**3D simple harmonics using Ladder operator**
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We discuss the quantum mechanics of three-dimensional simple harmonics by using the ladder operator method. This method is similar to that used for the derivation of wave function of hydrogen atom.

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## 1. Ladder operator

The Hamiltonian of 3D simple harmonics is given in terms of the radial momentum $p_r$ and the total orbital angular momentum $\boldsymbol{L}^2$ as

$$\begin{aligned} H_l &= \frac{1}{2\mu}(p_r{}^2 + \frac{\boldsymbol{L}^2}{r^2}) + \frac{1}{2}\mu\omega^2 r^2 \\ &= \frac{p_r{}^2}{2\mu} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2 \quad , \\ &= \frac{\hbar^2}{2\mu}[\frac{p_r{}^2}{\hbar^2} + \frac{l(l+1)}{r^2} + \frac{\mu^2\omega^2}{\hbar^2} r^2] \end{aligned}$$

where the radial momentum operator $p_r$ is given by

$$p_r = -i\hbar\frac{1}{r}\frac{\partial}{\partial r} r = -i\hbar(\frac{\partial}{\partial r} + \frac{1}{r}) .$$

$H_l$ is the Hamiltonian for a particle with mass $\mu$ that moves in one dimension in the effective potential

$$V_{eff}(r) = \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2 .$$

This potential can be rewritten as

$$V_{eff}(r) = V_{eff}(x) = \frac{\hbar\omega}{2}[\frac{l(l+1)}{x^2} + x^2]$$

with

$$x = \frac{r}{\sqrt{\hbar/(\mu\omega)}}.$$

Here we make a plot of $V_{eff}(r)/(\hbar\omega/2)$ as a function of $x$, where $l$ = 0,1, 2, 3, 4, and 5

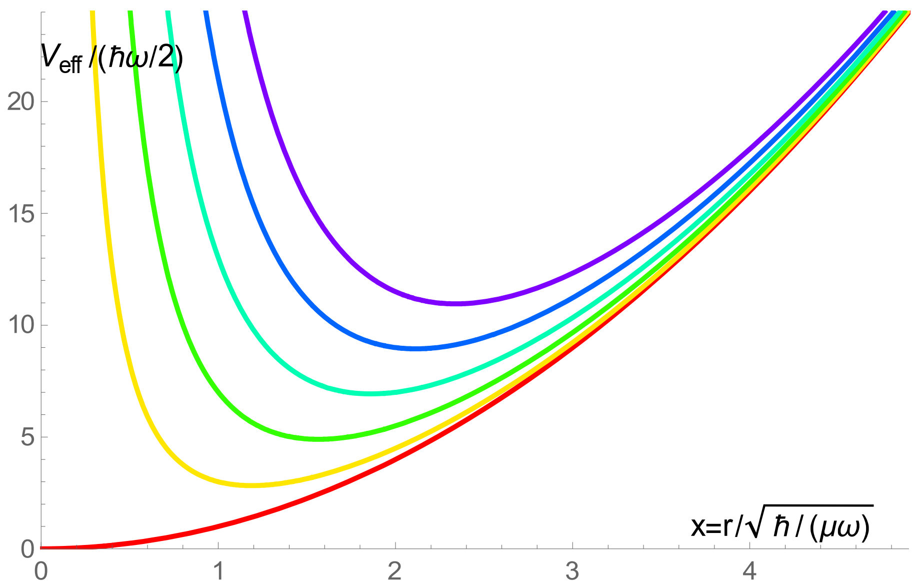


**Fig.** The effective potential for $l$ = 0 – 5 [from bottom ($l$ = 0, red) to top ($l$ = 5, purple)

Here we defined the ladder operator;

$$A_l = \frac{1}{\sqrt{2\mu\hbar\omega}}[ip_r - \frac{(l+1)\hbar}{r} + \mu\omega r],$$

The Hermite conjugate of $A_l$ is

$$A_l^+ = \frac{1}{\sqrt{2\mu\hbar\omega}}[-ip_r - \frac{(l+1)\hbar}{r} + \mu\omega r]$$

We now calculate $A_l^+ A_l$

$$
\begin{aligned}
A_l^+ A_l &= \frac{1}{2\mu\hbar\omega}[-ip_r - \frac{(l+1)\hbar}{r} + \mu\omega r][ip_r - \frac{(l+1)\hbar}{r} + \mu\omega r] \\
&= \frac{1}{2\mu\hbar\omega}\{p_r^{\ 2} + (-\frac{(l+1)\hbar}{r} + \mu\omega r)^2 + i[-\frac{(l+1)\hbar}{r} + \mu\omega r, p_r]\} \\
&= \frac{1}{2\mu\hbar\omega}\{p_r^{\ 2} + (-\frac{(l+1)\hbar}{r} + \mu\omega r)^2 - i\mu\omega[p_r, r] + i[p_r, \frac{(l+1)\hbar}{r}]\} \\
&= \frac{1}{2\mu\hbar\omega}\{p_r^{\ 2} + (-\frac{(l+1)\hbar}{r} + \mu\omega r)^2 - \mu\hbar\omega - \frac{(l+1)\hbar^2}{r^2}]\} \\
&= \frac{1}{2\mu\hbar\omega}\{p_r^{\ 2} + \frac{l(l+1)\hbar^2}{r^2} + \mu^2\omega^2 r^2 - \mu\hbar\omega(2l+3)]\} \\
&= \frac{1}{\hbar\omega}H_l - (l + \frac{3}{2})
\end{aligned}
$$

Then we get

$$
H_l = \hbar\omega\{A_l^+ A_l + (l + \frac{3}{2})\},
$$

Here we note that

(i)

$$
\begin{aligned}
\left[p_r, \frac{(l+1)}{r}\right]\psi(r) &= p_r \frac{l+1}{r}\psi - \frac{l+1}{r} p_r\psi \\
&= \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{l+1}{r}\psi) - \frac{l+1}{r}\frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}(r\psi) \\
&= \frac{\hbar}{i}\frac{(l+1)}{r}\frac{\partial}{\partial r}\psi - \frac{l+1}{r}\frac{\hbar}{i}\frac{1}{r}(\psi + r\frac{\partial}{\partial r}\psi) \\
&= -\frac{l+1}{r^2}\frac{\hbar}{i}\psi
\end{aligned}
$$

(ii)

$$
\begin{aligned}
[p_r, r]\psi(r) &= [p_r r\psi - r p_r \psi] \\
&= \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}(r^2\psi) - r\frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}(r\psi) \\
&= \frac{\hbar}{i}\frac{1}{r}(2r\psi + r^2\frac{\partial}{\partial r}\psi) - \frac{\hbar}{i}(\psi + r\frac{\partial}{\partial r}\psi) \\
&= \frac{\hbar}{i}\psi
\end{aligned}
$$

The commutation relation:

$$
\begin{aligned}
[A_l, A_l^+] &= \frac{1}{2\mu\hbar\omega}[ip_r - \frac{(l+1)\hbar}{r} + \mu\omega r, -ip_r - \frac{(l+1)\hbar}{r} + \mu\omega r] \\
&= \frac{i}{\mu\hbar\omega}[p_r, -\frac{(l+1)\hbar}{r} + \mu\omega r] \\
&= \frac{i}{\mu\hbar\omega}\{-[p_r, \frac{(l+1)\hbar}{r}] + \mu\omega[p_r, r]\}] \\
&= \frac{i}{\mu\hbar\omega}(\frac{l+1}{r^2}\frac{\hbar^2}{i} + \mu\omega\frac{\hbar}{i}) \\
&= 1 + \frac{(l+1)\hbar}{\mu\omega r^2}
\end{aligned}
$$

or

$$
[A_l, A_l^+] = \frac{H_{l+1} - H_l}{\hbar\omega} + 1,
$$

since

$$
H_{l+1} - H_l = \frac{\hbar^2}{2\mu r^2}[(l+2)(l+1) - l(l+1)] = \frac{\hbar^2}{\mu r^2}(l+1)].
$$

We also note that

$$
H_l = \hbar\omega\{A_l^+ A_l + (l + \frac{3}{2})\}
$$

and

$$
\begin{aligned}
[A_l, H_l] &= \hbar\omega[A_l, A_l^+ A_l] \\
&= \hbar\omega[A_l, A_l^+]A_l \\
&= (H_{l+1} - H_l + \hbar\omega)A_l
\end{aligned}
$$

## 2. Eigenvalue problem

Suppose that $|E,l\rangle$ is an eigenket of $H_l$ with eigenvalue $E$,

$$
H_l|E,l\rangle = E|E,l\rangle .
$$

Then we get

$$
\begin{aligned}
A_l H_l|E,l\rangle &= A_l E|E,l\rangle \\
&= ([A_l, H_l] + H_l A_l)|E,l\rangle . \\
&= (H_{l+1} + \hbar\omega)A_l|E,l\rangle
\end{aligned}
$$

or

$$
H_{l+1}A_l|E,l\rangle = (E - \hbar\omega)A_l|E,l\rangle
$$

This means that $A_l|E,l\rangle$ is the eigenket of $H_{l+1}$ with the eigenvalue $E-\hbar\omega$. Then we have

$$
A_l|E,l\rangle = \alpha_-|E-\hbar\omega, l+1\rangle .
$$

where

$$
H_{l+1}|E-\hbar\omega, l+1\rangle = (E-\hbar\omega)|E-\hbar\omega, l+1\rangle .
$$

## 3. Eigenstates

$A_l$ creates the radial wavefunction for a state has more orbital angular momentum and less energy that a state with which it started. In other words, $A_l$ diminishes the radial kinetic energy by some amount and adds a smaller quantity of energy to the tangential motion.

$$A_l|E,l\rangle \approx |E-\hbar\omega, l+1\rangle,$$

$$A_{l+1}|E-\hbar\omega, l+1\rangle \approx |E-2\hbar\omega, l+2\rangle,$$

$$A_{l+2}|E-2\hbar\omega, l+2\rangle \approx |E-3\hbar\omega, l+3\rangle.$$

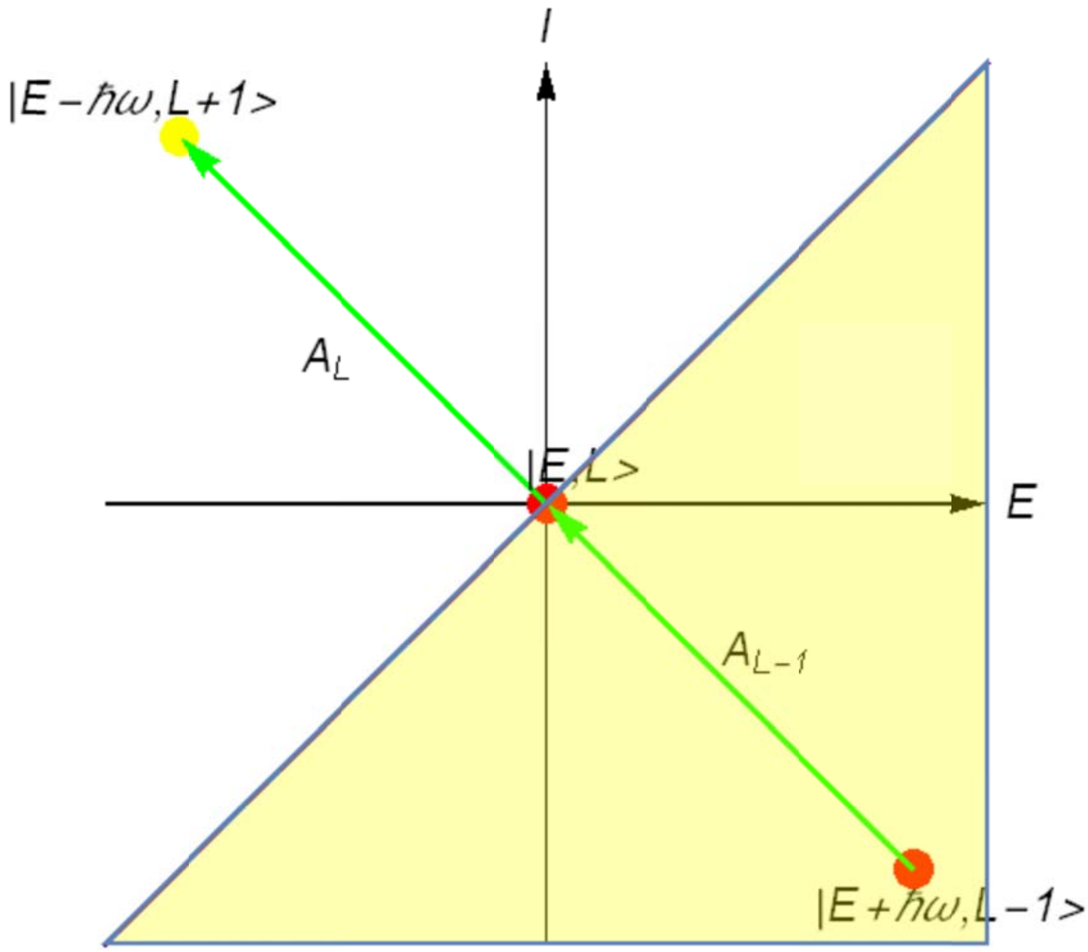


**Fig.** The boundary region [the allowed region below the straight line (purple line)] in the ($E$, $l$) plane with the action of $A_l$ and $A_{l-1}$ (denoted by green lines). The purple lined is expressed by $E_L = \hbar\omega(L+\frac{3}{2})$ . $A_L|E,L\rangle \approx |E-\hbar\omega, L+1\rangle = 0$ . $H_L|E,L\rangle = E|E,L\rangle$ . $A_{L-1}|E+\hbar\omega, L-1\rangle \approx |E,L\rangle$ , $H_{L-1}|E+\hbar\omega, L-1\rangle = E|E+\hbar\omega, L-1\rangle$

Suppose that

$$A_L|E,L\rangle = 0 \qquad \text{(a circuit orbit)}$$

where $L$ is the largest allowed value of l for energy *E*.

$$H_L|E_L,L\rangle = E_L|E_L,L\rangle = \hbar\omega(L+\frac{3}{2})|E_L,L\rangle .$$

where

$$H_L = \hbar\omega\{A_L{}^{+}A_L + (L+\frac{3}{2})\} .$$

So we get

$$E_L = \hbar\omega(L+\frac{3}{2}),$$

where

$$L=0, \qquad E_L = \frac{3}{2}\hbar\omega,$$

$$L=1, \qquad E_L = \frac{5}{2}\hbar\omega,$$

$$L=2, \qquad E_L = \frac{7}{2}\hbar\omega,$$

$$L=3, \qquad E_L = \frac{9}{2}\hbar\omega.$$

Since $L$ is a non-negative integer, it follows that the ground state is $3\hbar\omega/2$ and that the ground state has no angular momentum. In general, $\frac{E}{\hbar\omega}$ is any integer plus 3/2. These values of the allowed energies agree perfectly with what we could have deduced by treating H as a sum of 3D harmonic oscillator Hamiltonian.

## 4. A circuit orbit

$$A_L|E,L\rangle = 0$$

where

$$A_l = \frac{1}{\sqrt{2\mu\hbar\omega}}[ip_r - \frac{(L+1)\hbar}{r} + \mu\omega r]$$

$$(\frac{1}{r}\frac{\partial}{\partial r}r - \frac{L+1}{r} + \frac{\mu\omega}{\hbar}r)u_L(r) = 0$$

or

$$(\frac{\partial}{\partial r} - \frac{L}{r} + \frac{\mu\omega}{\hbar}r)u_L(r) = 0$$

where

$$\langle r|E,L\rangle = u_L(r),$$

$$l = \sqrt{\frac{\hbar}{2\mu\omega}}.$$ (characteristic length for the simple harmonics)

We solve the first-order differential equation

$$(\frac{d}{dr} - \frac{L}{r} + \frac{1}{2l^2}r)u_L(r) = 0.$$

The solution of this differential equation:

$$\frac{du}{u} = \frac{L}{r} - \frac{1}{2l^2}r$$

$$\ln u = L \ln r - \frac{r^2}{4l^2}$$

$$u_L(r) = Cr^L \exp(-\frac{r^2}{4l^2})$$

where $C$ is the normalization constant. The radial probability density is

$$\begin{aligned}
P(r,L)dr &= P(L,\xi)d\xi \\
&= r^2[u_L(r)]^2 dr \\
&= C^2 r^{2+2L} \exp(-\frac{r^2}{2l^2})dr \\
&= \frac{1}{2^{L+\frac{1}{2}}\Gamma(L+\frac{3}{2})} l^{-(2L+3)} r^{2+2L} \exp(-\frac{r^2}{2l^2})dr \\
&= \frac{1}{2^{L+\frac{1}{2}}\Gamma(L+\frac{3}{2})} \xi^{2+2L} \exp(-\frac{\xi^2}{2})d\xi
\end{aligned}$$

with

$$\xi = \frac{r}{l},$$

where

$$1 = \int_0^\infty dr r^2[u_L(r)]^2 = C^2 \int_0^\infty r^{2+2L} \exp(-\frac{r^2}{2l^2})$$

or

$$C = \frac{1}{\sqrt{2^{L+\frac{1}{2}}\Gamma(L+\frac{3}{2})}} l^{-(L+\frac{3}{2})}.$$

We make a plot of $P(L,\xi)$ as a function of $\xi = r/l$, where L = 0, 1, 2, 3,….,

$$P(L,\xi) = \frac{1}{2^{L+\frac{1}{2}}\Gamma(L+\frac{3}{2})}\xi^{2+2L}\exp(-\frac{\xi^2}{2}).$$

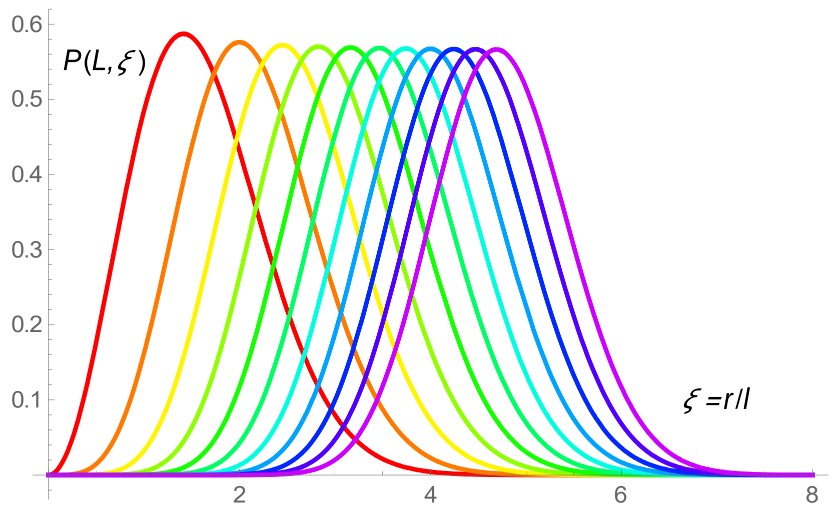


**Fig.** Radial probability distribution $P(L,\xi)$ of circular orbits in the 3D harmonic oscillator potential for $L$ = 0 (red) into 10 (purple), as s function of $\xi = r/l$, with $l = \sqrt{\hbar/(2\mu\omega)}$.

This function has a peak at

$$\xi = \frac{r}{l} = \sqrt{2(L+1)}.$$

## 5. Degeneracy

$$A_l H_l = H_{l+1} A_l + \hbar\omega A_l,$$

$$H_l A_l^+ = A_l^+ H_{l+1} + \hbar\omega A_l^+ ,$$

$$H_{l+1}|E,l+1\rangle = E|E,l+1\rangle ,$$

$$\begin{aligned} EA_l^+|E,l+1\rangle &= A_l^+ H_{l+1}|E,l+1\rangle \\ &= (H_l A_l^+ - \hbar\omega A_l^+)|E,l+1\rangle \\ &= (H_l - \hbar\omega) A_l^+|E,l+1\rangle \end{aligned}$$

or

$$H_l A_l^+|E,l+1\rangle = (E+\hbar\omega) A_l^+|E,l+1\rangle .$$

$A_l^+|E,l+1\rangle$ is the eigenket of $H_l$ with the eigenvalues $(E+\hbar\omega)$ and $l$.

$$A_l^+|E,l+1\rangle \propto |E+\hbar\omega,l\rangle,$$

or

$$A_{l-1}^+|E,l\rangle \propto |E+\hbar\omega,l-1\rangle.$$

$$A_{l-1}^+|E,l\rangle \propto |E+\hbar\omega,l-1\rangle,$$

$$A_l|E,l\rangle \propto |E-\hbar\omega,l+1\rangle,$$

$$H_{l+1}|E-\hbar\omega,l+1\rangle = (E-\hbar\omega)|E-\hbar\omega,l+1\rangle,$$

$$H_l|E,l\rangle = E|E,l\rangle,$$

$$H_{l-1}|E+\hbar\omega,l-1\rangle = (E+\hbar\omega)|E,l-1\rangle.$$

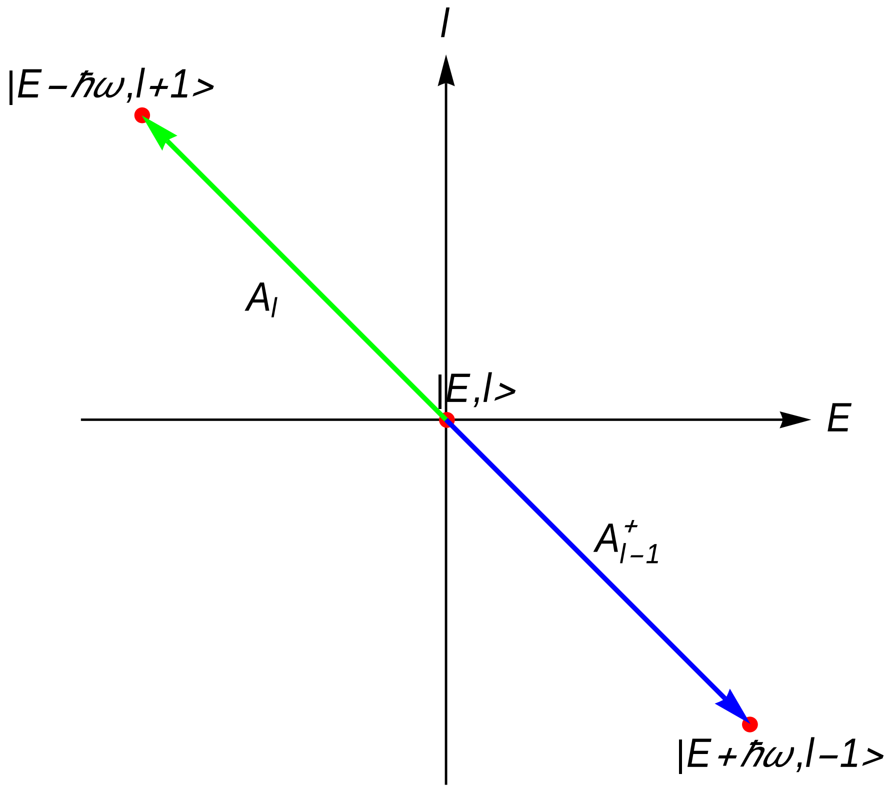


Figure helps to organize the generation of the radial wave functions. Each red circle represents the eigenstate. The degeneracy of the state with the quantum number $l$ is $(2l+1)$.

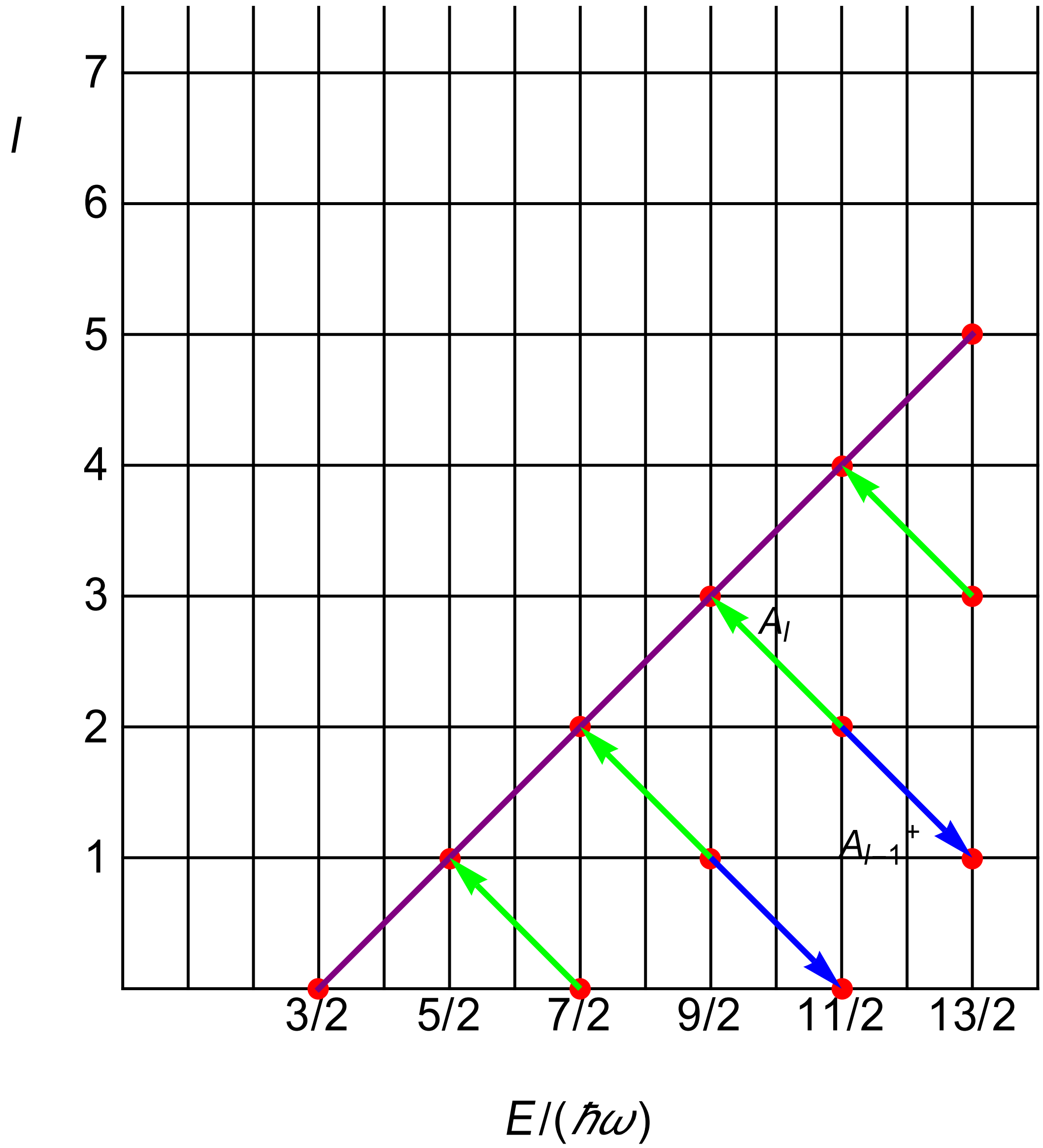


Fig. The allowed states denoted by red circles. in the ( $E/(\hbar\omega)$ , $l$) plane. The boundary line (purple line) is denoted by $E/(\hbar\omega) = (l + \frac{3}{2})$ .

((Degeneracy))

$E = \frac{3}{2}\hbar\omega$ degeneracy = *1* (l = 0)

| | | |
|---|---|---|
| $E=\frac{3}{2}\hbar\omega$ | degeneracy = 1 | ($l = 0$) |
| | | total states=1 |
| $E=\frac{5}{2}\hbar\omega$ | degeneracy = 3 | ($l = 1$) |
| | | total states=3 |
| $E=\frac{7}{2}\hbar\omega$ | degeneracy = 1 | ($l = 0$) |
| | degeneracy = 5 | ($l = 2$) |
| | | total states=6 |
| $E=\frac{9}{2}\hbar\omega$ | degeneracy = 3 | ($l = 1$) |
| | degeneracy = 7 | ($l = 3$) |
| | | total states=10 |
| $E=\frac{11}{2}\hbar\omega$ | degeneracy = 1 | ($l = 0$) |
| | degeneracy = 5 | ($l = 2$) |
| | degeneracy = 9 | ($l = 4$) |
| | | total states=15 |

**APPENDIX**
((**Mathematica**))

The quantum mechanical operator pr;
The operator A=AD;
The operator A+=AU;
The Hamiltonian of 3D simple harmonics. $\mu$ is a reduced mass.

$$\text{Clear["Global`*"]; pr} := -i\,\frac{\hbar}{r}\,D[r\,\#, r]\,\&;$$

$$\text{AD}[L\_] := \frac{1}{\sqrt{2\,\mu\,\hbar\,\omega}}\left(i\,\text{pr}[\#] - \frac{(L+1)\,\hbar}{r}\,\# + \mu\,\omega\,r\,\#\right)\&;$$

$$\text{AU}[L\_] := \frac{1}{\sqrt{2\,\mu\,\hbar\,\omega}}\left(-i\,\text{pr}[\#] - \frac{(L+1)\,\hbar}{r}\,\# + \mu\,\omega\,r\,\#\right)\&;$$

$$\text{H}[L\_] := \left(\frac{1}{2\,\mu}\,\text{pr}[\text{pr}[\#]] + \frac{L\,(L+1)\,\hbar^2}{2\,\mu\,r^2}\,\# + \frac{\mu\,\omega^2}{2}\,r^2\,\#\right)\&;$$

AI+ AI calculation

```
f1 = AU[L][AD[L][χ[r]]] // Simplify;
f2 = H[L][χ[r]] // Simplify;
```

$$\text{eq12} = \left(\text{f1} - \frac{\text{f2}}{\hbar\,\omega}\right) // \text{Simplify}$$

$$-\frac{1}{2}\,(3 + 2\,L)\,\chi[r]$$

The commutation relation
AD[L] AU[L] - AU[L]AD[L]

```
f3 = (AD[L][AU[L][χ[r]]] -  AU[L][AD[L][χ[r]]]) //
  Simplify
```

$$\frac{\left(r^2\,\mu\,\omega+\hbar+L\,\hbar\right)\,\chi[r]}{r^2\,\mu\,\omega}$$

```
f4 = (H[L + 1][χ[r]] - H[L][χ[r]] // Simplify)
```

$$\frac{(1+L)\,\hbar^2\,\chi[r]}{r^2\,\mu}$$

$$\text{eq34} = \text{f3} - \frac{\text{f4}}{\hbar\,\omega}\ //\ \text{Simplify}$$

$\chi[r]$

AD[L] H[L] - H[L]AD[L]

```
f5 = AD[L][H[L][χ[r]]] - H[L][AD[L][χ[r]]] //
   Simplify;

f6 = H[L + 1][AD[L][χ[r]]] - H[L][AD[L][χ[r]]] +
    ħ ω AD[L][χ[r]] // Simplify;

eq56 = f5 - f6 // Simplify
```

0

AD[L] H[L] = H[L+1]AD[L]+ħωAD[L]

```
f7 = AD[L][H[L][χ[r]]] - H[L + 1][AD[L][χ[r]]] -
   ħ ω AD[L][χ[r]] // Simplify
```

0