Wigner representation

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(Date: January 13, 2017)

Wigner distribution function in the phase space is a special representation of the density matrix. It provides a third, alternative, formulation of quantum mechanics, independent of the conventional Hilbert space or path integral formulations. It is useful in describing transport in quantum optics, quantum computing, de coherence, and chaos.

Here we discuss the fundamental properties of the Wigner distribution function.

1. Definition of Wigner function

The Wigner distribution function is a quasi-probability distribution function in phase space (x, p) and is defined by

$$W(x, p) = \frac{1}{\pi\hbar}\int dy \psi^*(x + y)e^{2ipy/\hbar}\psi(x - y)$$

$$W(x, p) = \frac{1}{\pi\hbar}\int dq \psi^*(p + q)e^{-2ixq/\hbar}\psi(p - q)$$

It is a generating function for all spatial autocorrelation function of a given quantum mechanica; wave function $\psi(x)$. We have different expressions for $W(x, p)$, but these expressions are equivalent to the above expression.

$$W(x, p) = \frac{1}{\pi\hbar}\int dy \psi^*(x - y)e^{-2ipy/\hbar}\psi(x + y)$$

and

$$W(x, p) = \frac{1}{2\pi\hbar}\int dy \psi^*(x - \frac{y}{2})e^{-ipy/\hbar}\psi(x + \frac{y}{2})$$

Note

$$
\begin{aligned}
W(x,p) &= \frac{1}{\pi\hbar}\int dy \langle\psi|x+y\rangle e^{2ipy/\hbar}\langle x-y|\psi\rangle \\
&= \frac{1}{\pi\hbar}\int dy \iint dp_1 dp_2 \langle\psi|p_1\rangle\langle p_1|x+y\rangle e^{2ipy/\hbar}\langle x-y|p_2\rangle\langle p_2|\psi\rangle \\
&= \frac{1}{\pi\hbar}\frac{1}{2\pi\hbar}\int dy \iint dp_1 dp_2 \langle\psi|p_1\rangle e^{-\frac{i}{\hbar}p_1(x+y)} e^{2ipy/\hbar} e^{\frac{i}{\hbar}p_2(x-y)}\langle p_2|\psi\rangle \\
&= \frac{1}{2\pi^2\hbar^2}\int dy \iint dp_1 dp_2 \langle\psi|p_1\rangle\langle p_2|\psi\rangle \exp[\frac{i}{\hbar}(-p_1+p_2)x+\frac{i}{\hbar}(2p-p_1-p_2)y]
\end{aligned}
$$

We note that

$$
\int dy \exp[\frac{i}{\hbar}(2p-p_1-p_2)y] = 2\pi\delta(\frac{2p-p_1-p_2}{\hbar}) = 2\pi\hbar\delta(2p-p_1-p_2)
$$

$$
W(x,p) = \frac{1}{\pi\hbar}\int dp_1 \langle\psi|p_1\rangle\langle 2p-p_1|\psi\rangle \exp[\frac{2i}{\hbar}(p-p_1)x]
$$

When we use $p_1 = p+q$, we have

$$
2p-p_1 = 2p-(p+q) = p-q, \qquad p-p_1 = -q
$$

Thus we get

$$
\begin{aligned}
W(x,p) &= \frac{1}{\pi\hbar}\int dq \langle\psi|p+q\rangle e^{-\frac{2i}{\hbar}qx}\langle p-q|\psi\rangle \\
&= \frac{1}{\pi\hbar}\int dq \psi^*(p+q) e^{-\frac{2i}{\hbar}qx}\psi(p-q)
\end{aligned}
$$

where the Fourier transforms are defined as

$$\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx e^{-\frac{i}{\hbar}px}\langle x|\psi\rangle$$

$$\langle x|\psi\rangle = \int dp \langle x|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp e^{\frac{i}{\hbar}px}\langle p|\psi\rangle$$

Formula:

$$\int dp e^{-iyp} = 2\pi\delta(y)$$

Using the density operator (which include mixed states), the Wigner distribution function can be rewritten as

$$\begin{aligned}
W(x,p) &= \frac{1}{\pi\hbar}\int e^{\frac{2ipy}{\hbar}} dy \psi^*(x+y)\psi(x-y) \\
&= \frac{1}{\pi\hbar}\int e^{\frac{2ipy}{\hbar}} dy \langle\psi|x+y\rangle\langle x-y|\psi\rangle \\
&= \frac{1}{\pi\hbar}\int e^{\frac{2ipy}{\hbar}} dy \langle x-y|\psi\rangle\langle\psi|x+y\rangle \\
&= \frac{1}{\pi\hbar}\int e^{\frac{2ipy}{\hbar}} dy \langle x-y|\hat{\rho}|x+y\rangle
\end{aligned}$$

or

$$W(x,p) = \frac{1}{\pi\hbar}\int e^{\frac{2ipy}{\hbar}} dy \langle x-y|\hat{\rho}|x+y\rangle$$

or

$$W(x,p) = \frac{1}{2\pi\hbar}\int e^{\frac{ipy}{\hbar}} dy \left\langle x-\frac{y}{2}\right|\hat{\rho}\left|x+\frac{y}{2}\right\rangle$$

When the variable y is replaced by $y' = -y$, the above integral can be rewritten as

$$W(x,p) = \frac{1}{\pi\hbar}\int_{\infty}^{-\infty} e^{-\frac{2ipy'}{\hbar}}(-dy')\langle x+y'|\hat{\rho}|x-y'\rangle$$
$$= \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{2ipy'}{\hbar}} dy'\langle x+y'|\hat{\rho}|x-y'\rangle$$

or

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{2ipy}{\hbar}} dy\langle x+y|\hat{\rho}|x-y\rangle$$

or

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{ipy}{\hbar}} dy\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle$$

or

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{ipy}{\hbar}} dy\left\langle x-\frac{y}{2}\right|\hat{\rho}\left|x+\frac{y}{2}\right\rangle$$

Note that the density operator is defined by

$\hat{\rho} = \overline{|\psi\rangle\langle\psi|}$ for the mixed state.

and

$\hat{\rho} = |\psi\rangle\langle\psi|$ for the pure state.

2. Momentum space expression

Similarly, we have

$$W(x,p) = \frac{1}{\pi\hbar}\int dq\langle\psi|p+q\rangle e^{-\frac{2i}{\hbar}qx}\langle p-q|\psi\rangle$$
$$= \frac{1}{\pi\hbar}\int e^{-\frac{2i}{\hbar}qx} dq\langle p-q|\hat{\rho}|p+q\rangle$$

or

$$W(x,p) = \frac{1}{\pi\hbar}\int e^{\frac{2i}{\hbar}qx} dq\langle p+q|\hat{\rho}|p-q\rangle$$

3. Expression of the density operator

The density operator can be expressed by

$$\hat{\rho} = \int ds\iint dxdp|x+s\rangle P(x,p)e^{\frac{2ips}{\hbar}}\langle x-s|$$

((**Proof**))

$$\langle x+y|\hat{\rho}|x-y\rangle = \int ds'\iint dx'dp\langle x+y|x'+s'\rangle W(x',p')e^{\frac{2ip's'}{\hbar}}\langle x'-s'|x-y\rangle$$
$$= \int dp'\iint W(x',p')e^{\frac{2ip's'}{\hbar}} dx'ds'\langle x+y|x'+s'\rangle\langle x'-s'|x-y\rangle$$
$$= \int dp'\iint dx'ds'W(x',p')e^{\frac{2ip's'}{\hbar}}\delta(x'-x)\delta(s'-y)$$
$$= \int dp'W(x,p')e^{\frac{2ip'y}{\hbar}}$$
$$= \int dpW(x,p)e^{\frac{2ipy}{\hbar}}$$

or

$$\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle = \int dp W(x,p) e^{\frac{ipy}{\hbar}}$$

or

$$\left\langle x-\frac{y}{2}\right|\hat{\rho}\left|x+\frac{y}{2}\right\rangle = \int dp W(x,p) e^{-\frac{ipy}{\hbar}}$$

or

$$\langle x|\hat{\rho}|x'\rangle = \int dp W(\frac{x+x'}{2}, p) e^{-\frac{ip(x-x')}{\hbar}}$$

In fact, we have the same expression such that

$$\begin{aligned}
\int dp W(x,p) e^{\frac{2ipy}{\hbar}} &= \int dp e^{\frac{2ipy}{\hbar}} \frac{1}{\pi\hbar}\int dy' e^{2ipy'}\langle x-y'|\hat{\rho}|x+y'\rangle \\
&= \frac{1}{\pi\hbar}\int dy'\langle x-y'|\hat{\rho}|x+y'\rangle \int dp e^{\frac{2ip(y+y')}{\hbar}} \\
&= \frac{1}{\pi\hbar}\int dy'\langle x-y'|\hat{\rho}|x+y'\rangle \pi\hbar\delta(y+y') \\
&= \langle x+y|\hat{\rho}|x-y\rangle
\end{aligned}$$

4. Mathematica properties

(i) The function $P(x,p)$ is a real valued function.

((**Proof**))

$$W(x,p)^* = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{2ipy}{\hbar}} dy \langle x-y|\hat{\rho}^+|x+y\rangle$$
$$= \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{2ipy}{\hbar}} dy \langle x-y|\hat{\rho}|x+y\rangle$$
$$= \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{2ipy}{\hbar}} dy \langle x+y|\hat{\rho}|x-y\rangle$$
$$= W(x,p)$$

since $\hat{\rho}^+ = \hat{\rho}$.

(ii) $\int_{-\infty}^{\infty} dp W(x,p) = \langle x|\hat{\rho}|x\rangle$

If the system can be described by a pure state

$$\int_{-\infty}^{\infty} dp W(x,p) = |\langle x|\psi\rangle|^2$$

((**Proof**))

$$\int_{-\infty}^{\infty} dp W(x,p) = \int_{-\infty}^{\infty} dp \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{2ipy}{\hbar}} dy \langle x+y|\hat{\rho}|x-y\rangle$$
$$= \int_{-\infty}^{\infty} dy \langle x+y|\hat{\rho}|x-y\rangle \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dp e^{-\frac{2ipy}{\hbar}}$$
$$= \int_{-\infty}^{\infty} dy \langle x+y|\hat{\rho}|x-y\rangle \delta(y)$$
$$= \langle x|\hat{\rho}|x\rangle$$

where

$$\frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dp e^{-\frac{2ipy}{\hbar}} = \frac{1}{\pi\hbar} 2\pi\delta\left(\frac{2y}{\hbar}\right) = \frac{1}{\pi\hbar} 2\pi \frac{\hbar}{2}\delta(y) = \delta(y)$$

(iii) $\int_{-\infty}^{\infty} dx W(x,p) = \langle p|\hat{\rho}|p\rangle$

If the system can be described by a pure state

$$\int_{-\infty}^{\infty} dx W(x,p) = |\langle p|\psi\rangle|^2 .$$

((**Proof**))

$$\begin{aligned}\int_{-\infty}^{\infty} dx W(x,p) &= \int_{-\infty}^{\infty} dx \int e^{\frac{2i}{\hbar}qx} dq \langle p+q|\hat{\rho}|p-q\rangle \\ &= \int dq \langle p+q|\hat{\rho}|p-q\rangle \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx e^{\frac{2i}{\hbar}qx} \\ &= \int dq \langle p+q|\hat{\rho}|p-q\rangle \delta(q) \\ &= \langle p|\hat{\rho}|p\rangle\end{aligned}$$

where

$$\frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx e^{\frac{2i}{\hbar}qx} = \frac{1}{\pi\hbar} 2\pi\delta(\frac{2x}{\hbar}) = \frac{1}{\pi\hbar} 2\pi \frac{\hbar}{2}\delta(x) = \delta(x)$$

(iv) $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W(x,p) = Tr[\hat{\rho}] = 1$

((**Proof**))

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx W(x,p) = \int_{-\infty}^{\infty} dp \langle p|\hat{\rho}|p\rangle = Tr[\hat{\rho}] = 1$$

(v) Time reversal operator

$$\langle x|\hat{\Theta}|\psi\rangle = \langle x|\psi\rangle^* = \langle\psi|x\rangle,$$

Under the time reversal, $W(x,p) \rightarrow P(x,-p)$

((**Proof**))

$$\begin{aligned}
W(x,p) &= \frac{1}{\pi\hbar}\int dy\,\psi^*(x+y)e^{2ipy/\hbar}\psi(x-y) \\
&\rightarrow \frac{1}{\pi\hbar}\int dy\,\psi(x+y)e^{2ipy/\hbar}\psi^*(x-y) \\
&= \frac{1}{\pi\hbar}\int dy\,\psi(x-y)e^{-2ipy/\hbar}\psi^*(x+y) \\
&= \frac{1}{\pi\hbar}\int dy\,\psi^*(x+y)e^{-2ipy/\hbar}\psi(x-y) \\
&= W(x,-p)
\end{aligned}$$

(vi) Parity operator

$$\langle x|\hat{\pi}|\psi\rangle = \langle -x|\psi\rangle$$

Under the parity reversal [$\psi(x) \rightarrow \psi(-x)$)

$$W(x,p) \rightarrow W(-x,-p)$$

((**Proof**))

$$W(x,p)=\frac{1}{\pi\hbar}\int dy\psi^*(x+y)e^{2ipy/\hbar}\psi(x-y)$$
$$\to\frac{1}{\pi\hbar}\int dy\psi^*(-x-y)e^{2ipy/\hbar}\psi(-x+y)$$
$$=\frac{1}{\pi\hbar}\int dy\psi^*(-x+y)e^{-2ipy/\hbar}\psi(-x-y)$$
$$=W(-x,-p)$$

(vii) Translation operator

$$\langle x|\hat{T}_{-a}|\psi\rangle=\langle x+a|\psi\rangle=\psi(x+a)$$

Under the translation operator

$$W(x,p)\to W(x+a,p)$$

((**Proof**))

$$W(x,p)=\frac{1}{\pi\hbar}\int dy\psi^*(x+y)e^{2ipy/\hbar}\psi(x-y)$$
$$\to\frac{1}{\pi\hbar}\int dy\psi^*(x+y+a)e^{2ipy/\hbar}\psi(-x+y+a)$$
$$=W(x+a,p)$$

((**Proof**))

$$\frac{\partial}{\partial t}\hat{\rho}=\frac{\partial}{\partial t}\overline{|\psi\rangle\langle\psi|}=-\frac{i}{\hbar}\hat{H}\overline{|\psi\rangle\langle\psi|}+\frac{i}{\hbar}\hat{H}\overline{|\psi\rangle\langle\psi|}=\frac{i}{\hbar}[\hat{H},\hat{\rho}]$$

$$\frac{\partial}{\partial t}P(x,p)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}e^{-2ipy/\hbar}dy\langle x+y|\frac{\partial\hat{\rho}}{\partial t}|x-y\rangle$$

$$=$$

$$\frac{\partial}{\partial x}W(x,p)=\frac{1}{\pi\hbar}\int\frac{2i}{\hbar}qe^{\frac{2i}{\hbar}qx}dq\langle p+q|\hat{\rho}|p-q\rangle$$

5. State overlap

The state overlap can be calculated as

$$2\pi\hbar\int dx\int dpW_{\psi}(x,p)W_{\theta}(x,p)==Tr[\hat{\rho}_{\psi}\hat{\rho}_{\theta}]$$

In the case of pure states,

$$2\pi\hbar\int dx\int dpW_{\psi}(x,p)W_{\theta}(x,p)=Tr[\hat{\rho}_{\psi}\hat{\rho}_{\theta}]=\left|\langle\psi|\theta\rangle\right|^2$$

((**Proof**))

$$I=2\pi\hbar\int dx\int dp[\frac{1}{\pi\hbar}\int dy_1e^{2ipy_1/\hbar}\langle x-y_1|\hat{\rho}_{\psi}|x+y_1\rangle][\frac{1}{\pi\hbar}\int dy_2e^{2ipy_2/\hbar}\langle x-y_2|\hat{\rho}_{\theta}|x+y_2\rangle]$$

$$\int dpe^{2ipy_1/\hbar}e^{2ipy_2/\hbar}=\int dpe^{2ip(y_1+y_2)/\hbar}=2\pi\delta(2\frac{(y_1+y_2)}{\hbar})=2\pi\frac{\hbar}{2}\delta(y_1+y_2)$$

$$I=2\int dx\int dy_1\langle x-y_1|\hat{\rho}_{\psi}|x+y_1\rangle\int dy_2\langle x-y_2|\hat{\rho}_{\theta}|x+y_2\rangle\delta(y_1+y_2)$$
$$=2\int dx\int dy_1\langle x-y_1|\hat{\rho}_{\psi}|x+y_1\rangle\langle x+y_1|\hat{\rho}_{\theta}|x-y_1\rangle$$

We use new variables such that

$$x-y_1=x_1,\qquad x+y_1=x_2$$

or

$$x = \frac{1}{2}(x_1 + x_2), \qquad y_1 = \frac{1}{2}(x_2 - x_1)$$

$$dxdy_1 = \frac{\partial(x, y_1)}{\partial(x_1, x_2)} dx_1 dx_2$$

Jacobian:

$$\frac{\partial(x, y_1)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Then we get

$$\begin{aligned} I &= \int dx_1 \int dx_2 \langle x_1 | \hat{\rho}_\psi | x_2 \rangle \langle x_2 | \hat{\rho}_\theta | x_1 \rangle \\ &= \int dx_1 \langle x_1 | \hat{\rho}_\psi \hat{\rho}_\theta | x_1 \rangle \\ &= Tr[\hat{\rho}_\psi \hat{\rho}_\theta] \\ &= Tr[|\psi\rangle\langle\psi|\theta\rangle\langle\theta|] \\ &= |\langle\psi|\theta\rangle|^2 \end{aligned}$$

6. Phase space average $\langle\psi|\hat{G}|\psi\rangle$

Operator expectation values (averages) are calculated as phase space averages of the respective Wigner transforms

$$g(x, p) = \int_{-\infty}^{\infty} dy \left\langle x - \frac{y}{2} \middle| \hat{G} \middle| x + \frac{y}{2} \right\rangle e^{i\frac{py}{\hbar}} \qquad \text{(definition)}$$

$$\langle\hat{G}\rangle = \langle\psi|\hat{G}|\psi\rangle = Tr[\hat{\rho}\hat{G}] = \int dx \int dp W_\psi(x,p) g(x,p)$$ (phase space average)

((**Proof**))

$$I = \int dx \int dp W_\psi(x,p) g(x,p)$$
$$= \int dx \int dp [\frac{1}{\pi\hbar}\int dy_1 e^{\frac{2ipy_1}{\hbar}/\hbar} \langle x - y_1|\hat{\rho}_\psi|x + y_1\rangle][\int_{-\infty}^{\infty} dy_2 \left\langle x - \frac{y_2}{2}\middle|\hat{G}\middle|x + \frac{y_2}{2}\right\rangle e^{i\frac{py_2}{\hbar}}]$$

$$\int dp e^{\frac{2ipy_1}{\hbar}/\hbar} e^{i\frac{py_2}{\hbar}} = \int dp e^{\frac{2ip(y_1 + \frac{y_2}{2})}{\hbar}} = 2\pi\delta(\frac{2y_1 + y_2)}{\hbar}) = \pi\hbar\delta(2y_1 + y_2)$$

$$I = \frac{1}{\pi\hbar}\int dx \int dy_1 [\int_{-\infty}^{\infty} dy_2 \langle x - y_1|\hat{\rho}_\psi|x + y_1\rangle \left\langle x - \frac{y_2}{2}\middle|\hat{G}\middle|x + \frac{y_2}{2}\right\rangle \pi\hbar\delta(2y_1 + y_2)$$
$$= \int dx \int dy_1 \langle x - y_1|\hat{\rho}_\psi|x + y_1\rangle\langle x + y_1|\hat{G}|x - y_1\rangle$$

We use new variables such that

$$x - y_1 = x_1, \qquad x + y_1 = x_2$$

or

$$x = \frac{1}{2}(x_1 + x_2), \qquad y_1 = \frac{1}{2}(x_2 - x_1)$$

$$dxdy_1 = \frac{\partial(x, y_1)}{\partial(x_1, x_2)} dx_1 dx_2$$

Jacobian:

$$\frac{\partial(x,y_1)}{\partial(x_1,x_2)} = \begin{vmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Then we have

$$I = \int dx_1 \int dx_2 \langle x_1 | \hat{\rho}_\psi | x_2 \rangle \langle x_2 | \hat{G} | x_1 \rangle = \int dx_1 \langle x_1 | \hat{\rho}_\psi \hat{G} | x_1 \rangle = Tr[\hat{\rho}_\psi \hat{G}]$$

Note that

$$Tr[\hat{\rho}_\psi \hat{G}] = Tr[|\psi\rangle\langle\psi|\hat{G}] = Tr[\langle\psi|\hat{G}|\psi\rangle] = \langle\psi|\hat{G}|\psi\rangle$$

7. Expression of $g(x,p)$ for the operator $\hat{G} = \exp[\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p})]$

We calculate $g(x,p)$ for the operator $\hat{G} = \exp[\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p})]$.

$$\begin{aligned}
g(x,p) &= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar}) \left\langle x - \frac{y}{2} \right| \exp[\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p})] \left| x + \frac{y}{2} \right\rangle \\
&= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar}) \left\langle x - \frac{y}{2} \right| \exp(\frac{i\sigma\tau}{2\hbar}) \exp(\frac{i\sigma\hat{x}}{\hbar}) \exp(\frac{i\tau\hat{p}}{\hbar}) \left| x + \frac{y}{2} \right\rangle \\
&= \exp(\frac{i\sigma\tau}{2\hbar}) \int_{-\infty}^{\infty} \exp(\frac{ipy}{\hbar}) \exp[\frac{i\sigma}{\hbar}(x - \frac{y}{2})] dy \left\langle x - \frac{y}{2} \right| \exp(\frac{i\tau\hat{p}}{\hbar}) \left| x + \frac{y}{2} \right\rangle \\
&= \exp(\frac{i\sigma\tau}{2\hbar}) \int_{-\infty}^{\infty} \exp(\frac{ipy}{\hbar}) \exp[\frac{i\sigma}{\hbar}(x - \frac{y}{2})] dy \left\langle x - \frac{y}{2} \middle| x + \frac{y}{2} - \tau \right\rangle \\
&= \exp(\frac{i\sigma\tau}{2\hbar}) \int_{-\infty}^{\infty} \exp(\frac{ipy}{\hbar}) \exp[\frac{i\sigma}{\hbar}(x - \frac{y}{2})] dy \delta(y - \tau) \\
&= \exp[\frac{i}{\hbar}(\frac{\sigma\tau}{2} + p\tau + \sigma(x - \frac{\tau}{2})]
\end{aligned}$$

or

$$g(x,p) = \exp[\frac{i}{\hbar}(\sigma x + p\tau)]$$

((**Note**))

(i) The Baker-Campbell-Haussdorff theorem

$$\begin{aligned}\hat{G} &= \exp[\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p}) \\ &= \exp(\frac{i\sigma\hat{x}}{\hbar})\exp(\frac{i\tau\hat{p}}{\hbar})\exp[\frac{\sigma\tau}{2\hbar^2}[\hat{x},\hat{p}] \\ &= \exp(\frac{i\sigma\tau}{2\hbar})\exp(\frac{i\sigma\hat{x}}{\hbar})\exp(\frac{i\tau\hat{p}}{\hbar})\end{aligned}$$

(ii) Translation operator $\exp(\frac{i\tau\hat{p}}{\hbar})$

$$\exp(\frac{i\tau\hat{p}}{\hbar})\left|x + \frac{y}{2}\right\rangle = \left|x + \frac{y}{2} - \tau\right\rangle$$

7. Fourier transform of the Wigner function

$$
\begin{aligned}
\langle \hat{G} \rangle &= \langle \psi | \hat{G} | \psi \rangle \\
&= \left\langle \exp[\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p}) \right\rangle \\
&= \iint dx dx' \langle \psi | x \rangle \langle x | \exp(\frac{i\sigma\tau}{2\hbar}) \exp(\frac{i\sigma\hat{x}}{\hbar}) \exp(\frac{i\tau\hat{p}}{\hbar}) | x' \rangle \langle x' | \psi \rangle \\
&= \exp(\frac{i\sigma\tau}{2\hbar}) \iint dx dx' \exp(\frac{i\sigma x}{\hbar}) \langle \psi | x \rangle \langle x | \exp(\frac{i\tau\hat{p}}{\hbar}) | x' \rangle \langle x' | \psi \rangle \\
&= \exp(\frac{i\sigma\tau}{2\hbar}) \iint dx dx' \exp(\frac{i\sigma x}{\hbar}) \langle \psi | x \rangle \langle x | x' - \tau \rangle \langle x' | \psi \rangle \\
&= \exp(\frac{i\sigma\tau}{2\hbar}) \int dx \exp(\frac{i\sigma x}{\hbar}) \langle \psi | x \rangle \langle x + \tau | \psi \rangle \\
&= \int dx \exp[(\frac{i\sigma}{\hbar}(x + \frac{\tau}{2}) \psi^*(x) \psi(x+\tau) \\
&= \int dx e^{\frac{i\sigma x}{\hbar}} \psi^*(x - \frac{\tau}{2}) \psi(x + \frac{\tau}{2})
\end{aligned}
$$

or

$$
\begin{aligned}
\langle \hat{G} \rangle &= \left\langle \exp[\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p}) \right\rangle \\
&= \int dx e^{\frac{i\sigma x}{\hbar}} \psi^*(x - \frac{\tau}{2}) \psi(x + \frac{\tau}{2}) \\
&= \iint dx dp \exp[\frac{i(\sigma x + \tau p)}{\hbar}] P(x, p) \\
&= C(\sigma, \tau)
\end{aligned}
$$

where

$\exp(\frac{i\tau\hat{p}}{\hbar})$ is the translation operator; $\exp(\frac{i\tau\hat{p}}{\hbar}) | x' \rangle = | x' - \tau \rangle$

$$\langle x | x' - \tau \rangle = \delta(x' - x - \tau)$$

7. Fourier transform of $W(x, p)$

$$\iint dxdp \exp[\frac{i(\sigma x+\tau p)}{\hbar}]W(x,p)=\int dx e^{\frac{i\sigma x}{\hbar}}\psi^*(x-\frac{\tau}{2})\psi(x+\frac{\tau}{2})$$

((**Proof**))

$$\begin{aligned}
&\iint dxdp \exp[\frac{i(\sigma x+\tau p)}{\hbar}]W(x,p) \\
&=\iint dxdp \exp[\frac{i(\sigma x+\tau p)}{\hbar}]\frac{1}{2\pi\hbar}\int dy \psi^*(x-\frac{y}{2})e^{-ipy/\hbar}\psi(x+\frac{y}{2}) \\
&=\frac{1}{2\pi\hbar}\iint dxdy \psi^*(x-\frac{y}{2})e^{\frac{i\sigma x}{\hbar}}\psi(x+\frac{y}{2})\int dp e^{-\frac{ip(y-\tau)}{\hbar}} \\
&=\frac{1}{2\pi\hbar}\iint dxdy \psi^*(x-\frac{y}{2})e^{\frac{i\sigma x}{\hbar}}\psi(x+\frac{y}{2})2\pi\hbar\delta(y-\tau) \\
&=\int dx e^{\frac{i\sigma x}{\hbar}}\psi^*(x-\frac{\tau}{2})\psi(x+\frac{\tau}{2})
\end{aligned}$$

where

$$W(x,p)=\frac{1}{2\pi\hbar}\int dy \psi^*(x-\frac{y}{2})e^{-ipy/\hbar}\psi(x+\frac{y}{2})$$

Thus we have

$$\begin{aligned}
\langle\hat{G}\rangle &= \left\langle \exp[\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})\right\rangle \\
&=\int dx e^{\frac{i\sigma x}{\hbar}}\psi^*(x-\frac{\tau}{2})\psi(x+\frac{\tau}{2}) \\
&=\iint dxdp \exp[\frac{i(\sigma x+\tau p)}{\hbar}]W(x,p)
\end{aligned}$$

8. The matrix element $\langle x|\hat{G}|y\rangle$

The **Wigner transformation** is a general invertible transformation of an operator $\hat{G}$ on a Hilbert space to a function $g(x,p)$ on phase space, and is given by

$$g(x,p)=\int_{-\infty}^{\infty} ds\left\langle x-\frac{s}{2}\right|\hat{G}\left|x+\frac{s}{2}\right\rangle e^{i\frac{ps}{\hbar}} \qquad \text{(definition)}$$

$$\begin{aligned}
I &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp e^{\frac{ip(x-y)}{\hbar}} g(\frac{x+y}{2},p) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} ds\left\langle \frac{x+y}{2}-\frac{s}{2}\right|\hat{G}\left|\frac{x+y}{2}+\frac{s}{2}\right\rangle \int_{-\infty}^{\infty} dp e^{\frac{ip(x-y)}{\hbar}} e^{i\frac{ps}{\hbar}} \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} ds\left\langle \frac{x+y}{2}-\frac{s}{2}\right|\hat{G}\left|\frac{x+y}{2}+\frac{s}{2}\right\rangle 2\pi\hbar\delta(x-y+s) \\
&= \int_{-\infty}^{\infty} ds\left\langle \frac{x+y}{2}-\frac{s}{2}\right|\hat{G}\left|\frac{x+y}{2}+\frac{s}{2}\right\rangle \delta(x-y+s) \\
&= \langle x|\hat{G}|y\rangle
\end{aligned}$$

or

$$\langle x|\hat{G}|y\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp e^{\frac{ip(x-y)}{\hbar}} g(\frac{x+y}{2},p).$$

Hermitian operators map to real functions. The inverse of this transformation, so from phase space to Hilbert space, is called the Weyl transformation.

9. Size of quantum state

When we consider two identical density operators, $\hat{\rho}_\psi = \hat{\rho}_\theta = \hat{\rho}$ and recall that

$$Tr[\hat{\rho}^2] \leq 1$$

where the equal sign holds only for a pure state. So we get from

$$2\pi\hbar\int dx\int dp[W(x,p)]^2 = Tr[\hat{\rho}^2]\le 1$$

or

$$2\pi\hbar \le \frac{1}{\int dx\int dp[W(x,p)]^2}$$

Note that the area A of x-p phase space where the function $P(x,p)$ takes on considerable values follows as

$$A = \frac{1}{\int dx\int dp[W(x,p)]^2}.$$

10. Upper bound of Wigner function

For the pure state,

$$\begin{aligned} W(x,p) &= \frac{1}{2\pi\hbar}\int dy\,\psi^*(x-\frac{y}{2})e^{-ipy/\hbar}\psi(x+\frac{y}{2}) \\ &= \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\phi_1^*(y)\phi_2(y) \\ &= \frac{1}{\pi\hbar}\langle\phi_1|\phi_2\rangle \end{aligned}$$

where

$$\phi_1(y) = \frac{1}{\sqrt{2}}e^{\frac{ipy}{\hbar}}\psi(x-\frac{y}{2}), \qquad \phi_2(y) = \frac{1}{\sqrt{2}}\psi(x+\frac{y}{2})$$

$$\langle\phi_1|\phi_1\rangle = \int_{-\infty}^{\infty}|\phi_1(y)|^2 dy = \int_{-\infty}^{\infty}\frac{1}{2}\left|\psi(x-\frac{y}{2})\right|^2 dy = \int_{-\infty}^{\infty}|\psi(\xi)|^2 d\xi = 1$$

$$\langle \phi_2 | \phi_2 \rangle = \int_{-\infty}^{\infty} |\phi_2(y)|^2 dy = \int_{-\infty}^{\infty} \frac{1}{2} \left| \psi(x + \frac{y}{2}) \right|^2 dy = \int_{-\infty}^{\infty} |\psi(\eta)|^2 d\eta = 1$$

These definitions allow one to estimate the Wigner function

$$|W(x,p)| = \frac{1}{\pi\hbar} |\langle \phi_1 | \phi_2 \rangle|$$

With the help of the Cauchy-Schwartz inequality

$$|\langle \phi_1 | \phi_2 \rangle|^2 \le \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle = 1$$

valid for two normalized states $|\phi_1\rangle$ and $|\phi_2\rangle$. The Wigner function of a pure normalized state cannot take on values larger than $\frac{1}{\pi\hbar}$.

11. Wigner function can take on negative value

For the case of two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ such that

$$Tr[\hat{\rho}_1 \hat{\rho}_2] = 0$$

This relation implies

$$2\pi\hbar \int dx \int dp W_\psi(x,p) W_\theta(x,p) == Tr[\hat{\rho}_\psi \hat{\rho}_\theta] = 0$$

That is the product of the two Wigner functions integrated over the whole phase space has to vanish. This condition enforces that the Wigner function $W_\psi(x,p)$ or / and $W_\theta(x,p)$ must take on negative values. This surprising feature makes it impossible to interpret the Wigner function as a true probability as a true probability distribution. Nevertheless, the Wigner function is useful to calculate the quantum mechanical expectation values.

12. Derivation of the wave function from the Wigner function

(i) Fourier transform of $W(x_1, p)$

$$\overline{W(x_1,s)} = \int_{-\infty}^{\infty} dpP(x_1,p)e^{-\frac{2ip}{\hbar}s}$$
$$= \int_{-\infty}^{\infty} dpe^{-\frac{2ip}{\hbar}s} \frac{1}{\pi\hbar}\int dy\psi^*(x_1+y)e^{\frac{2ipy}{\hbar}}\psi(x_1-y)$$
$$= \frac{1}{\pi\hbar}\int dy\psi^*(x_1+y)\psi(x_1-y)2\pi\delta[\frac{2}{\hbar}(s-y)]$$
$$= \frac{1}{\pi\hbar}\int dy\psi^*(x_1+y)\psi(x_1-y)\pi\hbar\delta(s-y)$$
$$= \psi^*(x_1+s)\psi(x_1-s)$$

where

$$W(x,p) = \frac{1}{\pi\hbar}\int dy\psi^*(x+y)e^{2ipy/\hbar}\psi(x-y)$$

When $x_1 = \frac{x+x_0}{2}$, $\quad s = \frac{-x+x_0}{2}$,

$$\overline{W(\frac{x+x_0}{2}, \frac{-x+x_0}{2})} = \psi^*(x_0)\psi(x)$$

or

$$\psi(x) = \frac{\overline{W(\frac{x+x_0}{2}, \frac{-x+x_0}{2})}}{\psi^*(x_0)} \qquad (1)$$

(ii) The choice of $x_1 = x_0$ and $s = 0$ yields

$$\overline{W(x_0,0)} = \psi^*(x_0)\psi(x_0)$$

or

$$\psi^*(x_0) = \frac{\overline{W(x_0,0)}}{\psi(x_0)} \qquad (2)$$

Using Eqs.(1) and (2),

$$\psi(x) = \frac{\overline{W(\frac{x+x_0}{2},\frac{-x+x_0}{2})}}{\overline{P(x_0,0)}}\psi(x_0) \qquad (3)$$

P is pure real while ψ is generally complex.

13. von Neumann equation in phase space (from Toda, Kubo, and Saito)

The equation of motion for the density operator;

$$\hat{\rho} = \overline{|\psi\rangle\langle\psi|}$$

$$\begin{aligned}
\frac{\partial\hat{\rho}}{\partial t} &= \overline{\frac{\partial|\psi\rangle}{\partial t}\langle\psi|} + \overline{|\psi\rangle\frac{\partial}{\partial t}\langle\psi|} \\
&= \overline{\frac{\partial|\psi\rangle}{\partial t}\langle\psi|} + \overline{|\psi\rangle\frac{\partial}{\partial t}\langle\psi|} \\
&= -\frac{i}{\hbar}\hat{H}\overline{|\psi\rangle\langle\psi|} + \frac{i}{\hbar}\overline{|\psi\rangle\langle\psi|}\hat{H} \\
&= -\frac{i}{\hbar}[\hat{H},\hat{\rho}]
\end{aligned}$$

Thus we have

$$\frac{\partial}{\partial t}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle = \left\langle x+\frac{y}{2}\right|\frac{\partial\hat{\rho}}{\partial t}\left|x-\frac{y}{2}\right\rangle = -\frac{i}{\hbar}\left\langle x+\frac{y}{2}\right|[\hat{H},\hat{\rho}]\left|x-\frac{y}{2}\right\rangle$$

We multiply both sides by $\frac{1}{2\pi\hbar}e^{-\frac{ipy}{\hbar}}$ and integrate over the variable y.

$$W(x,p) = \frac{1}{2\pi\hbar}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle$$

The equation of motion:

$$\begin{aligned}\frac{\partial W(x,p)}{\partial t} &= \frac{1}{2\pi\hbar}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\frac{\partial\hat{\rho}}{\partial t}\left|x-\frac{y}{2}\right\rangle \\ &= \frac{1}{2\pi\hbar}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|-\frac{i}{\hbar}[\hat{H},\hat{\rho}]\left|x-\frac{y}{2}\right\rangle \\ &= -\frac{i}{2\pi\hbar^2}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|(\hat{H}\hat{\rho}-\hat{\rho}\hat{H})\left|x-\frac{y}{2}\right\rangle\end{aligned}$$

We assume that the Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$$

$$\begin{aligned}\langle x'|[\hat{H},\hat{\rho}]|x''\rangle &= \langle x'|[\frac{1}{2m}\hat{p}^2 + V(\hat{x})]\hat{\rho} - \hat{\rho}[\frac{1}{2m}\hat{p}^2 + V(\hat{x}))]|x''\rangle \\ &= [-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x''^2}) + V(x') - V(x'')]\}\langle x'|\hat{\rho}|x''\rangle\end{aligned}$$

If we change the variables as

$$x' = x + \frac{y}{2}, \qquad x'' = x - \frac{y}{2}$$

or

$$x = \frac{x'+x''}{2}, \qquad y = x'-x''$$

Using

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial x''} = \frac{\partial x}{\partial x''}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x''}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

we get

$$\begin{aligned}
&-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x''^2}) + V(x') - V(x'') \\
&= -\frac{\hbar^2}{2m}[(\frac{1}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^2 - (\frac{1}{2}\frac{\partial}{\partial x} - \frac{\partial}{\partial y})^2)] + V(x+\frac{y}{2}) - V(x-\frac{y}{2}) \\
&= -\frac{\hbar^2}{m}\frac{\partial^2}{\partial x \partial y} + V(x+\frac{y}{2}) - V(x-\frac{y}{2})
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{\partial W}{\partial t} &= -\frac{i}{2\pi\hbar^2}\int dy e^{-\frac{ipy}{\hbar}} \left\langle x+\frac{y}{2}\right| [\frac{1}{2m}\hat{p}^2 + V(\hat{x})]\hat{\rho} - \hat{\rho}[\frac{1}{2m}\hat{p}^2 + V(\hat{x}))] \left| x-\frac{y}{2}\right\rangle \\
&= -\frac{i}{2\pi\hbar^2}\int dy e^{-\frac{ipy}{\hbar}} [-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x \partial y} + V(x+\frac{y}{2}) - V(x-\frac{y}{2})] \left\langle x+\frac{y}{2}\right| \hat{\rho} \left| x-\frac{y}{2}\right\rangle
\end{aligned}$$

We note that

$$\int dy e^{-\frac{ipy}{\hbar}} \frac{\partial}{\partial y}\frac{\partial}{\partial x}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle = \frac{ip}{\hbar}\frac{\partial}{\partial x}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle$$

$$e^{-\frac{ipy}{\hbar}} V(x+\frac{y}{2})\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle = V(x-\frac{\hbar}{2i}\frac{\partial}{\partial p})e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle$$

Thus we have

$$\begin{aligned}
\frac{\partial W}{\partial t} &= -\frac{i}{2\pi\hbar^2}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|[[\frac{1}{2m}\hat{p}^2+V(\hat{x})]\hat{\rho}-\hat{\rho}[\frac{1}{2m}\hat{p}^2+V(\hat{x})]]\left|x-\frac{y}{2}\right\rangle \\
&= -\frac{i}{2\pi\hbar^2}\int dy e^{-\frac{ipy}{\hbar}}[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x\partial y}+V(x+\frac{y}{2})-V(x-\frac{y}{2})]\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle \\
&= -\frac{i}{2\pi\hbar^2}\int dy e^{-\frac{ipy}{\hbar}}[-\frac{\hbar^2}{m}\frac{\partial}{\partial x}\frac{ip}{\hbar}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle] \\
&\quad -\frac{i}{2\pi\hbar^2}\int dy[V(x-\frac{\hbar}{2i}\frac{\partial}{\partial p})-V(x+\frac{\hbar}{2i}\frac{\partial}{\partial p})]e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle \\
&= -\frac{p}{m}\frac{\partial}{\partial x}\frac{1}{2\pi\hbar}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle \\
&\quad +\frac{1}{i\hbar}[V(x-\frac{\hbar}{2i}\frac{\partial}{\partial p})-V(x+\frac{\hbar}{2i}\frac{\partial}{\partial p})]\frac{1}{2\pi\hbar}\int dy e^{-\frac{ipy}{\hbar}}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle
\end{aligned}$$

or

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial}{\partial x}W + \frac{1}{i\hbar}[V(x-\frac{\hbar}{2i}\frac{\partial}{\partial p})-V(x+\frac{\hbar}{2i}\frac{\partial}{\partial p})]W \qquad (1)$$

which is the Liouville equation in the Wigner representation. If we take the classical limit , the operator $\hbar \to 0$, the operator on the right-hand side of Eq.(1) reduces to the Liouville operator, since

$$\frac{1}{i\hbar}[V(x-\frac{\hbar}{2i}\frac{\partial}{\partial p})-V(x+\frac{\hbar}{2i}\frac{\partial}{\partial p}) \to \frac{\partial V}{\partial x}\frac{\partial}{\partial p},$$

or

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial W}{\partial p}.$$

((**Note**))

The equation of motion for each point in the phase space is classical in the absence of forces

$$\frac{\partial}{\partial t}W(x,p) = -\frac{p}{m}\frac{\partial}{\partial x}W(x,p).$$

14. Wigner distribution function for simple harmonics

The wave function of the simple harmonics with state $|n\rangle$

$$\varphi_n(\xi) = \langle \xi | n \rangle = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}} e^{-\frac{\xi^2}{2}} H_n(\xi)$$

where

$$\varphi_n(\xi) = \langle \xi | n \rangle = \frac{1}{\sqrt{\beta}}\langle x | n \rangle = \frac{1}{\sqrt{\beta}}\varphi_n(x)$$

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}$$

$H_n(\xi)$ is the Hermite polynomial.

In the phase space variables *x* and *p* the Wigner function of the *n*-th energy eigenstates is given by

$$F_n(x,p) = \frac{(-1)^n}{\pi\hbar}\exp\{-[(\frac{p}{\hbar\beta})^2 + (\beta x)^2]\}L_n\{2[(\frac{p}{\hbar\beta})^2 + (\beta x)^2]\}$$

or

$$F_n(x,p) = \frac{(-1)^n}{\pi\hbar}\exp\{-[(\frac{p}{\hbar\beta})^2 + (\beta x)^2]\}L_n\{2[(\frac{p}{\hbar\beta})^2 + (\beta x)^2]\}$$

$L_n(x)$ is the Laguerre polynomial.

$\xi = \beta x$ (dimensionless)

$\eta = \frac{p}{\hbar\beta}$ (dimensionless)

since

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x} = \frac{\hbar}{i}\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} = \hbar\beta\frac{1}{i}\frac{\partial}{\partial\xi}$$

15. Operator Ordering in Quantum Mechanics: symmetrical operator

$$
\begin{aligned}
\langle \hat{x}\hat{p} \rangle &= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar}) \left\langle x - \frac{y}{2} \middle| \hat{x}\hat{p} \middle| x + \frac{y}{2} \right\rangle \\
&= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x - \frac{y}{2}) \left\langle x - \frac{y}{2} \middle| \hat{p} \middle| x + \frac{y}{2} \right\rangle \\
&= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x - \frac{y}{2}) \int p' dp' \left\langle x - \frac{y}{2} \middle| p' \right\rangle \left\langle p' \middle| x + \frac{y}{2} \right\rangle \\
&= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x - \frac{y}{2}) \int p' dp' \left\langle x - \frac{y}{2} \middle| p' \right\rangle \left\langle p' \middle| x + \frac{y}{2} \right\rangle \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x - \frac{y}{2}) \int p' dp' \exp[\frac{i}{\hbar} p'(x - \frac{y}{2})] \exp[-\frac{i}{\hbar} p'(x + \frac{y}{2})] \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x - \frac{y}{2}) \int p' dp' \exp(-\frac{ip' y}{\hbar}) \\
&= x \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \int p' dp' \exp[\frac{i(p - p')y}{\hbar}] - \frac{1}{2\pi\hbar} \frac{1}{2} \int_{-\infty}^{\infty} dy \int y p' dp' \exp[\frac{i(p - p')y}{\hbar}] \\
&= xp - \frac{1}{2} \frac{\hbar}{i} \frac{\partial}{\partial p} p \\
&= xp - \frac{1}{2} \frac{\hbar}{i}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \int p' dp' \exp[\frac{i(p - p')y}{\hbar}] &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} p' dp' \int dy \exp[\frac{i(p - p')y}{\hbar}] \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} p' dp' 2\pi\hbar \delta(p - p') \\
&= p
\end{aligned}
$$

$$\langle \hat{p}\hat{x} \rangle = \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar}) \left\langle x - \frac{y}{2} \middle| \hat{p}\hat{x} \middle| x + \frac{y}{2} \right\rangle$$

$$= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x + \frac{y}{2}) \left\langle x - \frac{y}{2} \middle| \hat{p} \middle| x + \frac{y}{2} \right\rangle$$

$$= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x + \frac{y}{2}) \int p' dp' \left\langle x - \frac{y}{2} \middle| p' \right\rangle \left\langle p' \middle| x + \frac{y}{2} \right\rangle$$

$$= \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x + \frac{y}{2}) \int p' dp' \left\langle x - \frac{y}{2} \middle| p' \right\rangle \left\langle p' \middle| x + \frac{y}{2} \right\rangle$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x + \frac{y}{2}) \int p' dp' \exp[\frac{i}{\hbar} p'(x - \frac{y}{2})] \exp[-\frac{i}{\hbar} p'(x + \frac{y}{2})]$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \exp(\frac{ipy}{\hbar})(x + \frac{y}{2}) \int p' dp' \exp(-\frac{ip' y}{\hbar})$$

$$= x \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \int p' dp' \exp[\frac{i(p - p')y}{\hbar}] + \frac{1}{2\pi\hbar} \frac{1}{2} \int_{-\infty}^{\infty} dy \int y p' dp' \exp[\frac{i(p - p')y}{\hbar}]$$

$$= xp + \frac{1}{2} \frac{\hbar}{i} \frac{\partial}{\partial p} p$$

$$= xp + \frac{1}{2} \frac{\hbar}{i}$$

Then we have

$$\left\langle \frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2} \right\rangle = \frac{1}{2}(xp - \frac{1}{2}\frac{\hbar}{i} + xp + \frac{1}{2}\frac{\hbar}{i}) = xp$$

Hence, for this example the Weyl-Wigner ordering corresponds to the symmetric ordering.

$$S(\hat{x}\hat{p}) = \frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2}$$

where $S(\hat{x}\hat{p})$ is summed over all possible ordering of $\hat{x}$ and $\hat{p}$.

$$S(\hat{x}\hat{p}) = xp$$

In general,

$$\left\langle S(\hat{x}^m, \hat{p}^n)\right\rangle = \int dx \int dp (x^m p^n) F(x, p)$$

For example, if $m = 2$, $n = 2$, then

$$S(\hat{x}^2, \hat{p}^2) = \frac{1}{6}(\hat{x}^2 \hat{p}^2 + \hat{p}^2 \hat{x}^2 + \hat{x}\hat{p}\hat{x}\hat{p} + \hat{p}\hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2 \hat{p} + \hat{x}\hat{p}^2 \hat{x})$$

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APPENDIX

*** product**

$$f(x, p) * g(x, p) = f(x + \frac{i\hbar}{2}\frac{\partial}{\partial p}, p - \frac{i\hbar}{2}\frac{\partial}{\partial x}) g(x, p)$$

The equivalent Fourier representation of the *-product is

$$f * g = \frac{1}{(\pi\hbar)^2}\int dp' dp'' dx' dx'' f(x', p')g(x'', p'')$$
$$\times \exp(\frac{-2i}{\hbar}[p(x'-x'') + p'(x''-x) + p''(x-x')])$$

An alternative integral representation of this product is

$$f * g = \frac{1}{(\pi\hbar)^2}\int dx' dx'' dp' dp'' f(x+x', p+p')g(x+x'', p+p'')$$
$$\times \exp[\frac{2i}{\hbar}(x' p'' - x'' p')]$$

Note that

$$\int dxdp f * g = \int dxdp g * f = \int dxdp fg$$

((**Proof**))

$$\int dxdp f * g = \int dxdp \frac{1}{(\pi\hbar)^2}\int dx' dx'' dp' dp'' f(x', p')g(x'', p'')$$
$$\times \exp(\frac{-2i}{\hbar}[p(x'-x'') + p'(x''-x) + p''(x-x')])$$

Note that

$$p(x'-x'') + p'(x''-x) + p''(x-x') = x(p''-p') + p(x'-x'') + (p' x'' - p'' x')$$

$$\int dp \exp[\frac{-2i}{\hbar} p(x'-x'')] = \pi\hbar\delta(x'-x''),$$

$$\int dx \exp[\frac{-2i}{\hbar} x(p''-p')] = \pi\hbar\delta(p''-p')$$

Then we get

$$\int dxdpf * g = \int dxdp \frac{1}{(\pi\hbar)^2} \int dx'dx''dp'dp'' f(x',p')g(x'',p'')$$
$$\times \exp(\frac{-2i}{\hbar}[x(p''-p') + p(x'-x'') + (p'x''-p''x')])$$
$$= \int dp'dp''dx'dx'' f(x',p')g(x'',p'')\exp[\frac{-2i}{\hbar}(p'x''-p''x')]$$
$$\times \delta(p''-p')\delta(x'-x'')$$
$$= \int dx'dp' f(x',p')g(x',p')$$

or

$$\int dxdpf * g = \int dxdpf(x,p)g(x,p)$$