Here we discuss the Zeeman effect using the perturbation theory. The discussion here consists of three parts, depending on the magnitude of magnetic field; (i) in the weak magnetic field limit where the spin-orbit interaction $H_{so}$ is dominant; (i) the intermediate magnetic field where the spin-orbit interaction is comparable with the Zeeman energy, and (iii) the strong magnetic field where the Zeeman energy $H_B$ is dominant.

Fig. $H_{so}$ is the spin-orbit interaction. $H_B$ is the Zeeman energy.

(()(Pieter Zeeman))
Pieter Zeeman (25 May 1865 – 9 October 1943) was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Hendrik Lorentz for his discovery of the Zeeman effect.

(Alfred Landé)

Alfred Landé (13 December 1888–30 October 1976) was a German-American physicist known for his contributions to quantum theory. He is responsible for the Landé g-factor and an explanation of the Zeeman Effect.
1. Orbital magnetic moment and spin magnetic moment

The total angular momentum \( \mathbf{J} \) is defined by

\[
\mathbf{J} = \mathbf{L} + \mathbf{S}.
\]

The total magnetic moment \( \mathbf{\mu} \) is given by

\[
\mathbf{\mu} = -\frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S}).
\]

The total magnetic moment along the direction of \( \mathbf{J} \), \( \mathbf{\mu}_j \), is defined by

\[
\mathbf{\mu}_j = -\frac{g_J \mu_B}{\hbar} \mathbf{J},
\]

where \( g_J \) is the Lande g-factor.
Basic classical vector model of orbital angular momentum ($L$), spin angular momentum ($S$), orbital magnetic moment ($\mu_L$), and spin magnetic moment ($\mu_S$). $J (= L + S)$ is the total angular momentum. $\mu$ is the component of the total magnetic moment ($\mu_L + \mu_S$) along the direction ($-J$).

Suppose that

$$L = aJ + L_\perp$$
$$S = bJ + S_\perp,$$

where $a$ and $b$ are constants, and the vectors $S_\perp$ and $L_\perp$ are perpendicular to $J$. 
Here we have the relation \( a + b = 1 \), and \( \mathbf{L}_\perp + \mathbf{S}_\perp = 0 \). The values of \( a \) and \( b \) are determined as follows.

\[
a = \frac{\mathbf{J} \cdot \mathbf{L}}{J^2}, \quad b = \frac{\mathbf{J} \cdot \mathbf{S}}{J^2}.
\]

Here we note that

\[
\mathbf{J} \cdot \mathbf{S} = (\mathbf{L} + \mathbf{S}) \cdot \mathbf{S} = \mathbf{S}^2 + \mathbf{L} \cdot \mathbf{S} = \mathbf{S}^2 + \frac{\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2}{2} = \frac{\mathbf{J}^2 - \mathbf{L}^2 + \mathbf{S}^2}{2},
\]

or

\[
\mathbf{J} \cdot \mathbf{S} = \frac{\mathbf{J}^2 - \mathbf{L}^2 + \mathbf{S}^2}{2} = \frac{\hbar^2}{2} [J(J + 1) - L(L + 1) + S(S + 1)],
\]

using the average in quantum mechanics. The total magnetic moment \( \mu \) is

\[
\mu = -\frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S}) = -\frac{\mu_B}{\hbar}[(a + 2b)\mathbf{J} + (L_\perp + 2S_\perp)].
\]

Thus we have

\[
\mu_j = -\frac{\mu_B}{\hbar}(a + 2b)\mathbf{J} = -\frac{\mu_B}{\hbar}(1 + b)\mathbf{J} = -\frac{\mu_B}{\hbar} \mathbf{J},
\]

with

\[
g_j = 1 + b = 1 + \frac{\mathbf{J} \cdot \mathbf{S}}{\mathbf{J}^2} = 1 + \frac{3}{2} + \frac{s(s + 1) - L(L + 1)}{2J(J + 1)}.
\]

2. **Derivation of Landé \( g \)-factor: approach from the classical model**

In the classical theory, the projection vectors of the spin angular momentum \( \mathbf{S} \) and the orbital angular momentum \( \mathbf{L} \) along the direction of the total angular momentum \( \mathbf{J} \)
\[ S_j = \frac{S \cdot J}{|J|^2} J, \quad L_j = \frac{L \cdot J}{|J|^2} J, \]

where

\[ J = L + S. \]

The total magnetic moment along the direction of \( J \) is given by

\[ \mu_j = -\frac{\mu_B}{\hbar} \left( \frac{L \cdot J}{|J|^2} J + 2 \frac{S \cdot J}{|J|^2} J \right). \]

Using \( L = J - S \), and squaring both sides, we get

\[ L^2 = J^2 + S^2 - 2J \cdot S, \]

or

\[ J \cdot S = \frac{J^2 + S^2 - L^2}{2}. \]

Using \( S = J - L \), and squaring both sides, we get

\[ S^2 = J^2 + L^2 - 2J \cdot L, \]

or

\[ J \cdot L = \frac{J^2 + L^2 - S^2}{2}. \]

Then we get
The Landé g-factor is defined by

\[ g_j = \frac{(3J^2 - L^2 + S^2)}{2J^2} \]

In quantum mechanics, we get

\[ g_j = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)}, \]

using the relations, \( J^2 \rightarrow \hbar^2 j(j + 1) \), \( S^2 \rightarrow \hbar^2 s(s + 1) \), and \( L^2 \rightarrow \hbar^2 L(L + 1) \). The total magnetic moment is given by

\[ \mu_j = -\frac{\mu_B}{\hbar} g_j J. \]

3. Derivation of Landé g-factor: approach from the Wigner-Eckhart

The specific formula we need from the Wigner-Eckhart theorem relates the matrix element of any general vector component \( \hat{V}_z \) to the matrix element of the total magnetic momentum, \( \hat{J}_z \):

\[ \langle jm | \hat{V}_z | jm' \rangle = \frac{\langle j | \hat{V} \cdot \hat{J} | jm \rangle}{\hbar^2 j(j+1)} \langle jm | \hat{J}_z | jm' \rangle. \]

For the Zeeman effect, \( L \) or \( S \) play the role of the vector \( V \). This equation is called the projection theorem because of the role of the projection \( V \cdot J \) in determining the constant of proportionality between the matrix of \( V_z \) and \( J_z \). Note that the matrix element of the projection \( V \cdot J \) is a diagonal element, but the \( V_z \) and \( J_z \) matrix elements are general matrix elements between different \( m \) states within a given \( j \) subspace.
\begin{align*}
\langle j | \hat{S} \cdot \hat{j} | j \rangle &= \frac{1}{2} \langle j | \hat{j}^2 + \hat{S}^2 - \hat{L}^2 | j \rangle \\
&= \frac{\hbar^2}{2} [j(j+1) + s(s+1) - l(l+1)]
\end{align*}

and

\begin{align*}
\langle j | \hat{L} \cdot \hat{j} | jm \rangle &= \frac{1}{2} \langle j | \hat{j}^2 + \hat{L}^2 - \hat{S}^2 | j \rangle \\
&= \frac{\hbar^2}{2} [j(j+1) + l(l+1) - s(s+1)]
\end{align*}

Then we have

\begin{align*}
\langle jm | \hat{S}_z | jm' \rangle &= \frac{\langle j | \hat{S} \cdot \hat{j} | j \rangle}{\hbar^2 j(j+1)} \langle jm | \hat{j}_z | jm' \rangle \\
&= \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} \langle jm | \hat{j}_z | jm' \rangle
\end{align*}

\begin{align*}
\langle jm | \hat{L}_z | jm' \rangle &= \frac{\langle j | \hat{L} \cdot \hat{j} | j \rangle}{\hbar^2 j(j+1)} \langle jm | \hat{j}_z | jm' \rangle \\
&= \frac{j(j+1) + l(l+1) - s(s+1)}{2j(j+1)} \langle jm | \hat{j}_z | jm' \rangle
\end{align*}

The total magnetic moment along the \( z \) axis is given by

\[ \hat{\mu}_z = -\frac{\mu_B}{\hbar} (\hat{L}_z + 2\hat{S}_z). \]

Then we have
\[
\langle jm|\hat{L}_z + 2\hat{S}_z|jm'\rangle = \\
= \left[ \frac{j(j+1) + l(l+1) - s(s+1)}{2j(j+1)} \right. \\
+ \left. \frac{j(j+1) + s(s+1) - l(l+1)}{j(j+1)} \right] \langle jm|\hat{J}_z|jm'\rangle \\
= \left[ \frac{3}{2} \right. + \left. \frac{s(s+1) - l(l+1)}{2j(j+1)} \right] \langle jm|\hat{J}_z|jm'\rangle \\
= g_J \langle jm|\hat{J}_z|jm'\rangle
\]

where \( g_J \) is the Landé g-factor given by

\[
g_J = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)}.
\]

Then magnetic moment is given by

\[
\hat{\mu}_z = -\frac{\mu_B}{\hbar} (\hat{L}_z + 2\hat{S}_z) = -\frac{\mu_B}{\hbar} g_J \hat{J}_z.
\]

4. Zeeman effect in the weak magnetic field

In the presence of the magnetic field \( B \) along the \( z \) axis, the Zeeman energy is given by

\[
\hat{H}_B = \frac{g_J \mu_B B}{\hbar} \hat{J}_z.
\]

The first-order Zeeman energy correction:

\[
E^{(1)}_B = \langle jm|\hat{H}_B|jm\rangle = \frac{g_J \mu_B B}{\hbar} \langle jm|\hat{J}_z|jm\rangle = g_J m_\mu_B B.
\]

For \( j = l + \frac{1}{2}, \ s = \frac{1}{2} \),

\[
g_J = \frac{3}{2} + \frac{3}{2j(j+1)} = 1 + \frac{1}{2l+1},
\]
$E_B^{(i)} = m(1 + \frac{1}{2l+1})\mu_B B$.

For $j = l - \frac{1}{2}$,

$$g_J = \frac{3}{2} + \frac{4}{2j(j + 1)} = 1 - \frac{1}{2l+1},$$

$E_B^{(i)} = m(1 - \frac{1}{2l+1})\mu_B B$.

**Fig.** Weak-field Zeeman structure of the hydrogen 2 P fine structure levels labeled with the quantum numbers of the coupled basis states. $n = 2$, $j = 3/2$ and $j = 1/2$, $l = 1$ and $s = 1/2$. The splitting of the levels between $2P_{3/2}$ and $2P_{1/2}$ at $B = 0$ is due to the spin orbit interaction.
5. **Zeeman effect in the intermediate magnetic field**

In the presence of magnetic field along the $z$ axis, the Zeeman energy is given by

$$
\hat{H}_B = -\mu \cdot B = \frac{\mu_B}{\hbar} (\hat{L} + 2\hat{S}) \cdot B = \frac{\mu_B}{\hbar} (\hat{J} + \hat{S}) \cdot B = \frac{\mu_B B}{\hbar} (\hat{J}_z + \hat{S}_z).
$$

The perturbed Hamiltonian is the sum of the spin orbit interaction $\hat{H}_{so}$ and the Zeeman energy $\hat{H}_B$ as

$$
\hat{H}_1 = \hat{H}_{so} + \hat{H}_B = \frac{e^2 h^2}{2m^2} \hat{L} \cdot \hat{S} + \mu_B B \frac{1}{\hbar} (L_z + 2S_z).
$$
Note that $\hat{H}_{so}$ is comparable to $\hat{H}_B$. The matrix of $\hat{L} \cdot \hat{S}$ under the basis of $|\phi_1\rangle = \left| m_i = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle$ and $|\phi_2\rangle = \left| m_i = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle$, is obtained as

$$\frac{2}{\hbar^2} \hat{L} \cdot \hat{S} = \begin{pmatrix}
\frac{(m - \frac{1}{2})}{\sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})}} & \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} \\
\frac{-1}{\sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})}} & -(m + \frac{1}{2})
\end{pmatrix}. $$

We note that

$$\frac{1}{\hbar}(L_z + 2S_z) \left| m_i = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle = \left| m_i = m + \frac{1}{2}, m_s = \frac{1}{2} \right\rangle,$$

$$\frac{1}{\hbar}(L_z + 2S_z) \left| m_i = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle = \left| m_i = m - \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle.$$

So the matrix of $\frac{1}{\hbar}(L_z + 2S_z)$ under the basis of $|\phi_1\rangle$ and $|\phi_2\rangle$ is diagonal,
\[
\frac{1}{\hbar}(L_z + 2S_z) = \begin{pmatrix}
m + \frac{1}{2} & 0 \\
0 & m - \frac{1}{2}
\end{pmatrix},
\]

Thus the resultant matrix is given by

\[
\hat{H}_1 = \frac{\hbar^2}{4} \begin{pmatrix}
\frac{(m - \frac{1}{2})}{\sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})}} & \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} \\
\frac{(l + m + \frac{1}{2})(l - m + \frac{1}{2})}{-(m + \frac{1}{2})}
\end{pmatrix} + \mu_B B \begin{pmatrix}
m + \frac{1}{2} \\
0
\end{pmatrix}
\]

We solve the eigenvalue problem. For simplicity we put

\[
\frac{\hbar^2}{4} = \frac{1}{2} \alpha, \quad \beta = \mu_B B.
\]

The matrix (2x2) is given by

\[
\hat{H}_1 = \frac{\alpha}{2} \begin{pmatrix}
m - \frac{1}{2} \\
\frac{(l + m + \frac{1}{2})(l - m + \frac{1}{2})}{-(m + \frac{1}{2})}
\end{pmatrix} + \beta \begin{pmatrix}
m + \frac{1}{2} \\
0
\end{pmatrix};
\]

or

\[
\hat{H}_1 = \frac{\alpha}{2} \begin{pmatrix}
m - \frac{1}{2} & k \\
k & -(m + \frac{1}{2})
\end{pmatrix} + \beta \begin{pmatrix}
m + \frac{1}{2} & 0 \\
0 & m - \frac{1}{2}
\end{pmatrix},
\]

where for simplicity we use
\[ k = \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})}. \]

We solve the eigenvalue problem of the matrix \( \hat{H}_1 \) (2 x 2 matrix). Thus the eigenvalues are

\[ \lambda_1 = -\frac{\alpha}{4} + \beta m + \frac{1}{2} \sqrt{\alpha^2 (l + \frac{1}{2})^2 + 2\alpha\beta m + \beta^2}, \]

\[ \lambda_2 = -\frac{\alpha}{4} + \beta m - \frac{1}{2} \sqrt{\alpha^2 (l + \frac{1}{2})^2 + 2\alpha\beta m + \beta^2}. \]

We introduce the ratio \( \zeta \) as

\[ \zeta = \frac{\beta}{\alpha}. \]

Then we have

\[ \lambda_1 = \frac{\alpha}{4} (-1 + 4\zeta m + \frac{1}{2} \sqrt{(l + \frac{1}{2})^2 + 2\alpha m + \zeta^2}), \]

and

\[ \lambda_2 = \frac{\alpha}{4} (-1 + 4\zeta m - \frac{1}{2} \sqrt{(l + \frac{1}{2})^2 + 2\alpha m + \zeta^2}). \]

For \( \zeta << 1 \) (weak field side)

\[ \lambda_1 = \frac{\alpha}{2} l + \beta m \frac{2l + 2}{2l + 1}, \]

\[ \lambda_2 = -\frac{\alpha}{2} (l + 1) + \beta m \frac{2l}{2l + 1}. \]
The first term is from the spin-orbit interaction and the second term is from the Zeeman effect.

For $\zeta \gg 1$ (strong field side)

$$\lambda_1 = \alpha \zeta (m + \frac{1}{2}) = \beta (m + \frac{1}{2}), \quad \lambda_2 = \alpha \zeta (m - \frac{1}{2}) = \beta (m - \frac{1}{2}).$$

((Mathematica))
Clear["Global`*"]

\[\text{rule1} = \left\{ \sqrt{k^2 + m^2} \rightarrow L + \frac{1}{2} \right\} \]

\[\text{rule2} = \left\{ k \rightarrow \sqrt{(L - m + \frac{1}{2})(L + m + \frac{1}{2})} \right\} \]

\[A_1 = \frac{\alpha}{2} \begin{pmatrix} m - \frac{1}{2} & k \\ k & -(m + \frac{1}{2}) \end{pmatrix} + \beta \begin{pmatrix} m + \frac{1}{2} & 0 \\ 0 & (m - \frac{1}{2}) \end{pmatrix} \]

\[\text{eq1} = \text{EigenSystem}[A_1];\]

\[\lambda_1 = \text{eq1}[[1, 2]] / . \text{rule1} / . \text{rule2} / \text{Simplify}[#1, L > 0] \& \]

\[\frac{1}{4} \left( -\alpha + 4 m \beta + \sqrt{(\alpha + 2 L \alpha)^2 + 8 m \alpha \beta + 4 \beta^2} \right) \]

\[\lambda_2 = \text{eq1}[[1, 1]] / . \text{rule1} / . \text{rule2} / \text{Simplify}[#1, L > 0] \& \]

\[-\frac{\alpha}{4} + m \beta - \frac{1}{4} \sqrt{(\alpha + 2 L \alpha)^2 + 8 m \alpha \beta + 4 \beta^2} \]

\[\lambda_{11} = \lambda_1 / . \{ \beta \rightarrow \xi \alpha \} / \text{Simplify}[#1, \alpha > 0] \& \]

\[\frac{1}{4} \alpha \left( -1 + 4 m \xi + \sqrt{1 + 4 L + 4 L^2 + 8 m \xi + 4 \xi^2} \right) \]

\[\lambda_{22} = \lambda_2 / . \{ \beta \rightarrow \xi \alpha \} / \text{Simplify}[#1, \alpha > 0] \& \]

\[-\frac{1}{4} \alpha \left( 1 - 4 m \xi + \sqrt{1 + 4 L + 4 L^2 + 8 m \xi + 4 \xi^2} \right) \]

\[\text{Series}[\lambda_{11}, \{ \xi, 0, 2 \}] / \text{Simplify}[#1, 2L + 1 > 0] \& \]

\[\frac{L \alpha}{2} + \frac{2 (1 + L) m \alpha \xi}{1 + 2 L} + \frac{(1 + 4 L + 4 L^2 - 4 m^2) \alpha \xi^2}{2 (1 + 2 L)^3} + O[\xi^3] \]

\[\text{Series}[\lambda_{22}, \{ \xi, 0, 2 \}] / \text{Simplify}[#1, 2L + 1 > 0] \& \]

\[-\frac{1}{2} (1 + L) \alpha + \frac{2 L m \alpha \xi}{1 + 2 L} - \frac{(1 + 4 L + 4 L^2 - 4 m^2) \alpha \xi^2}{2 (1 + 2 L)^3} + O[\xi^3] \]
6. **Zeeman effect in the strong magnetic field (Paschen Back effect)**

We consider the case where the magnetic field is strong enough that the Zeeman shifts are much larger than the fine-structure shifts. The perturbation assumption regarding the Zeeman effect is no longer valid. It is more appropriate to include the Zeeman Hamiltonian,

\[ \hat{H} = \hat{H}_0 + \hat{H}_B = \hat{H}_0 - \hat{\mu} \cdot \mathbf{B} = \hat{H}_0 + \frac{\mu_B}{\hbar} (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \mathbf{B}, \]

where

\[ \hat{\mu} = -\frac{\mu_B}{\hbar} (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}). \]
We note that

\[
\begin{align*}
[\hat{H}_0, \hat{L}_x] &= 0, \quad [\hat{H}_0, \hat{L}_y] = 0, \quad [\hat{H}_0, \hat{L}_z] = 0, \quad [\hat{H}_0, \hat{L}^2] = 0, \\
[\hat{H}_B, \hat{L}_x] &= 0, \quad [\hat{H}_B, \hat{L}^2] = 0, \\
[\hat{H}_0, \hat{S}_x] &= 0, \quad [\hat{H}_0, \hat{S}_y] = 0, \quad [\hat{H}_0, \hat{S}_z] = 0, \quad [\hat{H}_0, \hat{S}^2] = 0, \\
[\hat{H}_B, \hat{S}_x] &= 0, \quad [\hat{H}_B, \hat{S}^2] = 0
\end{align*}
\]

Then we have

\[
\begin{align*}
[\hat{H}_0, \hat{L}_x] &= 0, \quad [\hat{H}_0, \hat{L}^2] = 0, \quad [\hat{H}_0, \hat{S}_z] = 0, \quad [\hat{H}_0, \hat{S}^2] = 0, \\
[\hat{L}^2, \hat{L}_z] &= 0, \quad [\hat{S}^2, \hat{S}_z] = 0, \quad [\hat{H}_0, \hat{H}_B] = 0, \\
[\hat{H}_B, \hat{L}_z] &= 0, \quad [\hat{H}_B, \hat{L}^2] = 0, \\
[\hat{H}_B, \hat{S}_z] &= 0, \quad [\hat{H}_B, \hat{S}^2] = 0
\end{align*}
\]

We choose a simultaneous eigenket of \(\hat{H}_B, \hat{H}_0, \hat{L}^2, \hat{S}^2, \hat{L}_z, \) and \(\hat{S}_z\)

\[
|\psi\rangle = |nlm\rangle \otimes |sm_y\rangle = |nlm\rangle |sm_y\rangle
\]

Here the Zeeman Hamiltonian

\[
\hat{H}_B = \frac{\mu_B B}{\hbar} (\hat{L}_z + 2\hat{S}_z),
\]

in zeroth-order and treat the fine structure as a perturbation. We note that

\[
\hat{H}_0 |n;lm_z;sm_y\rangle = E_n |n;lm_z;sm_y\rangle,
\]
\[ \hat{H}_B | n; l m_i; s m_s \rangle = \frac{\mu_B B}{\hbar} (\hat{L}_z + 2 \hat{S}_z) | n; l m_i; s m_s \rangle = \mu_B B (m_i + 2m_s) | n; l m_i; s m_s \rangle. \]

The Zeeman energy is the expectation values,

\[ E_{\text{Zeeman}}^{(1)} = \frac{\mu_B B}{\hbar} \langle n; l m_i; s m_s | (\hat{L}_z + 2\hat{S}_z) | n; l m_i; s m_s \rangle = \mu_B B (m_i + 2m_s) \]

We now treat the spin-orbit interaction as a perturbation to the zeroth order state that include the Zeeman interaction.

\[ E_{\alpha 0}^{(1)} = \langle n; l m_i; s m_s | \hat{\mathcal{H}}_{\alpha 0} | n; l m_i; s m_s \rangle \]
\[ = \xi \langle n; l m_i; s m_s | \hat{\mathcal{L}} \cdot \hat{\mathcal{S}} | n; l m_i; s m_s \rangle \]
\[ = \xi \langle n; l m_i; s m_s | \frac{1}{2} (\hat{L}_z \hat{S}_+ + \hat{L}_- \hat{S}_z) | n; l m_i; s m_s \rangle \]
\[ = \xi \langle n; l m_i; s m_s | \hat{L}_z \hat{S}_z | n; l m_i; s m_s \rangle \]
\[ = \xi \hbar^2 m_i m_s \]
\[ = \frac{1}{2} m_c^2 \alpha^4 \frac{m_i m_s}{n^3 l(l + \frac{1}{2})(l + 1)} \]

where

\[ \xi = \frac{e^2 \hbar^2}{2m_e^2 c^2 n^3 a_B^3 l(l + 1/2)(l + 1)} = \frac{1}{2} m_c^2 \alpha^4 \frac{1}{n^3 l(l + 1/2)(l + 1)} \]

with

\[ \alpha = \frac{e^2}{\hbar c}, \quad a_B = \frac{\hbar^2}{mc^2}. \]

Note that \( E_{\alpha 0}^{(1)} \) is independent of the magnetic field. Thus we get
\[
\Delta E^{(1)} = E_{b}^{(1)} + E_{so}^{(1)} = \mu_B B (m_l + 2m_s) + \frac{1}{2} m_e c^2 \alpha^4 \frac{m_l m_s}{n^3 l \left( l + \frac{1}{2} \right) (l + 1)}
\]

or

\[
\frac{\Delta E^{(1)}}{\mu_B B_0} = (m_l + 2m_s) \frac{B}{B_0} + \frac{1}{2} \frac{m_e c^2 \alpha^4}{\mu_B B_0} \frac{m_l m_s}{n^3 l \left( l + \frac{1}{2} \right) (l + 1)}
\]

where \( B_0 \) is the characteristic magnetic field.

((Example))

The 2p state for the hydrogen. \( n = 2, l = 1, s = 1/2. m_l = 1, 0, -1. m_s = 1/2, -1/2. x = \frac{B}{B_0} \).

\[
k = \frac{1}{2} \frac{m_e c^2 \alpha^4}{\mu_B B_0}.
\]

We make a plot of \( \frac{\Delta E^{(1)}}{\mu_B B_0} \) as a function of \( x = \frac{B}{B_0} \);

\[
\frac{\Delta E^{(1)}}{\mu_B B_0} = (m_l + 2m_s) x + k \frac{m_l m_s}{n^3 l \left( l + \frac{1}{2} \right) (l + 1)}.
\]
Strong-field Zeeman structure of the $2p$ states of hydrogen. Solid lines show the Zeeman levels, while the dashed lines show the addition of the Zeeman contribution and the spin-orbit interaction. The quantum numbers indicate the uncoupled basis states. The vertical dashed line denotes $x = \frac{B}{B_0} = 1$.

$$|\phi_1\rangle = |m_i = 1, m_s = 1/2\rangle, \quad |\phi_2\rangle = |m_i = 0, m_s = 1/2\rangle,$$

$$|\phi_3\rangle = |m_i = 1, m_s = -1/2\rangle, \quad |\phi_4\rangle = |m_i = -1, m_s = 1/2\rangle,$$

$$|\phi_5\rangle = |m_i = 0, m_s = -1/2\rangle, \quad |\phi_6\rangle = |m_i = -1, m_s = -1/2\rangle.$$

REFERENCES


APPENDIX A. Landé g-factor
The magnetic moment is defined by

\[ \hat{\mu} = -\frac{\mu_B}{\hbar}(\hat{L} + 2\hat{S}) \]

The total angular momentum is

\[ \hat{J} = \hat{L} + \hat{S} \]

The expectation value of the \( m \)-th component of the magnetic moment \( \mu \) can be obtained from the projection theorem (decomposition theorem of the second kind, see Rose),

In

\[ \langle j, m | \hat{T}_q^{(1)} | j, m \rangle = \frac{\langle j \| \hat{J} \cdot \hat{T}^{(1)} \| j \rangle}{\hbar^2 j(j+1)} \langle j, m | \hat{T}_q | j, m \rangle \]

we put

\[ \hat{T}_q^{(1)} = \hat{\mu}_q, \quad \hat{J} \cdot \hat{T}^{(1)} = \hat{J} \cdot \hat{\mu} \]

Then we get

\[ \langle j, m | \hat{\mu}_q | j, m \rangle = \frac{\langle j \| \hat{J} \cdot \hat{\mu} \| j \rangle}{\hbar^2 j(j+1)} \langle j, m | \hat{T}_q | j, m \rangle \]

Now we have

\[ \hat{\mu} \cdot \hat{J} = -\frac{\mu_B}{\hbar}(\hat{L} + 2\hat{S}) \cdot (\hat{L} + \hat{S}) = -\frac{\mu_B}{\hbar} (\hat{L}^2 + 2\hat{S}^2 + 3\hat{L} \cdot \hat{S}) \cdot \]

and

\[ \hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{L} \cdot \hat{S} \]
or

\[ \hat{L} \cdot \hat{S} = \frac{\hat{J}^2 - \hat{L}^2 - \hat{S}^2}{2} \]

Then we get

\[ \hat{\mu} \cdot \hat{J} = \hat{J} \cdot \hat{\mu} = -\frac{\mu_B}{\hbar} [\hat{L}^2 + 2\hat{S}^2 + \frac{3}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)] \]

and

\[
< j \| \hat{J} \cdot \hat{\mu} \| j > = -\frac{\mu_B}{\hbar} \hbar^2 \{l(l+1) + 2s(s+1) + \frac{3}{2}[l(l+1) - l(l+1) - s(s+1)]\}
\]

Then the expectation value of the magnetic moment along the z axis is

\[
\langle j, m \| \hat{\mu}_0 \| j, m \rangle = \frac{< j \| \hat{J} \cdot \hat{\mu} \| j >}{\hbar^2 j(j+1)} \langle j, m \| \hat{J}_0 \| j, m \rangle
\]

\[= -\frac{\mu_B}{\hbar} \frac{m \hbar}{\hbar^2 j(j+1)} \hbar^2 \{l(l+1) + 2s(s+1) + \frac{3}{2}[l(l+1) - l(l+1) - s(s+1)]\}
\]

\[= -\frac{m \mu_B}{2 j(j+1)} [3(j(j+1) - l(l+1) + s(s+1)]
\]

since \( |j, m\rangle \) is a joint eigenstate of \( \hat{J}^2 \), \( \hat{L}^2 \), \( \hat{S}^2 \), and \( \hat{J}_0 = \hat{J}_z \) with eigenvalues \( \hbar^2 j(j+1) \), \( \hbar^2 l(l+1) \), \( \hbar^2 s(s+1) \), and \( \hbar m \), respectively.

Here we introduce the Landé g-factor as

\[ \hat{\mu} = -\frac{g_J \mu_B}{\hbar} \hat{J} \]

Then we have

\[ \langle j, m | \hat{\mu}_0 | j, m \rangle = -\frac{g_J \mu_B}{\hbar} \langle j, m | \hat{J}_0 | j, m \rangle = -\frac{g_J \mu_B}{\hbar} m \hbar = -m g_J \mu_B \]
and

\[
g_{j} = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)}
\]

**APPENDIX-B.** The expectation value of \( S_z \)

Since

\[
\hat{S} \cdot \hat{J} = \hat{S} \cdot (\hat{L} + \hat{S}) = \hat{S}^2 + \hat{L} \cdot \hat{S} = \hat{S}^2 + \frac{\hat{J}^2 - \hat{\mathcal{L}}^2 - \hat{S}^2}{2} = \frac{\hat{J}^2 - \hat{\mathcal{L}}^2 + \hat{S}^2}{2}
\]

we get the expectation of \( S_z \) as

\[
\langle j, m | \hat{S}_z | j, m \rangle = \frac{\langle j \| \hat{J} \cdot \hat{S} \| j \rangle}{\hbar^2 j(j+1)} \langle j, m | \hat{J}_0 | j, m \rangle \\
= \frac{\hbar m}{2\hbar^2 j(j+1)} \hbar^2 [j(j+1) - l(l+1) + s(s+1)] \\
= \frac{m\hbar}{2} \left[ 1 + \frac{s(s+1) - l(l+1)}{j(j+1)} \right]
\]

where we use the projection theorem.