Schwarz inequality and physical meaning of eigenvalue problem

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Here we discuss the Heisenberg's principle of uncertainty using the Schwarz inequality. We also discuss the physical meaning of the eigenvalue problem. When the measurement is done for the Hermitian operator $\hat{A}$, the state of the system collapses in to one of the eigenstates. This is equivalent to the solving of the eigenvalue problem, $\hat{A}|a\rangle = a|a\rangle$, where $|a\rangle$ is the eigenket of the operator $\hat{A}$ with the eigenvalue a. This implies that the uncertainty $(\Delta A)^2 = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2 = \langle\psi|(\hat{A} - a\hat{1})^2|\psi\rangle$ becomes zero when the measurement is completed.

Karl Hermann Amandus Schwarz (25 January 1843 – 30 November 1921) was a German mathematician, known for his work in complex analysis. Schwarz originally studied chemistry in Berlin but Kummer and Weierstraß persuaded him to change to Mathematics. Between 1867 and 1869 he worked in Halle, then in Zürich. From 1875 he worked at Göttingen University, dealing with the subjects of function theory, differential geometry and the calculus of variations. His works include *Bestimmung einer speziellen Minimalfläche*, which was crowned by the Berlin Academy in 1867 and printed in 1871, and *Gesammelte mathematische Abhandlungen* (1890). In 1892 he became a member of the Berlin Academy of Science and a professor at the University of Berlin, where his students included Lipót Fejér, Paul Koebe and Ernst Zermelo. He died in Berlin.

http://en.wikipedia.org/wiki/Hermann_Amandus_Schwarz

1. Theorems

((**Commuting observables**))

Two observables $\hat{A}$ and $\hat{B}$ commute if and only if they admit a common basis of eigenvectors.

((Simultaneous measurability))

The necessary and sufficient condition for two observables $\hat{A}$ and $\hat{B}$ to be simultaneously measured with arbitrary precision is that they commute;

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{0}.$$

2. Schwarz Inequality (proof-1)

The Schwarz inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2,$$

for any $|\alpha\rangle$ and $|\beta\rangle$.

((**Proof**))

We consider

$$|\chi\rangle = |\alpha\rangle + \lambda|\beta\rangle,$$

$\langle\chi|\chi\rangle \geq 0$ for any complex number λ.

or

$$\langle\chi|\chi\rangle = (\langle\alpha| + \lambda^*\langle\beta|)(|\alpha\rangle + \lambda|\beta\rangle \geq 0,$$

or

$$\langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda\lambda^*\langle\beta|\beta\rangle \geq 0.$$

The best inequality is obtained if λ is chosen so as to minimize the left-hand side. By differentiation, we get

$$\frac{\partial\langle\chi|\chi\rangle}{\partial\lambda^*} = \langle\beta|\alpha\rangle + \lambda\langle\beta|\beta\rangle = 0\,,$$

or

$$\frac{\partial\langle\chi|\chi\rangle}{\partial\lambda} = \langle\alpha|\beta\rangle + \lambda^*\langle\beta|\beta\rangle = 0\,.$$

Then we have

$$\lambda = \lambda_0 = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} = -\frac{\langle\alpha|\beta\rangle^*}{\langle\beta|\beta\rangle}\,.$$

Using this value of λ, we get the inequality

$$(\langle\alpha| - \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\langle\beta|)(|\alpha\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}|\beta\rangle \geq 0\,,$$

or

$$\langle\alpha|\alpha\rangle - 2\frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} + \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} \geq 0\,,$$

or

$$\langle\alpha|\alpha\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} \geq 0\,,$$

or

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2\,.$$

The equality holds if and only if

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = |\langle\alpha|\beta\rangle|^2\,.$$

This means that

$$\lambda_0 = -\frac{\langle\alpha|\alpha\rangle}{\langle\alpha|\beta\rangle} = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle},$$

3. Schwartz inequality (proof-2)

Proof of the Schwartz inequality.

$$A_{11} = \langle\alpha|\alpha\rangle \geq 0, \qquad A_{12} = \langle\alpha|\beta\rangle, \quad A_{21} = \langle\beta|\alpha\rangle, \quad A_{22} = \langle\beta|\beta\rangle \geq 0,$$

We assume that

$$\lambda = p + iq, \qquad (p, q \text{ are real})$$

$$\lambda^* = p - iq.$$

Then we consider the function given by

$$\begin{aligned} f(p,q) &= \langle\chi|\chi\rangle \\ &= (\langle\alpha| + \lambda^*\langle\beta|)(|\alpha\rangle + \lambda|\beta\rangle) \\ &= \langle\alpha|\alpha\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + |\lambda|^2\langle\beta|\beta\rangle \\ &= A_{11} + (p - iq)A_{21} + (p + iq)A_{12} + (p^2 + q^2)A_{22} \end{aligned}$$

In order to find the minimum value of $f(p,q)$, we calculate

$$\frac{\partial f(p,q)}{\partial p} = A_{21} + A_{12} + 2A_{22}p = 0,$$

$$\frac{\partial f(p,q)}{\partial q} = -iA_{21} + iA_{12} + 2A_{22}q = 0.$$

From these equations, we get

$$p = p_0 = -\frac{A_{12} + A_{21}}{2A_{22}}, \qquad q = q_0 = \frac{i(-A_{12} + A_{21})}{2A_{22}},$$

or

$$\lambda = \lambda_0 = p_0 + iq_0 = -\frac{(A_{12} + A_{21})}{2A_{22}} - \frac{(-A_{12} + A_{21})}{2A_{22}} = -\frac{A_{21}}{A_{22}}.$$

The substitution of the these values of p and q leads to

$$f(p = p_0, q = q_0) = A_{11} - \frac{A_{12}A_{21}}{A_{22}} \geq 0$$

or

$$A_{11}A_{22} \geq A_{12}A_{21},$$

or

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = \left|\langle\alpha|\beta\rangle\right|^2. \qquad \text{(Schwartz inequality)}$$

((**Note**))

$$\lambda_0 = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}.$$

When

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle,$$

λ_0 can be expressed by

$$\lambda_0 = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} = -\frac{\langle\alpha|\alpha\rangle}{\langle\alpha|\beta\rangle}.$$

((**Mathematica**))

```
f[p_, q_] := A11 + (p + ⅈ q) A12 + (p - ⅈ q) A21 + (p^2 + q^2) A22;

eq1 = D[f[p, q], p] == 0

A12 + A21 + 2 A22 p == 0

eq2 = D[f[p, q], q] == 0

ⅈ A12 - ⅈ A21 + 2 A22 q == 0

eq3 = Solve[{eq1, eq2}, {p, q}] // Simplify
```

$$\left\{\left\{p \to -\frac{A12 + A21}{2\,A22},\ q \to \frac{i\,(-A12 + A21)}{2\,A22}\right\}\right\}$$

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f1 = f[p, q] /. eq3[[1]] // FullSimplify
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$$A11 - \frac{A12\,A21}{A22}$$

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4. Physical meaning of eigenvalue problem

Suppose that $\hat{A}^{+} = \hat{A}$ (Hermitian operator). For simplicity, we put

$$|\alpha\rangle = |\psi\rangle, \qquad \text{and} \qquad |\beta\rangle = |\varphi\rangle = \hat{A}|\psi\rangle .$$

We use the Schwarz inequality

$$\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle \geq |\langle\psi|\varphi\rangle|^2 \qquad \text{(Schwarz inequality)}$$

or

$$\langle\psi|\psi\rangle \geq \frac{\langle\psi|\varphi\rangle^2}{\langle\varphi|\varphi\rangle} .$$

We assume the normalization condition; $\langle\psi|\psi\rangle = 1$.

First we show that $\langle\psi|\varphi\rangle$ is real.

$$\langle\psi|\varphi\rangle^{*} = \langle\varphi|\psi\rangle = \langle\psi|\hat{A}^{+}|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle = \langle\psi|\varphi\rangle$$

since $\langle\beta| = \langle\varphi| = \langle\psi|\hat{A}^{+} = \langle\psi|\hat{A}$ and $\hat{A}^{+} = \hat{A}$.

Then the above inequality can be rewritten as

$$1 = \langle\psi|\psi\rangle \geq \frac{\langle\psi|\hat{A}|\psi\rangle^2}{\langle\psi|\hat{A}^2|\psi\rangle},$$

or

$$\langle\psi|\hat{A}^2|\psi\rangle \geq \langle\psi|\hat{A}|\psi\rangle^2 .$$

We now consider the fluctuation defined by

$$(\Delta A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2 \geq 0 . \qquad (1)$$

The condition $\Delta A = 0$ (the absence of the fluctuation) corresponds to the case of Eq.(1) with the equality.

$$(\Delta A)^2 = [\langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2] = 0 .$$

This implies for the special state in which $\hat{A}$ has a sharp expectation value *a* that

$$(\Delta A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 = \left\langle\left(\hat{A} - \langle\hat{A}\rangle\right)^2\right\rangle = \left\langle\left(\hat{A} - a\hat{1}\right)^2\right\rangle = 0,$$

or

$$\left\langle\left(\hat{A} - a\hat{1}\right)^2\right\rangle = \langle\psi|(\hat{A} - a\hat{1})(\hat{A} - a\hat{1})|\psi\rangle = 0,$$

where *a* is real. Then we get

$$(\hat{A} - a\hat{1})|\psi\rangle = 0, \qquad \text{or} \qquad \hat{A}|\psi\rangle = a|\psi\rangle .$$

This corresponds to the eigenvalue problem. The condition for no fluctuation leads to the eigenvalue problem.

((**Note**))

The only possible result of a measurement of an observable $\hat{A}$ is one of the eigenvalue *a* of the corresponding observable $\hat{A}$;

$$\hat{A}|a\rangle = a|a\rangle .$$

5. The Heisenberg's principle of uncertainty

$\hat{A}$ and $\hat{B}$ are two Hermitian operators with the condition

$$[\hat{A}, \hat{B}] = i\hat{C} .$$

Then we have a Heisenberg's principle of uncertainty:

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle C\rangle\right| .$$

In other words, this principle is a direct consequence of the non-commutability between two observables. The proof of this theorem is given as follows.

((**Proof**))

Uncertainty product in a normalized state $|\psi\rangle$.

$$(\Delta A)^2 = \langle\psi|(\hat{A} - \langle A\rangle\hat{1})^2|\psi\rangle ,$$

$$(\Delta B)^2 = \langle\psi|(\hat{B} - \langle B\rangle\hat{1})^2|\psi\rangle ,$$

where

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$$

and

$$\langle B\rangle = \langle\psi|\hat{B}|\psi\rangle$$

Let us define

$$\delta\hat{A} = \hat{A} - \langle A \rangle \hat{1}\,, \qquad \delta\hat{B} = \hat{B} - \langle B \rangle \hat{1}\,,$$

$$\delta\hat{A}^{+} = \delta\hat{A}\,, \qquad \delta\hat{B}^{+} = \delta\hat{B}\,,$$

$$(\Delta A)^2 (\Delta B)^2 = \langle \psi | (\delta\hat{A})^2 | \psi \rangle \langle \psi | (\delta\hat{B})^2 | \psi \rangle = \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq \left| \langle \alpha | \beta \rangle \right|^2 = \left| \langle \beta | \alpha \rangle \right|^2 ,$$

where

$$|\alpha\rangle = \delta\hat{A}|\psi\rangle\,, \quad |\beta\rangle = \delta\hat{B}|\psi\rangle\,.$$

Then we get

$$\langle \beta | \alpha \rangle = \langle \psi | \delta\hat{B}\delta\hat{A} | \psi \rangle\,, \qquad \langle \beta | \beta \rangle = \langle \psi | \delta\hat{B}\delta\hat{B} | \psi \rangle\,,$$

We also note that

$$\delta\hat{B}\delta\hat{A} = \frac{1}{2}(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A}) - \frac{1}{2}\left[\delta\hat{A}, \delta\hat{B}\right],$$

Using the relation

$$[\delta\hat{A}, \delta\hat{B}] = [\hat{A} - \langle A \rangle \hat{1}, \hat{B} - \langle B \rangle \hat{1}] = [\hat{A}, \hat{B}] = i\hat{C}\,.$$

we have

$$\delta\hat{B}\delta\hat{A} = \hat{G} - \frac{1}{2} i\hat{C}\,,$$

where

$$\hat{G} = \frac{1}{2}(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A})\,.$$

Thus we get the inequality

$$(\Delta A)^2 (\Delta B)^2 \geq \left| \langle \psi | \hat{G} - \frac{1}{2} i\hat{C} | \psi \rangle \right|^2 = \left| \langle \psi | \hat{G} | \psi \rangle - \frac{1}{2} i \langle \psi | \hat{C} | \psi \rangle \right|^2 .$$

Note that $\langle\psi|\hat{G}|\psi\rangle$ and $\langle\psi|\hat{C}|\psi\rangle$ are real since $\hat{G}$ and $\hat{C}$ are Hermitian operators,

$$\hat{C} = -i[\hat{A}, \hat{B}] = -i(\hat{A}\hat{B} - \hat{B}\hat{A}),$$

$$\hat{C}^{+} = -(\hat{A}\hat{B} - \hat{B}\hat{A})^{+} i^{+} = -(\hat{B}\hat{A} - \hat{A}\hat{B}) i^{*} = i(\hat{B}\hat{A} - \hat{A}\hat{B}) = \hat{C}.$$

Then we have

$$(\Delta A)^2 (\Delta B)^2 \geq \left|\langle\psi|\hat{G}|\psi\rangle + \frac{1}{2} i \langle\psi|\hat{C}|\psi\rangle\right|^2 = \left|\langle\psi|\hat{G}|\psi\rangle\right|^2 + \frac{1}{4}\left|\langle\psi|\hat{C}|\psi\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|\hat{C}|\psi\rangle\right|^2,$$

or

$$(\Delta A)(\Delta B) \geq \frac{1}{2}\left|\langle\psi|\hat{C}|\psi\rangle\right|.$$

(i) The last inequality holds if and only if

$$\langle\psi|\hat{G}|\psi\rangle = 0,$$

or

$$\langle\psi|(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A})|\psi\rangle = 0. \tag{2}$$

(ii) From the definition,

$$[\delta\hat{A}, \delta\hat{B}] = i\hat{C},$$

we get

$$\langle\psi|(\delta\hat{A}\delta\hat{B} - \delta\hat{B}\delta\hat{A})|\psi\rangle = i\langle\psi|\hat{C}|\psi\rangle. \tag{3}$$

From Eqs.(2) and (3), we have

$$\langle\psi|\delta\hat{A}\delta\hat{B}|\psi\rangle = \frac{i}{2}\langle\psi|\hat{C}|\psi\rangle$$

and

$$\langle\psi|\delta\hat{B}\delta\hat{A}|\psi\rangle = -\frac{i}{2}\langle\psi|\hat{C}|\psi\rangle .$$

((Note))
It is found from the Schwarz inequality that

$$(\langle\alpha| + \lambda_0^*\langle\beta|)(|\alpha\rangle + \lambda_0|\beta\rangle = 0 ,$$

or

$$(|\alpha\rangle + \lambda_0|\beta\rangle = 0 ,$$

with the value of λ_0

$$\lambda_0 = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} = -\frac{\langle\alpha|\alpha\rangle}{\langle\alpha|\beta\rangle} .$$

In the above case, we have

$$\lambda_0 = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} = -\frac{\langle\psi|\delta\hat{B}\delta\hat{A})|\psi\rangle}{\langle\psi|\delta\hat{B}\delta\hat{B})|\psi\rangle} = \frac{\frac{i}{2}\langle\psi|\hat{C}|\psi\rangle}{\langle\psi|\delta\hat{B}\delta\hat{B})|\psi\rangle} ,$$

which means that λ_0 is a pure imaginary.

6. Gaussian wave packet

$$\hat{A} = \hat{p} \text{ ,and } \hat{B} = \hat{x} ,$$

$$[\hat{p}, \hat{x}] = i\hat{C} = -i\hbar\hat{1} ,$$

$$(\Delta x)(\Delta p) \geq \frac{1}{2}\left|\langle\psi|i\hbar\hat{1}|\psi\rangle\right| = \frac{\hbar}{2} ,$$

(Heisenberg's principle of uncertainty).

We now consider the case of $(\Delta x)(\Delta p) = \frac{\hbar}{2}$. First we consider the case for the Schwarz inequality when the equality holds,

$$\lambda_0 = -\frac{\langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle} = -\frac{\langle \psi | \delta\hat{\mathrm{B}}\delta\hat{\mathrm{A}}) | \psi \rangle}{\langle \psi | \delta\hat{\mathrm{B}}\delta\hat{\mathrm{B}}) | \psi \rangle} = \frac{\frac{i}{2}\langle \psi | \hat{C} | \psi \rangle}{\langle \psi | \delta\hat{\mathrm{B}}\delta\hat{\mathrm{B}}) | \psi \rangle} = \frac{-i\hbar}{2(\Delta x)^2},$$

$$\delta\hat{\mathrm{A}} | \psi \rangle = -\lambda_0 \delta\hat{\mathrm{B}} | \psi \rangle,$$

or

$$(\hat{p} - \langle p \rangle \hat{1}) | \psi \rangle = \frac{i\hbar}{2(\Delta x)^2} (\hat{x} - \langle x \rangle \hat{1}) | \psi \rangle,$$

$$\langle x | \hat{p} - \langle p \rangle \hat{1} | \psi \rangle = \frac{i\hbar}{2(\Delta x)^2} \langle x | \hat{x} - \langle x \rangle \hat{1} | \psi \rangle,$$

$$(\frac{\hbar}{i}\frac{\partial}{\partial x} - \langle p \rangle) \langle x | \psi \rangle = \frac{i\hbar}{2(\Delta x)^2} (x - \langle x \rangle) \langle x | \psi \rangle,$$

or

$$\frac{\hbar}{i}\frac{\partial}{\partial x} \langle x | \psi \rangle = [\frac{i\hbar}{2(\Delta x)^2} (x - \langle x \rangle) + \langle p \rangle] \langle x | \psi \rangle,$$

The normalized solution is

$$\langle x | \psi \rangle = \frac{1}{[2\pi(\Delta x)^2]^{1/4}} \exp[-\frac{1}{4(\Delta x)^2} (x - \langle x \rangle)^2 + \frac{i}{\hbar} \langle p \rangle x].$$

This is a minimum uncertainty wave packet (Gaussian wave packet). This state represents a plane wave that is modulated by a Gaussian amplitude function. Since i is imaginary, this equation is an eigenfunction of the non-Hermitian operator $\hat{A} + \lambda\hat{B} = \hat{p} + \lambda\hat{x}$.

REFERENCES

John S. Townsend , *A Modern Approach to Quantum Mechanics*, second edition (University Science Books, 2012).
David H. McIntyre, *Quantum Mechanics A Paradigms Approach* (Pearson Education, Inc., 2012).
J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics, second edition (Addison-Wesley, New York, 2011).
Eugen Merzbacher, *Quantum Mechanics*, third edition (John Wiley & Sons, New York, 1998).
Ramamurti Shankar, *Principles of Quantum Mechanics*, second edition (Springer, New York, 1994).
G. Auletta, M. Fortunato, and G. Parisi, *Quantum Mechanics* (Cambridge University Press, 2009.
D.J. Griffiths, Introduction to Quantum Mechanics (Prentice Hall, 1995).

APPENDIX

((Griffiths Problem 4-28))

An electron in the spin state is given by

$$|\chi\rangle = A\begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant *A*.

(b) Find the expectation values of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$.

(c) Find the uncertainties ΔS_x, ΔS_y, and ΔS_z.

(d) Confirm that your results are consistent with all three uncertainty principles.

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}\left|\langle \hat{S}_z \rangle\right|, \qquad \Delta S_y \Delta S_z \geq \frac{\hbar}{2}\left|\langle \hat{S}_x \rangle\right|, \qquad \Delta S_z \Delta S_x \geq \frac{\hbar}{2}\left|\langle \hat{S}_y \rangle\right|$$

((**Solution**))

(a)

$$\langle \chi | \chi \rangle = |A|^2 \begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = 25|A|^2 = 1$$

We choose *A*=1/5.

(b)

$$\langle S_x \rangle = \langle \chi | \hat{S}_x | \chi \rangle = \frac{\hbar}{2}\begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = 0$$

$$\langle S_y \rangle = \langle \chi | \hat{S}_y | \chi \rangle = \frac{\hbar}{2}\begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = -\frac{12}{25}\hbar$$

$$\langle S_z \rangle = \langle \chi | \hat{S}_z | \chi \rangle = \frac{\hbar}{2}\begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = -\frac{7\hbar}{50}$$

(c)

Since

$$\hat{S}_x{}^2 = \hat{S}_y{}^2 = \hat{S}_z{}^2 = \frac{\hbar^2}{4}\hat{1}$$

$$\langle \chi | \hat{S}_x{}^2 | \chi \rangle = \langle \chi | \hat{S}_y{}^2 | \chi \rangle = \langle \chi | \hat{S}_z{}^2 | \chi \rangle = \frac{\hbar^2}{4}\hat{1}$$

$$(\Delta S_x)^2 = \left\langle \hat{S}_x{}^2 \right\rangle - \left\langle \hat{S}_x \right\rangle^2 = \frac{\hbar^2}{4}$$

or

$$(\Delta S_x) = \frac{\hbar}{2}$$

$$(\Delta S_y)^2 = \left\langle \hat{S}_y{}^2 \right\rangle - \left\langle \hat{S}_y \right\rangle^2 = \frac{\hbar^2}{4} - (-\frac{24\hbar}{50})^2 = 0.0196\hbar^2$$

$$\Delta S_y = 0.14\hbar$$

$$(\Delta S_z)^2 = \left\langle \hat{S}_z{}^2 \right\rangle - \left\langle \hat{S}_z \right\rangle^2 = \frac{\hbar^2}{4} - (-\frac{7\hbar}{50})^2 = 0.23\hbar^2$$

$$\Delta S_z = 0.48\hbar$$

(d)

$$\Delta S_x \Delta S_y = 0.07\hbar^2$$

$$\frac{\hbar}{2}\left| < S_z > \right| = 0.07\hbar$$

$$\Delta S_y \Delta S_z = 0.0672\hbar^2$$

$$\frac{\hbar}{2}\left| < S_x > \right| = 0$$

$$\Delta S_z \Delta S_x = 0.24\hbar^2$$

$$\frac{\hbar}{2}\left| < S_y > \right| = 0.24\hbar^2$$