangle\langle 1,1; y| + \exp\left(-\frac{i}{\hbar} \hat{J}_y \theta\right) |1,0; y\rangle\langle 1,0; y| \\ &\quad + \exp\left(-\frac{i}{\hbar} \hat{J}_y \theta\right) |1,-1; y\rangle\langle 1,-1; y| \\ &= e^{-i\theta} |1,1; y\rangle\langle 1,1; y| + e^0 |1,0; y\rangle\langle 1,0; y| + e^{i\theta} |1,-1; y\rangle\langle 1,-1; y| \\ &= \hat{U}_y (e^{-i\theta} |1,1; z\rangle\langle 1,1; z| + e^0 |1,0; z\rangle\langle 1,0; z| + e^{i\theta} |1,-1; z\rangle\langle 1,-1; z|) \hat{U}_y^+ \\ &= \hat{U}_y \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \hat{U}_y^+ \end{aligned}$$

or

$$\begin{aligned}
\hat{R}_y(\theta) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}
\end{aligned}$$

#### 4. The matrix of $\hat{R}_z(\phi)$

$$\begin{aligned}
\hat{R}_z(\phi) &= \exp\left(-\frac{i}{\hbar} \hat{J}_z \phi\right) (|1,1; z\rangle\langle 1,1; z| + |1,0; z\rangle\langle 1,0; z| + |1,-1; z\rangle\langle 1,-1; z|) \\
&= e^{-i\phi} |1,1; z\rangle\langle 1,1; z| + e^0 |1,0; z\rangle\langle 1,0; z| + e^{i\phi} |1,-1; z\rangle\langle 1,-1; z| \\
&= \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}
\end{aligned}$$

#### 5. The matrix of $\hat{R} = \hat{R}_z(\phi)\hat{R}_y(\theta)$

$$\begin{aligned}
R(\theta, \phi) &= \hat{R}_z(\phi)\hat{R}_y(\theta) \\
&= \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\phi} \left(\frac{1+\cos\theta}{2}\right) & -e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} & e^{-i\phi} \left(\frac{1-\cos\theta}{2}\right) \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ e^{i\phi} \left(\frac{1-\cos\theta}{2}\right) & e^{i\phi} \frac{\sin\theta}{\sqrt{2}} & e^{i\phi} \left(\frac{1+\cos\theta}{2}\right) \end{pmatrix}
\end{aligned}$$

**((Mathematica))**

## Matrices $j = 1$

```
Clear["Global`*"]; j = 1;
exp_* := exp /. {Complex[re_, im_] :=> Complex[re, -im]};
Jx[j_, n_, m_] :=  $\frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jy[j_, n_, m_] :=  $-\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jz[j_, n_, m_] :=  $\hbar m$  KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

**Jx // MatrixForm**

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

**Jy // MatrixForm**

$$\begin{pmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

**Jz // MatrixForm**

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

**eq1 = Eigensystem[Jx]**

$$\{\{-\hbar, \hbar, 0\}, \{\{1, -\sqrt{2}, 1\}, \{1, \sqrt{2}, 1\}, \{-1, 0, 1\}\}\}$$

$\psi_{1x} = \text{Normalize}[\text{eq1}[[2, 2]]]; \psi_{1x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$\psi_{2x} = -\text{Normalize}[\text{eq1}[[2, 3]]]; \psi_{2x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\psi_{3x} = \text{Normalize}[\text{eq1}[[2, 1]]]; \psi_{3x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$U_{xT} = \{\psi_{1x}, \psi_{2x}, \psi_{3x}\}; U_x = \text{Transpose}[U_{xT}]; U_x // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

**UxH = UxT\* ; UxH // MatrixForm**

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

**UxH.Ux**

$$\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}\}$$

**eq2 = Eigensystem[Jy]**

$$\{\{-\hbar, \hbar, 0\}, \{\{-1, i\sqrt{2}, 1\}, \{-1, -i\sqrt{2}, 1\}, \{1, 0, 1\}\}\}$$

**$\psi_{1y} = -\text{Normalize}[eq2[[2, 2]]]$ ;  $\psi_{1y}$  // MatrixForm**

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

**UyH.Uy // Simplify**

$\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}\}$

$$\mathbf{Ry} = \mathbf{Uy} \cdot \begin{pmatrix} \mathbf{Exp}[-\mathbf{i} \theta] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{Exp}[\mathbf{i} \theta] \end{pmatrix} \cdot \mathbf{UyH} // \mathbf{ExpToTrig} // \mathbf{FullSimplify};$$

**Ry // MatrixForm**

$$\begin{pmatrix} \cos\left[\frac{\theta}{2}\right]^2 & -\frac{\sin[\theta]}{\sqrt{2}} & \sin\left[\frac{\theta}{2}\right]^2 \\ \frac{\sin[\theta]}{\sqrt{2}} & \cos[\theta] & -\frac{\sin[\theta]}{\sqrt{2}} \\ \sin\left[\frac{\theta}{2}\right]^2 & \frac{\sin[\theta]}{\sqrt{2}} & \cos\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

$$\mathbf{Rz} = \begin{pmatrix} \mathbf{Exp}[-\mathbf{i} \phi] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{Exp}[\mathbf{i} \phi] \end{pmatrix};$$

**R = Rz.Ry // Simplify; R // MatrixForm**

$$\begin{pmatrix} e^{-\mathbf{i} \phi} \cos\left[\frac{\theta}{2}\right]^2 & -\frac{e^{-\mathbf{i} \phi} \sin[\theta]}{\sqrt{2}} & e^{-\mathbf{i} \phi} \sin\left[\frac{\theta}{2}\right]^2 \\ \frac{\sin[\theta]}{\sqrt{2}} & \cos[\theta] & -\frac{\sin[\theta]}{\sqrt{2}} \\ e^{\mathbf{i} \phi} \sin\left[\frac{\theta}{2}\right]^2 & \frac{e^{\mathbf{i} \phi} \sin[\theta]}{\sqrt{2}} & e^{\mathbf{i} \phi} \cos\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$



## Direct calculation of the rotation matrix

$$\mathbf{R1} = \text{MatrixExp}\left[-\frac{i\hbar}{\hbar} \mathbf{J}_z \phi\right] \cdot \text{MatrixExp}\left[\frac{-i\hbar}{\hbar} \mathbf{J}_y \theta\right] // \text{TrigFactor};$$

$\mathbf{R1} // \text{MatrixForm}$

$$\begin{pmatrix} e^{i\phi} \cos\left[\frac{\theta}{2}\right]^2 & -\frac{e^{i\phi} \sin[\theta]}{\sqrt{2}} & e^{i\phi} \sin\left[\frac{\theta}{2}\right]^2 \\ \frac{\sin[\theta]}{\sqrt{2}} & \cos[\theta] & -\frac{\sin[\theta]}{\sqrt{2}} \\ e^{-i\phi} \sin\left[\frac{\theta}{2}\right]^2 & \frac{e^{-i\phi} \sin[\theta]}{\sqrt{2}} & e^{-i\phi} \cos\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

$\mathbf{R1}.\{1, 0, 0\} // \text{MatrixForm}$

$$\begin{pmatrix} e^{i\phi} \cos\left[\frac{\theta}{2}\right]^2 \\ \frac{\sin[\theta]}{\sqrt{2}} \\ e^{-i\phi} \sin\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

$\mathbf{R1}.\{0, 1, 0\} // \text{MatrixForm}$

$$\begin{pmatrix} -\frac{e^{i\phi} \sin[\theta]}{\sqrt{2}} \\ \cos[\theta] \\ \frac{e^{-i\phi} \sin[\theta]}{\sqrt{2}} \end{pmatrix}$$

## 7. Useful formula for $J = 1$

We note that these kets are the eigenket of  $\hat{J}_n = \hat{\mathbf{J}} \cdot \mathbf{n}$ , where  $\mathbf{n}$  is the unit vector given by

$$\mathbf{n} = (n_x, n_y, n_z) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

and

$$\hat{J}_n = \hat{\mathbf{J}} \cdot \mathbf{n} = \hat{J}_x n_x + \hat{J}_y n_y + \hat{J}_z n_z = \hbar \begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}} e^{-i\phi} & 0 \\ \frac{\sin\theta}{\sqrt{2}} e^{i\phi} & 0 & \frac{\sin\theta}{\sqrt{2}} e^{-i\phi} \\ 0 & \frac{\sin\theta}{\sqrt{2}} e^{i\phi} & -\cos\theta \end{pmatrix}$$

One can use  $S = 1$  instead of  $J = 1$ .

The eigenvalue problem can be written as

$$(\hat{\mathbf{J}} \cdot \mathbf{n})|1,1;\mathbf{n}\rangle_n = \hbar|1,1;\mathbf{n}\rangle,$$

$$(\hat{\mathbf{J}} \cdot \mathbf{n})|1,0;\mathbf{n}\rangle = 0,$$

$$(\hat{\mathbf{J}} \cdot \mathbf{n})|1,-1;\mathbf{n}\rangle = -\hbar|1,-1;\mathbf{n}\rangle$$

We calculate the rotation matrix with  $J = 1$  without the use of **Mathematica**. First we consider the Taylor expansion:

$$\exp\left(-\frac{i}{\hbar} \theta \hat{J}_y\right) = 1 + \frac{1}{1!} \left(-\frac{i}{\hbar} \theta \hat{J}_y\right) + \frac{1}{2!} \left(-\frac{i}{\hbar} \theta \hat{J}_y\right)^2 + \frac{1}{3!} \left(-\frac{i}{\hbar} \theta \hat{J}_y\right)^3 + \frac{1}{4!} \left(-\frac{i}{\hbar} \theta \hat{J}_y\right)^4 + \dots$$

where

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$$

Note that

$$\hat{J}_+|1,1\rangle = 0, \quad \hat{J}_+|1,0\rangle = \sqrt{2\hbar}|1,1\rangle, \quad \hat{J}_+|1,-1\rangle = \sqrt{2\hbar}|1,0\rangle$$

$$\hat{J}_-|1,1\rangle = \sqrt{2\hbar}|1,0\rangle, \quad \hat{J}_-|1,0\rangle = \sqrt{2\hbar}|1,-1\rangle, \quad \hat{J}_-|1,-1\rangle = 0$$

$$\hat{J}_y|1,1\rangle = \frac{i\hbar}{\sqrt{2}}|1,0\rangle, \quad \hat{J}_y|1,0\rangle = \frac{-i\hbar}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle), \quad \hat{J}_y|1,-1\rangle = -\frac{i\hbar}{\sqrt{2}}|1,0\rangle$$

Using the matrix

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad \hat{J}_y^2 = -\hbar^2 \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

we have

$$\hat{J}_y^3 = \hbar^2 \hat{J}_y, \quad \hat{J}_y^4 = \hat{J}_y^3 \hat{J}_y = \hbar^2 \hat{J}_y \hat{J}_y = \hbar^2 \hat{J}_y^2,$$

$$\hat{J}_y^5 = \hat{J}_y^4 \hat{J}_y = \hbar^2 \hat{J}_y^2 \hat{J}_y = \hbar^2 \hat{J}_y^3 = \hbar^4 \hat{J}_y$$

Therefore the Taylor expansion can be rewritten as

$$\begin{aligned}
\exp\left(-\frac{i}{\hbar} \theta \hat{J}_y\right) &= \hat{1} + \frac{\hat{J}_y}{\hbar} [(-\theta) + \frac{1}{3!}(-i\theta)^3 + \frac{1}{5!}(-i\theta)^5 + \dots] \\
&\quad + \frac{\hat{J}_y^2}{\hbar^2} \left[ \frac{1}{2!}(-i\theta)^2 + \frac{1}{4!}(-i\theta)^4 + \dots \right] \\
&= \hat{1} - \frac{\hat{J}_y}{\hbar} (i \sin \theta) + \frac{\hat{J}_y^2}{\hbar^2} (\cos \theta - 1) \\
&= \begin{pmatrix} \frac{1 + \cos \theta}{2} & -\frac{\sin \theta}{\sqrt{2}} & \frac{1 - \cos \theta}{2} \\ \frac{\sin \theta}{\sqrt{2}} & \cos \theta & -\frac{\sin \theta}{\sqrt{2}} \\ \frac{1 - \cos \theta}{2} & \frac{\sin \theta}{\sqrt{2}} & \frac{1 + \cos \theta}{2} \end{pmatrix}
\end{aligned}$$

For  $J = 1$ , we have the following formula

$$\exp\left[-\frac{i}{\hbar} \alpha (\hat{\mathbf{J}} \cdot \mathbf{n})\right] = \hat{1} - i \sin \alpha \frac{\hat{\mathbf{J}} \cdot \mathbf{n}}{\hbar} + (\cos \alpha - 1) \frac{(\hat{\mathbf{J}} \cdot \mathbf{n})^2}{\hbar^2}$$

((**Proof**)) The proof for this formula is given in terms of Mathematica.

## Matrix $j = 1$

```

Clear["Global`*"]; j = 1; I3 = IdentityMatrix[3];
exp_* := exp /. {Complex[re_, im_] :=> Complex[re, -im]};
Jx[j_, n_, m_] :=  $\frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jy[j_, n_, m_] :=  $-\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jz[j_, n_, m_] :=  $\hbar m$  KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jn = n1 Jx + n2 Jy + n3 Jz;
rule1 = {n3^2 -> 1 - n1^2 - n2^2};
f1 = MatrixExp[-i  $\frac{Jn}{\hbar} \alpha$ ] //. rule1 // FullSimplify;
f2 = I3 -  $\frac{Jn}{\hbar} i \sin[\alpha] + \frac{Jn.Jn}{\hbar^2} (\cos[\alpha] - 1)$  //. rule1 // Simplify;
f1 - f2 // Simplify
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```

((Note)) For  $J = 1/2$ , we have similar formula

$$\exp\left[-\frac{i}{2}\alpha(\hat{\sigma} \cdot \mathbf{n})\right] = \hat{1} \cos\left(\frac{\alpha}{2}\right) - i(\hat{\sigma} \cdot \mathbf{n}) \sin\left(\frac{\alpha}{2}\right)$$

Matrix  $j = 1/2$

```
Clear["Global`*"]; j = 1 / 2; I2 = IdentityMatrix[2];
exp_* := exp /. {Complex[re_, im_] => Complex[re, -im]};
σx = PauliMatrix[1]; σy = PauliMatrix[2];
σz = PauliMatrix[3];

σn = n1 σx + n2 σy + n3 σz;

rule1 = {n1 → Sin[θ] Cos[φ], n2 → Sin[θ] Sin[φ], n3 → Cos[θ]};

f1 = MatrixExp[-i  $\frac{\sigma_n}{2}$  α] //. rule1 // FullSimplify;

f2 = I2 Cos[ $\frac{\alpha}{2}$ ] - i σn Sin[ $\frac{\alpha}{2}$ ] //. rule1 // FullSimplify;
f1 - f2 // Simplify

{{0, 0}, {0, 0}}
```