Phase shift evaluated from Born approximation

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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In general, the Born approximation is valid for high energy limit, while the phase shift analysis is useful in the low energy limit. Here the formula for phase shift can be derived based on the Born approximation. The phase shift from the Born approximation is compared with that derived from the phase shift analysis. The advantage of the phase shift from the Born approximation is free from the consideration of the boundary condition of the wave function. We only need to evaluate the integral. Note that the Born approximation is valid for large incident energies and weak scattering potential.

1. Phase shift derived from the Born approximation

We consider the scattering amplitude in the Born approximation when the potential has spherical symmetry,

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int d^3\boldsymbol{r}' e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}} V(r') = -\frac{\mu}{2\pi\hbar^2}\int d^3\boldsymbol{r}' e^{-i(\boldsymbol{Q}\cdot\boldsymbol{r}} V(r'),$$

where $\boldsymbol{Q}$ is the scattering vector; $\boldsymbol{Q} = \boldsymbol{k}' - \boldsymbol{k}$. The scattering amplitude for a central field in the Born approximation is given by

$$f^{(B)}(\theta) = -\frac{2\mu}{\hbar^2 Q}\int_0^\infty dr r V(r)\sin(Qr), \qquad \text{(spherical symmetry)}$$

When we use the formula given by

$$\frac{\sin(Qr)}{Qr} = \sum_l (2l+1)[j_l(kr)]^2 P_l(\cos\theta),$$

where

$$Q = 2k\sin\frac{\theta}{2}.$$

Then we have

$$f^{(B1)}(\theta) = -\frac{2\mu}{\hbar^2}\int_0^\infty drr^2 V(r)\frac{\sin(Qr)}{Qr}$$

$$= -\frac{2\mu}{\hbar^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\int_0^\infty drr^2 V(r)[j_l(kr)]^2$$

((**Born approximation**)). This form should be equal to the result derived from the phase shift expansion,

$$f^{(B2)}(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)$$

We note that

$$\int_0^\pi f^{(B1)}(\theta)P_{l'}(\cos\theta)d(\cos\theta) = -\frac{2\mu}{\hbar^2}\sum_{l=0}^{\infty}\int_0^\infty drr^2 V(r)[j_l(kr)]^2(2l+1)\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta)$$

$$= -\frac{2\mu}{\hbar^2}\sum_{l=0}^{\infty}\int_0^\infty drr^2 V(r)[j_l(kr)]^2(2l+1)\frac{2\delta_{l,l'}}{2l+1}$$

$$= -\frac{4\mu}{\hbar^2}\int_0^\infty drr^2 V(r)[j_{l'}(kr)]^2$$

and

$$\int_0^\pi f^{(B2)}(\theta)P_{l'}(\cos\theta)d(\cos\theta) = \sum_{l=0}^{\infty}(2l+1)[\frac{e^{i\delta_l}\sin(\delta_l)}{k}]\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta)$$

$$= \sum_{l=0}^{\infty}(2l+1)[\frac{e^{i\delta_l}\sin(\delta_l)}{k}]\frac{2\delta_{l,l'}}{2l+1}$$

$$= 2\frac{e^{i\delta_{l'}}\sin(\delta_{l'})}{k}$$

where we use

$$\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta) = \frac{2\delta_{l,l'}}{2l+1}$$

Thus we have

$$e^{i\delta_l}\sin(\delta_l) = -\frac{2\mu}{\hbar^2}k\int_0^\infty drr^2V(r)[j_l(kr)]^2 . \qquad (1)$$

The advantage of this method is that we do not have to worry about the boundary condition which is essential to the phase shift analysis.

(a) The phase shift for the repulsive potential

In Eq.(1), we assume that δ_l is positive and very small. Then we have

$$e^{i\delta_l}\sin\delta_l = \delta_l = -\frac{2\mu}{\hbar^2}k\int_0^\infty drr^2V(r)[j_l(kr)]^2 = -k\int_0^\infty drr^2U(r)[j_l(kr)]^2 . \qquad (2)$$

with

$$U(r) = \frac{2\mu}{\hbar^2}V(r)$$

which means that $\delta_l > 0$ for the repulsive potential and $\delta_l < 0$ for the attractive potential

(b) The phase shift for attractive potential

In Eq.(1), we assume that $\delta_l = -\pi + \delta_l'$, with $\delta_l' > 0$

$$\begin{aligned}
e^{i\delta_l}\sin(\delta_l) &= e^{i(-\pi+\delta_l')}\sin(-\pi + \delta_l') \\
&= (-1)e^{i\delta_l'}(-\sin\delta_l') \\
&\approx \sin\delta_l' \\
&= -\frac{2\mu}{\hbar^2}k\int_0^\infty drr^2V(r)[j_l(kr)]^2
\end{aligned}$$

Thus we get

$$\sin\delta_l' = -\frac{2\mu}{\hbar^2}k\int_0^\infty drr^2V(r)[j_l(kr)]^2$$

which means that

$\delta_l' > 0$ for the attractive potential ($V < 0$).

2. Another approach for the determination of the scattering amplitude in the Born approximation

Here we show that

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int d^3\mathbf{r} e^{-i\mathbf{k}'\cdot\mathbf{r}} V(r) e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= -\frac{2\mu}{\hbar^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\int_0^{\infty} dr r^2 V(r)[j_l(kr)]^2$$

using a method which is different from the method used above

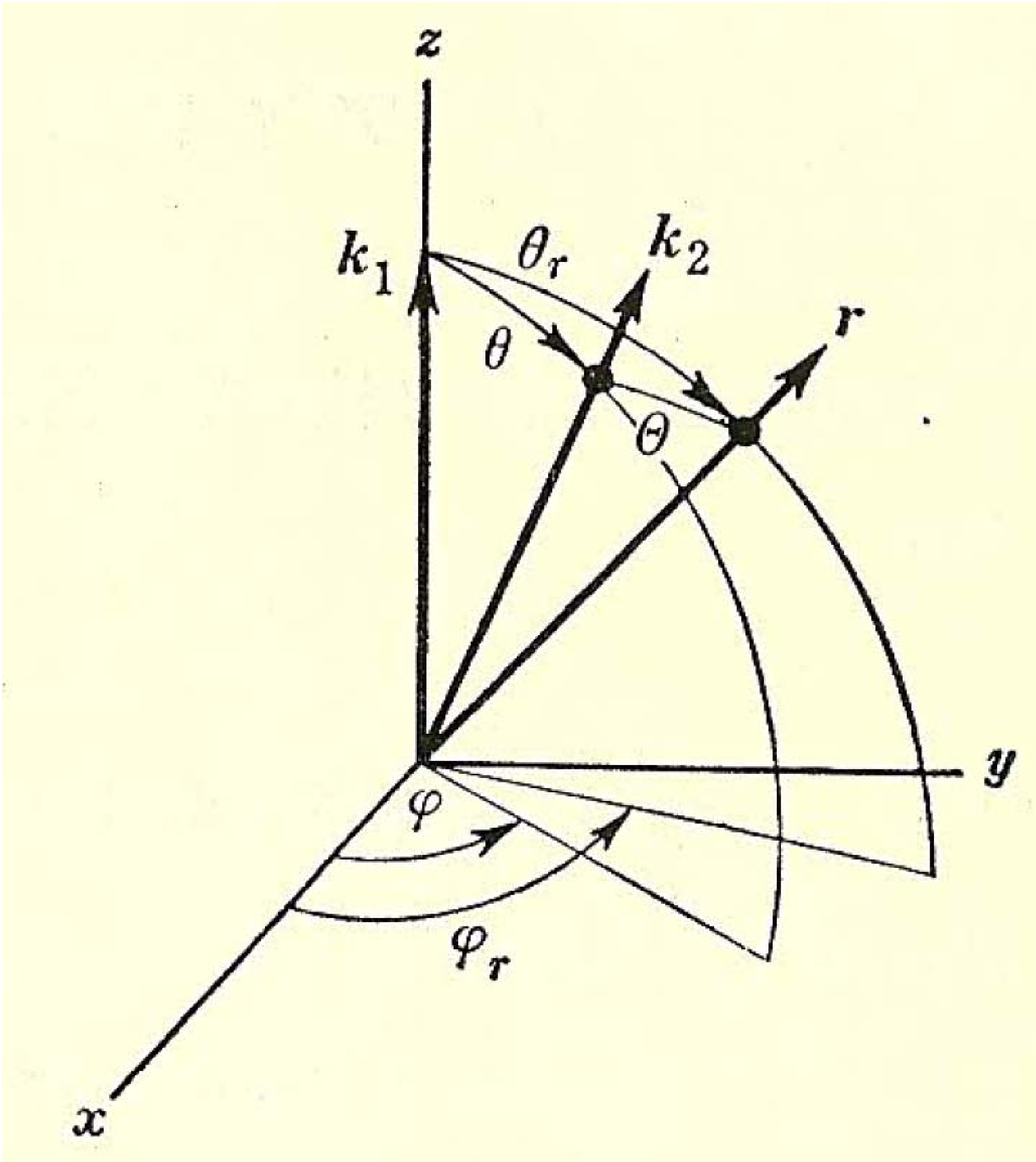


Fig. $\boldsymbol{k}_1 = \{k,0,0\}$ in the spherical coordinate (z axis). $\boldsymbol{k}_2 = \{k,\theta,\phi\}$ in the spherical coordinate. $\boldsymbol{r} = \{r,\theta_r,\phi_r\}$ in the spherical coordinate. Θ is the angle between $\boldsymbol{k}_2$ and $\boldsymbol{r}$.

We start with the Rayleigh's expansion formula

$$e^{i\boldsymbol{k}_1\cdot\boldsymbol{r}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta_r)\,,$$

and

$$e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}} = \sum_{l'=0}^{\infty}(2l'+1)i^{l'} j_{l'}(kr)P_{l'}(\cos\Theta)\,,$$

or

$$e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}} = \sum_{l'=0}^{\infty}(2l'+1)(-i)^{l'} j_{l'}(kr)P_{l'}(\cos\Theta)\,.$$

Then we get

$$\begin{aligned} e^{i\boldsymbol{k}_1\cdot\boldsymbol{r}}e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}} &= \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta_r)\sum_{l'=0}^{\infty}(2l'+1)(-i)^{l'} j_{l'}(kr)P_{l'}(\cos\Theta) \\ &= \sum_{l,l'}(2l+1)(2l'+1)i^l(-i)^{l'} j_l(kr)j_{l'}(kr)P_l(\cos\theta_r)P_{l'}(\cos\Theta) \end{aligned}$$

Here we use the addition theorem:

$$P_{l'}(\cos\Theta) = \frac{4\pi}{2l'+1}\sum_{m'=-l'}^{l'} Y_{l'}^{m'^*}(\theta_r,\phi_r)Y_{l'}^{m'}(\theta,\phi) \qquad \text{(addition theorem)}$$

Using this relation, we have

$$e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}} = 4\pi\sum_{l'=0}^{\infty}\sum_{m=-l'}^{l'}(-i)^{l'} j_{l'}(kr)Y_{l'}^{m'^*}(\theta_r,\phi_r)Y_{l'}^{m'}(\theta,\phi)\,.$$

The scattering amplitude can be rewritten as

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int V(r)r^2 dr\int \sin\theta_r d\theta_r d\phi_r e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

where

$$\int \sin\theta_r d\theta_r d\phi_r e^{-i\mathbf{k}'\cdot\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l,l'} (2l+1)(2l'+1) i^l (-i)^{l'} j_l(kr) j_{l'}(kr) \int \sin\theta_r d\theta_r d\phi_r P_l(\cos\theta_r) P_{l'}(\cos\Theta)$$

$$= \sum_{l,l',m'} (2l+1)(2l'+1) i^l (-i)^{l'} j_l(kr) j_{l'}(kr) \frac{4\pi}{2l+1} Y_{l'}^{m'}(\theta,\phi) \int \sin\theta_r d\theta_r d\phi_r Y_{l'}^{m'^*}(\theta_r,\phi_r) P_l(\cos\theta_r)$$

$$= 4\pi \sum_{l,l',m'} (2l'+1) i^l (-i)^{l'} j_l(kr) j_{l'}(kr) Y_{l'}^{m'}(\theta,\phi) \int \sin\theta_r d\theta_r d\phi_r Y_{l'}^{m'^*}(\theta_r,\phi_r) P_l(\cos\theta_r)$$

$$= 4\pi \sum_{l,l',m'} (2l'+1) i^l (-i)^{l'} j_l(kr) j_{l'}(kr) Y_{l'}^{m'}(\theta,\phi) \sqrt{\frac{4\pi}{2l+1}} \int \sin\theta_r d\theta_r d\phi_r Y_{l'}^{m'^*}(\theta_r,\phi_r) Y_l^0(\theta_r,\phi_r)$$

$$= 4\pi \sum_{l,} (2l+1)[j_l(kr)]^2 Y_l^0(\theta,\phi) \sqrt{\frac{4\pi}{2l+1}}$$

$$= 4\pi \sum_{l,} (2l+1)[j_l(kr)]^2 P_l(\cos\theta)$$

Note that

$$Y_l^0(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

$$\int \sin\theta_r d\theta_r d\phi_r Y_{l'}^{m'^*}(\theta_r,\phi_r) Y_l^0(\theta_r,\phi_r) = \delta_{l',l}\delta_{m',0} \qquad \text{(orthogonality)}$$

The final form of the scattering amplitude is given by

$$f^{(B)}(\theta) = -\frac{2\mu}{\hbar^2} \sum_l (2l+1) P_l(\cos\theta) \int_0^\infty r^2 dr V(r) [j_l(kr)]^2$$

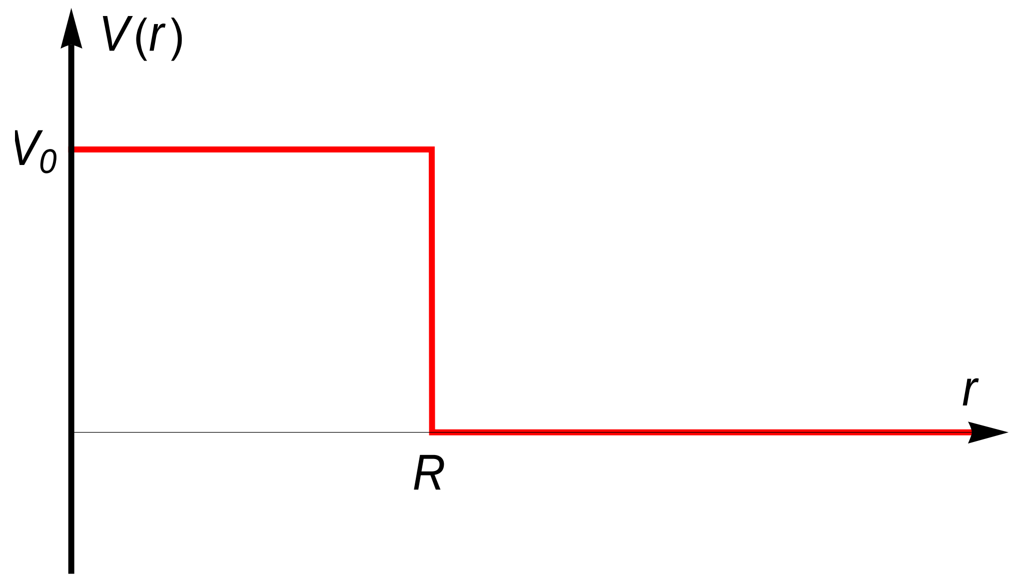
from the Born approximation. When $l = 0$ (S-wave), we have

$$f_0^{(B)}(\theta) = -\frac{2\mu}{\hbar^2} \int_0^\infty r^2 dr V(r) [j_0(kr)]^2 .$$

where

$$j_0(kr) = \frac{\sin(kr)}{kr} .$$

2. Repulsive square-well potential



A particle of mass μ is scattered from a spherical repulsive potential of radius *R*, we calculate the scattering amplitude using the Born approximation.

$$f^{(B)}(\theta) = -\frac{2\mu}{Q\hbar^2}\int_0^\infty dr rV(r)\sin(Qr) = -\frac{2\mu V_0}{Q\hbar^2}\int_0^R dr r\sin(Qr)$$

Noting that

$$\int_0^R dr r\sin(Qr) = \frac{\sin(QR) - QR\cos(QR)}{Q^2}.$$

we get

$$f^{(B)}(\theta) = -\frac{2\mu V_0}{\hbar^2 Q}[\frac{\sin(QR) - QR\cos(QR)}{Q^2}]$$

For $x \ll 1$,

$$\sin(x) - x\cos(x) = \frac{x^3}{3} + O(x^4)$$

Then in the limit $QR \ll 1$,

$$f^{(B)}(\theta) = -\frac{2\mu V_0}{\hbar^2 Q}\frac{1}{3Q^2}(QR)^3 = -\frac{2\mu V_0}{3\hbar^2}R^3$$

We now calculate the phase shift derived from the Born approximation

$$\delta_0^{(B)} = -\frac{2\mu}{\hbar^2}k\int_0^\infty dr r^2 V(r)[j_0(kr)]^2$$

where

$$j_0(kr) = \frac{\sin(kr)}{kr}$$

Then we get

$$\begin{aligned}
\delta_0^{(B)} &= -\frac{2\mu}{\hbar^2}k\int_0^\infty dr r^2 V(r)\frac{\sin^2(kr)}{k^2 r^2} \\
&= -\frac{2\mu V_0}{\hbar^2 k^2}k\int_0^R dr \sin^2(kr) \\
&= -\frac{\mu V_0 R}{\hbar^2 k^2}k[1-\frac{\sin(2kR)}{2kR}] \\
&\approx -\frac{\mu V_0 R}{\hbar^2 k^2}k\frac{2}{3}(kR)^2 < 0
\end{aligned}$$

For $kR << 1$, we have

$$\delta_0^{(B)} \approx \frac{\mu V_0 R}{\hbar^2 k^2}k\frac{2}{3}(kR)^2 = \frac{2\mu V_0 R^3}{3\hbar^2}k$$

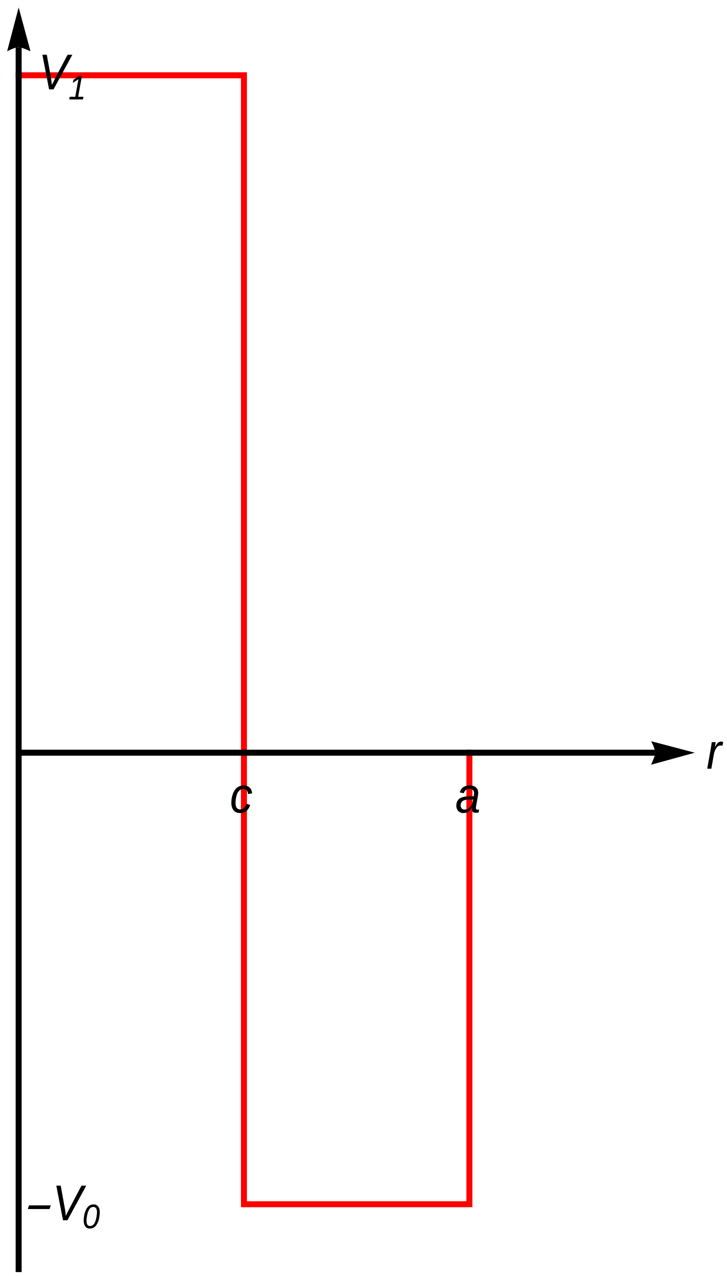
The total cross section is

$$\sigma_{tot} = \frac{4\pi}{k^2}\sin^2\delta_0^{(B)} = \frac{4\pi}{k^2}[\delta_0^{(B)}]^2 = \frac{16\pi\mu^2 V_0^2 R^6}{9\hbar^4}$$

which is the same as the result derived from the phase shift analysis

$$\sigma_{tot} = \frac{4}{9}\pi k_0^4 R^6 = \frac{16\pi\mu^2 V_0^2 R^6}{9\hbar^4}$$

3. Mixing of repulsive and attractive potential



$$\delta_0^{(B)} = -\frac{2\mu}{\hbar^2} k \int_0^\infty dr r^2 V(r) \frac{\sin^2(kr)}{k^2 r^2}$$

$$= -\frac{2\mu}{\hbar^2 k} [V_1 \int_0^c dr \sin^2(kr) - V_0 \int_c^a dr \sin^2(kr)$$

$$= -\frac{\mu}{\hbar^2 k} [-aV_0 + c(V_0 + V_1) - c(V_0 + V_1)\frac{\sin(2kc)}{2kc} + aV_0 \frac{\sin(2ka)}{2ka}]$$

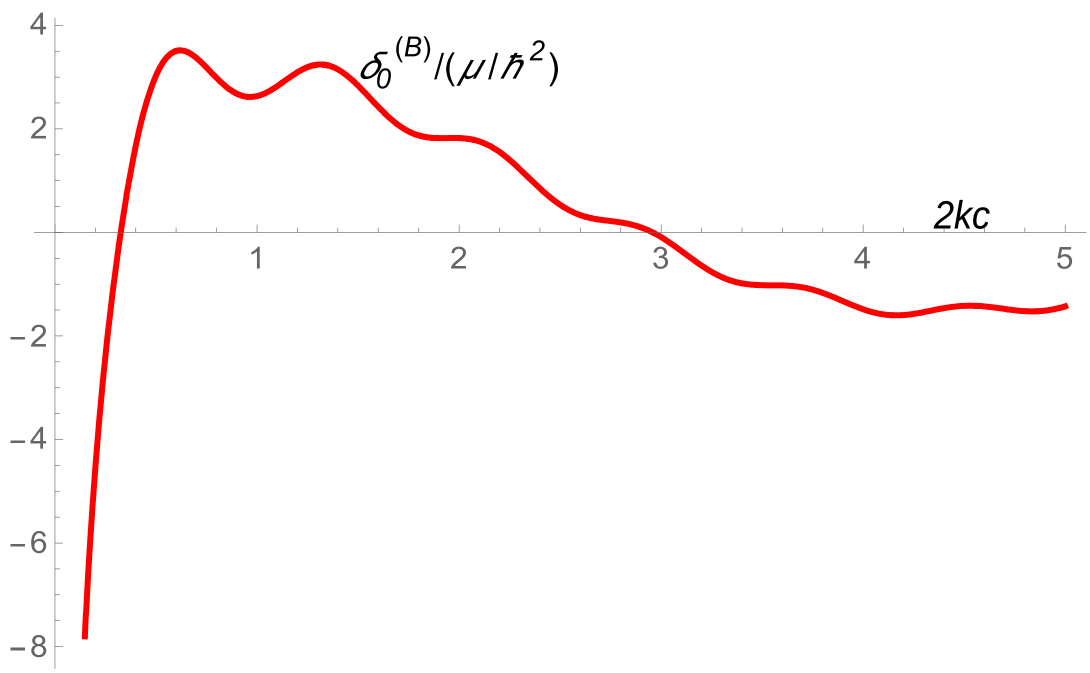
$$= -\frac{\mu}{\hbar^2 k} [(-bV_0 + cV_1) - c(V_0 + V_1)\frac{\sin(2kc)}{2kc} + aV_0 \frac{\sin(2ka)}{2ka}]$$

where $b = c - a$. For simplicity, we assume that

$$a >> c \qquad \text{and} \qquad V_1 > \frac{bV_0}{c}$$

((**Mathematica**))

$a = 4$, $c = 0.5$, $b = 3.5$, $V_0 = 1$, and $V_1 = 15\ V_0$;



The phase shift is negative at $2kc = 0$, become positive around $2kc = 0.3$. It decreases with increasing δ, and becomes again zero around $2kc = 3$. After that it becomes negative negative with further increasing δ.

4. Example-1: Phase shift analysis and Born approximation

(a) Determine the differential cross section $d\sigma / d\Omega$ in the Born approximation for scattering from the potential energy $U(r) = \frac{2\mu V(r)}{\hbar^2} = \gamma\delta(r-a)$. Show the explicit dependence of $d\sigma / d\Omega$ on θ.

(b) Evaluate $d\sigma / d\Omega$ in the low-energy limit. Show that the differential cross section is isotropic. What is the total cross section?

(c) Show that the validity of the Born approximation is given by $\gamma a << 1$.

Next we consider the spherically symmetric potential energy, $U(r) = \frac{2\mu V(r)}{\hbar^2} = \gamma\delta(r-a)$, where γ is a constant and $\delta(r-a)$ is a Dirac delta function that vanishes everywhere except on the spherical surface specified by $r = a$. We consider the differential equation with $l = 0$, in this differential equation

$$u''(r) + [k^2 - U(r) - \frac{l(l+1)}{r^2}]u(r) = 0, \qquad (1)$$

where $E = \frac{\hbar^2 k^2}{2\mu}$ is the kinetic energy of a particle with mass μ and $u(r) = rR(r)$.

(d) Find the form for $u(r)$ for $r>a$, where the phase shift is assumed to be δ_0.

(e) Find the form for $u(r)$ for $r<a$, where $u(r=0) = 0$.

(f) The function $u(r)$ is continuous at $r = a$. However the derivative of $u(r)$ with respect to r is not continuous. Use the boundary condition such that $u'(r)|_{r=a+0} - u'(r)|_{r=a+0} = \gamma u(a)$. Show that

$$\tan(ka + \delta_0) = \frac{k\sin(ka)}{k\cos(ka) + \gamma\sin(ka)} = \frac{\tan(ka)}{1 + \frac{\gamma}{k}\tan(ka)} \qquad (2)$$

(g) For $ka<<1$, find the expression for the phase shift δ_0.

(h) Find the expression of the total cross section.

(i) Show that the total cross section [result (g)] from the partial wave expansion at low energy agrees with that obtained in (b) from the Born approximation at low energy.

((**Solution**))

(a)

$$V(r) = \frac{\hbar^2\gamma}{2\mu}\delta(r-a), \qquad \text{or} \qquad U(r) = \gamma\delta(r-a)$$

The first order Born approximation:

$$\begin{aligned}
f^{(1)}(\theta) &= -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d\mathbf{r}' e^{-i\mathbf{Q}\cdot\mathbf{r}}V(\mathbf{r}') \\
&= -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int_0^\infty dr'\int_0^\pi d\theta' e^{-iQr'\cos\theta'}2\pi r'^2\sin\theta' V(r') \\
&= -\frac{\mu}{\hbar^2}\int_0^\infty dr'\int_0^\pi d\theta' e^{-iQr'\cos\theta'}r'^2\sin\theta'\frac{\gamma\hbar^2}{2\mu}\delta(r'-a) \\
&= -\frac{\gamma a^2}{2}\int_0^\pi d\theta' e^{-iQa\cos\theta'}\sin\theta'
\end{aligned}$$

where

$\boldsymbol{Q} = \boldsymbol{k}' - \boldsymbol{k}$; scattering vector

$$\int_0^\pi d\theta' e^{-iQa\cos\theta'}\sin\theta' = \frac{2}{Qa}\sin(Qa)$$

Then

$$f^{(1)}(\theta) = -\frac{\gamma a^2}{2}\frac{2}{Qa}\sin(Qa) = -\gamma a^2\frac{\sin(Qa)}{Qa}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left|f^{(1)}(\theta)\right|^2 = \left(\gamma a^2\right)^2\frac{\sin(Qa)^2}{(Qa)^2}.$$

where

$$Q = 2k\sin(\frac{\theta}{2}),$$

where θ is an angle between ***k***’ and ***k*** (Ewald’s sphere).

(b) $Qa << 1$

$$\frac{d\sigma}{d\Omega} = \left(\gamma a^2\right)^2\frac{\sin(Qa)^2}{(Qa)^2} \approx \left(\gamma a^2\right)^2,$$

which is isotropic, where we use the approximation

$$\frac{\sin^2 x}{x^2} = 1 - \frac{x^2}{3} + O(x^4).$$

Then we have the total cross section

$$\sigma_{tot} = 4\pi\gamma^2 a^4.$$

(c)

The validity of the Born approximation

$$\left|\frac{2\mu}{\hbar^2}\int d\mathbf{r}'\frac{e^{ikr'}}{4\pi r'}V(\mathbf{r}')\frac{e^{i\mathbf{k}\cdot\mathbf{r}'}}{(2\pi)^{3/2}}\right| << \frac{1}{(2\pi)^{3/2}}$$

or

$$\left|\frac{\mu}{2\pi\hbar^2}\int d\mathbf{r}'\frac{e^{ikr'}}{r'}V(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}\right| << 1$$

For spherical potential, we get

$$\left|\frac{2\mu}{\hbar^2 k}\int_0^\infty dr' e^{ikr'}V(r')\sin(kr')\right| << 1$$

When $V(r) = \frac{\hbar^2\gamma}{2\mu}\delta(r-a)$,

$$\left|\frac{\gamma}{k}\int_0^\infty dr' e^{ikr'}\delta(r'-a)\sin(kr')\right| << 1$$

or

$$\left|\frac{\gamma}{k}e^{ika}\sin(ka)\right| << 1$$

or

$$\frac{\gamma}{k}|\sin(ka)| << 1$$

Noting that $\sin(ka) \approx ka$ for the low energy limit, we have

$$\gamma a << 1$$

which is the condition for the validity of the Born approximation.

$$\sigma_{tot} = 4\pi a^2 \left(\frac{\gamma a}{1+\gamma a}\right)^2 \approx 4\pi\gamma^2 a^4$$

(d)

Here we derive the above solution directly from solving the differential equation.

$$u''(r) + [k^2 - U(r) - \frac{l(l+1)}{r^2}]u(r) = 0, \qquad (1)$$

where

$$u(r) = rR(r),$$

and

$$U(r) = \frac{2\mu}{\hbar^2}V(r) = \gamma\delta(r-a).$$

The boundary condition:

$$u'(r)|_{r=a+0} - u'(r)|_{r=a+0} = \gamma u(a)$$

Here we assume that $l = 0$. Then we have the differential equation

$$u''(r) + [k^2 - U(r)]u(r) = 0$$

For $r>a$

$$u''(r) + k^2 u(r) = 0$$

$$u = rR(r) = A\sin(kr + \delta_0)$$

(e)

For $r<a$

$$u''(r) + k^2 u(r) = 0$$

$$u = rR(r) = B\sin(kr)$$

(f) Since u is continuous at $r = a$,

$$A\sin(ka + \delta_0) = B\sin(ka) \,. \qquad (2)$$

Since

$$u'(r)|_{r=a+0} - u'(r)|_{r=a+0} = \gamma u(a) ,$$

we have

$$kA\cos(ka + \delta_0) = B[k\cos(ka) + \gamma\sin(ka)] \qquad (3)$$

From Eqs.(1) and (2), we have

$$\tan(ka + \delta_0) = \frac{k\sin(ka)}{k\cos(ka) + \gamma\sin(ka)} = \frac{\tan(ka)}{1 + \frac{\gamma}{k}\tan(ka)}$$

(g)

For ka<<1, we get

$$\tan(ka + \delta_0) \approx \frac{ka}{1 + \frac{\gamma}{k}ka} = \frac{ka}{1 + \gamma a}$$

Thus we have

$$ka + \delta_0 = \frac{ka}{1 + \gamma a}$$

$$\delta_0 = \frac{ka}{1+\gamma a} - ka = -ka\left(\frac{\gamma a}{1+\gamma a}\right)$$

(h)

The total cross section is

$$\sigma_{tot} = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\delta_0{}^2 = \frac{4\pi}{k^2}k^2a^4\left(\frac{\gamma a}{1+\gamma a}\right)^2 = 4\pi a^4\left(\frac{\gamma a}{1+\gamma a}\right)^2$$

(i) For $\gamma a << 1$, we have

$$\sigma_{tot} = 4\pi\gamma^2 a^4.$$

5. The phase shift from the Born approximation for the same example

Using the formula for the phase shift derived from the Born approximation

$$\delta_0^{(B)} = -k\int_0^\infty r^2 dr U(r)[j_0(kr)]^2$$

with $j_0(x) = \frac{\sin x}{x}$,

(a) calculate the phase shift $\delta_0^{(B)}$.

(b) calculate the total cross section for $ka<<1$.

(c) Suppose that $U(r) = \gamma_a\delta(r-a) + \gamma_b\delta(r-b)$ where $b>a$. calculate the cross section $\sigma_{tot}{}^{(B)}$

((**Solution**))

(a) We now calculate the phase shift derived from the Born approximation

$$\begin{aligned}\delta_0{}^{(B)} &= -\frac{2\mu}{\hbar^2}k\int_0^\infty drr^2V(r)[j_0(kr)]^2 \\ &= -\gamma k\int_0^\infty drr^2\delta(r-a)[j_0(kr)]^2 \\ &= -\gamma a^2k[j_0(ka)]^2 \\ &= -\gamma a^2k[\frac{\sin(ka)}{ka}]^2\end{aligned}$$

(b)

For $ka << 1$,

$$\delta_0^{(B)} \approx -\gamma a^2 k \ .$$

The total cross section is

$$\sigma_{tot} = \frac{4\pi}{k^2}\sin^2\delta_0^{(B)} = \frac{4\pi}{k^2}[\delta_0^{(B)}]^2 = \frac{4\pi}{k^2}(-\gamma a^2 k)^2 = 4\pi\gamma^2 a^4 \ ,$$

which is the same as the result derived from the phase shift analysis

$$\sigma_{tot} = 4\pi\gamma^2 a^4 .$$

(c)

$$\begin{aligned}\delta_0^{(B)} &= -\gamma_a k\int_0^\infty drr^2\delta(r-a)[j_0(kr)]^2 - \gamma_b k\int_0^\infty drr^2\delta(r-b)[j_0(kr)]^2 \\ &= -\gamma_a a^2 k[j_0(ka)]^2 - \gamma_b b^2 k[j_0(kb)]^2 - \\ &= -\gamma_a a^2 k[\frac{\sin(ka)}{ka}]^2 - \gamma_b b^2 k[\frac{\sin(kb)}{kb}]^2 \\ &\approx -k(\gamma_a a^2 + \gamma_b b^2)\end{aligned}$$

$$\sigma_{tot} = \frac{4\pi}{k^2}[\delta_0^{(B)}]^2 = 4\pi(\gamma_a{}^2 a^4 + \gamma_b{}^2 b^4)$$

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APPENDIX

Formula for scattering

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

$$j_0(x) = \frac{\sin x}{x}, \qquad n_0(x) = -\frac{\cos x}{x}$$

$$j_l(kr) = \sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(kr)$$

$$f^{(B)}(\theta) = -\frac{2\mu}{\hbar^2 Q}\int_0^\infty drrV(r)\sin(Qr) \qquad \text{(spherical symmetry)}$$

$$\int d(\cos\theta) P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{l',l}$$

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta) \qquad \text{(Rayleigh's expansion)}$$

$$\frac{\sin(Qr)}{Qr} = \sum_{l=0}^{\infty} (2l+1)[j_l(kr)]^2 P_l(\cos\theta)$$

where $Q = 2k\sin\frac{\theta}{2}$.

$$f^{(B)}(\theta) = -\frac{2\mu}{\hbar^2}\sum_l (2l+1) P_l(\cos\theta)\int_0^\infty r^2 drV(r)[j_l(kr)]^2 .$$

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)P_l(\cos\theta)$$

$$Y_l^0(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) .$$

$$P_0(\cos\theta) = 1$$