Translation operator and rotation operator for the 3D system

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Translation operators are linear and unitary. They are closely related to the momentum operator; for example, a translation operator that moves by an infinitesimal amount in the *x* direction has a simple relationship to the *x*-component of the momentum operator. Because of this, conservation of momentum holds when the translation operators commute with the Hamiltonian, i.e. when laws of physics are translation-invariant.

Here we discuss the properties of the translation operator for the 3D system.

1. Translation operator

The state vector:

$$|\psi\rangle = \int d\boldsymbol{r} |\boldsymbol{r}\rangle\langle\boldsymbol{r}|\psi\rangle,$$

where

$$\langle\boldsymbol{r}|\boldsymbol{r}'\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}'),$$

$$\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle = |\boldsymbol{r}+\boldsymbol{a}\rangle.$$

Translation operator:

$$|\psi\rangle \rightarrow |\psi'\rangle = \hat{T}(\boldsymbol{a})|\psi\rangle,$$

with

$$\hat{T}^{+}(\boldsymbol{a})\hat{T}(\boldsymbol{a}) = \hat{1}. \qquad \text{Unitary operator}$$

(a)

It is expected from the analogy of classical mechanics that

$$\langle\psi'|\hat{\boldsymbol{r}}|\psi'\rangle = \langle\psi|\hat{\boldsymbol{r}}|\psi\rangle + \langle\psi|\boldsymbol{a}|\psi\rangle.$$

or

$$\langle\psi|\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a})|\psi\rangle = \langle\psi|\hat{\boldsymbol{r}}+\boldsymbol{a}|\psi\rangle,$$

leading to the relation

$$\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a}) = \hat{\boldsymbol{r}} + \boldsymbol{a}\hat{1},$$

or

$$\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a}) = \hat{T}(\boldsymbol{a})\hat{\boldsymbol{r}} + \boldsymbol{a}\hat{T}(\boldsymbol{a}),$$

or

$$[\hat{\boldsymbol{r}}, \hat{T}(\boldsymbol{a})] = \boldsymbol{a}\hat{T}(\boldsymbol{a}).$$

Using the commutation relation, we get

$$\begin{aligned}\hat{\boldsymbol{r}}\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle &= \hat{T}(\boldsymbol{a})\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle + \boldsymbol{a}\hat{T}(\boldsymbol{a}))|\boldsymbol{r}\rangle \\ &= (\hat{T}(\boldsymbol{a})\boldsymbol{r}|\boldsymbol{r}\rangle + \boldsymbol{a}\hat{T}(\boldsymbol{a}))|\boldsymbol{r}\rangle, \\ &= (\boldsymbol{r} + \boldsymbol{a})\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle\end{aligned}$$

This implies that $\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle$ is the eigenket of $\hat{\boldsymbol{r}}$ with the eigenvalue $(\boldsymbol{r} + \boldsymbol{a})$,

$$\hat{T}(\boldsymbol{a})|\boldsymbol{r}\rangle = |\boldsymbol{r} + \boldsymbol{a}\rangle,$$

or

$$\langle \boldsymbol{r} + \boldsymbol{a}| = \langle \boldsymbol{r}|\hat{T}^{+}(\boldsymbol{a}).$$

Note that

$$|\boldsymbol{r}\rangle = \hat{T}^{+}(\boldsymbol{a})|\boldsymbol{r} + \boldsymbol{a}\rangle.$$

When $\boldsymbol{r} \to \boldsymbol{r} - \boldsymbol{a}$, we get

$$|\boldsymbol{r} - \boldsymbol{a}\rangle = \hat{T}^{+}(\boldsymbol{a})|\boldsymbol{r}\rangle,$$

or

$$\langle \boldsymbol{r}|\hat{T}(\boldsymbol{a}) = \langle \boldsymbol{r}-\boldsymbol{a}|$$

(b)

It is also expected from the analogy of classical mechanics that

$$\langle \psi'|\hat{\boldsymbol{p}}|\psi'\rangle = \langle \psi|\hat{\boldsymbol{p}}|\psi\rangle .$$

or

$$\langle \psi|\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{p}}\hat{T}(\boldsymbol{a})|\psi\rangle = \langle \psi|\hat{\boldsymbol{p}}|\psi\rangle ,$$

leading to the commutation relation

$$\hat{T}^{+}(\boldsymbol{a})\hat{\boldsymbol{p}}\hat{T}(\boldsymbol{a}) = \hat{\boldsymbol{p}} , \qquad \text{or} \qquad [\hat{\boldsymbol{p}},\hat{T}(\boldsymbol{a})] = 0 .$$

We assume that the infinitesimal translation operator is given by

$$\hat{T}(d\boldsymbol{r}) = \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}}\cdot d\boldsymbol{r} ,$$

where

$$\hat{T}^{+}(d\boldsymbol{r})\hat{T}(d\boldsymbol{r}) = \hat{1} ,$$

$$\hat{T}^{+}(d\boldsymbol{r})\hat{\boldsymbol{r}}\hat{T}(d\boldsymbol{r}) = \hat{\boldsymbol{r}} + d\boldsymbol{r}\hat{1} ,$$

$$\hat{T}^{+}(d\boldsymbol{r})\hat{\boldsymbol{p}}\hat{T}(d\boldsymbol{r}) = \hat{\boldsymbol{p}} .$$

(a) $\hat{\boldsymbol{G}}$ is a Hermitian operator.

$$\begin{aligned}\hat{T}^{+}(d\boldsymbol{r})\hat{T}(d\boldsymbol{r}) &= (\hat{1} + \frac{i}{\hbar}\hat{\boldsymbol{G}}^{+}\cdot d\boldsymbol{r})(\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}}\cdot d\boldsymbol{r}) \\ &= \hat{1} + \frac{i}{\hbar}(\hat{\boldsymbol{G}}^{+} - \hat{\boldsymbol{G}})\cdot d\boldsymbol{r} \\ &= \hat{1}\end{aligned}$$

Then we have

$$\hat{\boldsymbol{G}}^{+} = \hat{\boldsymbol{G}} \qquad \text{(Hermitian operator)}$$

(b) The commutation relation (I)

$$[\hat{\boldsymbol{p}}, \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r}] = 0,$$

or

$$[\hat{p}_{\alpha}, \hat{G}_{\beta}] = 0 \qquad \text{with } \alpha, \beta = x, y, z.$$

(c) The commutation relation (II)

$$(\hat{1} + \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r})\hat{\boldsymbol{r}}(\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{G}} \cdot d\boldsymbol{r} = \hat{\boldsymbol{r}} + d\boldsymbol{r}\hat{1},$$

or

$$(\hat{\boldsymbol{G}} \cdot d\boldsymbol{r})\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}(\hat{\boldsymbol{G}} \cdot d\boldsymbol{r}) = \frac{\hbar}{i} d\boldsymbol{r}\hat{1},$$

or

$$\sum_{\alpha}[\hat{G}_{\alpha}, \hat{x}_{\beta}]dx_{\alpha} = \frac{\hbar}{i} dx_{\beta}\hat{1} = \frac{\hbar}{i}\hat{1}\sum_{\alpha}\delta_{\alpha\beta}dx_{\alpha},$$

So we get the commutation relation

$$[\hat{G}_{\alpha}, \hat{x}_{\beta}] = \frac{\hbar}{i}\hat{1}.$$

From these results, it can be conclued that

$$\hat{\boldsymbol{G}} = \hat{\boldsymbol{p}}.$$

3. Infinitesimal translation operator

$$
\begin{aligned}
\hat{T}(d\boldsymbol{r})|\psi\rangle &= \hat{T}(d\boldsymbol{r})\int d\boldsymbol{r}'|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\psi\rangle \\
&= \int d\boldsymbol{r}'\hat{T}(d\boldsymbol{r})|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\psi\rangle \\
&= \int d\boldsymbol{r}'|\boldsymbol{r}'+d\boldsymbol{r}\rangle\langle\boldsymbol{r}'|\psi\rangle \\
&= \int d\boldsymbol{r}'|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'-d\boldsymbol{r}|\psi\rangle
\end{aligned}
$$

Using the Taylor expansion

$$
\langle\boldsymbol{r}'-d\boldsymbol{r}|\psi\rangle = \psi(\boldsymbol{r}'-d\boldsymbol{r}) = \psi(\boldsymbol{r}') - \frac{\partial\psi(\boldsymbol{r}')}{\partial\boldsymbol{r}'}d\boldsymbol{r}\ ,
$$

we have

$$
\begin{aligned}
\hat{T}(d\boldsymbol{r})|\psi\rangle &= \int d\boldsymbol{r}'|\boldsymbol{r}'\rangle[\psi(\boldsymbol{r}') - \frac{\partial\psi(\boldsymbol{r}')}{\partial\boldsymbol{r}'}d\boldsymbol{r}] \\
&= \int d\boldsymbol{r}'|\boldsymbol{r}'\rangle[\langle\boldsymbol{r}'|\psi\rangle - \frac{i}{\hbar}\langle\boldsymbol{r}'|\hat{\boldsymbol{p}}|\psi\rangle\cdot d\boldsymbol{r}] \\
&= (\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot d\boldsymbol{r})|\psi\rangle
\end{aligned}
$$

since

$$
\langle\boldsymbol{r}|\hat{\boldsymbol{p}}|\psi\rangle = \frac{\hbar}{i}\nabla_r\langle\boldsymbol{r}|\psi\rangle\ .
$$

Then we have

$$
\hat{T}(d\boldsymbol{r}) = 1 - \frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot d\boldsymbol{r}\ .
$$

4. Finite translation operator

The finite translation operator is given by

$$
\hat{T}(\boldsymbol{a}) = \lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\frac{\boldsymbol{a}}{N})^N = \exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a}),
$$

where we use the definition of e^{-x} as

$$e^{-x} = \lim_{N \to \infty} (1 - \frac{x}{N})^N .$$

5. Transformation function

Using the relation

$$\langle \boldsymbol{r} | \hat{\boldsymbol{p}} | \psi \rangle = \frac{\hbar}{i} \nabla \langle \boldsymbol{r} | \psi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \boldsymbol{r}} \langle \boldsymbol{r} | \psi \rangle .$$

we get the transformation function as

$$\langle \boldsymbol{r} | \boldsymbol{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar} \boldsymbol{p} \cdot \boldsymbol{r}) .$$

We note that

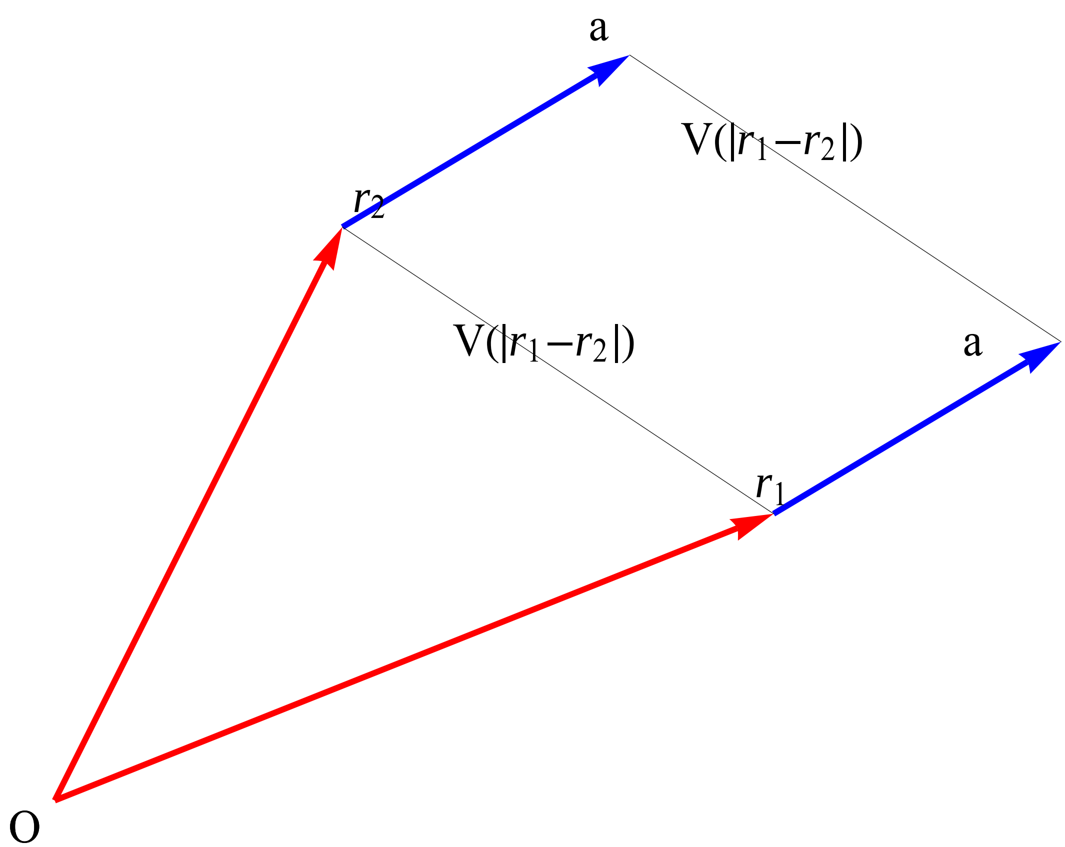
$$\langle \boldsymbol{p} | \boldsymbol{p}' \rangle = \delta(\boldsymbol{p} - \boldsymbol{p}')$$

Then we have

$$\langle \boldsymbol{p} | \boldsymbol{p}' \rangle = \int dr \langle \boldsymbol{p} | \boldsymbol{r} \rangle \langle \boldsymbol{r} | \boldsymbol{p}' \rangle = \frac{1}{(2\pi\hbar)^3} \int d\boldsymbol{r} \exp[\frac{i}{\hbar} (\boldsymbol{p}' - \boldsymbol{p}) \cdot \boldsymbol{r}] .$$

6. Translation operator for two-body problem

We consider a Hamiltonian of two particles at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. p_1 and p_2 are the momentum of particles 1 and 2, respectively.



The Hamiltonian is given by

$$\hat{H} = \frac{1}{2m_1}\hat{\boldsymbol{p}}_1^{\ 2} + \frac{1}{2m_2}\hat{\boldsymbol{p}}_2^{\ 2} + V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|),$$

where $V(|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2|)$ is the interaction between two particles with mass m_1 and m_2. This is so-called the central field problem.

((**Definition of Central-force Problem**))

In classical mechanics, the **central-force problem** is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

We consider the two particles (denoted by particle 1 and particle 2) located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. The position ket vector for these two particles is expressed by

$$|\boldsymbol{r}_1, \boldsymbol{r}_2\rangle = |\boldsymbol{r}_1\rangle_1 \otimes |\boldsymbol{r}_2\rangle_2,$$

using the Kronecker product $\otimes$. Note that we have the commutation relations,

$$[\hat{x}_{1i}, \hat{p}_{1j}] = i\hbar\delta_{ij}, \qquad [\hat{x}_{2i}, \hat{p}_{2j}] = i\hbar\delta_{ij},$$

$$[\hat{x}_{1i}, \hat{p}_{2j}] = 0, \quad [\hat{x}_{2i}, \hat{p}_{1j}] = 0.$$

which means that the operators for particle 1 and particle 2 are completely independent each other.

We introduce the translation operator (one particle) as

$$\hat{T}_1(\boldsymbol{a})\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle = \hat{T}_1(\boldsymbol{a})\left|\boldsymbol{r}_1\right\rangle_1 \otimes \left|\boldsymbol{r}_2\right\rangle_2 = \left|\boldsymbol{r}_1 + \boldsymbol{a}\right\rangle_1 \otimes \left|\boldsymbol{r}_2\right\rangle_2,$$

$$\hat{T}_2(\boldsymbol{a})\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle = \hat{T}_2(\boldsymbol{a})\left|\boldsymbol{r}_1\right\rangle_1 \otimes \left|\boldsymbol{r}_2\right\rangle_2 = \left|\boldsymbol{r}_1\right\rangle_1 \otimes \hat{T}_2(\boldsymbol{a})\left|\boldsymbol{r}_2\right\rangle_2 = \left|\boldsymbol{r}_1\right\rangle_1 \otimes \left|\boldsymbol{r}_2 + \boldsymbol{a}\right\rangle_2.$$

Here we assume that

$$[\hat{\boldsymbol{p}}_1, \hat{\boldsymbol{p}}_2] = 0,$$

Then we have

$$\begin{aligned}
\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}) &= \exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}_1 \cdot \boldsymbol{a})\exp(-\frac{i}{\hbar}\hat{\boldsymbol{p}}_2 \cdot \boldsymbol{a}) \\
&= \exp[-\frac{i}{\hbar}(\hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2) \cdot \boldsymbol{a}] \\
&= \exp[-\frac{i}{\hbar}\hat{\boldsymbol{P}} \cdot \boldsymbol{a}]
\end{aligned}$$

where we use the formula

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A} + \hat{B}),$$

when $[\hat{A}, \hat{B}] = 0$. We also define the total momentum as

$$\hat{\boldsymbol{P}} = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2.$$

Using the closure relation, we can define the wave function as

$$\left|\psi\right\rangle = \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle\left\langle \boldsymbol{r}_1, \boldsymbol{r}_2 \middle| \psi\right\rangle .$$

We now show that

$$[V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|), \hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})] = 0$$

((**Proof**))

$$\begin{aligned}
\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|)\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle &= \hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})V(\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|)\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle \\
&= V(\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|)\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle \\
&= V(\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|)\left|\boldsymbol{r}_1 + \boldsymbol{a}, \boldsymbol{r}_2 + \boldsymbol{a}\right\rangle
\end{aligned}$$

$$\begin{aligned}
V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|)\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle &= V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|)\left|\boldsymbol{r}_1 + \boldsymbol{a}, \boldsymbol{r}_2 + \boldsymbol{a}\right\rangle \\
&= V(\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|)\left|\boldsymbol{r}_1 + \boldsymbol{a}, \boldsymbol{r}_2 + \boldsymbol{a}\right\rangle
\end{aligned}.$$

Then we have

$$\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|)\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle = V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|)\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})\left|\boldsymbol{r}_1, \boldsymbol{r}_2\right\rangle$$

or

$$[\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}), V(\left|\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2\right|)] = 0.$$

Since

$$[\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}), \frac{1}{2m_1}\hat{\boldsymbol{p}}_1{}^2 + \frac{1}{2m_1}\hat{\boldsymbol{p}}_1{}^2] = 0 ,$$

we get the commutation relation

$$[\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a}), \hat{H}] = 0.$$

This means that there is a simultaneous eigenket $\left|\psi\right\rangle$ of both $\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})$ and $\hat{H}$, such that

$$H|\psi\rangle = E|\psi\rangle, \qquad \hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\psi\rangle = \lambda|\psi\rangle .$$

We also note that

$$\hat{T}_1(\delta\boldsymbol{a})\hat{T}_2(\delta\boldsymbol{a}) = \exp(-\frac{i}{\hbar}\hat{\boldsymbol{P}}\cdot\delta\boldsymbol{a}) = \hat{1} - \frac{i}{\hbar}\hat{\boldsymbol{P}}\cdot\delta\boldsymbol{a} .$$

For any $\delta\boldsymbol{a}$, we have

$$[\hat{H},\hat{\boldsymbol{P}}] = 0,$$

leading to

$$\frac{d}{dt}\langle\psi(t)|\hat{\boldsymbol{P}}|\psi(t)\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{\boldsymbol{P}}]|\psi(t)\rangle = 0 .$$

This implies the conservation of the total momentum.

((**In summary**))

What is the physical meaning of the above result?

From $H|\psi\rangle = E|\psi\rangle$, the wave function of the Schrödinger equation is given by

$$\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \langle\boldsymbol{r}_1,\boldsymbol{r}_2|\psi\rangle .$$

From $\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\psi\rangle = \lambda|\psi\rangle$, we have

$$\langle\boldsymbol{r}_1,\boldsymbol{r}_2|\hat{T}_1(\boldsymbol{a})\hat{T}_2(\boldsymbol{a})|\psi\rangle = \langle\boldsymbol{r}_1-\boldsymbol{a},\boldsymbol{r}_2-\boldsymbol{a}|\psi\rangle = \psi(\boldsymbol{r}_1-\boldsymbol{a},\boldsymbol{r}_2-\boldsymbol{a}) = \lambda\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) .$$

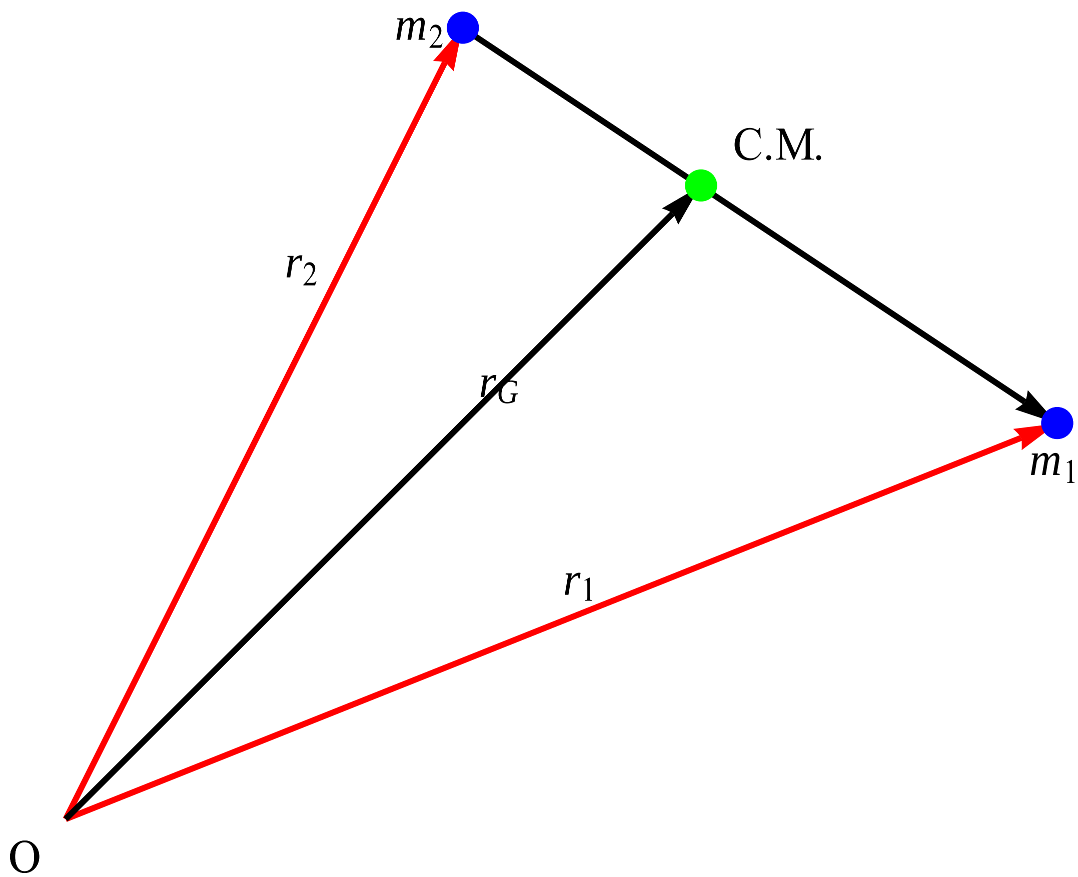
Suppose that $\boldsymbol{r}_2 - \boldsymbol{a} = 0$ ($\boldsymbol{a}$ can be chosen arbitrarily). Then we get

$$\psi(\boldsymbol{r}_1-\boldsymbol{r}_2,0) = \lambda\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) .$$

In other words, the wave function $\psi(\boldsymbol{r}_1,\boldsymbol{r}_2)$ is only dependent on the relative co-ordinate

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \qquad \psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \psi(\boldsymbol{r}).$$

6. Two-body problems: Classical mechanics



Lagrangian:

$$L = \frac{1}{2} m_1 \left(\frac{d\boldsymbol{r}_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{d\boldsymbol{r}_2}{dt}\right)^2 - V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|),$$

$$\boldsymbol{p}_1 = m_1 \frac{d\boldsymbol{r}_1}{dt}, \qquad \boldsymbol{p}_2 = m_2 \frac{d\boldsymbol{r}_2}{dt}.$$

Center of mass:

$$\boldsymbol{r}_G = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}.$$

Relative coordinate:

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 ,$$

$$\boldsymbol{r}_1 = \boldsymbol{r}_G + \frac{m_2 \boldsymbol{r}}{m_1 + m_2}, \qquad \boldsymbol{r}_2 = \boldsymbol{r}_G - \frac{m_1 \boldsymbol{r}}{m_1 + m_2}.$$

The Lagrangian L can be written in terms of $\boldsymbol{r}_G$ and $\boldsymbol{r}$

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, \boldsymbol{r}_G) = \frac{1}{2} m_1 (\frac{d\boldsymbol{r}_G}{dt} + \frac{m_2}{m_1 + m_2} \frac{d\boldsymbol{r}}{dt})^2 + \frac{1}{2} m_2 (\frac{d\boldsymbol{r}_G}{dt} - \frac{m_1}{m_1 + m_2} \frac{d\boldsymbol{r}}{dt})^2 - V(\boldsymbol{r}),$$

or

$$\begin{aligned} L(\boldsymbol{r}, \dot{\boldsymbol{r}}, \boldsymbol{r}_G) &= \frac{1}{2} M (\frac{d\boldsymbol{r}_G}{dt})^2 + \frac{1}{2} \mu (\frac{d\boldsymbol{r}}{dt})^2 - V(\boldsymbol{r}) \\ &= \frac{1}{2} M \dot{\boldsymbol{r}}_G{}^2 + \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}) \end{aligned}$$

where the total mass is defined by

$$M = m_1 + m_2 ,$$

and the reduced mass is defined by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} . \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} .$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_G}\right) = \frac{\partial L}{\partial \boldsymbol{r}_G} = 0 .$$

Since $L(\boldsymbol{r}, \dot{\boldsymbol{r}}, \boldsymbol{r}_G)$ is independent of $\boldsymbol{r}_G$, we find that the conjugate momentum

$$\boldsymbol{p}_G = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_G} = M \frac{d\boldsymbol{r}_G}{dt} = m_1 \frac{d\boldsymbol{r}_1}{dt} + m_2 \frac{d\boldsymbol{r}_2}{dt} = \boldsymbol{p}_1 + \boldsymbol{p}_2 ,$$

is a cyclic (time-independent) (which means the momentum conservation because of no external force). The conjugate momentum is given by

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = \mu \frac{d\boldsymbol{r}}{dt} = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 + m_2}.$$

Note that

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}_1}{dt} - \frac{d\boldsymbol{r}_2}{dt} = \frac{1}{m_1}\boldsymbol{p}_1 - \frac{1}{m_2}\boldsymbol{p}_2 = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 m_2},$$

$$\boldsymbol{p} = \mu \frac{d\boldsymbol{r}}{dt} = \frac{m_1 m_2}{m_1 + m_2} \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 m_2} = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 + m_2}.$$

Since the momentum of the center of mass is given by

$$\boldsymbol{p}_G = \boldsymbol{p}_1 + \boldsymbol{p}_2,$$

we get

$$\boldsymbol{p}_1 = \boldsymbol{p} + \frac{m_1}{m_1 + m_2}\boldsymbol{p}_G, \qquad \boldsymbol{p}_2 = -\boldsymbol{p} + \frac{m_2}{m_1 + m_2}\boldsymbol{p}_G.$$

The Hamiltonian H can be written as

$$H = \boldsymbol{p}_G \cdot \frac{d\boldsymbol{r}_G}{dt} + \boldsymbol{p} \cdot \frac{d\boldsymbol{r}}{dt} - L = \frac{\boldsymbol{p}_G^{\,2}}{2M} + \frac{\boldsymbol{p}^2}{2\mu} + V(\boldsymbol{r}) + const.$$

The total orbital angular momentum:

$$\begin{aligned}
\boldsymbol{L}_T &= \boldsymbol{L}_1 + \boldsymbol{L}_2 \\
&= \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \\
&= (\boldsymbol{r}_G + \frac{m_2 \boldsymbol{r}}{m_1 + m_2}) \times (\boldsymbol{p} + \frac{m_1}{m_1 + m_2}\boldsymbol{p}_G) + (\boldsymbol{r}_G - \frac{m_1 \boldsymbol{r}}{m_1 + m_2}) \times (-\boldsymbol{p} + \frac{m_2}{m_1 + m_2}\boldsymbol{p}_G)
\end{aligned}$$

or

$$\boldsymbol{L}_T = \boldsymbol{L}_G + \boldsymbol{L}$$

with

$$\boldsymbol{L}_G = \boldsymbol{r}_G \times \boldsymbol{p}_G . \qquad \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

7. Quantum Kepler problem

We now consider the quantum mechanics of the central force problem.

(i) The relative co-ordinate operator:

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2 ,$$

(ii) The relative momentum operator:

$$\hat{\boldsymbol{p}} = \frac{m_2 \hat{\boldsymbol{p}}_1 - m_1 \hat{\boldsymbol{p}}_2}{m_1 + m_2} .$$

(iii) The co-ordinate operator for the center of mass:

$$\hat{\boldsymbol{r}}_G = \frac{m_1 \hat{\boldsymbol{r}}_1 + m_2 \hat{\boldsymbol{r}}_2}{m_1 + m_2} .$$

(iv) The momentum operator for the center of mass:

$$\hat{\boldsymbol{p}}_G = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 .$$

(v) The total angular momentum operator for the system:

$$\hat{\boldsymbol{L}}_T = \hat{\boldsymbol{L}}_G + \hat{\boldsymbol{L}} ,$$

with

$$\hat{\boldsymbol{L}}_G = \hat{\boldsymbol{r}}_G \times \hat{\boldsymbol{p}}_G .$$

$$\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} . \qquad \text{(internal angular momentum)}$$

The reduced mass is defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} .$$

8. The commutation relation:

We assume that

$$[\hat{x}_{1i}, \hat{x}_{1j}] = 0, \qquad [\hat{x}_{2i}, \hat{x}_{2j}] = 0,$$

$$[\hat{p}_{1i}, \hat{p}_{1j}] = 0, \qquad [\hat{p}_{2i}, \hat{p}_{2j}] = 0,$$

$$[\hat{x}_{1i}, \hat{p}_{1j}] = i\hbar\delta_{ij}, \qquad [\hat{x}_{2i}, \hat{p}_{2j}] = i\hbar\delta_{ij},$$

for the same particle, and

$$[\hat{x}_{1i}, \hat{p}_{2j}] = 0, \qquad [\hat{x}_{2i}, \hat{p}_{1j}] = 0,$$

$$[\hat{x}_{1i}, \hat{x}_{2j}] = 0, \qquad [\hat{p}_{1i}, \hat{p}_{2j}] = 0,$$

for the different particles, where $i = x, y, z$, and $j = x, y, z$.

Based on the above relations, we discuss the commutation relations between $\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{r}}_G, \hat{\boldsymbol{p}}_G$, as follows.

$$\begin{aligned}
[\hat{x}_i, \hat{p}_j] &= [\hat{x}_{1i} - \hat{x}_{2i}, \frac{m_2\hat{p}_{1j} - m_1\hat{p}_{2j}}{m_1 + m_2}] \\
&= \frac{m_2}{m_1 + m_2}[\hat{x}_{1i}, \hat{p}_{1j}] + \frac{m_1}{m_1 + m_2}[\hat{x}_{2i}, \hat{p}_{2j}] \\
&= i\hbar\delta_{ij}\hat{1}
\end{aligned}$$

$$\begin{aligned}
[\hat{x}_i, \hat{p}_{Gj}] &= [\hat{x}_{1i} - \hat{x}_{2i}, \hat{p}_{1j} + \hat{p}_{2j}] \\
&= [\hat{x}_{1i}, \hat{p}_{1j}] - [\hat{x}_{2i}, \hat{p}_{2j}] \\
&= i\hbar\delta_{ij}\hat{1} - i\hbar\delta_{ij}\hat{1} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\hat{x}_{Gi}, \hat{p}_{Gj}] &= [\frac{m_1\hat{x}_{1i} + m_2\hat{x}_{2i}}{m_1 + m_2}, \hat{p}_{1j} + \hat{p}_{2j}] \\
&= \frac{m_1}{m_1 + m_2}[\hat{x}_{1i}, \hat{p}_{1j}] + \frac{m_2}{m_1 + m_2}[\hat{x}_{2i}, \hat{p}_{2j}] \\
&= i\hbar\delta_{ij}\hat{1}
\end{aligned}$$

$$[\hat{x}_{Gi},\hat{p}_j]=[\frac{m_1\hat{x}_{1i}+m_2\hat{x}_{2i}}{m_1+m_2},\frac{m_2\hat{p}_{1j}-m_1\hat{p}_{2j}}{m_1+m_2}]$$
$$=\frac{m_1m_2}{m_1+m_2}[\hat{x}_{1i},\hat{p}_{1j}]-\frac{m_1m_2}{m_1+m_2}[\hat{x}_{2i},\hat{p}_{2j}]$$
$$=0$$

$$[\hat{p}_{Gi},\hat{p}_j]=[\hat{p}_{1i}+\hat{p}_{2i},\frac{m_2\hat{p}_{1j}-m_1\hat{p}_{2j}}{m_1+m_2}]$$
$$=\frac{m_2}{m_1+m_2}[\hat{p}_{1i},\hat{p}_{1j}]-\frac{m_{12}}{m_1+m_2}[\hat{p}_{2i},\hat{p}_{2j}]$$
$$=0$$

We note that the original Hamiltonian

$$\hat{H}=\frac{1}{2m_1}\hat{\boldsymbol{p}}_1^{\ 2}+\frac{1}{2m_2}\hat{\boldsymbol{p}}_2^{\ 2}+V(|\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2|),$$

can be rewritten as

$$\hat{H}=\hat{H}_G+\hat{H}_{rel}=\frac{\hat{\boldsymbol{p}}_G^{\ 2}}{2M}+[\frac{\hat{\boldsymbol{p}}^2}{2\mu}+V(|\hat{\boldsymbol{r}}|)].$$

((**Mathematica**))
Using the commutation relations, we can directly show that

$$\frac{1}{2m_1}\hat{\boldsymbol{p}}_1^{\ 2}+\frac{1}{2m_2}\hat{\boldsymbol{p}}_2^{\ 2}=\frac{\hat{\boldsymbol{p}}_G^{\ 2}}{2M}+\frac{\hat{\boldsymbol{p}}^2}{2\mu}.$$

```
Clear["Global`*"]; p1 = {p1x, p1y, p1z};
p2 = {p2x, p2y, p2z}; μ = (m1 m2)/(m1 + m2); M1 = m1 + m2;
p = (m2 p1 - m1 p2)/(m1 + m2);
pG = p1 + p2;
K1 = (pG.pG)/(2 M1) + (p.p)/(2 μ) // FullSimplify;
K2 = (p1.p1)/(2 m1) + (p2.p2)/(2 m2) // Simplify;

K1 - K2 // Simplify

0
```

9. Reduction of the two-body problem

We note that

$$[\hat{\boldsymbol{p}}_G, \hat{H}_{rel}] = 0,$$

and

$$[\hat{H}, \hat{\boldsymbol{p}}_G] = [\hat{H}_G + \hat{H}_{rel}, \hat{\boldsymbol{p}}_G] = [\hat{H}_{rel}, \hat{\boldsymbol{p}}_G] = 0.$$

Then $\hat{H}_{rel}$, $\hat{H}$, and $\hat{\boldsymbol{p}}_G$ can all be simultaneously diagonalized. In other words, there exists a simultaneous eigenstate $|\boldsymbol{p}_G, E_r\rangle$.

$$\hat{H}_G|\boldsymbol{p}_G, E_r\rangle = E_G|E_G, E_r\rangle, \qquad \hat{H}_{rel}|\boldsymbol{p}_G, E_r\rangle = E_r|\boldsymbol{p}_G, E_r\rangle,$$

and

$$\hat{H}|\boldsymbol{p}_G, E_r\rangle = (\hat{H}_G + \hat{H}_{rel})|\boldsymbol{p}_G, E_r\rangle = (E_G + E_r)|\boldsymbol{p}_G, E_r\rangle.$$

We note that

$$\hat{H}_G \left| \boldsymbol{p}_G \right\rangle = \frac{\boldsymbol{p}_G{}^2}{2M} \left| \boldsymbol{p}_G \right\rangle = E_G \left| \boldsymbol{p}_G \right\rangle ,$$

where

$$E_G = \frac{\boldsymbol{p}_G{}^2}{2M} .$$

The wave function can be described by

$$\left|\psi\right\rangle = \left| \boldsymbol{p}_G \right\rangle \otimes \left| E_r \right\rangle = \left| \boldsymbol{p}_G \right\rangle \left| \psi_r \right\rangle ,$$

where

$$\left| E_r \right\rangle = \left| \psi_r \right\rangle .$$

10. The representation of $\left| \boldsymbol{r}_G, \boldsymbol{r} \right\rangle = \left| \boldsymbol{r}_G \right\rangle \otimes \left| \boldsymbol{r} \right\rangle$

Based on the commutation relations,

$$[\hat{x}_{Gi}, \hat{p}_{Gj}] = i\hbar\delta_{ij}\hat{1} , \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{1} ,$$

we can use the basis

$$\left| \boldsymbol{r}_G, \boldsymbol{r} \right\rangle = \left| \boldsymbol{r}_G \right\rangle \otimes \left| \boldsymbol{r} \right\rangle ,$$

for both the center-of mass co-ordinate and relative co-ordinate, corresponding to the basis for the momentum basis

$$\left| p_G, \boldsymbol{p} \right\rangle = \left| \boldsymbol{p}_G \right\rangle \otimes \left| \boldsymbol{p} \right\rangle .$$

The transformation functions are defined by

$$\left\langle \boldsymbol{r}_G \middle| \boldsymbol{p}_G \right\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar} \boldsymbol{p}_G \cdot \boldsymbol{r}_G) ,$$

and

$$\langle \boldsymbol{r} | \boldsymbol{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar} \boldsymbol{p} \cdot \boldsymbol{r}) .$$

The wave function in the position representation can be described by

$$|\psi\rangle = |\boldsymbol{p}_G\rangle |E_r\rangle = |\boldsymbol{p}_G\rangle |\psi_r\rangle .$$

The representation of the wave function in the positional representation

$$\langle \boldsymbol{r}_G, \boldsymbol{r} | \psi \rangle = \langle \boldsymbol{r}_G | \boldsymbol{p}_G \rangle \langle r | \psi_r \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar} \boldsymbol{p}_G \cdot \boldsymbol{r}_G) \langle \boldsymbol{r} | \psi_r \rangle .$$

11. Ehrenfest theorem for $\langle \hat{\boldsymbol{p}}_G \rangle$

We note that

$$[\hat{H}, \hat{\boldsymbol{p}}_G] = 0 .$$

From the Ehrenfest theorem, we have

$$\frac{d}{dt} \langle \hat{\boldsymbol{p}}_G \rangle = \frac{1}{i\hbar} \langle [\hat{\boldsymbol{p}}_G, \hat{H}] \rangle = 0 ,$$

leading to $\langle \hat{\boldsymbol{p}}_G \rangle$=constant of motion. For simplicity, we assume that

$$\hat{\boldsymbol{p}}_G = 0 .$$

The we have the final form of the Hamiltonian as

$$\hat{H} = \hat{H}_{rel} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(\hat{\boldsymbol{r}}) .$$

The Schrodinger equation is given by

$$[\frac{\hat{\boldsymbol{p}}^2}{2\mu} + V(\hat{\boldsymbol{r}})] |\psi_r\rangle = E_r |\psi_r\rangle$$

or

$$[-\frac{\hbar^2}{2\mu}\nabla_r^{\ 2} + V(\boldsymbol{r})]\langle \boldsymbol{r}|\psi_r\rangle = E_r\langle \boldsymbol{r}|\psi_r\rangle.$$

12. Rotation operator in Quantum mechanics

After the geometrical rotation;

$$\boldsymbol{r} \rightarrow \Re\boldsymbol{r} = \boldsymbol{r}', \qquad \text{(geometrical rotation)}$$

we assume that the state vector changes from the old state $|\psi\rangle$ to the new state $|\psi'\rangle$.

$$|\psi'\rangle = \hat{R}|\psi\rangle,$$

or

$$\langle\psi'| = \langle\psi|\hat{R}^+,$$

where $\hat{R}$ is a rotation operator in the quantum mechanics. It is natural to assume that

$$\langle\psi'|\hat{\boldsymbol{r}}|\psi'\rangle = \langle\psi|\hat{\boldsymbol{r}}'|\psi\rangle = \langle\psi|\Re\hat{\boldsymbol{r}}|\psi\rangle,$$

or

$$\langle\psi|\hat{R}^+\hat{\boldsymbol{r}}\hat{R}|\psi\rangle = \langle\psi|\Re\hat{\boldsymbol{r}}|\psi\rangle,$$

or

$$\hat{R}^+\hat{\boldsymbol{r}}\hat{R} = \Re\hat{\boldsymbol{r}}. \qquad (1)$$

The rotation operator is a unitary operator.

$$\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle,$$

or

$$\hat{R}^+\hat{R} = \hat{R}\hat{R}^+ = \hat{1} \quad \text{(Unitary operator)}$$

From Eq. (1),

$$\hat{\boldsymbol{r}}\hat{R} = \hat{R}\Re\hat{\boldsymbol{r}} .$$

Here we calculate

$$\hat{\boldsymbol{r}}\hat{R}\left|\boldsymbol{r}\right\rangle = \hat{R}\Re\hat{\boldsymbol{r}}\left|\boldsymbol{r}\right\rangle = \hat{R}\Re\boldsymbol{r}\left|\boldsymbol{r}\right\rangle = \Re\boldsymbol{r}\hat{R}\left|\boldsymbol{r}\right\rangle .$$

$\hat{R}\left|\boldsymbol{r}\right\rangle$ is the eigenket of $\hat{\boldsymbol{r}}$ with the eigenvalue $\Re\boldsymbol{r}$. So that we can write

$$\hat{R}\left|\boldsymbol{r}\right\rangle = \left|\Re\boldsymbol{r}\right\rangle .$$

When

$$\Re\boldsymbol{r} = \boldsymbol{r}_0 ,$$

or

$$\boldsymbol{r} = \Re^{-1}\boldsymbol{r}_0 ,$$

$$\hat{R}\left|\Re^{-1}\boldsymbol{r}_0\right\rangle = \left|\boldsymbol{r}_0\right\rangle ,$$

or

$$\left|\Re^{-1}\boldsymbol{r}_0\right\rangle = \hat{R}^{+}\left|\boldsymbol{r}_0\right\rangle .$$

For any $\boldsymbol{r}$,

$$\left|\Re^{-1}\boldsymbol{r}\right\rangle = \hat{R}^{+}\left|\boldsymbol{r}\right\rangle ,$$

$$\hat{R}\hat{R}^{+}\left|\boldsymbol{r}\right\rangle = \hat{R}\left|\Re^{-1}\boldsymbol{r}\right\rangle = \left|\Re\Re^{-1}\boldsymbol{r}\right\rangle = \left|\boldsymbol{r}\right\rangle .$$

In summary, we have

(1) $\quad \hat{R}^{+}\hat{R} = \hat{R}\hat{R}^{+} = \hat{1} .$

(2) $\hat{R}|\boldsymbol{r}\rangle = |\Re\boldsymbol{r}\rangle$.

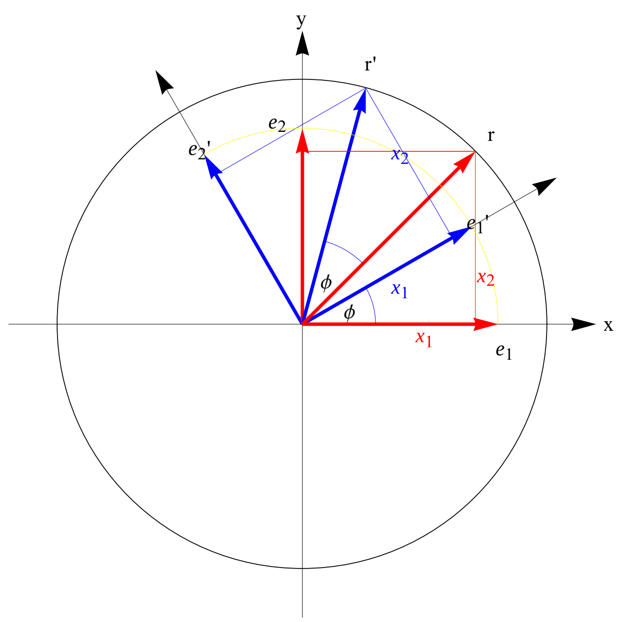
(3) $\langle\boldsymbol{r}|\hat{R}^{+} = \langle\Re\boldsymbol{r}|$.

(4) $\hat{R}^{+}|\boldsymbol{r}\rangle = |\Re^{-1}\boldsymbol{r}\rangle$.

(5) $\langle\boldsymbol{r}|\hat{R} = \langle\Re^{-1}\boldsymbol{r}|$.

13. Rotation matrix

Suppose that the vector $\boldsymbol{r}$ is rotated through θ (counter-clock wise) around the z axis. The position vector $\boldsymbol{r}$ is changed into $\boldsymbol{r}'$ in the same orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.



In this Fig, we have

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1' = \cos\phi \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_1' = \sin\phi$$
$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' = -\sin\phi, \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_2' = \cos\phi$$

We define $\boldsymbol{r}$ and $\boldsymbol{r}'$ as

$$\boldsymbol{r}' = x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2 = x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2',$$

and

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 .$$

Using the relation

$$\begin{aligned}
\boldsymbol{e}_1 \cdot \boldsymbol{r}' &= \boldsymbol{e}_1 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2') \\
\boldsymbol{e}_2 \cdot \boldsymbol{r}' &= \boldsymbol{e}_2 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_2 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2')
\end{aligned}$$

we have

$$\begin{aligned}
x_1' &= \mathbf{e}_1 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2') = x_1 \cos\phi - x_2 \sin\phi \\
x_2' &= \mathbf{e}_2 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2') = x_1 \sin\phi + x_2 \cos\phi
\end{aligned}$$

or including the x_3 axis,

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \Re_z(\phi) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We note that

$$\Re_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\Re_z^{-1}(\phi) = \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) & 0 \\ \sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

---

14. Infinitesimal rotation matrix around the *z* axis

We assume that $\phi = d\alpha$ (infinitesimally small angle);

$$\mathbf{r}'=\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}=\Re_z{}^{-1}(d\alpha)\mathbf{r}$$

$$=\begin{pmatrix} \cos(d\alpha) & \sin(d\alpha) & 0 \\ -\sin(d\alpha) & \cos(d\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\approx\begin{pmatrix} 1 & d\alpha & 0 \\ -d\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$=\begin{pmatrix} x+yd\alpha \\ -xd\alpha+y \\ z \end{pmatrix}$$

or

$$x'=x+yd\alpha$$
$$y'=y-xd\alpha$$
$$z'=z$$

Then we have

$$\begin{aligned}
\langle\mathbf{r}|\psi'\rangle &= \langle\mathbf{r}|\hat{R}_z(d\alpha)|\psi\rangle \\
&= \left\langle \Re_z{}^{-1}(d\alpha)\mathbf{r}\middle|\psi\right\rangle \\
&= \langle x+yd\alpha, y-xd\alpha, z|\psi\rangle \\
&= \psi(x+yd\alpha, y-xd\alpha, z) \\
&= \psi(x,y,z)-d\alpha(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x})\psi(x,y,z) \\
&= \psi+d\alpha(y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}) \\
&= \langle\mathbf{r}|\hat{1}-\frac{i}{\hbar}d\alpha\hat{L}_z|\psi\rangle
\end{aligned}$$

where we use the Taylor expansion and the angular (orbital) momentum is defined by

$$\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x.$$

Then we have the expression of the infinitesimal rotation operator as

$$\hat{R}_z(d\alpha) = \hat{1} - \frac{i}{\hbar} d\alpha \hat{L}_z \,.$$

((**Note**))

$$\langle \boldsymbol{r} | (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) | \psi \rangle = \langle r | \hat{L}_z | \psi \rangle = \frac{\hbar}{i} (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) \langle \boldsymbol{r} | \psi \rangle \,.$$

15. Positional-space representation of L in spherical co-ordinates

We also use the ket vector $|\boldsymbol{r}\rangle = |r, \theta, \phi\rangle$, where r, θ, and ϕ are the spherical coordinates.

$$\hat{R}_z(d\alpha) |r, \theta, \phi\rangle = |r, \theta, \phi + d\alpha\rangle,$$

$$\hat{R}_z^{+}(d\alpha) |r, \theta, \phi\rangle = |r, \theta, \phi - d\alpha\rangle.$$

$$\langle r, \theta, \phi - d\alpha | = \langle r, \theta, \phi | \hat{R}_z(d\alpha) \,.$$

thus we have

$$\langle r, \theta, \phi | \hat{R}_z(d\alpha) | \psi \rangle = \langle r, \theta, \phi - d\alpha | \psi \rangle = \langle r, \theta, \phi | \psi \rangle - d\alpha \frac{\partial}{\partial \phi} \langle r, \theta, \phi | \psi \rangle$$

On the other hand, we get

$$\langle r, \theta, \phi | \hat{R}_z(d\alpha) | \psi \rangle = \langle r, \theta, \phi | \hat{1} - \frac{i}{\hbar} \hat{L}_z d\alpha | \psi \rangle = \langle r, \theta, \phi | \psi \rangle - \frac{i}{\hbar} d\alpha \langle r, \theta, \phi | \hat{L}_z | \psi \rangle$$

Then we have

$$\langle r, \theta, \phi | \hat{L}_z | \psi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle r, \theta, \phi | \psi \rangle$$

or

$$L_z \frac{\partial}{\partial \phi} \psi(\mathbf{r}) = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \psi(\mathbf{r}).$$

16. Finite rotation

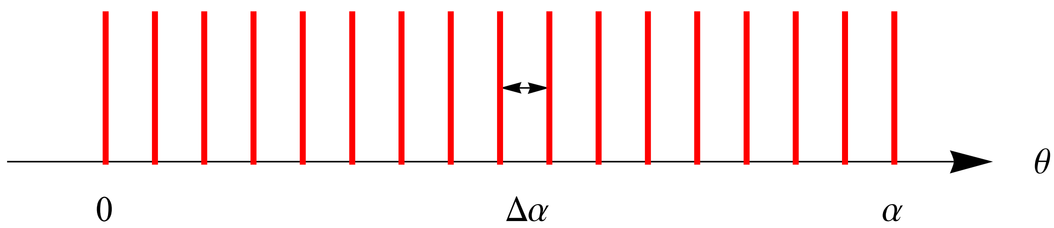


Fig. $\alpha = N\Delta\alpha$.

$$\hat{R}_z(\alpha = 0) = \hat{1},$$

$$\hat{R}_z(\alpha) = \lim_{N\to\infty}[\hat{R}_z(\Delta\alpha)]^N = \lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\Delta\alpha\hat{L}_z)^N = \lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\frac{\alpha}{N}\hat{L}_z)^N$$
$$= \exp(-\frac{i}{\hbar}\alpha\hat{L}_z)$$
,

((**Note**))

$$\lim_{N\to\infty}(\hat{1} - \frac{i}{\hbar}\frac{\alpha}{N}\hat{L}_z)^N = \lim_{N\to\infty}[(\hat{1} + \frac{\mu}{N})^{\frac{N}{\mu}}]^\mu = e^\mu ,$$

where

$$\mu = -\frac{i}{\hbar}\alpha\hat{L}_z .$$

In general, we have the rotation operator

$$\hat{R}_u(\alpha) = \exp(-\frac{i}{\hbar}\alpha\hat{\boldsymbol{L}}\cdot\boldsymbol{u}) .$$

In the case of an arbitrary quantum mechanical system, using the general angular momentum $\hat{\boldsymbol{J}}$ instead of $\hat{\boldsymbol{L}}$:

$$\hat{R}_u(\alpha) = \exp(-\frac{i}{\hbar}\alpha\hat{\boldsymbol{J}}\cdot\boldsymbol{u}).$$

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