Wigner-Eckart theorem

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The **Wigner–Eckart theorem** is a theorem of representation theory and quantum mechanics. It states that matrix elements of spherical tensor operators on the basis of angular momentum eigenstates can be expressed as the product of two factors, one of which is independent of angular momentum orientation, and the other a Clebsch-Gordan coefficient. The name derives from physicists Eugene Wigner and Carl Eckart who developed the formalism as a link between the symmetry transformation groups of space (applied to the Schrödinger equations) and the laws of conservation of energy, momentum, and angular momentum.
https://en.wikipedia.org/wiki/Wigner%E2%80%93Eckart_theorem

1. The Wigner and Eckart Theorem

We start with the formula given by

$$[\hat{J}_z,\hat{T}_q^{(k)}]=\hbar q\hat{T}_q^{(k)}$$

$$[\hat{J}_+,\hat{T}_q^{(k)}]=\hbar\sqrt{(k-q)(k+q+1)}\hat{T}_{q+1}^{(k)}$$

$$[\hat{J}_-,\hat{T}_q^{(k)}]=\hbar\sqrt{(k+q)(k-q+1)}\hat{T}_{q-1}^{(k)}$$

where

$$\hat{J}_+=\hat{J}_x+i\hat{J}_y,\qquad \hat{J}_-=\hat{J}_x-i\hat{J}_y$$

(a) The recursion relation from the relation $[\hat{J}_z,T_q^{(k)}]=\hbar q\hat{T}_q^{(k)}$

The matric element is

$$\langle\alpha';j',m'|[\hat{J}_z,T_q^{(k)}]|\alpha;j,m\rangle=\hbar q\langle\alpha;j',m'|\hat{T}_q^{(k)}|\alpha;j,m\rangle$$

or

$$\hbar(m'-m-q)\langle\alpha';j',m'|\hat{T}_q^{(k)}|\alpha;j,m\rangle=0$$

Then we have

$$\langle\alpha';j',m'|\hat{T}_q^{(k)}|\alpha;j,m\rangle\neq 0$$

only if

$$m' - m - q = 0 \qquad \text{(selection rule)}$$

(b) The recursion relation from the relation $[\hat{J}_+, \hat{T}_q^{(k)}] = \hbar\sqrt{(k-q)(k+q+1)}\hat{T}_{q+1}^{(k)}$

The matrix element:

$$\langle \alpha'; j', m' | [\hat{J}_+, \hat{T}_q^{(k)}] | \alpha; j, m \rangle = \hbar\sqrt{(k-q)(k+q+1)} \langle \alpha; j', m' | \hat{T}_{q+1}^{(k)} | \alpha; j, m \rangle$$

or

$$\langle \alpha'; j', m' | \hat{J}_+ T_q^{(k)} - T_q^{(k)} \hat{J}_+ | \alpha; j, m \rangle = \hbar\sqrt{(k-q)(k+q+1)} \langle \alpha; j', m' | \hat{T}_{q+1}^{(k)} | \alpha; j, m \rangle$$

or

$$\begin{aligned} &\langle \alpha'; j', m' | \hat{J}_+ \hat{T}_q^{(k)} | \alpha; j, m \rangle - \langle \alpha'; j', m' | \hat{T}_q^{(k)} \hat{J}_+ | \alpha; j, m \rangle \\ &= \hbar\sqrt{(k-q)(k+q+1)} \langle \alpha; j', m' | \hat{T}_{q+1}^{(k)} | \alpha; j, m \rangle \end{aligned}$$

Then we get

$$\begin{aligned} &\sqrt{(j'+m')(j'-m'+1)} \langle \alpha'; j', m'-1 | \hat{T}_q^{(k)} | \alpha; j, m \rangle \\ &- \sqrt{(j-m)(j+m+1)} \langle \alpha'; j', m' | \hat{T}_q^{(k)} | \alpha; j, m+1 \rangle \\ &= \sqrt{(k-q)(k+q+1)} \langle \alpha; j', m' | \hat{T}_{q+1}^{(k)} | \alpha; j, m \rangle \end{aligned}$$

(c) The recursion relation from the relation $[\hat{J}_-, \hat{T}_q^{(k)}] = \hbar\sqrt{(k+q)(k-q+1)}\hat{T}_{q-1}^{(k)}$

The matrix element

$$\begin{aligned} &\langle \alpha'; j', m' | \hat{J}_- \hat{T}_q^{(k)} | \alpha; j, m \rangle - \langle \alpha'; j', m' | \hat{T}_q^{(k)} \hat{J}_- | \alpha; j, m \rangle \\ &= \hbar\sqrt{(k+q)(k-q+1)} \langle \alpha; j', m' | \hat{T}_{q-1}^{(k)} | \alpha; j, m \rangle \end{aligned}$$

Then we get

$$\begin{aligned} &\sqrt{(j'-m')(j'+m'+1)} \langle \alpha'; j', m'+1 | \hat{T}_q^{(k)} | \alpha; j, m \rangle \\ &- \sqrt{(j+m)(j-m+1)} \langle \alpha'; j', m' | T_q^{(k)} | \alpha; j, m-1 \rangle \\ &= \sqrt{(k+q)(k-q+1)} \langle \alpha; j', m' | \hat{T}_{q+1}^{(k)} | \alpha; j, m \rangle \end{aligned}$$

So we have the recursion relation

$$\sqrt{(j'\pm m')(j'\mp m'+1)}\langle\alpha'; j', m'\mp 1|T_q^{(k)}|\alpha; j, m\rangle$$
$$=\sqrt{(j\mp m)(j\pm m+1)}\langle\alpha'; j', m'|T_q^{(k)}|\alpha; j, m\pm 1\rangle$$
$$+\sqrt{(k\mp q)(k\pm q+1)}\langle\alpha; j', m'|\hat{T}_{q\pm 1}^{(k)}|\alpha; j, m\rangle$$

We find the same recursion relations for the $\langle\alpha'; j', m'|\hat{T}_q^{(k)}|\alpha; j, m\rangle$ as we find for the Clebsch-Gordan coefficients $\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle = \langle j, k; m, q | j, k; j', m'\rangle$

with

$j_1 = j$, $\quad j_2 = k$,

$m_1 = m$, $\quad m_2 = q$,

$j = j'$ $\quad m = m'$

$$\sqrt{(j'\pm m')(j'\mp m'+1)}\langle j, k; m, q | j, k; j', m'\rangle$$
$$=\sqrt{(j\pm m)(j\mp m+1))}\langle j, k; m\mp 1, q | j_1, k; j', m'\mp 1\rangle$$
$$+\sqrt{(k\pm q)(k\mp q+1)}\langle j, k; m, q\mp 1 | j, k; j', m'\mp 1\rangle$$

By changing by $\pm\rightarrow\mp$ and $\mp\rightarrow\pm$, and $m'\rightarrow m'\mp 1$ we have

$$\sqrt{(j'\pm m')(j'\mp m'+1)}\langle j, k; m, q | j, k; j', m'\mp 1\rangle$$
$$=\sqrt{(j\mp m)(j\pm m+1))}\langle j, k; m\pm 1, q | j_1, k; j', m'\rangle$$
$$+\sqrt{(k\mp q)(k\pm q+1)}\langle j, k; m, q\pm 1 | j, k; j', m\rangle$$

Therefore the *m* behavior of the $\langle\alpha'; j', m'|T_q^{(k)}|\alpha; j, m\rangle$ must be the same as that of the Clebsch-Gordan coefficients. We see that

$\langle\alpha'; j', m'|\hat{T}_q^{(k)}|\alpha; j, m\rangle = \langle j, k; m, q | j, k; j', m'\rangle\times$ term independent of *m'*, *m*, and *q*.

or

$$\langle\alpha'; j', m'|\hat{T}_q^{(k)}|\alpha; j, m\rangle = \langle j, k; m, q | j, k; j', m'\rangle\langle\alpha' j'\|\hat{T}^{(k)}\|\alpha j\rangle$$

(Wigner-Eckart theorem)

$\langle \alpha' j' \| \hat{T}_k \| \alpha j \rangle$ is the reduced matrix element, independent of *m*', *m*, and *q*.

where

$$m' = m + q$$

$$j' = j + k, j + k - 1, \ldots\ldots., |j - k|$$

((Note)) As the Wigner-Eckart theorem, one can use conventionally the form

$$\langle \alpha'; j', m' | \hat{T}_q^{(k)} | \alpha; j, m \rangle = \langle j, k; m, q | j, k; j', m' \rangle \frac{\langle \alpha' j' \| \hat{T}^{(k)} \| \alpha j \rangle}{\sqrt{2j+1}}$$

The $\sqrt{2j+1}$ factor is arbitrary, but conventional. Here we use the formula which is used by Zettli.

((**Note**))

Recursion relation of the Clebsch-Gordan coefficients

$$| j_1, j_2; j, m \rangle = \sum_{m_1} \sum_{m_2} | j_1, j_2; m_1, m_2 \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$$

where

$$(m - m_1 - m_2) \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = 0$$

and

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1 \rangle \\
&= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \rangle \\
&+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \rangle
\end{aligned}$$

or

$$\begin{aligned}
&\sqrt{(j \pm m)(j \mp m + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
&= \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1))} \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \mp 1 \rangle \\
&+ \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)} \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \mp 1 \rangle
\end{aligned}$$

3. Definition of matrix element

We define the matrix element of the tensor using the Clebsch-Gordon coefficient as

$$\langle \alpha'; j', m' | \hat{T}_q^{(k)} | \alpha; j, m \rangle \propto \langle j, k; m, q | j, k; j', m' \rangle$$

$$\propto \text{ClebschGordan}[\{j,m\},\{k,q\},\{j',m'\}]$$

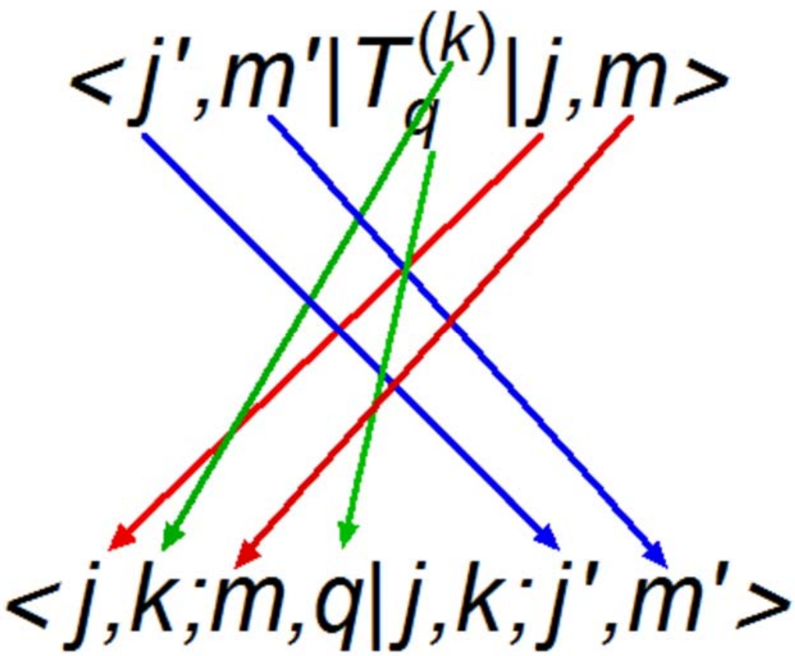


Fig. Definition of Clebsch-Gordan coefficient

where

$j' = j+k, j+k-1, \ldots\ldots, |j-k|.$

$m' = m + q$

with

$q = k, k-1, \ldots, -k.$

Note that the matrix $\langle \alpha'; j', m' | \hat{T}_q^{(k)} | \alpha; j, m \rangle$ is not a *n* x *n* types matrix since *j*' is not always equal to *j*.

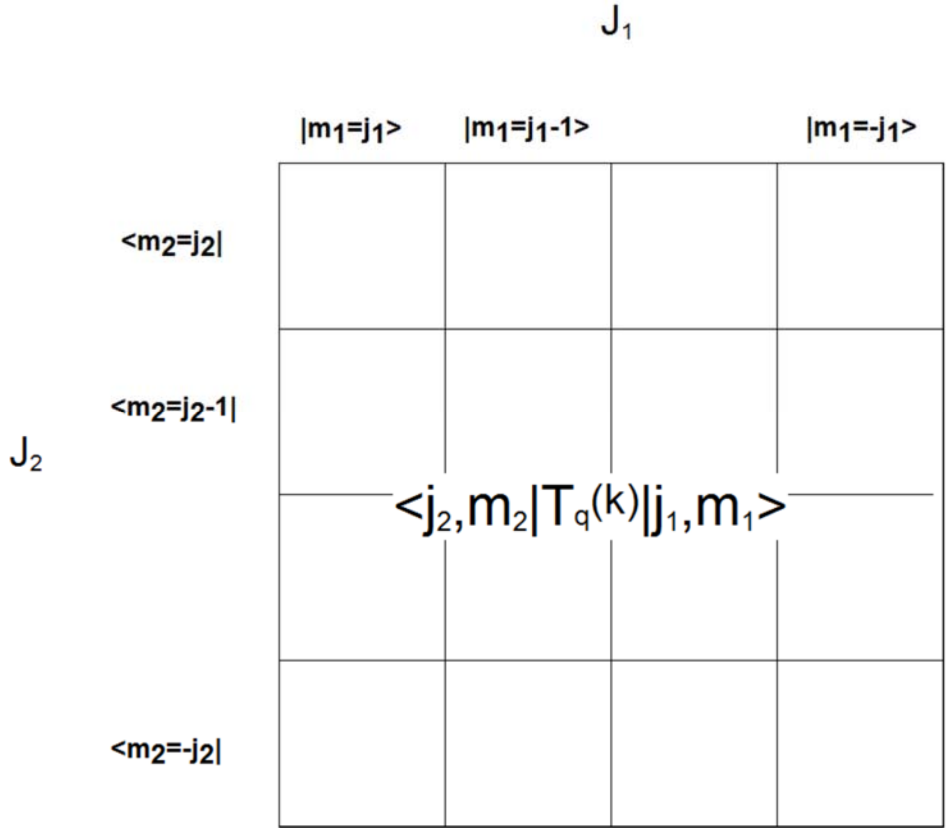


Fig. The matrix element: $\langle \alpha'; j', m' | \hat{T}_q^{(k)} | \alpha; j, m \rangle$. $j_2 = j'$. $m_2 = m'$. $j_1 = j$. $m_1 = m'$.

4. Numerical calculation of matrix using Mathematica

$$D_{j'} = D_j \times D_k = D_{j+k} + D_{j+k-1} + ... + D_{|j-k|}$$

$$j' = j + k, \qquad j + k - 1, \qquad \ldots.., \; |j - k|$$

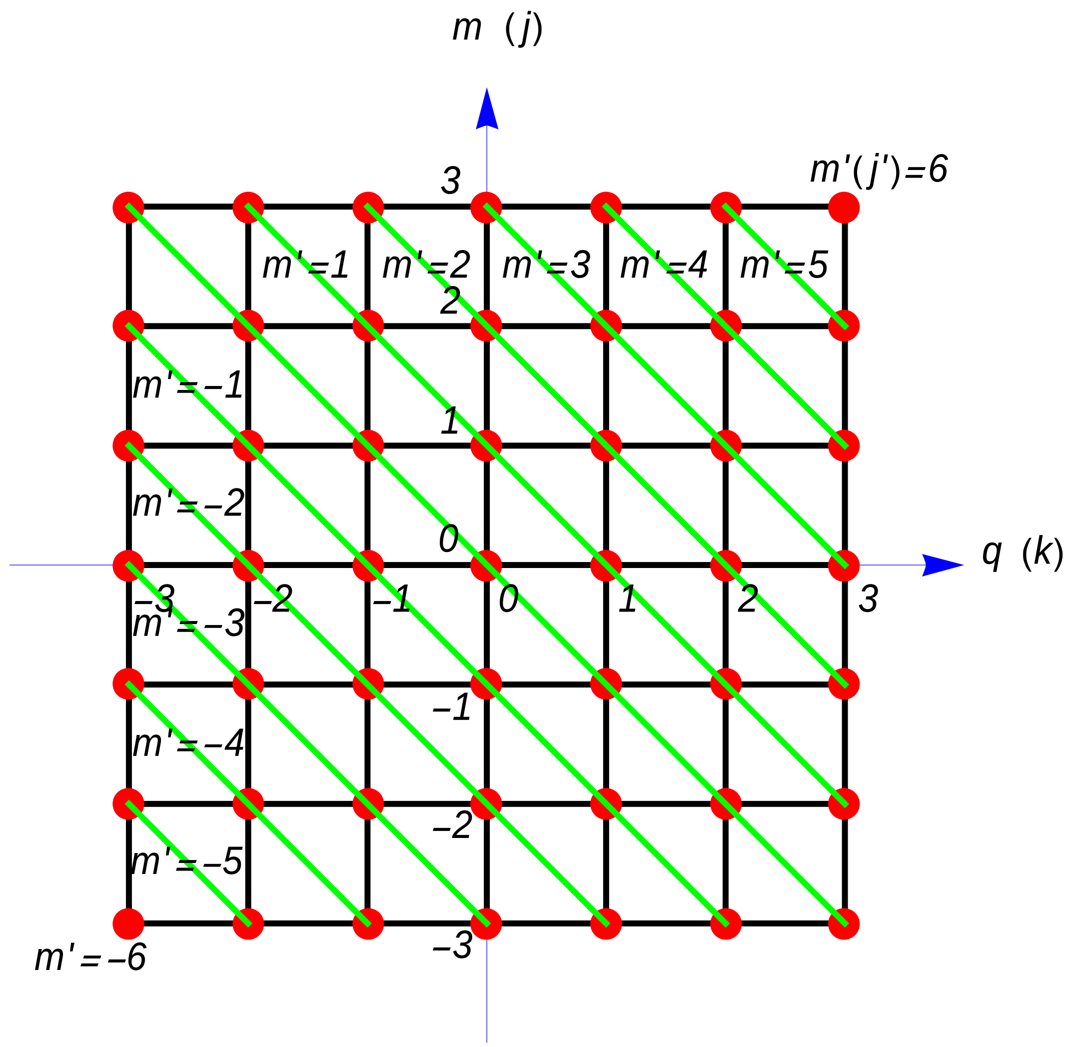


Fig. Example. j = 3 (m = 3, 2, 1, 0, -1, -2, -3). k = 3 (q = 3, 2, 1, 0, -1, -2, -3). j'=6 (m'=3,2, 1, 0, -1, -2, -3).

(a) Mathematica program I (standard)

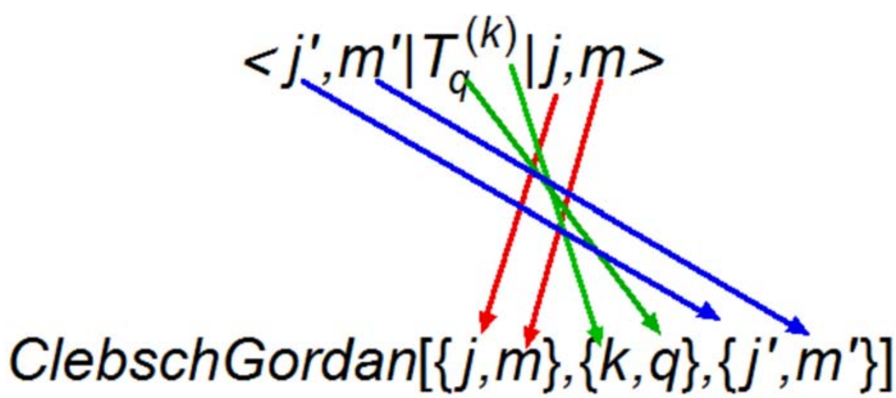


Fig. Mathyematica program for the clebsch-Gordan coefficient

Each point corresponds to the Clebsch-Gordan coefficient

ClebschGordan[{*j*,*m*},{*k*,*q*},{*j*',*m*'}] (in Mathematica)

which is equal to the matrix element

$$\langle j'm'|T_q^{(k)}|jm\rangle$$

where

$m' = m + q$, $j' = j+k, j+k-1, \ldots, |j-k|$.

(b) Mathematica program II

$$\langle j',m'|T_q^{(k)}|j,m\rangle$$

$\mathrm{CCG}[j',\{k,q\},j]$ = [{*m*', *m*}, the corresponding Clebsch-Gordan efficient]

((**Mathematica**))

```
Clear["Global`*"]; CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
     ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]];
CCG[j2_, {k1_, q1_}, j1_] :=
 Table[{{m1 + q1, m1}, CCGG[{j1, m1}, {k1, q1}, {j2, m1 + q1}]}, {m1, j1, -j1, -1}];
```

Using this program, one can get the table of the matrix elements of the tensor operator,

$$\langle j',m'|T_q^{(k)}|j,m\rangle$$

for given *k*, *q*, and *j*. Here the matrix can be obtained using the notation of matrix; The result from the Mathematica is as follows. The matrix element $\langle j',m'|T_q^{(k)}|j,m\rangle$ is given by

$\mathrm{CCG}[j',\{k,q\},j]$ = [{*m*', *m*}, the corresponding Clebsch-Gordan efficient]

with

$m' = m + q$,

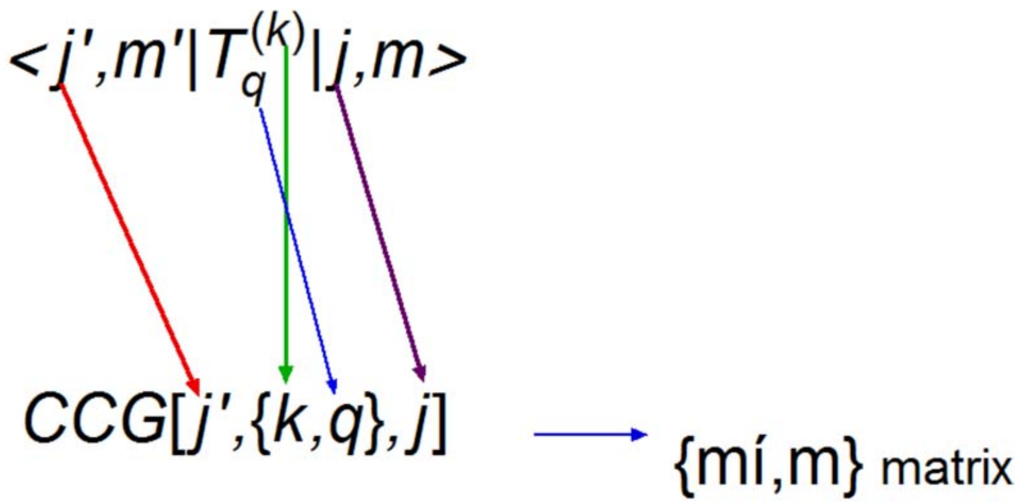


Fig. Mathematica program CCG for the matrix element {*m*', *m*} for given *j*', *k*, *q*, and *j*.

5. Example 1: Tensor of rank 0 (scalar)

By using the Mathematica, we calculate the matrix element

$\langle \alpha'; j', m' | \hat{T}_{q=0}^{(k=0)} | \alpha; j, m \rangle \neq 0$ at the point (*m*', *m*)

which is proportional to the Clebsch-Gordan coefficient

$\langle j, k; m, q | j, k; j', m' \rangle$;

ClebschGordan[{*j*,*m*},{*k*,*q*},{*j*', *m*'}] (in Mathematica)

or

$CCG[\{j', \{k, q\}, j]$ at the point (m', m) (program in this lecture)

only if

$m' = m$ and $j' = j$

((**Example**)) $CCG[\{j', \{k, q\}, j]$ at the point (m', m)

(a) $\langle j' = 0, m' | \hat{T}_{q=0}^{(k=0)} | j = 0, m \rangle$

$k = 0$, $q = 0$; $j = 0$, $j' = 0$

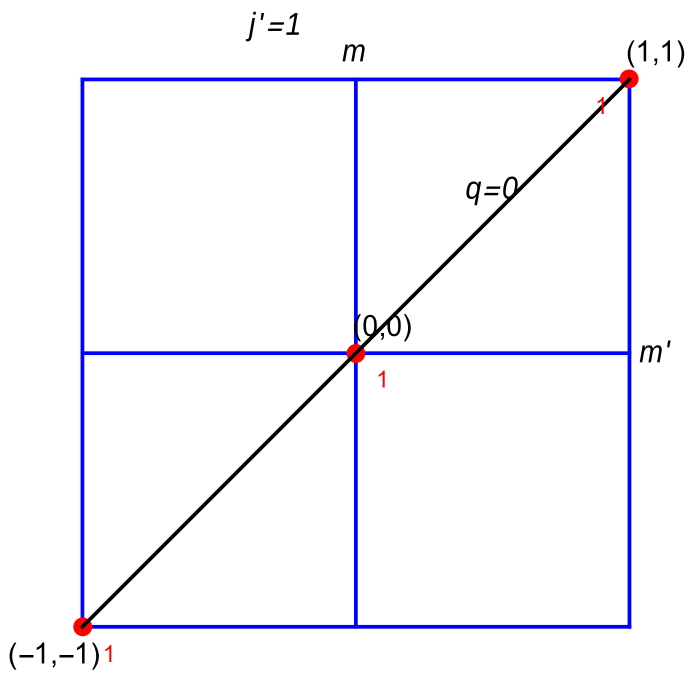
```
CCG[0, {0, 0}, 0]
{{{0, 0}, 1}}
```

(b)

$$\langle j'=1, m' | \hat{T}^{(k=0)}_{q=0} | j=1, m \rangle$$

$k = 0$, $q = 0$; $j = 1$, $j' = 1$

```
CCG[1, {0, 0}, 1]
{{{1, 1}, 1}, {{0, 0}, 1}, {{-1, -1}, 1}}
```



6. Example 2: Tensor of rank 1 (vector): $k = 1$.

We calculate the matrix element

$$\langle \alpha'; j', m' | \hat{T}^{(k=1)}_{q} | \alpha; j, m \rangle$$

which is proportional to the Clebsch-Gordan coefficient

$$\langle j, k; j', m' | j, k; m, q \rangle$$

only if

$m' = m + q$, $j' = j + 1, j, |j - 1|$

since $k = 1$. Note that $q = 1, 0, -1$

(a) $\langle j'=1, m' | \hat{T}_q^{(k=1)} | j=0, m \rangle$

$k = 1, q = 1$: $j = 0$, $j'=1$

```
CCG[1, {1, 1}, 0]
```

{{{1, 0}, 1}}

$k = 1, q = 0$: $j = 0$, $j'=1$

```
CCG[1, {1, 0}, 0]
```

{{{0, 0}, 1}}

$k = 1, q = -1$: $j = 0$, $j'=1$

```
CCG[1, {1, -1}, 0]
```

{{{-1, 0}, 1}}

(b) $\langle j'=2, m' | \hat{T}_q^{(k=1)} | j=1, m \rangle$

$k = 1, q = 1$: $j = 1$, $j'=2$

```
CCG[2, {1, 1}, 1]
```

$\left\{\{\{2, 1\}, 1\}, \left\{\{1, 0\}, \frac{1}{\sqrt{2}}\right\}, \left\{\{0, -1\}, \frac{1}{\sqrt{6}}\right\}\right\}$

$k = 1, q = 0$: $j = 1$, $j'=2$

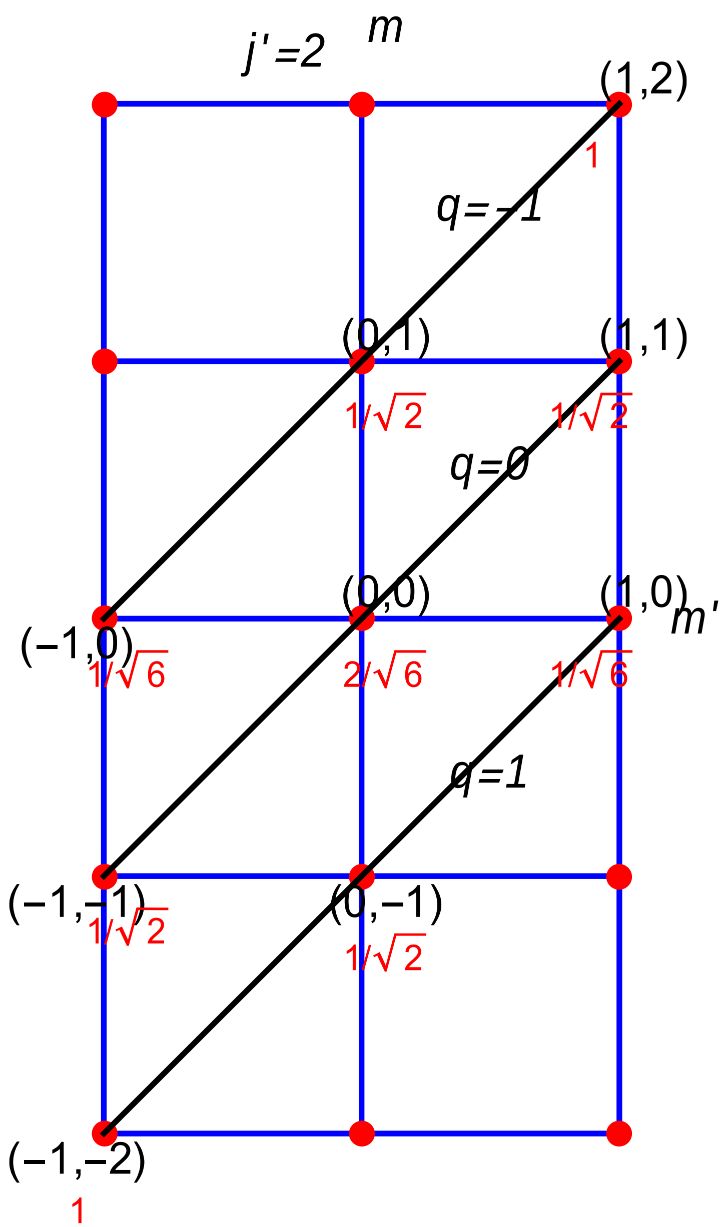
```
CCG[2, {1, 0}, 1]
```

$\left\{\left\{\{1, 1\}, \frac{1}{\sqrt{2}}\right\}, \left\{\{0, 0\}, \sqrt{\frac{2}{3}}\right\}, \left\{\{-1, -1\}, \frac{1}{\sqrt{2}}\right\}\right\}$

$k = 1, q = -1$: $j = 1$, $j'=2$

CCG[2, {1, -1}, 1]

$\left\{\left\{\{0, 1\}, \frac{1}{\sqrt{6}}\right\}, \left\{\{-1, 0\}, \frac{1}{\sqrt{2}}\right\}, \{\{-2, -1\}, 1\}\right\}$



(c) $\left\langle j'=1, m' \middle| \hat{T}_q^{(k=1)} \middle| j=1, m \right\rangle$

$k=1, q=1$: $j=1$, $j'=1$

```
CCG[1, {1, 1}, 1]
```

$\{\{\{2, 1\}, \text{Null}\}, \{\{1, 0\}, -\frac{1}{\sqrt{2}}\}, \{\{0, -1\}, -\frac{1}{\sqrt{2}}\}\}$

$k = 1, q = 0$: $j = 1$, j'=1

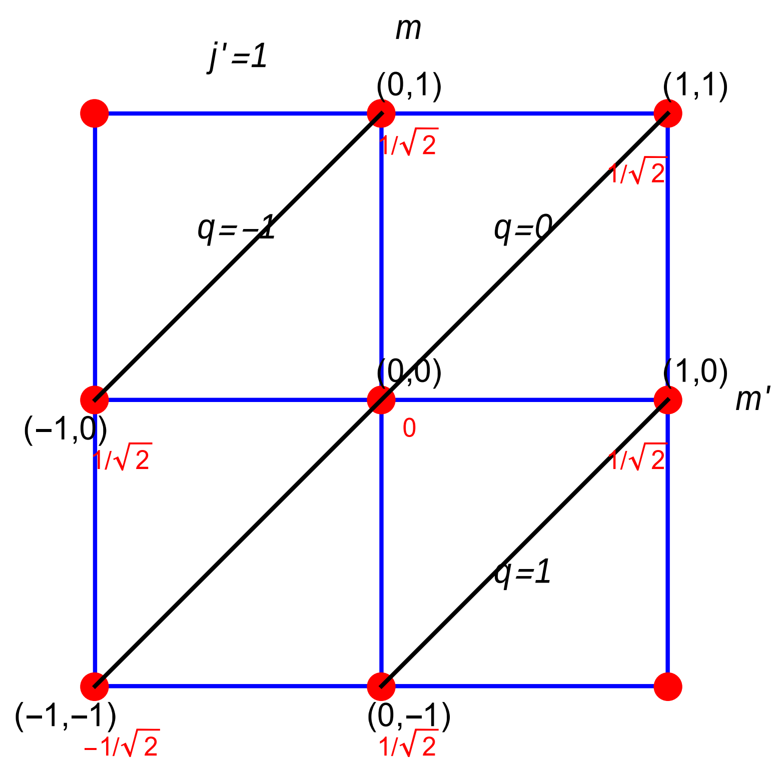
```
CCG[1, {1, 0}, 1]
```

$\{\{\{1, 1\}, \frac{1}{\sqrt{2}}\}, \{\{0, 0\}, 0\}, \{\{-1, -1\}, -\frac{1}{\sqrt{2}}\}\}$

$k = 1, q = -1$: $j = 1$, j'=1

```
CCG[1, {1, -1}, 1]
```

$\{\{\{0, 1\}, \frac{1}{\sqrt{2}}\}, \{\{-1, 0\}, \frac{1}{\sqrt{2}}\}, \{\{-2, -1\}, \text{Null}\}\}$



(d) $\langle j'=0, m' | \hat{T}_q^{(k=1)} | j=1, m \rangle$

$k = 1, q = 1$: $j = 1$, j'=0

```
CCG[0, {1, 1}, 1]
```

$\{\{\{2, 1\}, \text{Null}\}, \{\{1, 0\}, \text{Null}\}, \{\{0, -1\}, \frac{1}{\sqrt{3}}\}\}$

$k = 1, q = 0$: $j = 1$, $j'=0$

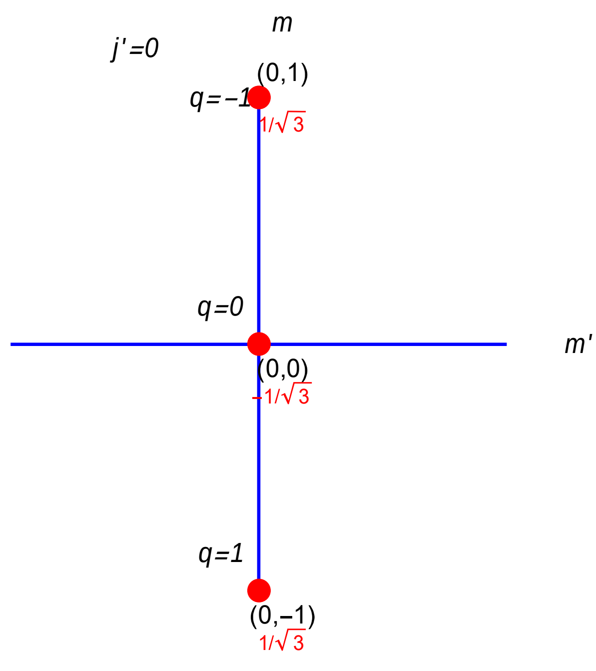
```
CCG[0, {1, 0}, 1]
```

$\{\{\{1, 1\}, \text{Null}\}, \{\{0, 0\}, -\frac{1}{\sqrt{3}}\}, \{\{-1, -1\}, \text{Null}\}\}$

$k = 1, q = -1$: $j = 1$, $j'=0$

```
CCG[0, {1, -1}, 1]
```

$\{\{\{0, 1\}, \frac{1}{\sqrt{3}}\}, \{\{-1, 0\}, \text{Null}\}, \{\{-2, -1\}, \text{Null}\}\}$



7. Selection rule for the rank 1 tensor (I)

We consider the case when

$$\hat{T}_{+1}^{(1)} = \hat{V}_{+1} = -\frac{1}{\sqrt{2}}\hat{V}_{+} = -\frac{1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y)$$

$$\hat{T}_{0}^{(1)} = \hat{V}_{0} = \hat{V}_z$$

$$\hat{T}_{-1}^{(1)} = \hat{V}_{-1} = \frac{1}{\sqrt{2}}\hat{V}_{-} = \frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y)$$

Then we have

(a)

$$\langle \alpha'; j', m'|\hat{V}_x|\alpha; j, m\rangle = \langle \alpha'; j', m'|\frac{\hat{T}_{-1}^{(1)} - \hat{T}_{1}^{(1)}}{\sqrt{2}}|\alpha; j, m\rangle = 0$$

$$\langle \alpha'; j', m'|\hat{V}_y|\alpha; j, m\rangle = \langle \alpha'; j', m'|\frac{\hat{T}_{-1}^{(1)} + \hat{T}_{1}^{(1)}}{-\sqrt{2}i}|\alpha; j, m\rangle = 0$$

unless $m' = m \pm 1$ and $j' = j+1, j, |j-1|$

(b)

$$\langle \alpha'; j', m'|\hat{V}_z|\alpha; j, m\rangle = \langle \alpha'; j', m'|\hat{T}_{0}^{(1)}|\alpha; j, m\rangle = 0$$

unless $m' = m$ and $j' = j+1, j, |j-1|$

8. Selection rule for the rank 1 tensor with the odd parity

Spherical tensor of rank 1

$$T_{1}^{(1)} = -\frac{\hat{x} + i\hat{y}}{\sqrt{2}}$$

$$T_{0}^{(1)} = \hat{z}$$

$$T_{-1}^{(1)} = \frac{\hat{x} - i\hat{y}}{\sqrt{2}}$$

From Wigner-Eckart theorem

$\langle n', l', m'|\hat{T}_q^{(1)}|n, l, m\rangle \neq 0$ for $m' = m + q$ and for $l' = l+1, l, |l-1|$.

where l is integer. For the parity operator, we have

$$\hat{\pi}\hat{T}_q^{(1)}\hat{\pi} = -\hat{T}_q^{(1)} \qquad \text{(odd parity)}$$

and

$$\hat{\pi}|n,l,m\rangle = (-1)^l |n,l,m\rangle, \qquad \langle n,l,m|\hat{\pi} = (-1)^l \langle n,l,m|$$

Thus the matrix element is equal to zero for $l' = l$, since

$$\langle n',l',m'|\hat{T}_q^{(1)}|n,l,m\rangle = -\langle n',l',m'|\hat{\pi}\hat{T}_q^{(1)}\hat{\pi}|n,l,m\rangle = (-1)^{l'+l-1}\langle n',l',m'|\hat{T}_q^{(1)}|n,l,m\rangle$$

Finally we have the selection rule

$$\langle n',l',m'|\hat{T}_q^{(1)}|n,l,m\rangle \neq 0 \text{ for } m' = m+q \text{ and for } l'-l = \pm 1$$

9. Selection rule for rank-2 tensor (even parity)

Spherical tensor of rank 2

$$\hat{x}^2 - \hat{y}^2 = \hat{T}_2^{(2)} + \hat{T}_{-2}^{(2)}$$

$$\hat{x}\hat{y} = \frac{\hat{T}_2^{(2)} - \hat{T}_{-2}^{(2)}}{2i}$$

$$\hat{y}\hat{z} = \frac{\hat{T}_1^{(2)} + \hat{T}_{-1}^{(2)}}{-2i}$$

$$\hat{x}\hat{z} = \frac{\hat{T}_1^{(2)} - \hat{T}_{-1}^{(2)}}{-2}$$

$$\left(\frac{\hat{x}^2 + \hat{y}^2 - 2\hat{z}^2}{\sqrt{6}}\right) = -\hat{T}_0^{(2)}$$

From Wigner-Eckart theorem

$$\langle n',l',m'|\hat{T}_q^{(2)}|n,l,m\rangle \neq 0$$

for $m'=m+q$ and for $l'=l+2,l+1,l,|l-1|,|l-2|$. For the parity operator, we have

$$\hat{\pi}\hat{T}_q^{(2)}\hat{\pi}=\hat{T}_q^{(2)} \qquad \text{(even parity)}$$

and

$$\hat{\pi}|n,l,m\rangle=(-1)^l|n,l,m\rangle, \qquad \langle n,l,m|\hat{\pi}=(-1)^l\langle n,l,m|$$

Thus the matrix element is equal to zero for $l'=l\pm1$ since

$$\langle n',l',m'|\hat{T}_q^{(2)}|n,l,m\rangle=\langle n',l',m'|\hat{\pi}\hat{T}_q^{(2)}\hat{\pi}|n,l,m\rangle=(-1)^{l'+l}\langle n',l',m'|\hat{T}_q^{(2)}|n,l,m\rangle.$$

Finally we have the selection rule;

$$\langle n',l',m'|\hat{T}_q^{(2)}|n,l,m\rangle\neq0 \text{ for } m'=m+q \text{ and for } l'-l=\pm2,0$$

10. Formula

$$[\hat{J}_\mu,T_q^{(k)}]=\sqrt{k(k+1)}\langle k,1;q,\mu|k,1;k,q+\mu\rangle\hat{T}_{q+\mu}^{(k)}$$

with

$$\mu=0,+1,-1.$$

((**Proof**))

We start with the formula

$$[\hat{J}_z,\hat{T}_q^{(k)}]=\hbar q\hat{T}_q^{(k)}$$

$$[\hat{J}_+,\hat{T}_q^{(k)}]=\hbar\sqrt{(k-q)(k+q+1)}\hat{T}_{q+1}^{(k)}$$

$$[\hat{J}_-,\hat{T}_q^{(k)}]=\hbar\sqrt{(k+q)(k-q+1)}\hat{T}_{q-1}^{(k)}$$

Using the Mathematica for the Clebsch-Gordan co-efficient

$$\langle j_1,j_2,m_1,m_2|j_1,j_2;j,m\rangle$$

we get

$$\langle k,1;q,0|k,1;k,q\rangle = \langle j_1 = k,\ j_2 = 1, m_1 = q, m_2 = 0|\ j_1 = k,\ j_2 = 1;\ j = k, m = q\rangle$$

$$= \frac{q}{\sqrt{k(k+1)}}$$

$$\langle k,1;q,1|k,1;k,q+1\rangle = \langle j_1 = k,\ j_2 = 1, m_1 = q, m_2 = 1|\ j_1 = k,\ j_2 = 1;\ j = k, m = q+1\rangle$$

$$= -\frac{\sqrt{(k-q)(k+q+1)}}{\sqrt{2}\sqrt{k(k+1)}}$$

$$\langle k,1;q,-1|k,1;k,q-1\rangle = \langle j_1 = k,\ j_2 = 1, m_1 = q, m_2 = -1|\ j_1 = k,\ j_2 = 1;\ j = k, m = q-1\rangle$$

$$= (-1)^{2(k+q)}\frac{\sqrt{(k+q)(k-q+1)}}{\sqrt{2}\sqrt{k(k+1)}}$$

$$= \frac{\sqrt{(k+q)(k-q+1)}}{\sqrt{2}\sqrt{k(k+1)}}$$

since $k+q$ is assumed to be integer.

((**Mathematica**))

```
Clear["Global`*"];
ClebschGordan[{k, q}, {1, 0}, {k, q}] // Simplify[#, k > 1 / 2] &
```

$$\begin{cases} \frac{q}{\sqrt{k\ (1+k)}} & k \geq q \ \&\&\ k+q \geq 0 \\ 0 & \text{True} \end{cases}$$

```
ClebschGordan[{k, q}, {1, 1}, {k, q + 1}] //
  Simplify[#, k > 1 / 2] & // FullSimplify
```

$$\begin{cases} -\frac{1}{\sqrt{2}\sqrt{\frac{k\ (1+k)}{k+k^2-q\ (1+q)}}} & k \geq 1+q \ \&\&\ k+q \geq 0 \\ 0 & \text{True} \end{cases}$$

```
ClebschGordan[{k, q}, {1, -1}, {k, q - 1}] //
  Simplify[#, k > 1 / 2] & // Factor
```

$$\begin{cases} \frac{(-1)^{2\ (k+q)}\sqrt{\frac{k+k^2+q-q^2}{k+k^2}}}{\sqrt{2}} & k \geq q \ \&\&\ k+q \geq 1 \\ 0 & \text{True} \end{cases}$$

```
f1[x_] := ClebschGordan[{1, -x}, {1, x}, {1, 0}] // Simplify
```

```
f1[0] + f1[-1] + f1[1]
```

0

Then we get the formula

$$\begin{aligned}[\hat{J}_z, \hat{T}_q^{(k)}] &= \hbar q \hat{T}_q^{(k)} \\ &= \hbar\sqrt{k(k+1)}\langle k,1;q,0|k,1;k,q\rangle \hat{T}_q^{(k)}\end{aligned}$$

$$\begin{aligned}[\hat{J}_+, \hat{T}_q^{(k)}] &= \hbar\sqrt{(k-q)(k+q+1)}\hat{T}_{q+1}^{(k)} \\ &= -\hbar\sqrt{2k(k+1)}\langle k,1;q,1|k,1;k,q+1\rangle \hat{T}_{q+1}^{(k)}\end{aligned}$$

$$\begin{aligned}[\hat{J}_-, \hat{T}_q^{(k)}] &= \hbar\sqrt{(k+q)(k-q+1)}\hat{T}_{q-1}^{(k)} \\ &= \hbar(-1)^{-2(k+q)}\sqrt{2k(k+1)}\langle k,1;q,-1|k,1;k,q-1\rangle \hat{T}_{q-1}^{(k)}\end{aligned}$$

We assume that

$$\hat{J}_1 = -\frac{1}{\sqrt{2}}(\hat{J}_x + i\hat{J}_y) = -\frac{1}{\sqrt{2}}\hat{J}_+, \qquad \hat{J}_{-1} = \frac{1}{\sqrt{2}}(\hat{J}_x - i\hat{J}_y) = \frac{1}{\sqrt{2}}\hat{J}_-.$$

Then the above formula can be rewritten as

$$[\hat{J}_z, \hat{T}_q^{(k)}] = \hbar\sqrt{k(k+1)}\langle k,1;q,0|k,1;k,q\rangle \hat{T}_q^{(k)}$$

$$[\hat{J}_{+1}, \hat{T}_q^{(k)}] = \hbar\sqrt{k(k+1)}\langle k,1;q,1|k,1;k,q+1\rangle \hat{T}_{q+1}^{(k)}$$

$$\begin{aligned}[\hat{J}_{-1}, \hat{T}_q^{(k)}] &= \frac{1}{\sqrt{2}}\hbar\sqrt{(k+q)(k-q+1)}\hat{T}_{q-1}^{(k)} \\ &= \hbar(-1)^{-2(k+q)}\sqrt{k(k+1)}\langle k,1;q,-1|k,1;k,q-1\rangle \hat{T}_{q-1}^{(k)} \\ &= \hbar(\sqrt{k(k+1)}\langle k,1;q,-1|k,1;k,q-1\rangle \hat{T}_{q-1}^{(k)}\end{aligned}$$

---

11. Projection theorem (Wigner-Eckart theorem for a scalar product $\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}$)

The projection theorem for the tensor of rank 1 is given by

$$\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)} = \sum_q (-1)^q \hat{J}_q \hat{T}_{-q}^{(1)}$$

(1) Decomposition theorem:

$$\langle j',m'|\hat{T}_q^{(1)}|j,m\rangle = \frac{\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)})|j,m\rangle}{\hbar^2 j(j+1)}\delta_{j,j'} \tag{1}$$

(2) Factorization theorem:

$$\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)})|j,m\rangle = \langle j',m'|\hat{J}_q|j,m\rangle < j\,||\,\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)}\,||\,j>. \tag{2}$$

(3) Decomposition theorem of the second kind (the projection theorem)

$$\langle j',m'|\hat{T}_q^{(1)}|j,m\rangle = \frac{\langle j',m'|\hat{J}_q|j,m\rangle < j\,||\,\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)}\,||\,j>}{\hbar^2 j(j+1)}\delta_{j,j'}. \tag{3}$$

((**Note**)) The expressions for the vector operator $\hat{V}_q$ insteads of $\hat{T}_q^{(1)}$;

$$\langle j',m'|\hat{V}_q|\alpha;j,m\rangle = \langle j,1;m,q|j,1;j',m'\rangle < j\,||\,\hat{\boldsymbol{V}}\,||\,j>$$

with $\langle j,1;m,q | j,1;j',m'\rangle \propto$ ClebschGordan[{j,m},{1,q},{j',m'}]

$$\langle j',m'|\hat{V}_q|j,m\rangle = \frac{\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}})|j,m\rangle}{\hbar^2 j(j+1)}\delta_{j,j'}$$

$$\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}})|j,m\rangle = \langle j',m'|\hat{J}_q|j,m\rangle < j \| \hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}} \| j >,$$

$$\langle j',m'|\hat{V}_q|j,m\rangle = \frac{\langle j',m'|\hat{J}_q|j,m\rangle < j \| \hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}} \| j >}{\hbar^2 j(j+1)}\delta_{j,j'}.$$

---

12. Proof of the decomposition theorem

The matrix element

$$M = \langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)})|j,m\rangle = \sum_{\mu}(-1)^{\mu}\langle j',m'|\hat{J}_q(\hat{J}_{\mu}\hat{T}_{-\mu}{}^{(1)})|j,m\rangle$$

Here we use the commutation relation,

$$\hat{J}_{\mu}\hat{T}_{-\mu}{}^{(1)} = [\hat{J}_{\mu},\hat{T}_{-\mu}{}^{(1)}] + \hat{T}_{-\mu}{}^{(1)}\hat{J}_{\mu}$$

Then we get

$$\begin{aligned} M &= \langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)})|j,m\rangle \\ &= \sum_{\mu}(-1)^{\mu}\langle j',m'|\hat{J}_q(\hat{T}_{-\mu}{}^{(1)}\hat{J}_{\mu})|j,m\rangle + \sum_{\mu}(-1)^{\mu}\langle j',m'|\hat{J}_q([\hat{J}_{\mu},\hat{T}_{-\mu}{}^{(1)}])|j,m\rangle \end{aligned}$$

We note that

$$[\hat{J}_{\mu},T_q^{(k)}] = \sqrt{k(k+1)}\langle k,1;q,\mu|k,1;k,q+\mu\rangle\hat{T}_{q+\mu}^{(k)}$$

For $q = -\mu$ and $k = 1$, we have

$$[\hat{J}_{\mu},T_{-\mu}^{(1)}] = \sqrt{2}\langle 1,1;-\mu,\mu|1,1;k,0\rangle\hat{T}_0^{(1)}$$

Using this commutation relation, we calculate the second term defined by

$$\sum_{\mu}(-1)^{\mu}\langle j',m'|\hat{J}_q([\hat{J}_{\mu},\hat{T}_{-\mu}^{(1)}])|j,m\rangle = \sqrt{2}\sum_{\mu}(-1)^{\mu}\langle j',m'|\hat{J}_q T_0^{(1)}|j,m\rangle\langle 1,1;-\mu,\mu|1,1;1,0\rangle$$
$$= \sqrt{2}\langle j',m'|\hat{J}_q T_0^{(1)}|j,m\rangle\sum_{\mu}(-1)^{\mu}\langle 1,1;-\mu,\mu|1,1;1,0\rangle$$
$$= 0$$

where

$$\sum_{\mu}(-1)^{\mu}\langle 1,1;-\mu,\mu|1,1;1,0\rangle = 0$$

(We check the final result using the Mathematica). Then we have

$$M = \sum_{\mu}(-1)^{\mu}\langle j',m'|\hat{J}_q\hat{T}_{-\mu}^{(1)}\hat{J}_{\mu}|j,m\rangle$$

Noting that

$$\langle j',m'|\hat{J}_{\mu}|j,m\rangle = \delta_{j,j'}\delta_{m',m+\mu}\langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle . \qquad (\mu = 1,0,-1),$$

we obtain

$$\langle j',m'|\hat{J}_{\mu}|j,m\rangle = \delta_{j,j'}\delta_{m',m+\mu}\langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle$$
$$=\delta_{j,j'}\delta_{m',m+\mu}\langle j,1;m,\mu|j,1;j,m+\mu\rangle < j\|\hat{\boldsymbol{J}}\|j>$$
$$=\delta_{j,j'}\delta_{m',m+\mu}\hbar\sqrt{j(j+1)}\langle j,1;m,\mu|j,1;j,m+\mu\rangle$$

using the Wigner-Eckart theorem, where

$$< j\|\hat{\boldsymbol{J}}\|j> = \hbar\sqrt{j(j+1)} .$$

$$\langle j,m+\mu|\hat{T}_{\mu}^{(1)}|j,m\rangle = \langle j,1;m,q|j,1;j,m+\mu\rangle\langle j'\|\hat{T}^{(1)}\|j\rangle$$

$$\langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle = \langle j,1;m,q|j,1;j,m+\mu\rangle\langle j'\|\hat{\boldsymbol{J}}\|j\rangle \quad \text{with } \hat{T}_{\mu}^{(1)} = \hat{J}_{\mu}$$

Then we have

$$\langle j',m'|\hat{J}_q\hat{T}_{-\mu}{}^{(1)}\hat{J}_\mu|j,m\rangle = \langle j',m'|\hat{J}_q|j',m'-q\rangle\sum_{j'',m''}\langle j',m'-q|\hat{T}_{-\mu}{}^{(1)}|j'',m''\rangle\langle j'',m''|\hat{J}_\mu|j,m\rangle$$

$$= \langle j',m'|\hat{J}_q|j',m'-q\rangle\langle j',m'-q|\hat{T}_{-\mu}{}^{(1)}|j,m+\mu\rangle\langle j,m+\mu|\hat{J}_\mu|j,m\rangle$$

$$= \langle j',m'|\hat{J}_q|j',m'-q\rangle\langle j,m+\mu|\hat{J}_\mu|j,m\rangle\langle j',m'-q|\hat{T}_{-\mu}{}^{(1)}|j,m+\mu\rangle$$

$$= \delta_{m'm+q}\langle j',m'|\hat{J}_q|j',m'-q\rangle\langle j,m+\mu|\hat{J}_\mu|j,m\rangle\langle j',m|\hat{T}_{-\mu}{}^{(1)}|j,m+\mu\rangle$$

$$= \delta_{m'm+q}\langle j',m'|\hat{J}_q|j',m'-q\rangle\langle j,m+\mu|\hat{J}_\mu|j,m\rangle$$

$$\times < j,1;m+\mu,-\mu\,|\,j,1;j',m>< j'\|\hat{T}^{(1)}\|\,j>$$

$$= \delta_{m'm+q}\hbar^2\sqrt{j'(j'+1)}\sqrt{j(j+1)} < j'\|\hat{T}^{(1)}\|\,j>\langle j',1;m'-q,q|j',1;j',m'\rangle$$

$$\langle j,1;m+\mu,-\mu|j,1;j',m\rangle\langle j,1;m,\mu|j,1;j,m+\mu\rangle$$

Here we note that

$$(-1)^\mu\langle j,1;m,\mu|j,1;j,m+\mu\rangle = \langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

(see the proof of this equation below). The sum over μ is

$$\sum_\mu(-1)^\mu\langle j,1;m+\mu,-\mu|j,1;j',m\rangle\langle j,1;m,\mu|j,1;j,m+\mu\rangle = \sum_\mu\langle j,1;m+\mu,-\mu|j,1;j',m\rangle$$

$$\times\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

$$= \delta_{j,j'}$$

(from the condition of orthogonality), where we use

$$\sum_{m_1,m_2}\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle\langle j_1,j_2;m_1,m_2|j_1,j_2;j',m'\rangle = \delta_{j,j'}\delta_{m,m'}$$

$$\langle j',m'|\hat{T}_q^{(k)}|j,m\rangle = \delta_{m',m+q}\langle j,k;m,q|j,k;j',m'\rangle < j'\|\hat{T}^{(k)}\|\,j>,$$

$$\langle j',m'|\hat{J}_\mu|j,m\rangle = \delta_{j,j'}\delta_{m',m+\mu}\hbar\sqrt{j(j+1)}\langle j,1;m,\mu|j,1;j',m+\mu\rangle$$

Then

$$\langle j',m+q|\hat{J}_q\hat{T}_{-\mu}{}^{(1)}\hat{J}_\mu|j,m\rangle = \hbar^2[j(j+1)]\delta_{j,j'} < j'\|\hat{T}^{(1)}\|\,j>\langle j,1;m,q|j,1;j,m+q\rangle$$

Here we put

$$m' = m + q$$

we have the form such that

$$\langle j',m'|\hat{J}_q\hat{T}_{-\mu}{}^{(1)}\hat{J}_\mu|j,m\rangle = \hbar^2 j(j+1)\delta_{j,j'} < j'\|\hat{T}^{(1)}\| j > \langle j,1;m,q|j,1;j',m'\rangle$$
$$= \hbar^2 j(j+1)\delta_{j,j'}\langle j',m'|\hat{T}_q^{(1)}|j,m\rangle$$

Then we get the final result

$$\langle j',m'|\hat{T}_q{}^{(1)}|j,m\rangle = \frac{\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)})|j,m\rangle}{\hbar^2 j(j+1)}\delta_{j,j'}$$

((**Note-1**))
Proof of

$$\langle j,1;m,\mu|j,1;j,m+\mu\rangle = (-1)^\mu\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

$$\langle j,m+\mu|\hat{J}_\mu|j,m\rangle = \hbar\sqrt{j(j+1)}\langle j,1;m,\mu|j,1;j,m+\mu\rangle$$

When μ is changed into -μ,

$$\langle j,m-\mu|\hat{J}_{-\mu}|j,m\rangle = \hbar\sqrt{j(j+1)}\langle j,1;m,-\mu|j,1;j,m-\mu\rangle$$

When $m' = m - \mu$, or $m = m' + \mu$,

$$\langle j,m'|\hat{J}_{-\mu}|j,m'+\mu\rangle = \hbar\sqrt{j(j+1)}\langle j,1;m'+\mu,-\mu|j,1;j,m'\rangle$$

Replacing μ' into μ,

$$\langle j,m|\hat{J}_{-\mu}|j,m+\mu\rangle = \hbar\sqrt{j(j+1)}\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

Here we note that

$$\hat{J}_\mu{}^+ = (-1)^\mu\hat{J}_{-\mu}, \qquad \text{or} \qquad \hat{J}_{-\mu} = (-1)^\mu\hat{J}_\mu{}^+$$

Then we get

$$\langle j,m|\hat{J}_{-\mu}|j,m+\mu\rangle = (-1)^{\mu}\langle j,m|\hat{J}_{\mu}^{+}|j,m+\mu\rangle$$
$$= \hbar\sqrt{j(j+1)}\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

Then we have

$$\langle j,m|\hat{J}_{\mu}^{+}|j,m+\mu\rangle = \langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle^{*}$$
$$= (-1)^{\mu}\hbar\sqrt{j(j+1)}\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

Since

$$\langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle = \langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle^{*}$$

from the definition, we finally obtain the relation

$$\langle j,m+\mu|\hat{J}_{\mu}|j,m\rangle = (-1)^{\mu}\hbar\sqrt{j(j+1)}\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

or

$$\langle j,1;m,\mu|j,1;j,m+\mu\rangle = (-1)^{\mu}\langle j,1;m+\mu,-\mu|j,1;j,m\rangle$$

((**Note-2**))
We show that

$$\sum_{\mu}(-1)^{\mu}\langle 1,1;-\mu,\mu|1,1;1,0\rangle = 0$$

using the Mathematica.

((**Mathematica**)) We use the Mathematica to calculate the Clebsch-Gordan coefficient.

```
Clear["Global`*"];

ClebschGordan[{1, -1}, {1, 1}, {1, 0}]
```

$-\frac{1}{\sqrt{2}}$

```
ClebschGordan[{1, 1}, {1, -1}, {1, 0}]
```

$\frac{1}{\sqrt{2}}$

```
ClebschGordan[{1, 0}, {1, 0}, {1, 0}]
```

0

```
Sum[(-1)^μ ClebschGordan[{1, -μ}, {1, μ}, {1, 0}], {μ, -1, 1, 1}]
```

0

13. Proof of the factorization theorem

$$\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)})|j,m\rangle = \sum_{j'',m''}\langle j',m'|\hat{J}_q|j'',m''\rangle\langle j'',m''|\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}|j,m\rangle$$

We use the fact that $\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}$ is a tensor of rank 0 and

$$\langle j'',m''|\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}|j,m\rangle = \delta_{j'',j}\delta_{m'',m} < j\,||\,\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}\,||\,j>$$

Then we have

$$\begin{aligned}\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)})|j,m\rangle &= \sum_{j'',m''}\langle j',m'|\hat{J}_q|j'',m''\rangle\delta_{j'',j}\delta_{m'',m} < j\,||\,\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}\,||\,j> \\ &= \langle j',m'|\hat{J}_q|j,m\rangle < j\,||\,\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}\,||\,j>\end{aligned}$$

14. Proof of the theorem III

$$\begin{aligned}\langle j',m'|\hat{T}_q^{(1)}|j,m\rangle &= \frac{\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)})|j,m\rangle}{\hbar^2 j(j+1)}\delta_{j,j'} \\ &= \frac{\langle j',m'|\hat{J}_q|j,m\rangle < j\,||\,\hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)}\,||\,j>}{\hbar^2 j(j+1)}\delta_{j,j'}\end{aligned}$$

15 Calculation of the reduced matrix $\langle j' \| \hat{\boldsymbol{J}}^2 \| j \rangle$

In the projection theorem

$$\langle j',m'|\hat{T}_q^{(1)}|j,m\rangle = \frac{\langle j',m'|\hat{J}_q|j,m\rangle < j \| \hat{\boldsymbol{J}}\cdot\boldsymbol{T}^{(1)} \| j >}{\hbar^2 j(j+1)}\delta_{j,j'}$$

we put

$$\hat{T}_q^{(1)} = \hat{J}_q$$

Then we get

$$\langle j',m'|\hat{J}_q|j,m\rangle = \frac{\langle j',m'|\hat{J}_q|j,m\rangle < j \| \hat{\boldsymbol{J}}^2 \| j >}{\hbar^2 j(j+1)}\delta_{j,j'}$$

or

$$< j' \| \hat{\boldsymbol{J}}^2 \| j > = \delta_{j,j'}\hbar^2 j(j+1)$$

---

16 Calculation of the reduced matrix $\langle j \| \hat{\boldsymbol{J}} \| j \rangle$

In the Wigner-Eckart theorem,

$$\langle j,m'|\hat{T}_q^{(k=1)}|j,m\rangle = \langle j,k=1;m,q|j,k=1;j,m'\rangle\langle j\|\hat{T}^{(k=1)}\|j\rangle$$

we put

$$\hat{T}_q^{(k=1)} = \hat{J}_0, \qquad (q=0)$$

$$\hat{T}^{(k=1)} = \hat{\boldsymbol{J}}$$

Then we get

$$\langle j,m'|\hat{J}_0|j,m\rangle = \langle j,k=1;m,q=0|j,k=1;j,m'\rangle\langle j\|\hat{\boldsymbol{J}}\|j\rangle$$

Here we note that

$$\langle j',m'|\hat{J}_0|j,m\rangle = m\hbar\delta_{m'm'}\delta_{j,j'}$$

$$\langle j,k=1;m,q=0|\, j,k=1;\, j,m'\rangle = \frac{m}{\sqrt{j(j+1)}}\delta_{m,m'}$$

Then we have

$$1 = \frac{1}{\hbar\sqrt{j(j+1)}}\langle j\|\hat{\boldsymbol{J}}\|\, j\rangle$$

or

$$\langle j\|\hat{\boldsymbol{J}}\|\, j\rangle = \hbar\sqrt{j(j+1)}$$

((**Proof**))

The proof of the formula

$$\langle j,k=1;m,q=0|\, j,k=1;\, j,m\rangle = \frac{m}{\sqrt{j(j+1)}}$$

is given by Mathematica as follows.

((**Mathematica**))

```
Clear["Global`*"]; CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]];
```

```
j1 = 1;
```

```
Table[{m1, CCGG[{j1, m1}, {1, 0}, {j1, m1}], m1/Sqrt[j1 (j1 + 1)]}, {m1, j1, -j1, -1}]
```

$\{\{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}, \{0, 0, 0\}, \{-1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}\}$

```
j1 = 2;
```

```
Table[{m1, CCGG[{j1, m1}, {1, 0}, {j1, m1}], m1/Sqrt[j1 (j1 + 1)]}, {m1, j1, -j1, -1}]
```

$\{\{2, \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\}, \{1, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\}, \{0, 0, 0\}, \{-1, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\}, \{-2, -\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\}\}$

```
j1 = 3;
```

```
Table[{m1, CCGG[{j1, m1}, {1, 0}, {j1, m1}], m1/Sqrt[j1 (j1 + 1)]}, {m1, j1, -j1, -1}]
```

$\{\{3, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\}, \{2, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}, \{1, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\}, \{0, 0, 0\},$
$\{-1, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\}, \{-2, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\}, \{-3, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\}\}$

17. Landè g-factor

The magnetic moment is defined by

$$\hat{\boldsymbol{\mu}} = -\frac{\mu_B}{\hbar}(\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}})$$

The total angular momentum is

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$$

The expectation value of the *m*-th component of the magnetic moment $\boldsymbol{\mu}$ can be obtained from the projection theorem (decomposition theorem of the second kind, see Rose),

In

$$\langle j,m|\hat{T}_q^{(1)}|j,m\rangle = \frac{\langle j,m|\hat{J}_q|j,m\rangle < j\,||\,\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{T}}^{(1)}\,||\,j>}{\hbar^2 j(j+1)}$$

we put

$$\hat{T}_q^{(1)} = \hat{\mu}_q, \qquad \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{T}}^{(1)} = \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\mu}}$$

Then we get

$$\langle j,m|\hat{\mu}_q|\, j,m\rangle = \frac{< j \,\|\, \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\mu}} \,\|\, j >}{\hbar^2 j(j+1)} \langle j,m|\hat{J}_q|; j,m\rangle$$

Now we have

$$\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{J}} = -\frac{\mu_B}{\hbar}(\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}}) \cdot (\hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}) = -\frac{\mu_B}{\hbar}(\hat{\boldsymbol{L}}^2 + 2\hat{\boldsymbol{S}}^2 + 3\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}) \cdot$$

and

$$\hat{\boldsymbol{J}}^2 = \hat{\boldsymbol{L}}^2 + \hat{\boldsymbol{S}}^2 + 2\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}$$

or

$$\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}} = \frac{\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2}{2}$$

Then we get

$$\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{J}} = \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\mu}} \cdot = -\frac{\mu_B}{\hbar}[\hat{\boldsymbol{L}}^2 + 2\hat{\boldsymbol{S}}^2 + \frac{3}{2}(\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2)] \cdot$$

and

$$< j \,\|\, \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\mu}} \,\|\, j > = -\frac{\mu_B}{\hbar}\hbar^2 \{l(l+1) + 2s(s+1) + \frac{3}{2}[(j(j+1) - l(l+1) - s(s+1)]\}$$

Then the expectation value of the magnetic moment along the z axis is

$$\begin{aligned}
\langle j,m|\hat{\mu}_0|\, j,m\rangle &= \frac{< j \,\|\, \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\mu}} \,\|\, j >}{\hbar^2 j(j+1)} \langle j,m|\hat{J}_0|\, j,m\rangle \\
&= -\frac{\mu_B}{\hbar}\frac{m\hbar}{\hbar^2 j(j+1)}\hbar^2 \{l(l+1) + 2s(s+1) \\
&\quad + \frac{3}{2}[(j(j+1) - l(l+1) - s(s+1)]\} \\
&= -\frac{m\mu_B}{2j(j+1)}[3(j(j+1) - l(l+1) + s(s+1)]
\end{aligned}$$

since $|j,m\rangle$ is a joint eigenstate of $\hat{\boldsymbol{J}}^2$, $\hat{\boldsymbol{L}}^2$, $\hat{\boldsymbol{S}}^2$, and $\hat{J}_0 = \hat{J}_z$ with eigenvalues $\hbar^2 j(j+1)$, $\hbar^2 l(l+1)$, $\hbar^2 s(s+1)$, and $\hbar m$, respectively.

Here we introduce the Landè g-factor as

$$\hat{\boldsymbol{\mu}} = -\frac{g_J \mu_B}{\hbar}\hat{\boldsymbol{J}}$$

Then we have

$$\langle j,m|\hat{\mu}_0|j,m\rangle = -\frac{g_J \mu_B}{\hbar}\langle j,m|\hat{J}_0|j,m\rangle = -\frac{g_J \mu_B}{\hbar} m\hbar = -m g_J \mu_B$$

and

$$g_J = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)}$$

18. The expectation value of S_z

Since

$$\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{J}}\cdot = \hat{\boldsymbol{S}}\cdot(\hat{\boldsymbol{L}}+\hat{\boldsymbol{S}}) = \hat{\boldsymbol{S}}^2 + \hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}^2 + \frac{\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2}{2} = \frac{\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 + \hat{\boldsymbol{S}}^2}{2}$$

we get the expectation of S_z as

$$\begin{aligned}
\langle j,m|\hat{S}_0|j,m\rangle &= \frac{<j\|\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{S}}\|j>}{\hbar^2 j(j+1)}\langle j,m|\hat{J}_0|;j,m\rangle \\
&= \frac{m\hbar}{2\hbar^2 j(j+1)}\hbar^2[j(j+1) - l(l+1) + s(s+1)] \\
&= \frac{m\hbar}{2}[1 + \frac{s(s+1) - l(l+1)}{j(j+1)}]
\end{aligned}$$

where we use the projection theorem.

19. Spin-orbit interaction: example of equivalent operators

The idea of operator equivalents finds applications in many branches of physics. As one of typical examples, we consider the effect of the spin-orbit interaction.

$$H_{SO} = \sum_{i=1}^{N}\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}_i\cdot\hat{\boldsymbol{s}}_i = \lambda\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}$$

where λ is the spin-orbit interaction constant. We use the equivalent operator

$$\langle j,m'|\hat{T}_q^{(1)}|j,m\rangle = \frac{\langle J,m'|\hat{J}_q|J,m\rangle}{\hbar\sqrt{j(j+1)}} < j \| \hat{T}^{(1)} \| j >$$

for the rank-1 tensor (vector): operator equivalents.

$$\begin{aligned}\langle LM_L';SM_S'|\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}_i\cdot\hat{\boldsymbol{s}}_i|LM_L;S,M_S\rangle &= [[\langle LM_L'|\hat{\boldsymbol{L}}|LM_L\rangle\frac{< L \| \xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}} \| L >}{\hbar\sqrt{L(L+1)}}]\cdot[\langle SM_s'|\boldsymbol{S}|SM_s\rangle\frac{< S \| \hat{\boldsymbol{s}}_i \| S >}{\hbar\sqrt{S(S+1)}}] \\ &= \langle L\|\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}\|L\rangle\langle S\|\hat{\boldsymbol{s}}_i\|S\rangle\frac{\langle LM_L'|\hat{\boldsymbol{L}}|LM_L\rangle}{\hbar\sqrt{L(L+1)}}\cdot\frac{\langle SM_s'|\boldsymbol{S}|SM_s\rangle}{\hbar\sqrt{S(S+1)}}\end{aligned}$$

If we sum over all the electrons (i) , we get

$$\begin{aligned}\langle LM_L';SM_S'|H_{SO}|LM_L;S,M_S\rangle &= \langle LM_L';SM_S'|\sum_i\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}_i\cdot\hat{\boldsymbol{s}}_i|LM_L;S,M_S\rangle \\ &= \sum_i\langle L\|\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}\|L\rangle\langle S\|\hat{\boldsymbol{s}}_i\|S\rangle\frac{\langle LM_L'|\hat{\boldsymbol{L}}|LM_L\rangle}{\hbar\sqrt{L(L+1)}}\cdot\frac{\langle SM_s'|\boldsymbol{S}|SM_s\rangle}{\hbar\sqrt{S(S+1)}}\end{aligned}$$

Noting that

$$\begin{aligned}\langle LM_L';SM_S'|H_{SO}|LM_L;S,M_S\rangle &= \langle LM_L';SM_S'|\lambda\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}|LM_L;S,M_S\rangle \\ &= \lambda\langle LM_L'|\hat{\boldsymbol{L}}|LM_L\rangle\cdot\langle SM_s'|\hat{\boldsymbol{S}}|SM_s\rangle\end{aligned}$$

Then we have

$$\begin{aligned}\lambda\langle LM_L'|\hat{\boldsymbol{L}}|LM_L\rangle\cdot\langle SM_s'|\hat{\boldsymbol{S}}|SM_s\rangle &= \sum_i\langle L\|\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}\|L\rangle\langle S\|\hat{\boldsymbol{s}}_i\|S\rangle \\ &\quad\times\frac{\langle LM_L'|\hat{\boldsymbol{L}}|LM_L\rangle}{\hbar\sqrt{L(L+1)}}\cdot\frac{\langle SM_s'|\boldsymbol{S}|SM_s\rangle}{\hbar\sqrt{S(S+1)}}\end{aligned}$$

or

$$\lambda = \sum_i\frac{\langle L\|\xi(\boldsymbol{r}_i)\hat{\boldsymbol{l}}\|L\rangle\langle S\|\hat{\boldsymbol{s}}_i\|S\rangle}{\hbar^2\sqrt{L(L+1)}\sqrt{S(S+1)}}$$

20. Magnetic form factor

Here we use the projection theorem,

$$\langle j',m'|\hat{\boldsymbol{V}}_q|j,m\rangle = \langle j',m'|\hat{\boldsymbol{J}}_q|j,m'\rangle \frac{< j \| \hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}} \| j >}{\hbar^2 j(j+1)}$$

where $\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}$ is a scalar so its expectation value is independent of *m*.

$$\hat{\boldsymbol{V}} = \sum_\nu e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}_\nu}\hat{\boldsymbol{s}}_\nu = \sum_\nu f_\nu \hat{\boldsymbol{s}}_\nu \qquad \hat{\boldsymbol{J}} = \hat{\boldsymbol{S}}_{ld} = \sum_\nu \hat{\boldsymbol{s}}_\nu$$

where the atom site is denoted by $j = \{l, d\}$, and $\boldsymbol{\nu}$ is the vector connecting atom at the site $\{l, d\}$ and electrons surrounding the nucleus. We also put

$$f_\nu = e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}_\nu},$$

for the simplicity. $\boldsymbol{\kappa}$ is the wave vector. $\hat{\boldsymbol{S}}_{ld}$ is the resultant spin determined from the Hund rule. Then we get

$$\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}} = \left(\sum_\nu f_\nu \hat{\boldsymbol{s}}_\nu\right)\cdot\boldsymbol{S}_{ld} = \boldsymbol{S}_{ld}\cdot\left(\sum_\nu f_\nu \hat{\boldsymbol{s}}_\nu\right).$$

Here we use the projection theorem,

$$\begin{aligned}\langle\lambda'|\sum_\nu f_\nu \hat{\boldsymbol{s}}_\nu|\lambda\rangle &= \langle\lambda'|\hat{\boldsymbol{S}}_{ld}|\lambda\rangle \frac{< \lambda \| \boldsymbol{S}_{ld}\cdot\sum_\nu f_\nu \hat{\boldsymbol{s}}_\nu \| \lambda >}{\hbar^2 S(S+1)} \qquad (1)\\ &= F_d(\boldsymbol{\kappa})\langle\lambda'|\hat{\boldsymbol{S}}_{ld}|\lambda\rangle\end{aligned}$$

where

$$F_d(\boldsymbol{\kappa}) = \frac{< \lambda \| \boldsymbol{S}_{ld}\cdot\sum_\nu f_\nu \hat{\boldsymbol{s}}_\nu \| \lambda >}{\hbar^2 S(S+1)}$$

is called the magnetic form factor. It is obtained by the Fourier transform of the normalized spin density at the site *j* (or denoted by *l*, *d*).

21. Quadrupole interaction

Using the Wigner-Eckart thorem, we get

$$\left\langle j,m'\left|\hat{Q}_q^{(k=2)}\right|j,m\right\rangle = \left\langle j,k=2;m,q\right|j,k=2;j,m'\rangle\left\langle j\left\|\hat{Q}^{(k=2)}\right\|j\right\rangle$$

for $\hat{T}_q^{(k=2)} = \hat{Q}_q^{(k=2)}$ (spherical tensor of rank-2)

and

$$\left\langle j,m'\left|\hat{Q}_{q=0}^{(k=2)}\right|j,m\right\rangle = \left\langle j,k=2;m,q=0\right|j,k=2;j,m'\rangle\left\langle j\left\|\hat{Q}^{(k=2)}\right\|j\right\rangle$$

In this last equation we put $m' = j$ and $m = j$

$$\left\langle j,m'=j\left|\hat{Q}_{q=0}^{(k=2)}\right|j,m=j\right\rangle = \left\langle j,k=2;m=j,q=0\right|j,k=2;j,m'=j\rangle\left\langle j\left\|\hat{Q}^{(k=2)}\right\|j\right\rangle$$

Then we get

$$\left\langle j\left\|\hat{Q}^{(k=2)}\right\|j\right\rangle = \frac{\left\langle j,m'=j\left|\hat{Q}_{q=0}^{(k=2)}\right|j,m=j\right\rangle}{\left\langle j,k=2;m=j,q=0\right|j,k=2;j,m'=j\rangle} =$$

Using this relation, we have

$$\begin{aligned}\left\langle j,m'\left|\hat{Q}_q^{(k=2)}\right|j,m\right\rangle &= \frac{\left\langle j,k=2;m,q\right|j,k=2;j,m'\rangle\left\langle j,m'=j\left|\hat{Q}_{q=0}^{(k=2)}\right|j,m=j\right\rangle}{\left\langle j,k=2;m=j,q=0\right|j,k=2;j,m'=j\rangle} \\ &= \frac{1}{2}eQ\frac{\left\langle j,k=2;m,q\right|j,k=2;j,m'\rangle}{\left\langle j,k=2;m=j,q=0\right|j,k=2;j,m'=j\rangle}\end{aligned}$$

where we define

$$\left\langle j,m'=j\left|\hat{Q}_{q=0}^{(k=2)}\right|j,m=j\right\rangle = \frac{1}{2}eQ$$

22. Summary: formula

$$\begin{aligned}\left\langle j',m'\left|\hat{J}_\mu\right|j,m\right\rangle &= \left\langle j,1;m,\mu\right|j,k=1;j',m'\rangle < j'\|\hat{\boldsymbol{J}}\|j> \\ &= \delta_{j,j'}\delta_{m',m+\mu}\left\langle j,1;m,\mu\right|j,1;j,m+\mu\rangle < j\|\hat{\boldsymbol{J}}\|j> \\ &= \delta_{j,j'}\delta_{m',m+\mu}\sqrt{j(j+1)}\left\langle j,1;m,\mu\right|j,1;j',m+\mu\rangle\end{aligned}$$

$$\left\langle j',m'\left|\hat{T}_q^{(k)}\right|j,m\right\rangle = \delta_{m',m+q}\left\langle j,k;m,q\right|j,k;j',m'\rangle\left\langle j'\left\|\hat{T}^{(k)}\right\|j\right\rangle$$

$$[\hat{J}_\mu, T_q^{(k)}] = \sqrt{k(k+1)}\left\langle k,1;q,\mu\right|k,1;k,q+\mu\rangle\hat{T}_{q+\mu}^{(k)}$$

$$[\hat{J}_\mu, T^{(1)}_{-\mu}] = \sqrt{2}\langle 1,1;-\mu,\mu|1,1;k,0\rangle \hat{T}^{(1)}_0$$

$$(-1)^\mu \langle j,1;m,\mu | j,1;j,m+\mu\rangle = \langle j,1;m+\mu,-\mu | j,1;j,m\rangle$$

$$< j' \| \hat{\boldsymbol{J}}^2 \| j >= \delta_{j,j'}\hbar j(j+1)$$

$$\hat{\boldsymbol{J}} \cdot \boldsymbol{T}^{(1)} = \sum_q (-1)^q \hat{J}_q \hat{T}_{-q}{}^{(1)}$$

$$\langle j,k=1;m,q=0 | j,k=1;j,m'\rangle = \frac{m}{\sqrt{j(j+1)}}\delta_{m,m'}$$

$$\langle j \| \hat{\boldsymbol{J}} \| j\rangle = \hbar\sqrt{j(j+1)}$$

$$\langle j,m|\hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{V}}| j,m\rangle = \hbar\sqrt{j(j+1)} < j \| \hat{\boldsymbol{V}} \| j >>$$

$$\langle j,m|\hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{V}}| j,m\rangle =< j \| \hat{\boldsymbol{J}} \| j >>< j \| \hat{\boldsymbol{V}} \| j >>$$

The construction of tensor with rank k

$$\hat{T}^{(k)}_q = \sum_{q_1,q_2} \langle k_1,k_2;q_1,q_2 | k_1,k_2;k,q\rangle \hat{X}^{(k_1)}_{q_1} \hat{Z}^{(k_2)}_{q_2}$$

Decomposition theorem

$$\langle j',m'|\hat{T}_q{}^{(1)}| j,m\rangle = \frac{\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}} \cdot \boldsymbol{T}^{(1)})| j,m\rangle}{\hbar^2 j(j+1)}\delta_{j,j'}$$

Factorization theorem

$$\langle j',m'|\hat{J}_q(\hat{\boldsymbol{J}} \cdot \boldsymbol{T}^{(1)})| j,m\rangle = \langle j',m'|\hat{J}_q| j,m\rangle < j \| \hat{\boldsymbol{J}} \cdot \boldsymbol{T}^{(1)} \| j > .$$

Decomposition theorem of the second kind (the projection theorem)

$$\langle j',m'|\hat{T}_q{}^{(1)}| j,m\rangle = \frac{\langle j',m'|\hat{J}_q| j,m\rangle < j \| \hat{\boldsymbol{J}} \cdot \boldsymbol{T}^{(1)} \| j >}{\hbar^2 j(j+1)}\delta_{j,j'} .$$

$$\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)=\sum_q(-1)^q\hat{T}_q^{(k)}(1)\hat{T}_{-q}^{(k)}(2)$$

$$\begin{aligned}&\left\langle j_1', j_2'; j'm'\right|\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)\left|j_1, j_2; j, m\right\rangle\\&=\delta_{m,m'}\delta_{j,j'}(-1)^{j_1'+j_2-j}\times\sqrt{2j_1'+1}\sqrt{2j_2'+1}\\&< j_1'\|\hat{T}^{(k)}(1)\| j_1 >< j_2'\|\hat{T}^{(k)}(2)\| j_2 > W(j_1, j_2, j_1', j_2'; j, k)\end{aligned}$$

Equivalent operator

$$\left\langle J,M'\right|\hat{T}_q^{(1)}\left|J,M\right\rangle=\left\langle J,M'\right|\hat{J}_q\left|J,M\right\rangle\frac{\left\langle J\left\|\hat{T}^{(k)}\right\|J\right\rangle}{\left\langle J\left\|\hat{\boldsymbol{J}}\right\|J\right\rangle}=\frac{\left\langle J,M'\right|\hat{J}_q\left|J,M\right\rangle}{\hbar\sqrt{j(j+1)}}\left\langle J\left\|\hat{T}^{(k)}\right\|J\right\rangle.$$

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APPENDIX

A.1. Orthogonality condition

The CG coefficients form an unitary matrix. Furthermore, the matrix elements are taken to be real by convention. A real unitary matrix is orthogonal. We have the orthogonal condition.

$$\langle j_1, j_2, j, m | j_1, j_2, m_1, m_2 \rangle = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle^* = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$$

Closure relation

$$\begin{aligned}\langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1', m_2' \rangle &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; j, m | j_1, j_2; m_1', m_2' \rangle \\ &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\ &= \delta_{m_1, m_1'} \delta_{m_2, m_2'}\end{aligned}$$

or

$$\sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

Similarly,

$$\begin{aligned}\langle j_1, j_2; j, m | j_1, j_2; j', m' \rangle &= \sum_{j,m} \langle j_1, j_2; j, m | j_1, j_2; m_1, m_2 \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j', m' \rangle \\ &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j', m' \rangle \\ &= \delta_{j, j'} \delta_{m, m'}\end{aligned}$$

or

$$\sum_{m_1, m_2} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j', m' \rangle = \delta_{j, j'} \delta_{m, m'}$$

As a special case of this, we may set $j = j'$, $m' = m = m_1 + m_2$.

$$\sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} \left| \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \right|^2 = 1$$

which is just the normalization condition of $| j_1, j_2; j, m \rangle$.

A.2. Clebsch-Gordon series

$$D^{(k_1)}_{q_1' q_1}(\hat{R}) D^{(k_2)}_{q_2' q_2}(\hat{R}) = \sum_{k'' q'' q'} \langle k_1, k_2; q_1', q_2' | k_1, k_2; k'', q' \rangle \langle k_1, k_2; q_1, q_2 | k_1, k_2; k'', q' \rangle D^{(k'')}_{q' q''}(\hat{R})$$

((**proof**)) Sakurai Modern Quantum Mechanics

$$D^{(j_1)}_{m_1 m_1'}(\hat{R}) D^{(j_2)}_{m_2 m_2'}(\hat{R}) = \sum_{j,m,m''} \langle j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' \mid j_1, j_2; j, m' \rangle D^{(j)}_{mm'}(\hat{R}) \quad (1)$$

We start with the notation given by

$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + ... + D_{|j_1-j_2|}$$

This means that a similarity transformation must exist which reduces $D_{j_1} \times D_{j_2}$ to the block form

$$\begin{pmatrix} D_{|j_1-j_2|}(\hat{R}) & 0 & \cdots & 0 & 0 \\ 0 & D_{|j_1-j_2|+1}(\hat{R}) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_{|j_1+j_2-1}(\hat{R}) & 0 \\ 0 & 0 & \cdots & 0 & D_{j_1+j_2}(\hat{R}) \end{pmatrix}$$

First we note that the left-hand side of Eq.(1) is the same as

$$\langle j_1, j_2; m_1, m_2 | \hat{R} | j_1, j_2; m_1', m_2' \rangle = \langle j_1, m_1 | \hat{R} | j_1, m_1' \rangle \langle j_2, m_2 | \hat{R} | j_2, m_2' \rangle = D^{(j_1)}_{m_1 m_1'}(\hat{R}) D^{(j_2)}_{m_2 m_2'}(\hat{R})$$

((**Note**)) This expression can be understood from the following consideration.

$$\hat{R}(\theta) = \hat{R}_y(\theta) = \exp(-\frac{i}{\hbar}\theta \hat{J}_y) = \exp[-\frac{i}{\hbar}\theta(\hat{J}_{1y} + \hat{J}_{2y})] = \exp(-\frac{i}{\hbar}\theta \hat{J}_{1y}) \exp(-\frac{i}{\hbar}\theta \hat{J}_{2y})$$

$$\begin{aligned} &\langle j_1, j_2; m_1, m_2 | \exp(-\frac{i}{\hbar}\theta \hat{J}_{1y}) \exp(-\frac{i}{\hbar}\theta \hat{J}_{2y}) | j_1, j_2; m_1', m_2' \rangle \\ &= \langle j_1, m_1 | \exp(-\frac{i}{\hbar}\theta \hat{J}_{1y}) | j_1, m_1' \rangle \langle j_2, m_2 | \exp(-\frac{i}{\hbar}\theta \hat{J}_{2y}) | j_2, m_2' \rangle \\ &= D^{(j_1)}_{m_1 m_1'}(\theta) D^{(j_2)}_{m_2 m_2'}(\theta) = D^{(j_1)}_{m_1 m_1'}(\hat{R}) D^{(j_2)}_{m_2 m_2'}(\hat{R}) \end{aligned}$$

This matrix element can be also calculated as

$$
\begin{aligned}
\langle j_1, j_2; m_1, m_2 | \hat{R} | j_1, j_2; m_1', m_2' \rangle = & \sum_{j,m,j,m'} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; j, m | \hat{R} | j_1, j_2; j', m' \rangle \\
& \times \langle j_1, j_2; j'm' | j_1, j_2; m_1' m_2' \rangle \\
= & \sum_{j,m,j,m'} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle D^{(j)}_{mm'}(\hat{R}) \delta_{j,j'} \\
& \times \langle j_1, j_2; j'm' | j_1, j_2; m_1' m_2' \rangle \\
= & \sum_{j,m,m'} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; j, m' | j_1, j_2; m_1' m_2' \rangle D^{(j)}_{mm'}(\hat{R}) \\
= & \sum_{j,m,m'} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1' m_2' | j_1, j_2; j, m \rangle D^{(j)}_{mm'}(\hat{R})
\end{aligned}
$$

using the closure relations. Note that all the Clebsch-Gordon coefficients are real.

A.3. Formula for tensor product

We define

$$
\hat{T}^{(k)}(1) \cdot \hat{T}^{(k)}(2) = \sum_q (-1)^q \hat{T}_q^{(k)}(1) \hat{T}_{-q}^{(k)}(2)
$$

which represents an interaction between two independent subsystems 1 and 2. Here we discuss the matrix element of the type

$$
\langle j_1', j_2'; j'm' | \hat{T}^{(k)}(1) \cdot \hat{T}^{(k)}(2) | j_1, j_2; jm \rangle
$$

where $\hat{T}^{(k)}(1)$ is the tensor operator of the rank k for the subsystem 1 and $\hat{T}^{(k)}(2)$ is the tensor operator of the rank k for the subsystem 2. Here we note that

$$
| j_1, j_2; j, m \rangle = \sum_{m_1, m_2} | j_1, j_2; m_1, m_2 \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle
$$

and

$$
| j_1', j_2'; j', m' \rangle = \sum_{m_1', m_2'} | j_1', j_2'; m_1', m_2' \rangle \langle j_1', j_2'; m_1', m_2' | j_1', j_2'; j', m' \rangle
$$

By using these, the matrix elements becomes

$$
\begin{aligned}
\langle j_1', j_2'; j'm' | \hat{T}^{(k)}(1) \cdot \hat{T}^{(k)}(2) | j_1, j_2; jm \rangle = & \sum_{\substack{q, m_1', m_2' \\ m_1, m_2}} (-1)^q \langle j_1', j_2'; m_1', m_2' | j_1', j_2'; j', m' \rangle^* \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
& \times \langle j_1', j_2'; m_1', m_2' | \hat{T}_q^{(k)}(1) \hat{T}_{-q}^{(k)}(2) | j_1, j_2; m_1, m_2 \rangle
\end{aligned}
$$

Here we have

$$\left\langle j_1', j_2'; m_1', m_2' \right| \hat{T}_q^{(k)}(1) \hat{T}_{-q}^{(k)}(2) \left| j_1, j_2; m_1, m_2 \right\rangle = \left\langle j_1', m_1' \right| \hat{T}_q^{(k)}(1) \left| j_1, m_1 \right\rangle \left\langle j_2', m_2' \right| \hat{T}_{-q}^{(k)}(2) \left| j_2, m_2 \right\rangle$$

since two factors operate in separate decoupled systems. According to the Wigner-Eckart theorem,

$$\left\langle j_1', m_1' \right| \hat{T}_q^{(k)}(1) \left| j_1, m_1 \right\rangle = \left\langle j_1, k; m_1, q \right| j_1, k; j_1', m_1' \rangle < j_1' \| \hat{T}^{(k)}(1) \| j_1 >$$

$$\left\langle j_2', m_2' \right| \hat{T}_{-q}^{(k)}(2) \left| j_2, m_2 \right\rangle = \left\langle j_2, k; m_2, -q \right| j_2, k; j_2', m_2' \rangle < j_2' \| \hat{T}^{(k)}(1) \| j_2 >$$

Then we get

$$\left\langle j_1', j_2'; m_1', m_2' \right| \hat{T}_q^{(k)}(1) \hat{T}_q^{(k)}(2) \left| j_1, j_2; m_1, m_2 \right\rangle = \left\langle j_1, k; m_1, q \right| j_1, k; j_1', m_1' \rangle \left\langle j_2, k; m_2, -q \right| j_2, k; j_2', m_2' \rangle$$
$$< j_1' \| \hat{T}^{(k)}(1) \| j_1 >< j_2' \| \hat{T}^{(k)}(2) \| j_2 >$$

Using this relation, we have

$$\begin{aligned} \frac{\left\langle j_1', j_2'; j', m' \right| \hat{T}_q^{(k)}(1) \hat{T}_{-q}^{(k)}(2) \left| j_1, j_2; j, m \right\rangle}{< j_1' \| \hat{T}^{(k)}(1) \| j_1 >< j_2' \| \hat{T}^{(k)}(2) \| j_2 >} &= \sum_{\substack{m_1', m_2' \\ m_1, m_2}} \left\langle j_1', j_2'; m_1', m_2' \right| j_1', j_2'; j', m' \rangle^* \left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \rangle \\ &\quad \times \left\langle j_1, k; m_1, q \right| j_1, k; j_1', m_1' \rangle \left\langle j_2, k; m_2, -q \right| j_2, k; j_2', m_2' \rangle \\ &= \sum_{\substack{m_1', m_2' \\ m_1, m_2}} \left\langle j_1', j_2'; j', m' \right| j_1', j_2'; m_1', m_2' \rangle \left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \rangle \\ &\quad \times \left\langle j_1, k; m_1, q \right| j_1, k; j_1', m_1' \rangle \left\langle j_2, k; m_2, -q \right| j_2, k; j_2', m_2' \rangle \end{aligned}$$

since

$$\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j', m' \rangle^* = \left\langle j_1, j_2; j', m' \right| j_1, j_2; m_1, m_2 \rangle$$

$$\begin{aligned} \frac{\left\langle j_1', j_2'; j', m' \right| \hat{T}^{(k)}(1) \cdot \hat{T}^{(k)}(2) \left| j_1, j_2; j, m \right\rangle}{< j_1' \| \hat{T}^{(k)}(1) \| j_1 >< j_2' \| \hat{T}^{(k)}(2) \| j_2 >} &= \sum_{\substack{q, m_1', m_2' \\ m_1, m_2}} (-1)^q \left\langle j_1', j_2'; m_1', m_2' \right| j_1', j_2'; j', m' \rangle^* \left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \rangle \\ &\quad \times \left\langle j_1, k; m_1, q \right| j_1, k; j_1', m_1' \rangle \left\langle j_2, k; m_2, -q \right| j_2, k; j_2', m_2' \rangle \\ &= \delta_{m,m'} \sum_{m_1, m_1'} (-1)^{m_1' - m_1} \left\langle j_1', j_2'; m_1', m' - m_1' \right| j_1', j_2'; j', m' \rangle \\ &\quad \times \left\langle j_1, j_2; m_1, m - m_1 \right| j_1, j_2; j, m \rangle \left\langle j_1, k; m_1, m_1' - m_1 \right| j_1, k; j_1', m_1' \rangle \\ &\quad \times \left\langle j_2, k; m - m_1, m_1 - m_1' \right| j_2, k; j_2', m - m_1' \rangle \end{aligned}$$

We may also apply the Wigner-Eckart theorem to the entire matrix element

$$\begin{aligned}\langle j'm'|\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)|j,m\rangle &= \delta_{m,m'}\delta_{j,j'}\langle j,k=0;m,q=0|j,k=0;j,m\rangle \\ &\quad < j\|\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)\|j> \\ &= \delta_{m,m'}\delta_{j,j'} < j\|\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)\|j>\end{aligned}$$

where

$$\langle j,k=0;m,q=0|j,k=0;j,m\rangle = 1.$$

Using the above two equations, we get

$$\begin{aligned}\langle j_1',j_2';j'm'|\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)|j_1,j_2;j,m\rangle &= \delta_{m,m'}\delta_{j,j'} < j_1'\|\hat{T}^{(k)}(1)\|j_1>< j_2'\|\hat{T}^{(k)}(2)\|j_2> \\ &\quad \sum_{m_1,m_1'}(-1)^{m_1'-m_1}\langle j_1',j_2';m_1',m'-m_1'|j_1',j_2';j',m'\rangle \\ &\quad \times\langle j_1,j_2;m_1,m-m_1|j_1,j_2;j,m\rangle\langle j_1,k;m_1,m_1'-m_1|j_1,k;j_1',m_1'\rangle \\ &\quad \times\langle j_2,k;m-m_1,m_1-m_1'|j_2,k;j_2',m-m_1'\rangle \\ &= \delta_{m,m'}\delta_{j,j'} < j_1'\|\hat{T}^{(k)}(1)\|j_1>< j_2'\|\hat{T}^{(k)}(2)\|j_2> \\ &\quad \sum_{m_1,m_1'}(-1)^{m_1'-m_1}\langle j_1',j_2';m_1',m'-m_1'|j_1',j_2';j',m'\rangle \\ &\quad \times\langle j_1,j_2;m_1,m-m_1|j_1,j_2;j,m\rangle\langle j_1,k;m_1,m_1'-m_1|j_1,k;j_1',m_1'\rangle \\ &\quad \times\langle j_2,k;m-m_1,m_1-m_1'|j_2,k;j_2',m-m_1'\rangle \\ &= \delta_{m,m'}\delta_{j,j'}(-1)^{j_1'+j_2-j}\times\sqrt{2j_1'+1}\sqrt{2j_2'+1} \\ &\quad < j_1'\|\hat{T}^{(k)}(1)\|j_1>< j_2'\|\hat{T}^{(k)}(2)\|j_2> W(j_1,j_2,j_1',j_2';j,k)\end{aligned}$$

or

$$\begin{aligned}\langle j_1',j_2';j'm'|\hat{T}^{(k)}(1)\cdot\hat{T}^{(k)}(2)|j_1,j_2;j,m\rangle &= \delta_{m,m'}\delta_{j,j'}(-1)^{j_1'+j_2-j}\times\sqrt{2j_1'+1}\sqrt{2j_2'+1} \\ &\quad < j_1'\|\hat{T}^{(k)}(1)\|j_1>< j_2'\|\hat{T}^{(k)}(2)\|j_2> W(j_1,j_2,j_1',j_2';j,k)\end{aligned}$$

where the Racah coefficient *W* is defined by (Tinkham, Rose)

$$W(j_1, j_2, j_1', j_2'; j, k) = \frac{(-1)^{-j_1' - j_2 + j}}{\sqrt{(2j_1'+1)(2j_2'+1)}} \sum_{m_1, m_1'} (-1)^{m_1' - m_1} \langle j_1, j_2; m_1, m - m_1 | j_1, j_2; j, m \rangle$$
$$\times \langle j_2, k; m - m_1, m_1 - m_1' | j_2, k; j_2', m - m_1' \rangle$$
$$\times \langle j_1, k; m_1, m_1' - m_1 | j_1, k; j_1', m_1' \rangle$$
$$\times \langle j_1', j_2'; m_1', m - m_1' | j_1', j_2'; j, m \rangle$$

A.4 Wigner 3*j* coefficient

The Clebsch-Gordan coefficients are sometimes expressed using the Wigner 3*j* symbol,

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

Connection among these two is given by

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = (-1)^{j_1 - j_2 + m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

or

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{m - j_1 + j_2} \frac{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, -m \rangle}{\sqrt{2j+1}}$$

They have the symmetry

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = (-1)^{j_1 + j_2 - j} \langle j_2, j_1; m_2, m_1 | j_2, j_2; j, m \rangle$$

A.5 Property of the Wigner 3j symmbol

We have that an even permutation of the column leaves the numerical value unchanged

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

An odd permutation is equivalent to multiplication by $(-1)^{j_1 + j_2 + j_3}$

$$(-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$$
$$= \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$
$$= \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}$$

We also have

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

A6. The orthogonality property

$$\sum_{j,m}\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle\langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'}$$

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle = (-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$
$$\langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle = (-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

$$\delta_{m_1,m_1'}\delta_{m_2,m_2'} = \sum_{j,m}\langle j_1, j_2; j, m | j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle$$
$$= \sum_{j,m}(-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}(-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & -m \end{pmatrix}$$
$$= \sum_{j,m}(-1)^{2j_1-2j_2+2m}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & -m \end{pmatrix}$$
$$= \sum_{j,m}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix}$$

Similarly, we have

$$\delta_{j,j'}\delta_{m,m'} = \sum_{m_1,m_2}\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle\langle j_1,j_2;m_1,m_2|j_1,j_2;j',m'\rangle$$

$$= \sum_{m_1,m_2}(-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m\end{pmatrix}(-1)^{j_1-j_2+m'}\sqrt{2j'+1}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & -m'\end{pmatrix}$$

$$= (2j+1)\sum_{m_1,m_2}(-1)^{2j_1-2j_2+m+m'}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & -m'\end{pmatrix}$$

$$= (2j+1)\sum_{m_1,m_2}(-1)^{2j_1-2j_2+2m}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & -m'\end{pmatrix}$$

$$= (2j+1)\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & -m'\end{pmatrix}$$

((**Note**))

$$(-1)^{2j_1-2j_2+2m} = 1$$

(a) Suppose that j_1=1/2 and j_2=1. j = 3/2 or 1/2. Then m is a half-integer. m = 3/2, 1/2, -1/2, or -3/2. Then j_1 - j_2 + m=integer.

(b) Suppose that j_1=1/2 and j_2=3/2. j = 2 or 1. Then m is an integer. m = 2, 1, 0, -1, -2. Then j_1 - j_2 + m=integer.

A.7 Mathematica for Wigner 3j coefficient

((**Mathematica**)) Calculation of the Wigner 3*j* coefficient

W3J[{j_1,m_1}m{j_2,m_2},{j_3,m_3}]→$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}$

```
Clear["Global`*"]; W3J[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    (-1)^(j1-j2-m)/Sqrt[2 j + 1] ClebschGordan[{j1, m1}, {j2, m2}, {j, -m}], Null]];
```

```
W3J[{2, 1}, {2, 1}, {2, -2}]
```

$-\sqrt{\frac{3}{35}}$

```
W3J[{3/2, 1/2}, {3/2, 1/2}, {2, -1}]
```

0

```
W3J[{2, 1}, {2, 1}, {3, -2}]
```

0

```
W3J[{4, 0}, {4, 0}, {0, 0}]
```

$\frac{1}{3}$

```
W3J[{3, 0}, {2, 0}, {3, 0}]
```

$\frac{2}{\sqrt{105}}$

```
W3J[{j, -m}, {0, 0}, {j, m}] // Simplify[#, Abs[m] ≤ j && Abs[m] ≤ j] &
```

$\frac{(-1)^{j-m}}{\sqrt{1+2j}}$

A.8 Matrix elements of vector operator

Spherical tensor of rank 1

$$T_1^{(1)} = -\frac{V_x + iV_y}{\sqrt{2}} = -\frac{V_+}{\sqrt{2}}$$

$$T_0^{(1)} = V_z$$

$$T_{-1}^{(1)} = \frac{V_x - iV_y}{\sqrt{2}} = \frac{V_-}{\sqrt{2}}$$

Using the Wigner-Eckart theorem, we have

$$\left\langle J',M'\left|\hat{T}_q^{(1)}\right|J,M\right\rangle = \left\langle J,1;M,q\,\middle|\,J,1;J',M'\right\rangle\left\langle J'\left\|\hat{T}^{(k)}\right\|J\right\rangle$$

where

$J'=J+1,\ J,\ J-1, \qquad M' = M+q$

For $q = 1$,

$$\begin{aligned}\left\langle J+1,M+1\left|\hat{T}_1^{(1)}\right|J,M\right\rangle &= \left\langle J,1;M,q=1\,\middle|\,J,1;J+1,M+1\right\rangle\left\langle J+1\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= (-1)^{2(J+M)}\sqrt{\frac{(1+J+M+1)(J+M+2)}{2(1+J)(1+2J)}}\left\langle J+1\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= \sqrt{\frac{(1+J+M+1)(J+M+2)}{2(1+J)(1+2J)}}\left\langle J+1\left\|\hat{T}^{(1)}\right\|J\right\rangle\end{aligned}$$

$$\begin{aligned}\left\langle J,M+1\left|\hat{T}_1^{(1)}\right|J,M\right\rangle &= \left\langle J,1;M,q=1\,\middle|\,J,1;J,M+1\right\rangle\left\langle J\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= -\sqrt{\frac{(J-M)(J+M+1)}{2J(J+1)}}\left\langle J\left\|\hat{T}^{(1)}\right\|J\right\rangle\end{aligned}$$

$$\begin{aligned}\left\langle J-1,M+1\left|\hat{T}_1^{(1)}\right|J,M\right\rangle &= \left\langle J,1;M,q=1\,\middle|\,J,1;J-1,M+1\right\rangle\left\langle J-1\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= \sqrt{\frac{(J-M)(J-M-1)}{2J(2J+1)}}\left\langle J-1\left\|\hat{T}^{(1)}\right\|J\right\rangle\end{aligned}$$

For $q = 0$,

$$\begin{aligned}\left\langle J+1,M\left|\hat{T}_0^{(1)}\right|J,M\right\rangle &= \left\langle J,1;M,q=0\,\middle|\,J,1;J+1,M\right\rangle\left\langle J+1\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= (-1)^{2(J+M)}\sqrt{\frac{(J-M+1)(J+M+1)}{(2J+1)(J+1)}}\left\langle J+1\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= \sqrt{\frac{(J-M+1)(J+M+1)}{(2J+1)(J+1)}}\left\langle J+1\left\|\hat{T}^{(1)}\right\|J\right\rangle\end{aligned}$$

$$\begin{aligned}\left\langle J,M\left|\hat{T}_0^{(1)}\right|J,M\right\rangle &= \left\langle J,1;M,q=0\,\middle|\,J,1;J,M\right\rangle\left\langle J\left\|\hat{T}^{(1)}\right\|J\right\rangle \\ &= \frac{M}{\sqrt{J(J+1)}}\left\langle J\left\|\hat{T}^{(1)}\right\|J\right\rangle\end{aligned}$$

$$\langle J-1, M|\hat{T}_0^{(1)}|J, M\rangle = \langle J, 1; M, q=0|J, 1; J-1, M\rangle\langle J-1\|\hat{T}^{(1)}\|J\rangle$$
$$= \sqrt{\frac{(J+M)(J-M)}{J(2J+1)}}\langle J-1\|\hat{T}^{(1)}\|J\rangle$$

For $q = -1$,

$$\langle J+1, M-1|\hat{T}_{-1}^{(1)}|J, M\rangle = \langle J, 1; M, q=-1|J, 1; J+1, M-1\rangle\langle J+1\|\hat{T}^{(1)}\|J\rangle$$
$$= (-1)^{2(J+M)}\sqrt{\frac{(J-M+1)(J-M+2)}{2(J+1)(2J+1)}}\langle J+1\|\hat{T}^{(1)}\|J\rangle$$
$$= \sqrt{\frac{(J-M+1)(J-M+2)}{2(J+1)(2J+1)}}\langle J+1\|\hat{T}^{(1)}\|J\rangle$$

$$\langle J, M-1|\hat{T}_{-1}^{(1)}|J, M\rangle = \langle J, 1; M, q=-1|J, 1; J, M-1\rangle\langle J\|\hat{T}^{(1)}\|J\rangle$$
$$= (-1)^{2(J+M)}\sqrt{\frac{(J+M)(J-M+1)}{2J(J+1)}}\langle J\|\hat{T}^{(1)}\|J\rangle$$
$$= \sqrt{\frac{(J+M)(J-M+1)}{2J(J+1)}}\langle J\|\hat{T}^{(1)}\|J\rangle$$

$$\langle J-1, M-1|\hat{T}_{-1}^{(1)}|J, M\rangle = \langle J, 1; M, q=-1|J, 1; J-1, M-1\rangle\langle J-1\|\hat{T}^{(1)}\|J\rangle$$
$$= \sqrt{\frac{(J+M)(J+M-1)}{2J(2J+1)}}\langle J-1\|\hat{T}^{(1)}\|J\rangle$$

Here we note that

$$(-1)^{2(J+M)} = 1$$

when J is either a positive integer or a half-integer.

A.9 Matrix elements of vector operator ($J' = J$)

We now calculate the matrix element with $J' = J$,

$$\langle J, M'|\hat{T}_q^{(1)}|J, M\rangle = \langle J, 1; M, q|J, 1; J, M'\rangle\langle J'\|\hat{T}^{(k)}\|J\rangle$$

where $M' = M + q$

For $q = 1$

$$\left\langle J, M+1 \right| \hat{T}_1^{(1)} \left| J, M \right\rangle = \left\langle J,1; M,1 \right| J,1; J, M+1 \rangle \left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle$$

$$= -\frac{\sqrt{(J-M)(J+M+1)}}{\sqrt{2J(J+1)}} \left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle$$

$$= \left\langle J, M+1 \right| -\frac{\hat{J}_+}{\sqrt{2}} \left| J, M \right\rangle \frac{\left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle}{\hbar \sqrt{J(J+1)}}$$

For q=0

$$\left\langle J, M \right| \hat{T}_0^{(1)} \left| J, M \right\rangle = \left\langle J,1; M,0 \right| J,1; J, M \rangle \left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle$$

$$= \frac{M}{\sqrt{J(J+1)}} \left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle$$

$$= \left\langle J, M \right| \hat{J}_0 \left| J, M \right\rangle \frac{\left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle}{\sqrt{J(J+1)}}$$

For q = -1

$$\left\langle J, M-1 \right| \hat{T}_{-1}^{(1)} \left| J, M \right\rangle = \left\langle J,1; M,-1 \right| J,1; J, M-1 \rangle \left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle$$

$$= (-1)^{2(J+M)} \frac{\sqrt{(J+M)(J-M+1)}}{\sqrt{2J(J+1)}} \left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle$$

$$= \left\langle J, M-1 \right| (-1)^{2(J+M)} \frac{\hat{J}_+}{\sqrt{2}} \left| J, M \right\rangle \frac{\left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle}{\hbar \sqrt{J(J+1)}}$$

$$= \left\langle J, M-1 \right| \frac{\hat{J}_+}{\sqrt{2}} \left| J, M \right\rangle \frac{\left\langle J \left\| \hat{T}^{(k)} \right\| J \right\rangle}{\hbar \sqrt{J(J+1)}}$$

where we use the relations

$$\left\langle J, M+1 \right| \hat{J}_+ \left| J, M \right\rangle = \hbar \sqrt{(J-M)(J+M+1)}$$

$$\hat{J}_- \left| J, M \right\rangle = \hbar \sqrt{(J+M)(J-M+1)} \left| J, M-1 \right\rangle$$

$$\left\langle J, M \right| \hat{J}_0 \left| J, M \right\rangle = \hbar M \left| J, M \right\rangle$$

Here we note that

$$(-1)^{2(J+M)} = 1$$

when J is either a positive integer or a half-integer. Since

$$V_+ = -\sqrt{2}T_1^{(1)}$$

$$V_z = T_0^{(1)}$$

$$V_- = \sqrt{2}T_{-1}^{(1)}$$

we have

$$\langle J, M+1|\hat{V}_+|J, M\rangle = \langle J, M+1|\hat{J}_+|J, M\rangle \frac{\langle J\|\hat{\boldsymbol{V}}\|J\rangle}{\hbar\sqrt{J(J+1)}}$$

$$\langle J, M|\hat{V}_0|J, M\rangle = \langle J, M|\hat{J}_0|J, M\rangle \frac{\langle J\|\hat{\boldsymbol{V}}\|J\rangle}{\hbar\sqrt{J(J+1)}}$$

$$\langle J, M-1|\hat{V}_-|J, M\rangle = \langle J, M-1|\hat{J}_-|J, M\rangle \frac{\langle J\|\hat{\boldsymbol{V}}\|J\rangle}{\hbar\sqrt{J(J+1)}}$$

A.10 Operator equivalents

We start with

$$\langle J, M'|\hat{T}_q^{(1)}|J, M\rangle = \langle J, 1; M, q|J, 1; J, M'\rangle\langle J'\|\hat{T}^{(1)}\|J\rangle$$

Suppose that $\hat{T}_q^{(1)} = \hat{J}_q$. Then we get

$$\langle J, M'|\hat{J}_q|J, M\rangle = \langle J, 1; M, q|J, 1; J, M'\rangle\langle J'\|\hat{\boldsymbol{J}}\|J\rangle$$

or

$$\langle J, M+1|\hat{J}_1|J, M\rangle = \langle J, 1; M, 1|J, 1; J, M+1\rangle\langle J'\|\hat{\boldsymbol{J}}\|J\rangle$$

$$\langle J, M|\hat{J}_0|J, M\rangle = \langle J, 1; M, 0|J, 1; J, M\rangle\langle J'\|\hat{\boldsymbol{J}}\|J\rangle$$

$$\langle J, M-1|\hat{J}_{-1}|J, M\rangle = \langle J, 1; M, -1|J, 1; J, M-1\rangle\langle J'\|\hat{\boldsymbol{J}}\|J\rangle$$

From the above equations we have the following relations

$$\frac{\langle J,M'|\hat{T}_q^{(1)}|J,M\rangle}{\langle J,M'|\hat{J}_q|J,M\rangle} = \frac{\langle J,1;M,q|J,1;J,M'\rangle\langle J'\|\hat{T}^{(1)}\|J\rangle}{\langle J,1;M,q|J,1;J,M'\rangle\langle J'\|\hat{\boldsymbol{J}}\|J\rangle} = \frac{\langle J'\|\hat{T}^{(1)}\|J\rangle}{\langle J'\|\hat{\boldsymbol{J}}\|J\rangle}$$

or

$$\langle J,M'|\hat{T}_q^{(1)}|J,M\rangle = \langle J,M'|\hat{J}_q|J,M\rangle\frac{\langle J\|\hat{T}^{(1)}\|J\rangle}{\langle J\|\hat{\boldsymbol{J}}\|J\rangle} = \frac{\langle J,M'|\hat{J}_q|J,M\rangle}{\hbar\sqrt{j(j+1)}}\langle J\|\hat{T}^{(1)}\|J\rangle$$

In other words, $\hat{T}_q^{(1)}$ may be replaced by $c\hat{J}_q$, the angular operator times a constant *c*.

$$\hat{T}_q^{(1)} = c\hat{J}_q$$

A.11 Calculation of the scalar product

$$\hat{\boldsymbol{V}}\cdot\hat{\boldsymbol{W}} = -\hat{V}_-\hat{W}_+ + \hat{V}_0\hat{W}_0 - \hat{V}_+\hat{W}_-$$

$$\begin{aligned}\langle J,M|\hat{V}_+\hat{W}_-|J,M\rangle &= \sum_{M'}\langle J,M|\hat{V}_+|J,M'\rangle\langle J,M'|\hat{W}_-|J,M\rangle \\ &= \langle J,M|\hat{V}_+|J,M-1\rangle\langle J,M-1|\hat{W}_-|J,M\rangle \\ &= \langle J,M|\hat{J}_+|J,M-1\rangle\langle J,M-1|\hat{J}_-|J,M\rangle\frac{\langle J\|\hat{\boldsymbol{V}}\|J\rangle}{\hbar\sqrt{J(J+1)}}\frac{\langle J\|\hat{\boldsymbol{W}}\|J\rangle}{\hbar\sqrt{J(J+1)}}\end{aligned}$$

$$\begin{aligned}\langle J,M|\hat{V}_-\hat{W}_+|J,M\rangle &= \sum_{M'}\langle J,M|\hat{V}_-|J,M'\rangle\langle J,M'|\hat{W}_+|J,M\rangle \\ &= \langle J,M|\hat{V}_-|J,M+1\rangle\langle J,M+1|\hat{W}_+|J,M\rangle \\ &= \langle J,M|\hat{J}_-|J,M+1\rangle\langle J,M+1|\hat{J}_-|J,M\rangle\frac{\langle J\|\hat{\boldsymbol{V}}\|J\rangle}{\hbar\sqrt{J(J+1)}}\frac{\langle J\|\hat{\boldsymbol{W}}\|J\rangle}{\hbar\sqrt{J(J+1)}}\end{aligned}$$

$$\begin{aligned}\langle J,M|\hat{V}_0\hat{W}_0|J,M\rangle &= \sum_{M'}\langle J,M|\hat{V}_0|J,M'\rangle\langle J,M'|\hat{W}_0|J,M\rangle \\ &= \langle J,M|\hat{V}_0|J,M\rangle\langle J,M|\hat{W}_0|J,M\rangle \\ &= \langle J,M|\hat{J}_0|J,M\rangle\langle J,M|\hat{J}_0|J,M\rangle\frac{\langle J\|\hat{\boldsymbol{V}}\|J\rangle}{\hbar\sqrt{J(J+1)}}\frac{\langle J\|\hat{\boldsymbol{W}}\|J\rangle}{\hbar\sqrt{J(J+1)}}\end{aligned}$$

((**Mathematica**))

```
Clear["Global`*"];

ClebschGordan[{J, M}, {1, 1}, {J, M + 1}] // FullSimplify[#, {2 J > 1}] &
```

$$-\frac{\sqrt{\frac{(J-M)\,(1+J+M)}{J\,(1+J)}}}{\sqrt{2}}$$

```
ClebschGordan[{J, M}, {1, 0}, {J, M}] // FullSimplify[#, {2 J > 1}] &
```

$$\frac{M}{\sqrt{J\,(1+J)}}$$

```
ClebschGordan[{J, M}, {1, -1}, {J, M - 1}] // FullSimplify[#, {2 J > 1}] &
```

$$\frac{(-1)^{2\,(J+M)}\sqrt{\frac{J+J^2+M-M^2}{J+J^2}}}{\sqrt{2}}$$

```
J + J^2 + M - M^2 // Factor
```

$$(1 + J - M)\,(J + M)$$

A.12

$$(-1)^{m_1'-m_1}\langle j_1',j_2';m_1',m'-m_1'|j_1',j_2';j',m'\rangle\times\langle j_1,j_2;m_1,m-m_1|j_1,j_2;j,m\rangle$$
$$\langle j_1,k;m_1,m_1'-m_1|j_1,k;j_1',m_1'\rangle\times\langle j_2,k;m-m_1,m_1-m_1'|j_2,k;j_2',m-m_1'\rangle$$
$$=(-1)^{m_1'-m_1}(-1)^{j_1'-j_2'+m'}\sqrt{2j'+1}\begin{pmatrix} j_1' & j_2' & j' \\ m_1' & m'-m_1' & -m' \end{pmatrix}(-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m-m_1 & -m \end{pmatrix}$$
$$\times(-1)^{j_1-k+m_1'}\sqrt{2j_1'+1}\begin{pmatrix} j_1 & k & j_1' \\ m_1 & m_1'-m_1 & -m_1' \end{pmatrix}(-1)^{j_2-k+m-m_1'}\sqrt{2j_2'+1}\begin{pmatrix} j_2 & k & j_2' \\ m-m_1 & m_1-m_1' & -m+m_1' \end{pmatrix}$$
$$=(-1)^{2j_1-2k+j_1'-j_2'+2m-m_1+m'+m_1'}\sqrt{2j'+1}\sqrt{2j+1}\sqrt{2j_1'+1}\sqrt{2j_2'+1}$$
$$\times\begin{pmatrix} j_1' & j_2' & j' \\ m_1' & m'-m_1' & -m' \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m-m_1 & -m \end{pmatrix}$$
$$\times\begin{pmatrix} j_1 & k & j_1' \\ m_1 & m_1'-m_1 & -m_1' \end{pmatrix}\begin{pmatrix} j_2 & k & j_2' \\ m-m_1 & m_1-m_1' & -m+m_1' \end{pmatrix}$$

(Wigner-Eckart theorem)

Appendix B Application

B.1 Tinkham

$$V_1 = -\left(\frac{V_x + iV_y}{\sqrt{2}}\right) = -\frac{V_+}{\sqrt{2}} \qquad \boldsymbol{e}_1 = -\left(\frac{\boldsymbol{e}_x + i\boldsymbol{e}_y}{\sqrt{2}}\right)$$

$$V_0 = V_z \qquad \boldsymbol{e}_0 = \boldsymbol{e}_z$$

$$V_{-1} = \left(\frac{V_x - iV_y}{\sqrt{2}}\right) = \frac{V_-}{\sqrt{2}} \qquad \boldsymbol{e}_{-1} = \left(\frac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}}\right)$$

where

$$\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = (-1)^\mu \delta_{\mu,-\nu}$$

Then the vector $\boldsymbol{V}$ can be expresses by

$$\boldsymbol{V} = V_x\boldsymbol{e}_x + V_y\boldsymbol{e}_y + V_z e_z = -V_{-1}\boldsymbol{e}_1 + V_0\boldsymbol{e}_0 - V_1\boldsymbol{e}_{-1} = \sum_\mu (-1)^\mu V_{-\mu}\boldsymbol{e}_\mu$$

The scalar product of two vectors has the form

$$\begin{aligned}
\boldsymbol{V}\cdot\boldsymbol{W} &= \sum_{\mu,\nu}(-1)^{\mu+\nu}V_{-\mu}\boldsymbol{e}_\mu \cdot W_{-\nu}\boldsymbol{e}_\nu \\
&= \sum_{\mu,\nu}(-1)^{\mu+\nu}V_{-\mu}W_{-\nu}(-1)^\mu\delta_{\mu,-\nu} \\
&= \sum_\mu(-1)^\mu V_{-\mu}W_\mu \\
&= -V_{-1}W_1 + V_0W_0 - V_1W_{-1} \\
&= -V_-W_+ + V_0W_0 - V_+W_-
\end{aligned}$$

B.2 Spherical Harmonics as rotator matrices

Using the relation

$$|\Re\boldsymbol{r}\rangle = \hat{R}|\boldsymbol{r}\rangle$$

$$\begin{aligned}
|\boldsymbol{n}\rangle &= |\Re\boldsymbol{r}\rangle \\
&= \hat{R}|\boldsymbol{e}_z\rangle \\
&= \hat{R}_z(\phi)\hat{R}_y(\theta)|\boldsymbol{e}_z\rangle \\
&= \sum_{m'}\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\langle lm'|\boldsymbol{e}_z\rangle
\end{aligned}$$

Then

$$\langle lm|\boldsymbol{n}\rangle = \sum_{m'} \langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\langle lm'|\boldsymbol{e}_z\rangle$$

Here note that

$$\langle \boldsymbol{n}|lm\rangle = Y_\ell^m(\boldsymbol{n}) = Y_\ell^m(\theta,\phi)$$

or

$$\langle lm|\boldsymbol{n}\rangle = [Y_\ell^m(\theta,\phi)]^*$$

$\langle lm|\boldsymbol{e}_z\rangle = [Y_\ell^m(\theta,\phi)]^*$ evaluated at $\theta = 0$ with ϕ undetermined. At $\theta = 0$, $Y_\ell^m(\theta,\phi)$ is known to vanish for $m\neq 0$. Then we get

$$\begin{aligned}\langle lm|\boldsymbol{e}_z\rangle &= [Y_\ell^m(\theta=0,\phi)]^*\delta_{m,0} \\ &= \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta=1)\delta_{m,0} \\ &= \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,0}\end{aligned}$$

$$\begin{aligned}[Y_\ell^m(\theta,\phi)]^* &= \sum_{m'} \langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\langle lm'|\boldsymbol{e}_z\rangle \\ &= \sqrt{\frac{2\ell+1}{4\pi}}\sum_{m'} \langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\delta_{m',0} = \sqrt{\frac{2\ell+1}{4\pi}}\langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|l0\rangle\end{aligned}$$

or

$$\langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|l0\rangle = \sqrt{\frac{4\pi}{2\ell+1}}[Y_\ell^m(\theta,\phi)]^*$$

Since

$$\hat{R}_z(\phi) = \exp[-\frac{i}{\hbar}\hat{J}_z\phi]$$

we have

$$\langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|l0\rangle = \langle lm|\exp[-\frac{i}{\hbar}\hat{J}_z\phi]\hat{R}_y(\theta)|l0\rangle = e^{-im\phi}\langle lm|\hat{R}_y(\theta)|l0\rangle$$

or

$$e^{-im\phi}\langle lm|\hat{R}_y(\theta)|l0\rangle = \sqrt{\frac{4\pi}{2\ell+1}}[Y_\ell^m(\theta,\phi)]^*$$

or

$$\langle lm|\hat{R}_y(\theta)|l0\rangle = e^{im\phi}\sqrt{\frac{4\pi}{2\ell+1}}[Y_\ell^m(\theta,\phi)]^*$$

B.3 Nuclear quadrupole field (Yosida)

The nucleus is not just a point, but has a finite size. If we define the nuclear charge distribution function $\rho(\boldsymbol{r})$ and the electrostatic potential due to the electrons around the nucleus by $V(\boldsymbol{r})$.

$$H = \int \rho(\boldsymbol{r})V(\boldsymbol{r})d\boldsymbol{r} ,$$

where d$\boldsymbol{r}$ denotes the volume elements. Expanding $V(\boldsymbol{r})$ about the origin, we get

$$H = ZeV_0 + \sum_j P_j (\frac{\partial V}{\partial x_j})_0 + \frac{1}{2}\sum_{j,k} Q_{jk}'(\frac{\partial^2 V}{\partial x_j \partial x_k})_0 + \dots$$

Here Ze, P_j and Q_{jk}' are defined by

$$Ze = \int \rho(\boldsymbol{r})d\boldsymbol{r} \qquad \text{(nuclear charge)}$$

$$P_j = \int \rho(\boldsymbol{r})x_j d\boldsymbol{r} \qquad \text{(electric dipole moment)}$$

$$Q_{jk}' = \int \rho(\boldsymbol{r})x_j x_k d\boldsymbol{r} \qquad \text{(electric quadrupole moment)}$$

The electric dipole moment P_j vanishes if the nuclear charge distribution has inversion symmetry with respect to the origin, as is assumed here. The first term is the energy of the nucleus when the nucleus is regarded as a point charge. Neglecting this term, we get

$$H_Q = H - ZeV_0 = \frac{1}{2}\sum_{j,k} Q_{jk}'V_{jk}$$

where

$$V_{jk} = (\frac{\partial^2 V}{\partial x_j \partial x_k})_0$$

The Hamiltonian H_Q is the interaction of electric field gradient and the quadrupole moment. We introduce the traceless tensor as

$$Q_{jk} = 3Q_{jk}' - \delta_{jk}\sum_i Q_{ii}' = \begin{pmatrix} 3Q_{11}' - \sum_i Q_{ii}' & 3Q_{12}' & 3Q_{13}' \\ 3Q_{21}' & 3Q_{22}' - \sum_i Q_{ii}' & 3Q_{23}' \\ 3Q_{31}' & 3Q_{32}' & 3Q_{33}' - \sum_i Q_{ii}' \end{pmatrix}$$

Then we get

$$H_Q = \frac{1}{6}\sum_{j,k} Q_{jk}V_{jk} + \frac{1}{6}\left(\sum_i Q_{ii}'\right)\left(\sum_j V_{jj}\right)$$

with

$$Q_{jk} = \int \rho(\boldsymbol{r})(3x_j x_k - \delta_{jk}\boldsymbol{r}^2)d\boldsymbol{r}$$

Suppose that

$$\rho(\boldsymbol{r}) = \sum_i e\delta(\boldsymbol{r} - \boldsymbol{r}_i)$$

Then we have

$$Q_{jk} = \sum_i e(3x_{ij}x_{ik} - \delta_{jk}\boldsymbol{r}_i^2)$$

((**Slichter**))

We are in general concerned only with the ground state of a nucleus or perhaps with an excited state when the excited state is sufficiently long-lived. The eigenstate of nucleus are characterized by the state $\left|I, m\right\rangle$ with $m = I$, I-1, I-2,…,-I (2I +1 states). Then we need only the matrix elements of the quadrupole operator,

$$\left\langle I, m' \middle| \hat{Q}_{jk} \middle| I, m \right\rangle$$

According to the Wigner-Eckart theorem these can be shown to obey the equation

$$\langle I,m'|\hat{Q}_{jk}|I,m\rangle = c\langle I,m'|\frac{3}{2}(\hat{I}_\alpha\hat{I}_\beta + \hat{I}_\beta\hat{I}_\alpha) - \delta_{\alpha\beta}\boldsymbol{I}^2|I,m\rangle$$

where c is a constant. We will show you later how to derive this form.

The Hamiltonian is then given by

$$H_Q = \frac{eQ}{6I(2I-1)}\sum_{\alpha,\beta} V_{\alpha\beta}[\frac{3}{2}(\hat{I}_\alpha\hat{I}_\beta + \hat{I}_\beta\hat{I}_\alpha) - \delta_{\alpha\beta}\boldsymbol{I}^2]$$

We choose a set of principal axes such that

$$V_{\alpha\beta} = 0 \qquad \text{for } \alpha \neq \beta$$

Then we get a simplified Hamiltonian

$$H_Q = \frac{eQ}{6I(2I-1)}[V_{xx}(3\hat{I}_x{}^2 - \boldsymbol{I}^2) + V_{yy}(3\hat{I}_y{}^2 - \boldsymbol{I}^2) + V_{zz}(3\hat{I}_z{}^2 - \boldsymbol{I}^2)].$$

Since

$$V_{xx} + V_{yy} + V_{zz} = 0 \qquad \text{(from the Laplace's equation)}$$

we have

$$H_Q = \frac{eQ}{4I(2I-1)}[V_{zz}(3\hat{I}_z{}^2 - \boldsymbol{I}^2) + (V_{xx} - V_{yy})(\hat{I}_x{}^2 - \hat{I}_y{}^2)]$$

We define

$$eq = V_{zz}$$

$$\eta = \frac{V_{xx} - V_{yy}}{V_{zz}}$$

where η is called the asymmetry parameter and q is called the field gradient. then we have

$$H_Q = \frac{e^2qQ}{4I(2I-1)}[(3\hat{I}_z{}^2 - \boldsymbol{I}^2) + \eta)(\hat{I}_x{}^2 - \hat{I}_y{}^2)].$$

B-4 **(Equivalent operator)**

$$\hat{T}_2^{(2)} = \hat{U}_1{}^2 = \frac{1}{2}\hat{U}_+{}^2$$

$$\hat{T}_1^{(2)} = \frac{\hat{U}_1\hat{U}_0 + \hat{U}_0\hat{U}_1}{\sqrt{2}} = -\frac{1}{2}(\hat{U}_+\hat{U}_0 + \hat{U}_0\hat{U}_+)$$

$$\hat{T}_0^{(2)} = \frac{\hat{U}_1\hat{U}_{-1} + 2\hat{U}_0\hat{U}_0 + \hat{U}_{-1}\hat{U}_1}{\sqrt{6}} = \frac{-\frac{(\hat{U}_+\hat{U}_- + \hat{U}_-\hat{U}_+)}{2} + 2\hat{U}_0{}^2}{\sqrt{6}}$$

$$\hat{T}_{-1}^{(2)} = \frac{\hat{U}_0\hat{U}_{-1} + \hat{U}_{-1}\hat{U}_1}{\sqrt{2}} = \frac{1}{2}(\hat{U}_-\hat{U}_0 + \hat{U}_0\hat{U}_-)$$

$$\hat{T}_{-2}^{(2)} = \hat{U}_{-1}\hat{U}_{-1} = \frac{1}{2}\hat{U}_-{}^2$$

$$\hat{T}_2^{(2)} = \frac{\hat{I}_+{}^2}{2} = \frac{1}{2}(\hat{I}_x + i\hat{I}_y)(\hat{I}_x + i\hat{I}_y) = \frac{\hat{I}_x{}^2 - \hat{I}_y{}^2}{2} + i\,\frac{(\hat{I}_x\hat{I}_y + \hat{I}_y\hat{I}_x)}{2}$$

$$\hat{T}_1^{(2)} = -\left(\frac{\hat{I}_+\hat{I}_0 + \hat{I}_0\hat{I}_+}{2}\right) = -\left(\frac{\hat{I}_z\hat{I}_x + \hat{I}_x\hat{I}_z}{2}\right) - i\left(\frac{\hat{I}_y\hat{I}_z + \hat{I}_z\hat{I}_y}{2}\right)$$

$$\hat{T}_0^{(2)} = \frac{-\left(\frac{\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+}{2}\right) + 2\hat{I}_0{}^2}{\sqrt{6}} = \frac{2\hat{I}_z{}^2 - (\hat{I}_x{}^2 + \hat{I}_y{}^2)}{\sqrt{6}}$$

$$\hat{T}_{-1}^{(2)} = \frac{\hat{I}_0\hat{I}_- + \hat{I}_-\hat{I}_0}{2} = \left(\frac{\hat{I}_z\hat{I}_x + \hat{I}_x\hat{I}_z}{2}\right) - i\left(\frac{\hat{I}_y\hat{I}_z + \hat{I}_z\hat{I}_y}{2}\right)$$

$$\hat{T}_{-2}^{(2)} = \frac{\hat{I}_-{}^2}{2} = \frac{1}{2}(\hat{I}_x - i\hat{I}_y)(\hat{I}_x - i\hat{I}_y) = \left(\frac{\hat{I}_x{}^2 - \hat{I}_y{}^2}{2}\right) - i\,\frac{(\hat{I}_x\hat{I}_y + \hat{I}_y\hat{I}_x)}{2}$$

or

$$\hat{I}_x\hat{I}_y + \hat{I}_y\hat{I}_x = -i(\hat{T}_2^{(2)} - \hat{T}_{-2}^{(2)})$$

$$-\hat{I}_x^{\ 2} - \hat{I}_y^{\ 2} + 2\hat{I}_z^{\ 2} = \sqrt{6}\hat{T}_0^{(2)}$$

$$\hat{I}_z\hat{I}_x + \hat{I}_x\hat{I}_z = i(\hat{T}_{-1}^{(2)} - \hat{T}_1^{(2)})$$

B.5 The operator equivalent (Thompson p.321)

(a)

$$\langle I, m|T_{q=0}^{\ (k=2)}|I, m\rangle = \langle I, k=2; m, q=0|I, k=2; I, m\rangle < I \| T^{(k=2)} \| I >$$
$$= \frac{3m^2 - I(I+1)}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$

$$\langle I, m|\frac{-\left(\frac{\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+}{2}\right) + 2\hat{I}_0^{\ 2}}{\sqrt{6}}|I, m\rangle = \frac{1}{\sqrt{6}}[-\frac{(I-m+1)(I+m)}{2} + 2m^2 + -\frac{(I+m+1)(I-m)}{2}]$$
$$= \frac{1}{\sqrt{6}}[3m^2 - I(I+1)]$$

where

$$\langle I, m+1|\hat{I}_+|I, m\rangle = \sqrt{(I-m)(I+m+1)}$$
$$\langle I, m|\hat{I}_+|I, m-1\rangle = \sqrt{(I-m+1)(I+m)}$$

$$\langle I, m-1|\hat{I}_-|I, m\rangle = \sqrt{(I+m)(I-m+1)}$$
$$\langle I, m|\hat{I}_-|I, m+1\rangle = \sqrt{(I+m+1)(I-m)}$$

Then we have

$$\langle I, m|T_{q=0}^{\ (k=2)}|I, m\rangle = \frac{\langle I, m|-\left(\frac{\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+}{2}\right) + 2\hat{I}_0^{\ 2}|I, m\rangle}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$

The atomic spectroscopic quadrupole moment Q is defined by

$$Q = \langle I, I | T_{q=0}{}^{(k=2)} | I, I \rangle$$
$$= \frac{3I^2 - I(I+1)}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$
$$= \sqrt{\frac{I(2I-1)}{(I+1)(2I+3)}} < I \| T^{(k=2)} \| I >$$

which is consistent with vanishing matrix elements for $I = 0$ and $I = 1/2$.

$$< I \| T^{(k=2)} \| I >= \sqrt{\frac{(I+1)(2I+3)}{I(2I-1)}} Q$$

Suppose that

$$\langle I, m | \hat{T}_{q=0}^{(2)} | I, m \rangle = c \langle I, m | 3\hat{I}_z{}^2 - \boldsymbol{I}^2 | I, m \rangle = c[3m^2 - I(I+1)]$$

((**Equivalent operator**))

Then we have

$$\langle I, m | T_{q=0}{}^{(k=2)} | I, m \rangle = \frac{3m^2 - I(I+1)}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$
$$= \frac{3m^2 - I(I+1)}{I(2I-1)} Q$$

(b)

$$\langle I, m+2 | T_{q=2}{}^{(k=2)} | I, m \rangle = \langle I, k=2; m, q=2 | I, k=2; I, m+2 \rangle < I \| T^{(k=2)} \| I >$$
$$= \sqrt{\frac{3}{2}} \frac{\sqrt{(I-m-1)(I-m)(I+m+1)(I+m+2)}}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$

and

$$\langle I, m+2 | \hat{I}_+{}^2 | I, m \rangle = \sum_{m'} \langle I, m+2 | \hat{I}_+ | I, m' \rangle \langle I, m' | \hat{I}_+ | I, m \rangle$$
$$= \langle I, m+2 | \hat{I}_+ | I, m+1 \rangle \langle I, m+1 | \hat{I}_+ | I, m \rangle$$
$$= \sqrt{(I-m-1)(I-m)(I+m+1)(I+m+2)}$$

Then we have

$$\langle I,m+2|T_{q=2}^{(k=2)}|I,m\rangle = \sqrt{6}\frac{\langle I,m+2|\frac{\hat{I}_+{}^2}{2}|I,m\rangle}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$

(c)

$$\langle I,m+1|T_{q=1}^{(k=2)}|I,m\rangle = \langle I,k=2;m,q=1|I,k=2;I,m+1\rangle < I \| T^{(k=2)} \| I >$$
$$= -\sqrt{\frac{3}{2}}\frac{\sqrt{(I-m)(I+m+1)}}{\sqrt{I(1+I)(2I-1)(2I+3)}}(2m+1) < I \| T^{(k=2)} \| I >$$

$$\langle I,m-1|-\frac{\hat{I}_-\hat{I}_0+\hat{I}_0\hat{I}_-}{\sqrt{2}}|I,m\rangle = -\frac{1}{\sqrt{2}}[m\langle I,m-1|\hat{I}_-|I,m\rangle + (m-1)\langle I,m-1|\hat{I}_-|I,m\rangle]$$
$$= -\frac{1}{\sqrt{2}}(2m-1)\langle I,m-1|\hat{I}_-|I,m\rangle$$
$$= -\frac{1}{\sqrt{2}}(2m-1)\sqrt{(I-m+1)(I+m)}$$

Then

$$\langle I,m+1|T_{q=1}^{(k=2)}|I,m\rangle = \sqrt{3}\frac{\langle I,m-1|-\frac{\hat{I}_-\hat{I}_0+\hat{I}_0\hat{I}_-}{\sqrt{2}}|I,m\rangle}{\sqrt{I(1+I)(2I-1)(2I+3)}} < I \| T^{(k=2)} \| I >$$

(d)

$$\langle I,m-1|T_{q=-1}^{(k=2)}|I,m\rangle = \langle I,k=2;m,q=-1|I,k=2;I,m-1\rangle < I \| T^{(k=2)} \| I >$$
$$= (-1)^{2(I+m)}\sqrt{\frac{3}{2}}\frac{\sqrt{(I-m+1)(I+m)}}{\sqrt{I(1+I)(2I-1)(2I+3)}}(2m-1) < I \| T^{(k=2)} \| I >$$

$$\langle I,m-1|\frac{\hat{I}_-\hat{I}_0+\hat{I}_0\hat{I}_-}{\sqrt{2}}|I,m\rangle = \frac{1}{\sqrt{2}}[m\langle I,m-1|\hat{I}_-|I,m\rangle + (m-1)\langle I,m-1|\hat{I}_-|I,m\rangle]$$
$$= \frac{1}{\sqrt{2}}(2m-1)\langle I,m-1|\hat{I}_-|I,m\rangle$$
$$= \frac{1}{\sqrt{2}}(2m-1)\sqrt{(I-m+1)(I+m)}$$

or

$$\left\langle I,m-1\right|T_{q=-1}{}^{(k=2)}\left|I,m\right\rangle=(-1)^{2(I+m)}\sqrt{3}\frac{\left\langle I,m-1\right|\frac{\hat{I}_-\hat{I}_0+\hat{I}_0\hat{I}_-}{\sqrt{2}}\left|I,m\right\rangle}{\sqrt{I(1+I)(2I-1)(2I+3)}}<I\parallel T^{(k=2)}\parallel I>$$

(e)

$$\left\langle I,m-2\right|T_{q=-2}{}^{(k=2)}\left|I,m\right\rangle=\left\langle I,k=2;m,q=-2\right|I,k=2;I,m-2\rangle<I\parallel T^{(k=2)}\parallel I>$$
$$=(-1)^{2(I+m)}\sqrt{\frac{3}{2}}\frac{\sqrt{(I+m-1)(I-m+2)(I+m)(I-m+1)}}{\sqrt{I(1+I)(2I-1)(2I+3)}}<I\parallel T^{(k=2)}\parallel I>$$

and

$$\left\langle I,m-2\right|\hat{I}_-{}^2\left|I,m\right\rangle=\sum_{m'}\left\langle I,m-2\right|\hat{I}_-\left|I,m'\right\rangle\left\langle I,m'\right|\hat{I}_-\left|I,m\right\rangle$$
$$=\left\langle I,m-2\right|\hat{I}_-\left|I,m-1\right\rangle\left\langle I,m-1\right|\hat{I}_-\left|I,m\right\rangle$$
$$=\sqrt{(I+m-1)(I-m+2)(I+m)(I-m+1)}$$

or

$$\left\langle I,m-2\right|T_{q=-2}{}^{(k=2)}\left|I,m\right\rangle=(-1)^{2(I+m)}\sqrt{6}\frac{\left\langle I,m-2\right|\frac{\hat{I}_-{}^2}{2}\left|I,m\right\rangle}{\sqrt{I(1+I)(2I-1)(2I+3)}}<I\parallel T^{(k=2)}\parallel I>$$

where we use the relations

$$\left\langle I,m+1\right|\hat{I}_+\left|I,m\right\rangle=\sqrt{(I-m)(I+m+1)}$$

$$\left\langle I,m-1\right|\hat{I}_-\left|I,m\right\rangle=\sqrt{(I+m)(I-m+1)}$$

$$\left\langle I,m\right|\hat{I}_0\left|I,m\right\rangle=m$$

((**Mathematica**))

Calculation of the Clebsch-Gordan coefficients for the rank 2 tensors

```
Clear["Global`*"];

ClebschGordan[{J, m}, {2, 2}, {J, m + 2}] // FullSimplify[#, {J > 1}] &
```

$$\sqrt{\frac{3}{2}}\ \sqrt{\frac{(-1+J-m)\ (J-m)\ (1+J+m)\ (2+J+m)}{J\ (1+J)\ (-1+2\ J)\ (3+2\ J)}}$$

```
ClebschGordan[{J, m}, {2, 1}, {J, m + 1}] // FullSimplify[#, {J > 1}] &
```

$$-\sqrt{\frac{3}{2}}\ \sqrt{\frac{(J-m)\ (1+J+m)}{J\ (-3+J\ (1+4\ J\ (2+J)))}}\ (1+2\ m)$$

```
ClebschGordan[{J, m}, {2, 0}, {J, m}] // FullSimplify[#, {J > 1}] &
```

$$\frac{-J\ (1+J)+3\ m^2}{\sqrt{J\ (-3+J\ (1+4\ J\ (2+J)))}}$$

```
ClebschGordan[{J, m}, {2, -1}, {J, m - 1}] // FullSimplify[#, {J > 1}] &
```

$$(-1)^{2\ (J+m)}\ \sqrt{\frac{3}{2}}\ \sqrt{\frac{(1+J-m)\ (J+m)}{J\ (-3+J\ (1+4\ J\ (2+J)))}}\ (-1+2\ m)$$

```
ClebschGordan[{J, m}, {2, -2}, {J, m - 2}] // FullSimplify[#, {J > 1}] &
```

$$(-1)^{2\ (J+m)}\ \sqrt{\frac{3}{2}}\ \sqrt{\frac{(1+J-m)\ (2+J-m)\ (-1+J+m)\ (J+m)}{J\ (1+J)\ (-1+2\ J)\ (3+2\ J)}}$$

```
(-3 + J (1 + 4 J (2 + J))) // Factor
```

$$(1+J)\ (-1+2\ J)\ (3+2\ J)$$

$$\hat{T}_2^{(2)} = \hat{U}_1{}^2 = \frac{1}{2}\hat{U}_+{}^2$$

$$\hat{T}_1^{(2)} = \frac{\hat{U}_1\hat{U}_0 + \hat{U}_0\hat{U}_1}{\sqrt{2}} = -\frac{1}{2}(\hat{U}_+\hat{U}_0 + \hat{U}_0\hat{U}_+)$$

$$\hat{T}_0^{(2)} = \frac{\hat{U}_1\hat{U}_{-1} + 2\hat{U}_0\hat{U}_0 + \hat{U}_{-1}\hat{U}_1}{\sqrt{6}} = \frac{-\dfrac{(\hat{U}_+\hat{U}_- + \hat{U}_-\hat{U}_+)}{2} + 2\hat{U}_0{}^2}{\sqrt{6}}$$

$$\hat{T}_{-1}^{(2)} = \frac{\hat{U}_0\hat{U}_{-1} + \hat{U}_{-1}\hat{U}_1}{\sqrt{2}} = \frac{1}{2}(\hat{U}_-\hat{U}_0 + \hat{U}_0\hat{U}_-)$$

$$\hat{T}_{-2}^{(2)} = \hat{U}_{-1}\hat{U}_{-1} = \frac{1}{2}\hat{U}_-{}^2$$

where

$$\hat{U}_1 = -\frac{\hat{U}_x + i\hat{U}_y}{\sqrt{2}} = -\frac{\hat{U}_+}{\sqrt{2}}, \qquad \hat{U}_0 = \hat{U}_z, \qquad \hat{U}_{-1} = \frac{\hat{U}_-}{\sqrt{2}}$$

or

$$\hat{I}_x\hat{I}_y + \hat{I}_y\hat{I}_x = -i(\hat{T}_2^{(2)} - \hat{T}_{-2}^{(2)})$$

$$-\hat{I}_x{}^2 - \hat{I}_y{}^2 + 2\hat{I}_z{}^2 = \sqrt{6}\hat{T}_0^{(2)}$$

$$\hat{I}_z\hat{I}_x + \hat{I}_x\hat{I}_z = i(\hat{T}_{-1}^{(2)} - \hat{T}_1^{(2)})$$

((**Comment))**

$$\hat{T}(i)_{q=0}^{k=2} = (2\hat{z}_i{}^2 - \hat{x}_i{}^2 - \hat{y}_i{}^2)$$

Wigner-Eckart theorem

$$\langle I', m'|\hat{T}(i)_{q=0}^{k=2}|I, m\rangle = \langle I, k=2; m, q=0|I, k=2; I', m'\rangle < I'\|\hat{T}(i)^{k=2}\| I >$$

$$\langle I', m'|\sum_i \hat{T}(i)_{q=0}^{k=2}|I, m\rangle = \langle I, k=2; m, q=0|I, k=2; I', m'\rangle \sum_i < I'\|\hat{T}(i)^{k=2}\| I >$$

((Slichter page 169 – 170))

The last term of the right-hand side is independent of m and m'.

C.1

$$\langle j',m'|\hat{J}_q|j,m\rangle = \langle j,k=1;m,q|j,k=1;j',m'\rangle < j\,||\,\hat{\boldsymbol{J}}\,||\,j >$$

((**Proof**))

(a) $q = 1$;

$$\begin{aligned}\langle j',m'|\hat{J}_1|j,m\rangle &= -\frac{1}{\sqrt{2}}\langle j',m'|(\hat{J}_x + i\hat{J}_y)|j,m\rangle \\ &= -\frac{1}{\sqrt{2}}\langle j',m'|\hat{J}_+|j,m\rangle \\ &= -\frac{\hbar}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}\langle j',m'|j,m+1\rangle \\ &= -\frac{\hbar}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}\delta_{j',j}\delta_{m',m+1}\end{aligned}$$

$$\langle j,1;m,q=1|j,1;j'=j,m'=m+1\rangle = -\frac{\hbar}{\sqrt{2}}\frac{\sqrt{(j-m)(j+m+1)}}{\sqrt{j(j+1)}} \qquad \text{(CG)}$$

Then we have

$$\langle j,1;m,q=1|j,1;j'=j,m'=m+1\rangle\sqrt{j(j+1)} = -\frac{\hbar}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}\,.$$

(b) $q = -1$

$$\begin{aligned}\langle j',m'|\hat{J}_{-1}|j,m\rangle &= \frac{1}{\sqrt{2}}\langle j',m'|(\hat{J}_x - i\hat{J}_y)|j,m\rangle \\ &= \frac{1}{\sqrt{2}}\langle j',m'|\hat{J}_-|j,m\rangle \\ &= \frac{\hbar}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\langle j',m'|j,m-1\rangle \\ &= \frac{\hbar}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\delta_{j',j}\delta_{m',m-1}\end{aligned}$$

$$\left\langle j,1;m,q=-1\right| j,1;j'=j,m'=m-1\rangle = \frac{1}{\sqrt{2}}(-1)^{2(j+m)}\frac{\sqrt{(j+m)(j-m+1)}}{\sqrt{j(j+1)}}$$

Then we have

$$\left\langle j,1;m,q=-1\right| j,1;j'=j,m'=m-1\rangle\sqrt{j(j+1)} = \frac{\hbar}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

since $2(j+m) = \text{even}$.

(c) $q=0$

$$\left\langle j',m'\right|\hat{J}_0\left| j,m\right\rangle = \left\langle j',m'\right|\hat{J}_z\left| j,m\right\rangle = m\hbar\left\langle j',m'\right| j,m\rangle = m\hbar\delta_{m',m}$$

$$\left\langle j,1;m,q=0\right| j,1;j'=j,m'=m\rangle = \frac{m\hbar}{\sqrt{j(j+1)}}$$

Then we have

$$\left\langle j,1;m,q=0\right| j,1;j'=j,m'=m\rangle\sqrt{j(j+1)} = m\hbar\ .$$

C.2 Matrix of scalar

$\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}$ is a scalar

$$\left\langle j,m\right|\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}\left| j,m\right\rangle = \hbar\sqrt{j(j+1)} < j\,||\,\hat{\boldsymbol{V}}\,||\, j >>$$

where

$$\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}} = \sum_{\mu}(-1)^{\mu}\hat{J}_{-\mu}\hat{V}_{\mu}$$

((Proof))

$$\begin{aligned}\langle j,m|\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}|j,m\rangle &= \sum_{\mu}(-1)^{\mu}\langle j,m|\hat{J}_{-\mu}\hat{V}_{\mu}|j,m\rangle \\ &= \sum_{m''}\sum_{\mu}(-1)^{\mu}\langle j,m|\hat{J}_{-\mu}|j,m''\rangle\langle j,m''|\hat{V}_{\mu}|j,m\rangle \\ &= \sum_{m''}[\langle j,m|\hat{J}_0|j,m''\rangle\langle j,m''|\hat{V}_0|j,m\rangle - \langle j,m|\hat{J}_1|j,m''\rangle\langle j,m''|\hat{V}_{-1}|j,m\rangle \\ &\quad -\langle j,m|\hat{J}_{-1}|j,m''\rangle\langle j,m''|\hat{V}_1|j,m\rangle] \\ &= [\langle j,m|\hat{J}_0|j,m\rangle\langle j,m|\hat{V}_0|j,m\rangle - \langle j,m|\hat{J}_1|j,m-1\rangle\langle j,m-1|\hat{V}_{-1}|j,m\rangle - \langle j,m|\hat{J}_{-1}|j,m+1\rangle\langle j,m+1|\hat{V}\end{aligned}$$

where

$$\langle j,m|\hat{J}_1|j,m-1\rangle = -\frac{\hbar}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

$$\langle j,m|\hat{J}_0|j,m\rangle = m\hbar$$

$$\langle j,m|\hat{J}_{-1}|j,m+1\rangle = \frac{\hbar}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}$$

$$\langle j',m'|\hat{V}_{\mu}|j,m\rangle = \langle j,k=1;m,\mu|j,k=1;j',m'\rangle < j\,||\,\hat{\boldsymbol{V}}\,||\,j >$$

$$\begin{aligned}\langle j,m|\hat{V}_0|j,m\rangle &= \langle j,k=1;m,\mu=0|j,k=1;j,m\rangle < j\,||\,\hat{\boldsymbol{V}}\,||\,j > \delta_{\mu,0} \\ &= \frac{m}{\sqrt{j(j+1)}} < j\,||\,\hat{\boldsymbol{V}}\,||\,j >\end{aligned}$$

$$\begin{aligned}\langle j,m+1|\hat{V}_1|j,m\rangle &= \langle j,k=1;m,1|j,k=1;j,m+1\rangle < j\,||\,\hat{\boldsymbol{V}}\,||\,j > \\ &= -\frac{1}{\sqrt{2}}\sqrt{\frac{(j-m)(j+m+1)}{j(j+1)}} < j\,||\,\hat{\boldsymbol{V}}\,||\,j >\end{aligned}$$

$$\begin{aligned}\langle j,m-1|\hat{V}_{-1}|j,m\rangle &= \langle j,k=1;m,-1|j,k=1;j,m-1\rangle < j\,||\,\hat{\boldsymbol{V}}\,||\,j > \\ &= \frac{1}{\sqrt{2}}\sqrt{\frac{(j+m)(j-m+1)}{j(j+1)}} < j\,||\,\hat{\boldsymbol{V}}\,||\,j >\end{aligned}$$

Then we have

$$\langle j,m|\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}|j,m\rangle = [\frac{\hbar m^2}{\sqrt{j(j+1)}} + \frac{1}{2}\frac{(j-m)(j+m+1)}{\sqrt{j(j+1)}} + \frac{1}{2}\frac{(j+m)(j-m+1)}{\sqrt{j(j+1)}}] < j$$

$$=$$

$$\langle j,m|\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}|j,m\rangle = \hbar\sqrt{j(j+1)} < j \| \hat{\boldsymbol{V}} \| j >>$$

or

$$\langle j,m|\hat{\boldsymbol{J}}\cdot\hat{\boldsymbol{V}}|j,m\rangle = < j \| \hat{\boldsymbol{J}} \| j >> < j \| \hat{\boldsymbol{V}} \| j >>$$

where

$$< j \| \hat{\boldsymbol{J}} \| j >> = \hbar\sqrt{j(j+1)}$$