Solution of Chapter 8 (Sakurai and Napolitano, Pearson)
J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics, 2nd edition (Pearson 2011).

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8.1 These exercises are to give you some practice with natural units.

(a) Express the proton mass $m_p = 1.67262158 \times 10^{-27}$ kg in units of GeV.

(b) Assume a particle with negligible mass is confined to a box the size of the proton, around 1 fm $= 10^{-15}$ m. Use the uncertainty principle estimate the energy of the confined particle. You might be interested to know that the mass, in natural units, of the pion, the lightest strongly interacting particle, is $m_\pi = 135$ MeV.

(c) String theory concerns the physics at a scale that combines gravity, relativity, and quantum mechanics. Use dimensional analysis to find the "Planck mass" M_P, which is formed from G, $\hbar$, and c, and express the result in GeV.

((**Solution**))

NIST Physics constant

cgs units

```
Clear["Global`*"];
rule1 = {μB → 9.27400915 × 10^-21, kB → 1.3806504 × 10^-16,
   NA → 6.02214179 × 10^23, c → 2.99792 × 10^10,
   ħ → 1.054571628 10^-27, me → 9.10938215 10^-28,
   mp → 1.672621637 × 10^-24, mn → 1.674927211 × 10^-24,
   qe → 4.8032068 × 10^-10, eV → 1.602176487 × 10^-12,
   meV → 1.602176487 × 10^-15, keV → 1.602176487 × 10^-9,
   MeV → 1.602176487 × 10^-6, GeV → 1.602176487 × 10^-3,
   Å → 10^-8, THz → 10^12, α → 7.2973525376 × 10^-3,
   σSB → 5.670400 × 10^-5, rB → 0.52917720859 × 10^-8,
   amu → 1.660538782 × 10^-24, G → 6.6740831 × 10^-8};
```

$$\frac{mp\, c^2}{GeV} \text{ //. rule1}$$

0.938269

$$E1 = \frac{c\, \hbar}{\Delta x\, MeV} \text{ //. rule1}$$

$$\frac{1.97327 \times 10^{-11}}{\Delta x}$$

$$M1 = \sqrt{\frac{\hbar\, c}{G}} \text{ //. rule1}$$

0.0000217647

$$\frac{M1\, c^2}{GeV} \text{ //. rule1}$$

1.22091×10^{19}

8-2

8.2 Show that a matrix $\eta^{\mu\nu}$ with the same elements as the metric tensor $\eta_{\mu\nu}$ used in this chapter has the property that $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^{\mu}_{\nu}$, the identity matrix. Thus, show that the natural relationship $\eta^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma}$ in fact holds with this definition. Show also that $a^{\mu}b_{\mu} = a_{\mu}b^{\mu}$ for two four-vectors a^{μ} and b^{μ}.

((**Solution**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = (1  0  0  0
      0 -1  0  0
      0  0 -1  0
      0  0  0 -1);
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0; γu[1] = γux;
γu[2] = γuy; γu[3] = γuz ; γu[5] = ⅈ γu0 .γux.γuy.γuz;
γd[μ_] := Sum[Gd[μ, ν] γu[ν], {ν, 0, 3, 1}];
σu[μ_, ν_] := ⅈ/2 (γu[μ].γu[ν] - γu[ν].γu[μ]);
σd[μ_, ν_] := ⅈ/2 (γd[μ].γd[ν] - γd[ν].γd[μ]);
γd[5] = -ⅈ γd[3] .γd[2] .γd[1] .γd[0];
```

```
Gum // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
Gdm // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
Gum.Gdm // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8-3

8.3 Show that (8.1.11) is in fact a conserved current when $\Psi(\mathbf{x}, t)$ satisfies the Klein-Gordon equation.

$$j^{\mu} = \frac{i}{2m}\left[\Psi^{*}\partial^{\mu}\Psi - \left(\partial^{\mu}\Psi\right)^{*}\Psi\right], \tag{8.1.11}$$

((**Solution**))

$$\begin{aligned}
\partial_\mu j^\mu &= \frac{i}{2m}[\partial_\mu(\psi^*\partial^\mu\psi) - \partial_\mu((\partial^\mu\psi)^*\psi)] \\
&= \frac{i}{2m}[\partial_\mu\psi^*(\partial^\mu\psi) + \psi^*(\partial^2\psi) - (\partial^2\psi)^*\psi - (\partial^\mu\psi)^*\partial_\mu\psi)] \\
&= \frac{i}{2m}[\psi^*(\partial^2\psi) - (\partial^2\psi)^*\psi] \\
&= \frac{i}{2m}[\psi^*(-m^2\psi) - (-m^2\psi^*)\psi] \\
&= 0
\end{aligned}$$

8-4

8.4 Show that (8.1.14) follows from (8.1.8).

$$\left[\partial_\mu\partial^\mu + m^2\right]\Psi(\mathbf{x},t) = 0. \qquad (8.1.8)$$

$$\left[D_\mu D^\mu + m^2\right]\Psi(\mathbf{x},t) = 0, \qquad (8.1.14)$$

where $D_\mu \equiv \partial_\mu + ieA_\mu$. We refer to D_μ as the covariant derivative.

((**Solution**))

$$p^\mu \to p^\mu - \frac{e}{c}A^\mu$$ (e is the charge of particle).

Since $p^\mu = i\hbar\partial^\mu$, we have

$$i\hbar\partial^\mu \to i\hbar\partial^\mu - \frac{e}{c}A^\mu,$$

or

$$\partial^\mu \to \partial^\mu + \frac{ie}{c\hbar}A^\mu = D^\mu$$

Then we have

$$(\partial_\mu \partial^\mu + m^2)\psi(\boldsymbol{r},t) = 0 \qquad \rightarrow \qquad (D_\mu D^\mu + m^2)\psi(\boldsymbol{r},t) = 0$$

8-5

8.5 Derive (8.1.16a), (8.1.16b), and (8.1.18).

$$i D_t \phi = -\frac{1}{2m}\mathbf{D}^2(\phi + \chi) + m\phi \qquad (8.1.16a)$$

$$i D_t \chi = +\frac{1}{2m}\mathbf{D}^2(\phi + \chi) - m\chi, \qquad (8.1.16b)$$

$$i D_t \Upsilon = \left[-\frac{1}{2m}\mathbf{D}^2(\tau_3 + i\tau_2) + m\tau_3\right]\Upsilon. \qquad (8.1.18)$$

((**Solution**))

$$iD_t\begin{pmatrix}\phi \\ \chi\end{pmatrix} = \begin{pmatrix} -\frac{1}{2m}\boldsymbol{D}^2 + m & -\frac{1}{2m}\boldsymbol{D}^2 \\ \frac{1}{2m}\boldsymbol{D}^2 & \frac{1}{2m}\boldsymbol{D}^2 - m \end{pmatrix}\begin{pmatrix}\phi \\ \chi\end{pmatrix}$$

where

$$\begin{pmatrix} -\frac{1}{2m}\boldsymbol{D}^2 + m & -\frac{1}{2m}\boldsymbol{D}^2 \\ \frac{1}{2m}\boldsymbol{D}^2 & \frac{1}{2m}\boldsymbol{D}^2 - m \end{pmatrix} = (-\frac{1}{2m}\boldsymbol{D}^2 + m)\tau_3 - i\frac{1}{2m}\boldsymbol{D}^2\tau_2$$

$$= -\frac{1}{2m}\boldsymbol{D}^2(\tau_3 + i\tau_2) + m\tau_3$$

where

$$\tau_3 = \sigma_z, \qquad \tau_2 = \sigma_y$$

8-6

8.6 Show that the free-particle energy eigenvalues of (8.1.18) are $E = \pm E_p$ and that the eigenfunctions are indeed given by (8.1.21), subject to the normalization that $\Upsilon^\dagger \tau_3 \Upsilon = \pm 1$ for $E = \pm E_p$.

$$iD_t\Upsilon = \left[-\frac{1}{2m}\mathbf{D}^2(\tau_3 + i\tau_2) + m\tau_3 \right]\Upsilon. \qquad (8.1.18)$$

$$\Upsilon(\mathbf{x},t) = \frac{1}{2(mE_p)^{1/2}} \begin{pmatrix} E_p + m \\ m - E_p \end{pmatrix} e^{-iE_p t + i\mathbf{p}\cdot\mathbf{x}} \text{ for } E = +E_p \qquad (8.1.21a)$$

and $$\Upsilon(\mathbf{x},t) = \frac{1}{2(mE_p)^{1/2}} \begin{pmatrix} m - E_p \\ E_p + m \end{pmatrix} e^{+iE_p t + i\mathbf{p}\cdot\mathbf{x}} \text{ for } E = -E_p, \qquad (8.1.21b)$$

((**Solution**))

```
Clear["Global`*"];
```

$$A1 = \frac{p^2}{2m}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

```
eq1 = Eigensystem[A1] // Simplify[#, {p > 0, m > 0}] &
```

$$\left\{\left\{-\sqrt{m^2+p^2},\ \sqrt{m^2+p^2}\right\},\ \left\{\left\{-\frac{2m^2+p^2-2m\sqrt{m^2+p^2}}{p^2},\ 1\right\},\ \left\{-\frac{2m^2+p^2+2m\sqrt{m^2+p^2}}{p^2},\ 1\right\}\right\}\right\}$$

$$eq12 = eq1 \;/\!/.\; \left\{p^2 \to E1^2 - m^2\right\} \;/\!/\; \text{Simplify}[\#, E1 > 0]\ \&$$

$$\left\{\{-E1,\ E1\},\ \left\{\left\{-\frac{(E1-m)^2}{p^2},\ 1\right\},\ \left\{-\frac{(E1+m)^2}{p^2},\ 1\right\}\right\}\right\}$$

```
ψ1 = eq12[[2, 2]]; ψ2 = eq12[[2, 1]];
```

```
ϕ1 = Normalize[ψ1] // Simplify[#, {E1 > 0, m > 0, p > 0}] &
```

$$\left\{-\frac{(E1+m)^2}{\sqrt{1+\frac{(E1+m)^4}{p^4}}\ p^2},\ \frac{1}{\sqrt{1+\frac{(E1+m)^4}{p^4}}}\right\}$$

```
ϕ2 = Normalize[ψ2] // Simplify[#, {E1 > 0, m > 0, p > 0}] &
```

$$\left\{-\frac{(E1-m)^2}{\sqrt{1+\frac{(E1-m)^4}{p^4}}\ p^2},\ \frac{1}{\sqrt{1+\frac{(E1-m)^4}{p^4}}}\right\}$$

8.7 This problem is taken from *Quantum Mechanics II: A Second Course in Quantum Theory*, 2nd ed., by Rubin H. Landau (1996). A spinless electron is bound by the Coulomb potential $V(r) = -Ze^2/r$ in a stationary state of total energy $E \leq m$. You can incorporate this interaction into the Klein-Gordon equation by using the covariant derivative with $V = -e\Phi$ and $\mathbf{A} = 0$.

(a) Assume that the radial and angular parts of the equation separate and that the wave function can be written as $e^{-iEt}[u_l(r)/r]Y_{lm}(\theta,\phi)$. Show that the radial equation becomes

$$\frac{d^2u}{d\rho^2} + \left[\frac{2EZ\alpha}{\gamma\rho} - \frac{1}{4} - \frac{l(l+1)-(Z\alpha)^2}{\rho^2}\right]u_l(\rho) = 0,$$

where $\alpha = e^2$, $\gamma^2 = 4(m^2 - E^2)$, and $\rho = \gamma r$.

(b) Assume that this equation has a solution of the usual form of a power series times the $\rho \to \infty$ and $\rho \to 0$ solutions, that is,

$$u_l(\rho) = \rho^k(1 + c_1\rho + c_2\rho^2 + \cdots)e^{-\rho/2},$$

and show that

$$k = k_\pm = \frac{1}{2} \pm \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2}$$

and that only for k_+ is the expectation value of the kinetic energy finite and that this solution has a nonrelativistic limit that agrees with the solution found for the Schrödinger equation.

(c) Determine the recurrence relation among the c_i's for this to be a solution of the Klein-Gordon equation, and show that unless the power series terminates, the wave function will have an incorrect asymptotic form.

(d) In the case where the series terminates, show that the energy eigenvalue for the k_+ solution is

$$E = \frac{m}{\left(1+(Z\alpha)^2\left[n-l-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-(Z\alpha)^2}\right]^{-2}\right)^{1/2}},$$

where n is the principal quantum number.

(e) Expand E in powers of $(Z\alpha)^2$ and show that the first-order term yields the Bohr formula. Connect the higher-order terms with relativistic corrections, and discuss the degree to which the degeneracy in l is removed.

Jenkins and Kunselman, in *Phys. Rev. Lett.* **17** (1966) 1148, report measurements of a large number of transition energies for π^- atoms in large-Z nuclei. Compare some of these to the calculated energies, and discuss the accuracy of the prediction. (For example, consider the $3d \to 2p$ transition in ^{59}Co, which emits a photon with energy 384.6 ± 1.0 keV.) You will probably need either to use a computer to carry out the energy differences with high enough precision, or else expand to higher powers of $(Z\alpha)^2$.

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((**Solution**))

Problem 8-7

Series expansion: Klein-Gordan hydrogen atom

```
Clear["Global`*"];
```

$$eq1 = u''[\rho] + \left(\frac{2\,E1\,Z\,\alpha}{\gamma\,\rho} - \frac{1}{4} - \frac{L1\,(L1+1) - (Z\,\alpha)^2}{\rho^2}\right) u[\rho];$$

$$rule1 = \left\{u \to \left(Exp\left[-\frac{\#}{2}\right] F1[\#]\ \&\right)\right\};$$

eq2 = eq1 /. rule1 // Simplify

$$\frac{1}{\gamma\,\rho^2} e^{-\rho/2} \left(\left(-L1\,\gamma - L1^2\,\gamma + Z\,\alpha\,(Z\,\alpha\,\gamma + 2\,E1\,\rho)\right) F1[\rho] + \gamma\,\rho^2\,(-F1'[\rho] + F1''[\rho])\right)$$

$$eq3 = \left(-L1\,\gamma - L1^2\,\gamma + Z\,\alpha\,(Z\,\alpha\,\gamma + 2\,E1\,\rho)\right) F1[\rho] + \gamma\,\rho^2\,(-F1'[\rho] + F1''[\rho]);$$

$$rule2 = \left\{F1 \to \left(\sum_{s=0}^{10} C[s]\ \#^{k+s}\ \&\right)\right\};$$

$$eq4 = \frac{eq3}{\rho^k}\ /.\ rule2\ //\ Expand;$$

list1 = Table[{s, Coefficient[eq4, ρ, s]}, {s, 0, 5}] // Simplify; list1 // TableForm

0	$\left(-k + k^2 - L1 - L1^2 + Z^2\,\alpha^2\right)\gamma\,C[0]$
1	$2\,E1\,Z\,\alpha\,C[0] + \gamma\left(k^2\,C[1] - \left(L1 + L1^2 - Z^2\,\alpha^2\right) C[1] + k\,(-C[0] + C[1])\right)$
2	$2\,E1\,Z\,\alpha\,C[1] - \gamma\left((1+k)\,C[1] + \left(-2 - 3\,k - k^2 + L1 + L1^2 - Z^2\,\alpha^2\right) C[2]\right)$
3	$2\,E1\,Z\,\alpha\,C[2] + \gamma\left(-(2+k)\,C[2] + \left(6 + 5\,k + k^2 - L1 - L1^2 + Z^2\,\alpha^2\right) C[3]\right)$
4	$2\,E1\,Z\,\alpha\,C[3] + \gamma\left(-(3+k)\,C[3] + \left(12 + 7\,k + k^2 - L1 - L1^2 + Z^2\,\alpha^2\right) C[4]\right)$
5	$2\,E1\,Z\,\alpha\,C[4] + \gamma\left(-(4+k)\,C[4] + \left(20 + 9\,k + k^2 - L1 - L1^2 + Z^2\,\alpha^2\right) C[5]\right)$

Determination of recursion formula

$$rule3 = \left\{F1 \to \left(\sum_{s=q-3}^{q+3} C[s]\ \#^{s+k}\ \&\right)\right\};$$

$$eq5 = \frac{eq3}{\rho^{-3+k+q}}\ /.\ rule3\ //\ Expand;$$

list2 = Table[{s, Coefficient[eq5, ρ, s]}, {s, 2, 7}] // Simplify; list2 // TableForm

2	$(2\,E1\,Z\,\alpha - (-2+k+q)\,\gamma)\,C[-2+q] + \left(2+k^2-L1-L1^2-3\,q+q^2+k\,(-3+2\,q)+Z^2\,\alpha^2\right)\gamma\,C[-1+q]$
3	$(2\,E1\,Z\,\alpha - (-1+k+q)\,\gamma)\,C[-1+q] + \left(k^2-L1-L1^2-q+q^2+k\,(-1+2\,q)+Z^2\,\alpha^2\right)\gamma\,C[q]$
4	$(2\,E1\,Z\,\alpha - (k+q)\,\gamma)\,C[q] + \left(k+k^2-L1-L1^2+q+2\,k\,q+q^2+Z^2\,\alpha^2\right)\gamma\,C[1+q]$
5	$(2\,E1\,Z\,\alpha - (1+k+q)\,\gamma)\,C[1+q] + \left(2+k^2-L1-L1^2+3\,q+q^2+k\,(3+2\,q)+Z^2\,\alpha^2\right)\gamma\,C[2+q]$
6	$(2\,E1\,Z\,\alpha - (2+k+q)\,\gamma)\,C[2+q] + \left(6+k^2-L1-L1^2+5\,q+q^2+k\,(5+2\,q)+Z^2\,\alpha^2\right)\gamma\,C[3+q]$
7	$(2\,E1\,Z\,\alpha - (3+k+q)\,\gamma)\,C[3+q]$

```
eq6 = list2[[3, 2]]
```

$$(2\,E1\,Z\,\alpha - (k+q)\,\gamma)\,C[q] + \left(k+k^2-L1-L1^2+q+2\,k\,q+q^2+Z^2\,\alpha^2\right)\gamma\,C[1+q]$$

```
Solve[eq6 == 0, C[q + 1]]
```

$$\left\{\left\{C[1+q] \to -\left(\left((2\,E1\,Z\,\alpha - k\,\gamma - q\,\gamma)\,C[q]\right) \Big/ \left(\left(k+k^2-L1-L1^2+q+2\,k\,q+q^2+Z^2\,\alpha^2\right)\gamma\right)\right)\right\}\right\}$$

Determination of k from the first term of the series expansion

```
s1 = (-k + k^2 - L1 - L1^2 + Z^2 α^2); s11 = Solve[s1 == 0, k] // Simplify
```

$$\left\{\left\{k \to \frac{1}{2}\left(1-\sqrt{1+4\,L1+4\,L1^2-4\,Z^2\,\alpha^2}\right)\right\}, \left\{k \to \frac{1}{2}\left(1+\sqrt{1+4\,L1+4\,L1^2-4\,Z^2\,\alpha^2}\right)\right\}\right\}$$

```
kp = k /. s11[[2]]; km = k /. s11[[1]];
```

$$\frac{1}{2}\left(1-\sqrt{1+4\,L1+4\,L1^2-4\,Z^2\,\alpha^2}\right)$$

We choose kp as k. Determination of the energy eigenvalue under the condition that $C[1+q] = -\frac{(2\,E1\,Z\,\alpha - k\,\gamma - q\,\gamma)\,C[q]}{\left(k+k^2-L1-L1^2+q+2\,k\,q+q^2+Z^2\,\alpha^2\right)\gamma}$ =0 when q=nr (integer)

```
s2p = (2 E1 Z α - k γ - q γ) /. {q → nr} //. {γ → 2 √(m^2 - E1^2)} // FullSimplify
```

$$-2\left(\sqrt{-E1^2+m^2}\,(k+nr) - E1\,Z\,\alpha\right)$$

8-8

8.8 Prove that the traces of the γ^μ, $\boldsymbol{\alpha}$, and β are all zero.

((Solution))

Sakurai 8-8

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = (1   0   0   0
      0  -1   0   0
      0   0  -1   0
      0   0   0  -1);
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0; γu[1] = γux;
γu[2] = γuy; γu[3] = γuz ; γu[5] = ⅈ γu0 .γux.γuy.γuz;
γd[μ_] := Sum[Gd[μ, ν] γu[ν], {ν, 0, 3, 1}];
σu[μ_, ν_] := ⅈ/2 (γu[μ].γu[ν] - γu[ν].γu[μ]);
σd[μ_, ν_] := ⅈ/2 (γd[μ].γd[ν] - γd[ν].γd[μ]);
γd[5] = -ⅈ γd[3] .γd[2] .γd[1] .γd[0];
β = γu0;
```

```
Tr[β]
```

0

```
{Tr[αx], Tr[αy], Tr[αz]}
```

{0, 0, 0}

```
Table[Tr[γu[μ]], {μ, 0, 3}]
```

{0, 0, 0, 0}

```
s3 = Solve[s2p == 0, E1] // FullSimplify
```

$$\left\{\left\{E1 \to -\frac{m\,(k+nr)}{\sqrt{(k+nr)^2+Z^2\,\alpha^2}}\right\},\ \left\{E1 \to \frac{m\,(k+nr)}{\sqrt{(k+nr)^2+Z^2\,\alpha^2}}\right\}\right\}$$

nr=n-(l+1); principal quantum number

```
s4 = E1 /. s3[[2]] /. {k → kp} /. nr → (n - L1 - 1) // Simplify
```

$$\left(m\left(-1-2\,L1+2\,n+\sqrt{1+4\,L1+4\,L1^2-4\,Z^2\,\alpha^2}\right)\right)\Big/ \left(2\sqrt{\left(Z^2\,\alpha^2+\frac{1}{4}\left(-1-2\,L1+2\,n+\sqrt{\left(1+4\,L1+4\,L1^2-4\,Z^2\,\alpha^2\right)}\right)^2\right)}\right)$$

$$\text{Series}[s4,\ \{\alpha,\ 0,\ 7\}]\ //\ \text{Simplify}\left[\#,\ \left\{\left(-1-2\,L1+\sqrt{(1+2\,L1)^2}+2\,n\right)>0,\ L1>0\right\}\right]\ \&$$

$$m-\frac{\left(m\,Z^2\right)\alpha^2}{2\,n^2}+\frac{m\,(3+6\,L1-8\,n)\,Z^4\,\alpha^4}{8\,(1+2\,L1)\,n^4}-\frac{1}{16\left((1+2\,L1)^3\,n^6\right)}\left(m\left(5+40\,L1^3+L1^2\,(60-96\,n)-24\,n+24\,n^2+16\,n^3+6\,L1\left(5-16\,n+8\,n^2\right)\right)Z^6\right)\alpha^6+O[\alpha]^8$$

8.9 **(a)** Derive the matrices γ^μ from (8.2.10) and show that they satisfy the Clifford algebra (8.2.4).

(b) Show that

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I \otimes \tau_3$$

and $$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \sigma^i \otimes i\tau_2,$$

where I is the 2×2 identity matrix, and σ^i and τ_i are the Pauli matrices. (The $\otimes$ notation is a formal way to write our 4×4 matrices as 2×2 matrices of 2×2 matrices.)

$$\left(\gamma^0\right)^2 = 1 \qquad (8.2.4a)$$

$$\left(\gamma^i\right)^2 = -1 \qquad i = 1, 2, 3 \qquad (8.2.4b)$$

and $$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \qquad \text{if } \mu \neq \nu. \qquad (8.2.4c)$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (8.2.10)$$

((**Solution**)) Mathematica

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = {{1, 0, 0, 0},
      {0, -1, 0, 0},
      {0, 0, -1, 0},
      {0, 0, 0, -1}};
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0; γu[1] = γux;
γu[2] = γuy; γu[3] = γuz ; γu[5] = ⅈ γu0 .γux.γuy.γuz;
γd[μ_] := Sum[Gd[μ, ν] γu[ν], {ν, 0, 3, 1}];
σu[μ_, ν_] := ⅈ/2 (γu[μ].γu[ν] - γu[ν].γu[μ]);
σd[μ_, ν_] := ⅈ/2 (γd[μ].γd[ν] - γd[ν].γd[μ]);
γd[5] = -ⅈ γd[3] .γd[2] .γd[1] .γd[0];
```

```
γu[0] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[0].γu[0] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[1].γu[1] // MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[2].γu[2] // MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[3].γu[3] // MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
ⅈ/4! Sum[Signature[{a0, a1, a2, a3}]
    γu[a0].γu[a1].γu[a2].γu[a3], {a0, 0, 3, 1},
   {a1, 0, 3, 1}, {a2, 0, 3, 1}, {a3, 0, 3, 1}] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
Au[μ_, ν_] := γu[μ].γu[ν] + γu[ν].γu[μ];
```

```
MatrixForm
```

```
Table[{μ, ν, Au[μ, ν]}, {μ, 0, 3}, {ν, 0, 3}] // TableForm
```

```
0            0              0              0
0            1              2              3
2  0  0  0   0  0  0  0     0  0  0  0     0  0  0  0
0  2  0  0   0  0  0  0     0  0  0  0     0  0  0  0
0  0  2  0   0  0  0  0     0  0  0  0     0  0  0  0
0  0  0  2   0  0  0  0     0  0  0  0     0  0  0  0
1            1              1              1
0            1              2              3
0  0  0  0   -2  0   0   0  0  0  0  0     0  0  0  0
0  0  0  0   0   -2  0   0  0  0  0  0     0  0  0  0
0  0  0  0   0   0   -2  0  0  0  0  0     0  0  0  0
0  0  0  0   0   0   0  -2  0  0  0  0     0  0  0  0
2            2              2              2
0            1              2              3
0  0  0  0   0  0  0  0     -2  0   0   0   0  0  0  0
0  0  0  0   0  0  0  0     0   -2  0   0   0  0  0  0
0  0  0  0   0  0  0  0     0   0   -2  0   0  0  0  0
0  0  0  0   0  0  0  0     0   0   0  -2   0  0  0  0
3            3              3              3
0            1              2              3
0  0  0  0   0  0  0  0     0  0  0  0     -2  0   0   0
0  0  0  0   0  0  0  0     0  0  0  0     0   -2  0   0
0  0  0  0   0  0  0  0     0  0  0  0     0   0   -2  0
0  0  0  0   0  0  0  0     0  0  0  0     0   0   0  -2
```

8-10

8.10 Prove the continuity equation (8.2.11) for the Dirac equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad (8.2.11)$$

((Solution))

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0 \qquad \text{(Dirac equation)} \qquad (1)$$

where

$$p_{\mu} = i\hbar\partial_{\mu}$$

Now we take the adjoint of the Dirac equation

$$(i\partial_{\mu}\psi^{+}\gamma^{\mu +} + \frac{mc}{\hbar}\psi^{+}) = 0$$

or

$$(i\partial_{\mu}\psi^{+}\gamma^{0}\gamma^{\mu}\gamma^{0} + \frac{mc}{\hbar}\psi^{+}) = 0 \qquad (2)$$

since

$$(\gamma^{\mu})^{+} = \gamma^{0}\gamma^{\mu}\gamma^{0}.$$

Multiplying Eq.(2) from the right by γ^{0},

$$(i\partial_{\mu}\psi^{+}\gamma^{0}\gamma^{\mu}(\gamma^{0})^{2} + \frac{mc}{\hbar}\psi^{+}\gamma_{0} = 0$$

We define

$$\psi^{+}\gamma^{0} = \overline{\psi} \qquad \text{(Dirac conjugate)}$$

Noting that $(\gamma^{0})^{2} = 1$, we get

$$(i\partial_{\mu}\overline{\psi}\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi} = 0 \qquad (3)$$

In order to get a probability current, we multiply Eq.(1) from the left by $\overline{\psi}$ and Eq.(3) from the right by ψ and add to obtain

$$\overline{\psi}(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi + (i\partial_{\mu}\overline{\psi}\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi})\psi = 0$$

or

$$\overline{\psi}\gamma^{\mu}(\partial_{\mu}\psi)+(\partial_{\mu}\overline{\psi})\gamma^{\mu}\psi=0$$

or simply

$$\partial_{\mu}(\overline{\psi}\gamma^{\mu}\psi)=0$$

The probability four current is given by

$$j^{\mu}=c\overline{\psi}\gamma^{\mu}\psi=(c\rho,\boldsymbol{J})$$

This satisfies the equation of continuity

$$\partial_{\mu}j^{\mu}=0$$

8-11

8.11 Find the eigenvalues for the free-particle Dirac equation (8.2.20).

$$H=\begin{pmatrix} mc^2 & 0 & cp_z & c(p_x-ip_y) \\ 0 & mc^2 & c(p_x+ip_y) & -cp_z \\ cp_z & c(p_x-ip_y) & -mc^2 & 0 \\ c(p_x+ip_y) & -cp_z & 0 & -mc^2 \end{pmatrix}$$

When $p_x=p_y=0$, $p_z=p$

$$H=\begin{pmatrix} mc^2 & 0 & cp & 0 \\ 0 & mc^2 & 0 & -cp_z \\ cp & 0 & -mc^2 & 0 \\ 0 & -cp & 0 & -mc^2 \end{pmatrix}$$

We use c = 1 units.

$$\begin{bmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = E \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}. \qquad (8.2.20)$$

((**Solution**))

$$\hat{H} = \begin{pmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{pmatrix}$$

$$\hat{H}|1\rangle = m|1\rangle + p|3\rangle, \qquad \hat{H}|3\rangle = p|1\rangle - m|3\rangle$$

$$\hat{H}|2\rangle = m|2\rangle - p|4\rangle, \qquad \hat{H}|4\rangle = -p|2\rangle - m|4\rangle$$

Under the basis of $\{|1\rangle, |3\rangle\}$,we have

$$\begin{aligned} \hat{H}_{sub1} &= \begin{pmatrix} m & p \\ p & -m \end{pmatrix} \\ &= m\hat{\sigma}_z + p\hat{\sigma}_x \\ &= \sqrt{p^2+m^2}\left(\frac{m}{\sqrt{p^2+m^2}}\hat{\sigma}_z + \frac{p}{\sqrt{p^2+m^2}}\hat{\sigma}_x\right) \\ &= \sqrt{p^2+m^2}(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}_1) \end{aligned}$$

where

$$\boldsymbol{n}_1 = \frac{m}{\sqrt{p^2+m^2}}\boldsymbol{e}_z + \frac{p}{\sqrt{p^2+m^2}}\boldsymbol{e}_x$$

$$n_{1z} = \cos\alpha = \frac{m}{\sqrt{p^2+m^2}}, \qquad n_{1x} = \sin\alpha = \frac{p}{\sqrt{p^2+m^2}}$$

The eigenket $|+\boldsymbol{n}_1\rangle$, with the eigenvalue $\sqrt{p^2+m^2}$.

The eigenket $|-\boldsymbol{n}_1\rangle$, with the eigenvalue $-\sqrt{p^2+m^2}$.

Under the basis of $\{|2\rangle, |4\rangle\}$,we have

$$\begin{aligned}
\hat{H}_{sub1} &= \begin{pmatrix} m & -p \\ -p & -m \end{pmatrix} \\
&= m\hat{\sigma}_x - p\hat{\sigma}_x \\
&= \sqrt{p^2+m^2}\,(\frac{m}{\sqrt{p^2+m^2}}\hat{\sigma}_z + \frac{-p}{\sqrt{p^2+m^2}}\hat{\sigma}_x) \\
&= \sqrt{p^2+m^2}\,(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n}_2)
\end{aligned}$$

The eigenket $|+\boldsymbol{n}_2\rangle$, with the eigenvalue $\sqrt{p^2+m^2}$.

The eigenket $|-\boldsymbol{n}_2\rangle$, with the eigenvalue $-\sqrt{p^2+m^2}$.

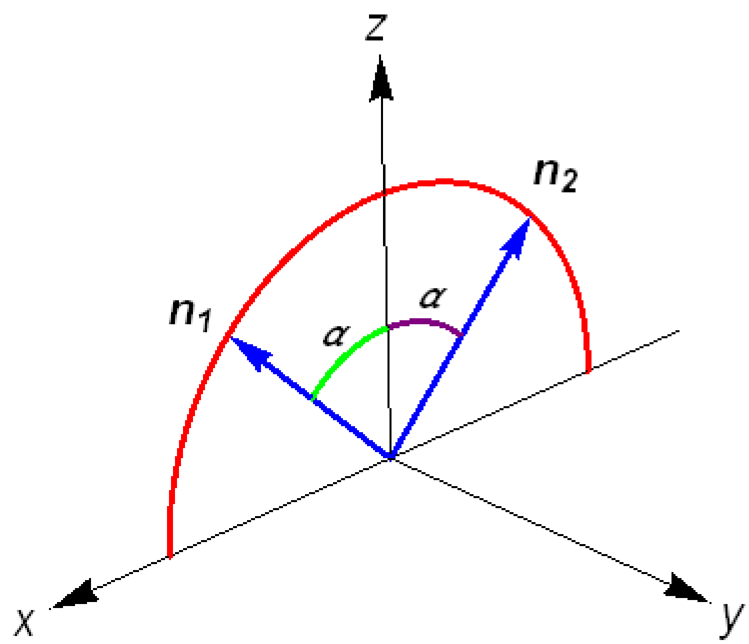
where

$$\boldsymbol{n}_2 = \frac{m}{\sqrt{p^2+m^2}}\boldsymbol{e}_z - \frac{p}{\sqrt{p^2+m^2}}\boldsymbol{e}_x$$

where

$$\boldsymbol{n}_2 = \frac{m}{\sqrt{p^2+m^2}}\boldsymbol{e}_z - \frac{p}{\sqrt{p^2+m^2}}\boldsymbol{e}_x$$

$$n_{2z} = \cos\alpha = \frac{m}{\sqrt{p^2+m^2}}, \qquad n_{2x} = -\sin\alpha = -\frac{p}{\sqrt{p^2+m^2}}$$



$$|+\boldsymbol{n}_1\rangle = \begin{pmatrix} \cos(\frac{\alpha}{2}) \\ 0 \\ \sin(\frac{\alpha}{2}) \\ 0 \end{pmatrix}, \qquad \sqrt{p^2+m^2} \qquad \text{(helicity 1)}$$

$$|-\boldsymbol{n}_1\rangle = \begin{pmatrix} -\sin(\frac{\alpha}{2}) \\ 0 \\ \cos(\frac{\alpha}{2}) \\ 0 \end{pmatrix}. \qquad -\sqrt{p^2+m^2} \qquad \text{(helicity 1)}$$

$$|+\boldsymbol{n}_2\rangle = \begin{pmatrix} 0 \\ \cos(\frac{\alpha}{2}) \\ 0 \\ -\sin(\frac{\alpha}{2}) \end{pmatrix}, \qquad \sqrt{p^2+m^2} \qquad \text{(helicity -1)}$$

$$\left|-\boldsymbol{n}_2\right\rangle = \begin{pmatrix} 0 \\ \sin(\frac{\alpha}{2}) \\ 0 \\ \cos(\frac{\alpha}{2}) \end{pmatrix}. \qquad -\sqrt{p^2+m^2}, \qquad \text{(helicity -1)}$$

We confirm that these eigenkets are also the eigenket of the helicity operator

$$\Lambda_s = \frac{\hbar}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that

$$\Lambda_s H = H\Lambda_s$$

$$\Lambda_s\left|+\boldsymbol{n}_1\right\rangle = \left|+\boldsymbol{n}_1\right\rangle, \qquad \Lambda_s\left|-\boldsymbol{n}_1\right\rangle = \left|-\boldsymbol{n}_1\right\rangle \qquad \text{(helicity 1)}$$

$$\Lambda_s\left|+\boldsymbol{n}_2\right\rangle = -\left|+\boldsymbol{n}_2\right\rangle, \qquad \Lambda_s\left|-\boldsymbol{n}_2\right\rangle = -\left|-\boldsymbol{n}_2\right\rangle \qquad \text{(helicity -1)}$$

8-12

8.12 Insert one of the four solutions $u_{R,L}^{(\pm)}(p)$ from (8.2.22) into the four-vector probability current (8.2.13) and interpret the result.

$$u_R^{(+)}(p) = \begin{bmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{bmatrix} \qquad u_L^{(+)}(p) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_p+m} \end{bmatrix} \qquad \text{for } E = +E_p, \tag{8.2.22a}$$

$$u_R^{(-)}(p) = \begin{bmatrix} \frac{-p}{E_p+m} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad u_L^{(-)}(p) = \begin{bmatrix} 0 \\ \frac{p}{E_p+m} \\ 0 \\ 1 \end{bmatrix} \quad \text{for } E = -E_p. \quad (8.2.22b)$$

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi \qquad (8.2.13)$$

((**Solution**)) Mathematica

Sakurai 8-12

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0; γu[1] = γux;
γu[2] = γuy; γu[3] = γuz ; γu[5] = i γu0 .γux.γuy.γuz;
γd[μ_] := Sum[Gd[μ, ν] γu[ν], {ν, 0, 3, 1}];
σu[μ_, ν_] := i/2 (γu[μ].γu[ν] - γu[ν].γu[μ]);
σd[μ_, ν_] := i/2 (γd[μ].γd[ν] - γd[ν].γd[μ]);
γd[5] = -i γd[3] .γd[2] .γd[1] .γd[0]; N1 = Sqrt[1 + p^2/(E1 + m)^2];
rule1 = {p -> Sqrt[E1 - m^2]};
rule2 = {p^2 -> E1^2 - m^2};
```

$$uRp = \frac{1}{N1}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E1+m} \\ 0 \end{pmatrix};\ uLp = \frac{1}{N1}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E1+m} \end{pmatrix};\ uRm = \frac{1}{N1}\begin{pmatrix} \frac{-p}{E1+m} \\ 0 \\ 1 \\ 0 \end{pmatrix};$$

$$uLm = \frac{1}{N1}\begin{pmatrix} 0 \\ \frac{p}{E1+m} \\ 0 \\ 1 \end{pmatrix};$$

Probability current density for uRp

```
j01 = Transpose[uRp].γu0.γu0.uRp /. rule1 // Simplify
```

{{1}}

```
j11 = Transpose[uRp].γu0.γu[1].uRp /. rule1
```

{{0}}

```
j21 = Transpose[uRp].γu0.γu[2].uRp /. rule1
```

{{0}}

```
j31 = Transpose[uRp].γu0.γu[3].uRp /. rule2 // FullSimplify
```

$$\left\{\left\{\frac{p}{E1}\right\}\right\}$$

Probability current density for uLp

```
j02 = Transpose[uLp].γu0.γu0.uLp /. rule1 // Simplify
```

{{1}}

```
j12 = Transpose[uLp].γu0.γu[1].uLp
```

{{0}}

```
j22 = Transpose[uLp].γu0.γu[2].uLp
```

{{0}}

```
j32 = Transpose[uLp].γu0.γu[3].uLp /. rule2 // FullSimplify
```

$\left\{\left\{\frac{p}{E1}\right\}\right\}$

Probability current density for uRm

```
j03 = Transpose[uRm].γu0.γu0.uRm /. rule2 // FullSimplify
```

{{1}}

```
j13 = Transpose[uRm].γu0.γu[1].uRm
```

{{0}}

```
j23 = Transpose[uRm].γu0.γu[2].uRm
```

{{0}}

```
j33 = Transpose[uRm].γu0.γu[3].uRm /. rule2 // FullSimplify
```

$\left\{\left\{-\frac{p}{E1}\right\}\right\}$

Probability current density for uLm

```
j04 = Transpose[uLm].γu0.γu0.uLp
```

{{0}}

```
j14 = Transpose[uLm].γu0.γu[1].uLm
```

{{0}}

```
j24 = Transpose[uLm].γu0.γu[2].uLm
```

{{0}}

```
j34 = Transpose[uLm].γu0.γu[3].uLm /. rule2 // FullSimpli
```

$\left\{\left\{-\frac{p}{E1}\right\}\right\}$

8-13

8.13 Make use of Problem 8.9 to show that U_T as defined by (8.3.28) is just $\sigma^2 \otimes I$, up to a phase factor.

$$U_T = \gamma^1 \gamma^3 \qquad (8.3.28)$$

((**Solution**))

$$S = i\gamma^5\gamma^0 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$I_2 \otimes \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$U_T = \gamma^1 \gamma^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$i\sigma^2 \otimes I_2 = i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

8-14

8.14 Write down the positive-helicity, positive-energy free-particle Dirac spinor wave function $\Psi(\mathbf{x}, t)$.

(a) Construct the spinors $\mathcal{P}\Psi$, $\mathcal{C}\Psi$, $\mathcal{T}\Psi$.

(b) Construct the spinor $\mathcal{CPT}\Psi$ and interpret it using the discussion of negative-energy solutions to the Dirac equation.

((Solution))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1]; σy = PauliMatrix[2];
σz = PauliMatrix[3]; I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
```

```
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0;
γu[1] = γux; γu[2] = γuy; γu[3] = γuz ;
γu[5] = ⅈ γu0 .γux.γuy.γuz;
γd[μ_] := Sum[Gd[μ, ν] γu[ν], {ν, 0, 3, 1}];
σu[μ_, ν_] := ⅈ/2 (γu[μ].γu[ν] - γu[ν].γu[μ]);
σd[μ_, ν_] := ⅈ/2 (γd[μ].γd[ν] - γd[ν].γd[μ]);
γd[5] = -ⅈ γd[3] .γd[2] .γd[1] .γd[0];
Σu[3] = σu[1, 2]; Σu[2] = σu[3, 1]; Σu[1] = σu[2, 3];
αu[1] = αx; αu[2] = αy; αu[3] = αz;
β = γu[0];
```

$$\psi[z\_, t\_] := \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E1+m} \\ 0 \end{pmatrix} \text{Exp}[ⅈ (p\, z - E1\, t)];$$

Hamiltonain and helicity: eigenvalue problem

```
H = αz p + β m ;
```

$$H.\psi[z, t] - E1\,\psi[z, t] \;/.\; \left\{E1 \to \sqrt{m^2 + p^2}\right\} \;//\; \text{Simplify}$$

```
{{0}, {0}, {0}, {0}}
```

```
Σu[3].ψ[z, t] - ψ[z, t]
```

```
{{0}, {0}, {0}, {0}}
```

Parity operator

```
P1ψ = γu[0].ψ[-z, t] // Simplify;
P1ψ // MatrixForm
```

$$\begin{pmatrix} e^{-i\,(E1\,t+p\,z)} \\ 0 \\ -\frac{e^{-i\,(E1\,t+p\,z)}\,p}{E1+m} \\ 0 \end{pmatrix}$$

Charge conjugate

```
C1ψ = i γu[2].ψ[z, t]* // Simplify;
C1ψ // MatrixForm
```

$$\begin{pmatrix} 0 \\ -\frac{e^{i\,(E1\,t-p\,z)}\,p}{E1+m} \\ 0 \\ e^{i\,(E1\,t-p\,z)} \end{pmatrix}$$

CP

```
CP1ψ = i γu[2].P1ψ // Simplify;
CP1ψ // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{e^{-i\,(E1\,t+p\,z)}\,p}{E1+m} \\ 0 \\ e^{-i\,(E1\,t+p\,z)} \end{pmatrix}$$

Time reversal

```
Tψ1 = γu[1].γu[3].ψ[z, t]* // Simplify;
Tψ1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ -e^{i\,(E1\,t-p\,z)} \\ 0 \\ -\frac{e^{i\,(E1\,t-p\,z)}\,p}{E1+m} \end{pmatrix}$$

CPT

```
PTψ1 = γu[0].Tψ1 /. z → -z;
PTψ1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ -e^{i\,(E1\,t+p\,z)} \\ 0 \\ \frac{e^{i\,(E1\,t+p\,z)}\,p}{E1+m} \end{pmatrix}$$

```
CPTψ1 = i γu[2].PTψ1*; CPTψ1 // MatrixForm
```

$$\begin{pmatrix} \frac{e^{-i\,(E1\,t+p\,z)}\,p}{E1+m} \\ 0 \\ e^{-i\,(E1\,t+p\,z)} \\ 0 \end{pmatrix}$$

8-16

8.16 Expand the energy eigenvalues given by (8.4.43) in powers of $Z\alpha$, and show that the result is equivalent to including the relativistic correction to kinetic energy (5.3.10) and the spin-orbit interaction (5.3.31) to the nonrelativistic energy eigenvalues for the one-electron atom (8.4.44).

$$E = \frac{mc^2}{\left[1 + \frac{(Z\alpha)^2}{\left[\sqrt{(j+1/2)^2-(Z\alpha)^2}+n'\right]^2}\right]^{1/2}}. \qquad (8.4.43)$$

$$\Delta_{nl}^{(1)} = E_n^{(0)}\left[\frac{Z^2\alpha^2}{n^2}\left(-\frac{3}{4}+\frac{n}{l+\frac{1}{2}}\right)\right] \tag{5.3.10a}$$

$$= -\frac{1}{2}m_e c^2 Z^4\alpha^4\left[-\frac{3}{4n^4}+\frac{1}{n^3\left(l+\frac{1}{2}\right)}\right]. \tag{5.3.10b}$$

$$\Delta_{nlj} = -\frac{Z^2\alpha^2}{2nl(l+1)(l+1/2)}E_n^{(0)}\begin{Bmatrix} l & j=l+\frac{1}{2} \\ -(l+1) & j=l-\frac{1}{2} \end{Bmatrix}. \tag{5.3.31}$$

$$E = mc^2 - \frac{1}{2}\frac{mc^2(Z\alpha)^2}{n^2}, \tag{8.4.44}$$

((**Solution**)) using Mathematica

Sakurai problem 8-16

```
Clear["Global`*"]; E1[n1_, j_] := me c^2 / Sqrt[1 + Z^2 α^2 / (n1 - j - 1/2 + Sqrt[(j + 1/2)^2 - Z^2 α^2])^2];
```

Taylor expansion of the energy for the Dirac theory as a function of the fine structure constant α

```
Series[E1[n, j], {α, 0, 5}] // FullSimplify[#, j > 0] &
```

$$c^2\,\text{me} - \frac{(c^2\,\text{me}\,Z^2)\,\alpha^2}{2\,n^2} + \frac{c^2\,\text{me}\,(3+6j-8n)\,Z^4\,\alpha^4}{8\,(1+2j)\,n^4} + O[\alpha]^6$$

```
Series[E1[n, j], {α, 0, 7}] // FullSimplify[#, j > 0] &
```

$$c^2\,\text{me} - \frac{(c^2\,\text{me}\,Z^2)\,\alpha^2}{2\,n^2} + \frac{c^2\,\text{me}\,(3+6j-8n)\,Z^4\,\alpha^4}{8\,(1+2j)\,n^4} - \frac{\left(c^2\,\text{me}\left(5\,(1+2j)^3-24\,(1+2j)^2\,n+24\,(1+2j)\,n^2+16\,n^3\right)Z^6\right)\alpha^6}{16\left((1+2j)^3\,n^6\right)} + O[\alpha]^8$$

```
Series[E1[n, j], {α, 0, 9}] // FullSimplify[#, j > 0] &
```

$$c^2\,me - \frac{(c^2\,me\,Z^2)\,\alpha^2}{2\,n^2} + \frac{c^2\,me\,(3+6\,j-8\,n)\,Z^4\,\alpha^4}{8\,(1+2\,j)\,n^4} - \frac{\left(c^2\,me\,\left(5\,(1+2\,j)^3 - 24\,(1+2\,j)^2\,n + 24\,(1+2\,j)\,n^2 + 16\,n^3\right)\,Z^6\right)\,\alpha^6}{16\,\left((1+2\,j)^3\,n^6\right)} + \frac{1}{128\,(1+2\,j)^5\,n^8}\,c^2\,me\,\left(35\,(1+2\,j)^5 - 240\,(1+2\,j)^4\,n + 480\,(1+2\,j)^3\,n^2 - 64\,(1+2\,j)^2\,n^3 - 384\,(1+2\,j)\,n^4 - 256\,n^5\right)\,Z^8\,\alpha^8 + O[\alpha]^{10}$$

Definition of the the energy E0 and ΔE (Dirac theory)

$E0 = -\frac{(c^2\,me\,Z^2)\,\alpha^2}{2\,n^2}$; $\Delta E = \frac{c^2\,me\,(3+6\,j-8\,n)\,Z^4\,\alpha^4}{8\,(1+2\,j)\,n^4}$

$$\Delta E = E0\,\frac{\frac{c^2\,me\,(3+6\,j-8\,n)\,Z^4\,\alpha^4}{8\,(1+2\,j)\,n^4}}{-\frac{(c^2\,me\,Z^2)\,\alpha^2}{2\,n^2}} \text{ // FullSimplify}$$

$$-\frac{E0\,(3+6\,j-8\,n)\,Z^2\,\alpha^2}{4\,(1+2\,j)\,n^2}$$

Definition of Δrel (relativistic correction)

```
Δrel = E0 Z^2 α^2 (-3/(4 n^2) + 1/(n (L1 + 1/2)));

Δrelp = Δrel /. {L1 → j - 1/2} // Simplify
```

$$\frac{E0\,(-3\,j+4\,n)\,Z^2\,\alpha^2}{4\,j\,n^2}$$

```
Δrelm = Δrel /. {L1 → j + 1/2} // Simplify
```

$$\frac{E0\left(-3+\frac{4\,n}{1+j}\right)Z^2\,\alpha^2}{4\,n^2}$$

Definition of Δso (spin-orbit interaction)

```
Δsop = -E0 ((Z α)^2 (L1))/(2 n L1 (L1 + 1) (L1 + 1/2)) /. {L1 → j - 1/2} // Simplify;
Δsom = -E0 ((Z α)^2 (-L1 - 1))/(2 n L1 (L1 + 1) (L1 + 1/2)) /. {L1 → j + 1/2} // Simplify;

p1 = Δrelp + Δsop // FullSimplify
```

$$-\frac{E0\,(3+6\,j-8\,n)\,Z^2\,\alpha^2}{4\,(1+2\,j)\,n^2}$$

```
p2 = Δrelm + Δsom // FullSimplify
```

$$-\frac{E0\,(3+6\,j-8\,n)\,Z^2\,\alpha^2}{4\,(1+2\,j)\,n^2}$$

```
p1 - p2 // FullSimplify
```

0

```
ΔE - p1 // Simplify
```

0

In summary, ΔE=p1=p2= $\frac{E0\left(-3+\frac{4\,n}{1+j}\right)Z^2\,\alpha^2}{4\,n^2}$

8.17 The National Institute of Standards and Technology (NIST) maintains a web site with up-to-date high-precision data on the atomic energy levels of hydrogen and deuterium:

The accompanying table of data was obtained from that web site. It gives the energies of transitions between the $(n,l,j) = (1,0,1/2)$ energy level and the energy level indicated by the columns on the left.

n	l	j	$[E(n,l,j) - E(1,0,1/2)]/hc$ (cm^{-1})
2	0	1/2	82 258.954 399 2832(15)
2	1	1/2	82 258.919 113 406(80)
2	1	3/2	82 259.285 001 249(80)
3	0	1/2	97 492.221 724 658(46)
3	1	1/2	97 492.211 221 463(24)
3	1	3/2	97 492.319 632 775(24)
3	2	3/2	97 492.319 454 928(23)
3	2	5/2	97 492.355 591 167(23)
4	0	1/2	102 823.853 020 867(68)
4	1	1/2	102 823.848 581 881(58)
4	1	3/2	102 823.894 317 849(58)
4	2	3/2	102 823.894 241 542(58)
4	2	5/2	102 823.909 486 535(58)
4	3	5/2	102 823.909 459 541(58)
4	3	7/2	102 823.917 081 991(58)

(The number in parentheses is the numerical value of the standard uncertainty referred to the last figures of the quoted value.) Compare these values to those predicted by (8.4.43). (You may want to make use of Problem 8.16.) In particular:

(a) Compare fine-structure splitting between the $n = 2, j = 1/2$ and $n = 2, j = 3/2$ states to (8.4.43).

(b) Compare fine-structure splitting between the $n = 4, j = 5/2$ and $n = 4, j = 7/2$ states to (8.4.43).

(c) Compare the $1S \rightarrow 2S$ transition energy to the first line in the table. Use as many significant figures as necessary in the values of the fundamental constants, to compare the results within standard uncertainty.

(d) How many examples of the Lamb shift are demonstrated in this table? Identify one example near the top and another near the bottom of the table, and compare their values.

((**Solution**)) Mathematica

Sakurai Problem 8-17

```
Clear["Global`*"];
rule1 = {c → 2.99792 × 10^10,
   ℏ → 1.054571628 10^-27, me → 9.10938215 10^-28,
   qe → 4.8032068 × 10^-10, eV → 1.602176487 × 10^-12,
   meV → 1.602176487 × 10^-15,
   keV → 1.602176487 × 10^-9,
   MeV → 1.602176487 × 10^-6}; Z = 1;
```

$$\alpha = \frac{qe^2}{\hbar\, c} \;/.\; \text{rule1};$$

$$\text{E1}[n1\_, j\_] := \frac{me\, c^2}{\sqrt{1 + \frac{Z^2 \alpha^2}{\left(n1 - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2}\right)^2 - Z^2 \alpha^2}\right)^2}}};$$

(a) Compare fine-structure splitting between the $n = 2, j = 1/2$ and $n = 2, j = 3/2$ states to (8.4.43).

Z=1. A1 is the energy $E\left(2, \frac{3}{2}\right)$ in the Dirac theory. Unit (eV)

$$\mathbf{A1} = \frac{\mathbf{E1}\left[2,\ \frac{3}{2}\right]}{\mathbf{eV}}\ //.\ \mathbf{rule1}$$

510 994.

A2 is the energy $E\left(2, \frac{1}{2}\right)$ in the Dirac theory. Unit (eV)

$$\mathbf{A2} = \frac{\mathbf{E1}\left[2,\ \frac{1}{2}\right]}{\mathbf{eV}}\ //.\ \mathbf{rule1}$$

510 994.

The difference between A1 and A2 (eV)

```
A1 - A2 // Simplify // ScientificForm
```

4.52845×10^{-5}

Experimental results forthe difference between $E\left(2, \frac{3}{2}\right)$ and $E\left(2, \frac{1}{2}\right)$ (eV)

$$(82\,259.285001249 - 82\,258.919113406)\ \frac{2\,\pi\,\hbar\,c}{\mathbf{eV}}\ //.\ \mathbf{rule1}\ //\ \mathbf{ScientificForm}$$

4.53642×10^{-5}

(b) Compare fine-structure splitting between the $n = 4, j = 5/2$ and $n = 4, j = 7/2$ states to (8.4.43).

B1 is the energy $E\left(4, \frac{7}{2}\right)$ in the Dirac theory. Unit (eV)

```
B1 = E1[4, 7/2] / eV //. rule1
```

$$B1 = \frac{E1\left[4, \frac{7}{2}\right]}{eV} \text{ //. rule1}$$

510996.

B2 is the energy $E\left(4, \frac{5}{2}\right)$ in the Dirac theory. Unit (eV)

$$B2 = \frac{E1\left[4, \frac{5}{2}\right]}{eV} \text{ //. rule1}$$

510996.

The difference between A1 and A2 (eV)

```
B1 - B2 // Simplify
```

9.43313×10^{-7}

Experimental results forthe difference between $E\left(4, \frac{7}{2}\right)$ and $E\left(4, \frac{5}{2}\right)$ (eV)

$$(102823.917081991 - 102823.909459541) \frac{2 \pi \hbar c}{eV} \text{ //. rule1 // ScientificForm}$$

9.45062×10^{-7}

(c) Compare the $1S \to 2S$ transition energy to the first line in the table. Use as many significant figures as necessary in the values of the fundamental constants, to compare the results within standard uncertainty.

The energy difference, $E\left(2, \frac{1}{2}\right)$ and $E\left(1, \frac{1}{2}\right)$ in the Dirac theory. Unit (cm^{-1})

$$C1 = \frac{E1\left[2, \frac{1}{2}\right] - E1\left[1, \frac{1}{2}\right]}{2 \pi \hbar c} \text{ //. rule1 // ScientificForm}$$

8.23043×10^{4}

The experimental result for the energy difference, $E\left(2, \frac{1}{2}\right)$ and $E\left(1, \frac{1}{2}\right)$.
Unit (cm^{-1})

82258.9543492832 // ScientificForm

8.2259×10^{4}

(d) How many examples of the Lamb shift are demonstrated in this table? Identify one example near the top and another near the bottom of the table, and compare their values.

The lamb shift : E(2,0,1/2) and E(2,1,1/2)

82259.285001249 - 82258.919113406 // ScientificForm

3.65888×10^{-1}

The lamb shift : E(3,0,1/2) and E(3,1,1/2)

```
97492.221724658 - 97492.211221463 //
 ScientificForm
```

1.05032×10^{-2}

The lamb shift : E (3, 1, 3/2) and E (3, 2, 3/2)

```
97492.319632775 - 97492.319454928 //
 ScientificForm
```

1.77847×10^{-4}

The lamb shift : E (4, 0, 1/2) and E (4, 1, 1/2)

```
102823.853020867 - 102823.848581881 //
 ScientificForm
```

4.43899×10^{-3}

((Pearson))

The lamb shift : E (4, 1, 3/2) and E (4, 2, 3/2)

```
102823.894317849 - 102823.894241542 //
 ScientificForm
```

7.6307×10^{-5}

The lamb shift : E (4,3, 5/2) and E (4,2, 5/2)

```
102823.909486535 - 102823.909459541 //
 ScientificForm
```

2.6994×10^{-5}