Stern-Gerlach experiment for the angular momentum j = 1

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The Stern-Gerlach experiment can be performed on a variety of atoms or particles Suxh experiments always results in a finite number of discrete beams exiting the analyzer. For spin 1/2 particles, there are two output beams. For spin 1 particles, there are three output beams. The deflections are consistent with the magnetic moments from spin angular momentum components of $m\hbar$ with m = 1, 0, and -1. For an analyzer aligned along the z axis, the three output states are labelled as $|1,z\rangle$, $|0,z\rangle$, and $|-1,z\rangle$. Here we consider the Stern-Gerlach experiment with spin 1 particle.

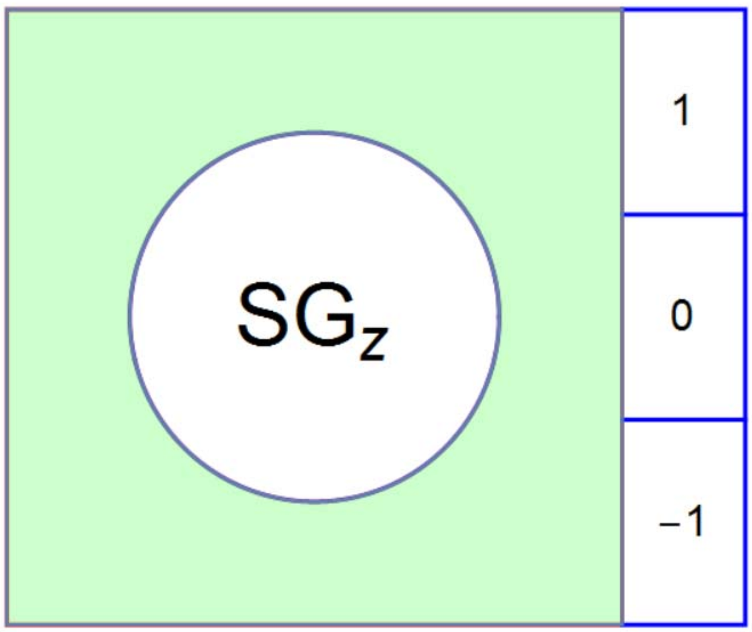


Fig. Spin-1 Stern-Gerlach experiment. The magnetic field is along the z axis.

1. $j = 1$

The angular momentum (3x3 matrices), under the basis of $|1,z\rangle$, $|0,z\rangle$, and $|-1,z\rangle$, can be expressed by

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where

$$|1,1\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

2. Stern-Gerlach experiment with $J = 1$

$$\hat{R}_y(\theta) = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & e^{i\phi}\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}.$$

Using the Mathematica, one can get the matrix representation

$$\hat{J}_{\boldsymbol{n}} = \hat{\boldsymbol{J}}\cdot\boldsymbol{n} = \hat{J}_x\cdot\boldsymbol{n}_x + \hat{J}_y\cdot\boldsymbol{n}_y + \hat{J}_z\cdot\boldsymbol{n}_z = \hbar\begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi i} & 0 \\ \frac{\sin\theta}{\sqrt{2}}e^{i\phi i} & 0 & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi i} \\ 0 & \frac{\sin\theta}{\sqrt{2}}e^{i\phi i} & -\cos\theta \end{pmatrix}.$$

The above result can be obtained by solving the eigenvalue problem:

$$(\hat{J}\cdot\boldsymbol{n})|1,\boldsymbol{n}\rangle = \hbar|1,\boldsymbol{n}\rangle, \quad (\hat{J}\cdot\boldsymbol{n})|0,\boldsymbol{n}\rangle = 0, \qquad (\hat{J}\cdot\boldsymbol{n})|-1,\boldsymbol{n}\rangle = -\hbar|-1,\boldsymbol{n}\rangle.$$

Use the Mathematica to obtain the eigenkets and the eigenvalues:
Eigensystem[$J_{\boldsymbol{n}}$]

$$|1,z\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |0,z\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |-1,z\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $\theta = \pi/2$ and $\phi = 0$ (corresponding to the x axis)

$$|1,x\rangle = \hat{R}_y(\frac{\pi}{2})|1,z\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2}|1,z\rangle + \frac{1}{\sqrt{2}}|0,z\rangle + \frac{1}{2}|-1,z\rangle,$$

$$|0,x\rangle = \hat{R}_y(\frac{\pi}{2})|0,z\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}|1,z\rangle + \frac{1}{\sqrt{2}}|-1,z\rangle,$$

$$|-1,x\rangle = \hat{R}_y(\frac{\pi}{2})|-1,z\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2}|1,z\rangle - \frac{1}{\sqrt{2}}|0,z\rangle + \frac{1}{2}|-1,z\rangle.$$

More generally for the unit vector $\boldsymbol{n}$ in the x-z plane,

$$|1,\boldsymbol{n}\rangle_{\boldsymbol{n}} = \frac{1+\cos\theta}{2}|1,z\rangle + \frac{\sin\theta}{\sqrt{2}}|0,z\rangle + \frac{1-\cos\theta}{2}|-1,z\rangle,$$

$$|0,\boldsymbol{n}\rangle = -\frac{\sin\theta}{\sqrt{2}}|1,z\rangle + \cos\theta|0,z\rangle + \frac{\sin\theta}{\sqrt{2}}|-1,z\rangle,$$

$$|-1,\boldsymbol{n}\rangle = \frac{1-\cos\theta}{2}|1,z\rangle - \frac{\sin\theta}{\sqrt{2}}|0,z\rangle + \frac{1+\cos\theta}{2}|-1,z\rangle,$$

or, inversely

$$\begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}.$$

When $\theta = \pi/2$, this matrix is expressed by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix},$$

or

$$|1,z\rangle = \frac{1}{2}|1,x\rangle - \frac{1}{\sqrt{2}}|0,x\rangle + \frac{1}{2}|-1,x\rangle,$$

$$|0,z\rangle = \frac{1}{\sqrt{2}}|1,x\rangle - \frac{1}{2}|-1,x\rangle,$$

$$|-1,z\rangle = \frac{1}{2}|1,x\rangle + \frac{1}{\sqrt{2}}|0,x\rangle + \frac{1}{2}|-1,x\rangle,$$

more generally

$$|1,z\rangle = \frac{1+\cos\theta}{2}|1,\boldsymbol{n}\rangle - \frac{\sin\theta}{\sqrt{2}}|0,\boldsymbol{n}\rangle + \frac{1-\cos\theta}{2}|-1,\boldsymbol{n}\rangle,$$

$$|0,z\rangle = \frac{\sin\theta}{\sqrt{2}}|1,\boldsymbol{n}\rangle + \cos\theta|0,\boldsymbol{n}\rangle - \frac{\sin\theta}{\sqrt{2}}|-1,\boldsymbol{n}\rangle,$$

$$|-1,z\rangle = \frac{1-\cos\theta}{2}|1,\boldsymbol{n}\rangle + \frac{\sin\theta}{\sqrt{2}}|0,\boldsymbol{n}\rangle + \frac{1+\cos\theta}{2}|-1,\boldsymbol{n}\rangle.$$

For $\theta = \pi/2$ and $\phi = \pi/2$ (corresponding to the y axis)

$$\hat{R}|1,z\rangle = \begin{pmatrix} -i/2 \\ 1/\sqrt{2} \\ i/2 \end{pmatrix} = -i\begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix},$$

$$\hat{R}|0,z\rangle = \begin{pmatrix} i/\sqrt{2} \\ 0 \\ i/\sqrt{2} \end{pmatrix} = i\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix},$$

Conventionally we use

$$|1,y\rangle = \begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix},$$

$$|0,y\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix},$$

$$|-1,y\rangle = \begin{pmatrix} 1/2 \\ -i/\sqrt{2} \\ -1/2 \end{pmatrix}$$

or inversely,

$$|1,z\rangle = \frac{1}{2}|1,y\rangle + \frac{1}{\sqrt{2}}|0,y\rangle + \frac{1}{2}|-1,y\rangle$$

$$|0,z\rangle = -\frac{i}{\sqrt{2}}|1,y\rangle + \frac{i}{\sqrt{2}}|-1,y\rangle$$

$$|-1,z\rangle = -\frac{1}{2}|1,y\rangle + \frac{1}{\sqrt{2}}|0,y\rangle - \frac{1}{2}|-1,y\rangle$$

3. Calculation of the rotation matrix with J = 1 without the use of Mathematica

Taylor expansion:

$$\exp(-\frac{i}{\hbar}\theta\hat{J}_y) = 1 + \frac{1}{1!}(-\frac{i}{\hbar}\theta\hat{J}_y) + \frac{1}{2!}(-\frac{i}{\hbar}\theta\hat{J}_y)^2 + \frac{1}{3!}(-\frac{i}{\hbar}\theta\hat{J}_y)^3 + \frac{1}{4!}(-\frac{i}{\hbar}\theta\hat{J}_y)^4 + ..$$

where

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}.$$

Note that

$$\hat{J}_+|1,z\rangle = 0, \qquad \hat{J}_+|0,z\rangle = \sqrt{2}\hbar|1,z\rangle, \qquad \hat{J}_+|-1,z\rangle = \sqrt{2}\hbar|0,z\rangle,$$

$$\hat{J}_-|1,z\rangle = \sqrt{2}\hbar|0,z\rangle, \quad \hat{J}_-|0,z\rangle = \sqrt{2}\hbar|-1,z\rangle, \quad \hat{J}_-|-1,z\rangle = 0,$$

$$\hat{J}_y|1,z\rangle = \frac{i\hbar}{\sqrt{2}}|0,z\rangle, \quad \hat{J}_y|0,z\rangle = \frac{-i\hbar}{\sqrt{2}}(|1,z\rangle - |-1,z\rangle), \quad \hat{J}_y|-1,z\rangle = -\frac{i\hbar}{\sqrt{2}}|0,z\rangle,$$

$$\hat{J}_y = \hbar\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \qquad \hat{J}_y{}^2 = -\hbar^2\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

$$\hat{J}_y{}^3 = \hbar^2\hat{J}_y, \qquad \hat{J}_y{}^4 = \hat{J}_y{}^3\hat{J}_y = \hbar^2\hat{J}_y\hat{J}_y = \hbar^2\hat{J}_y{}^2,$$

$$\hat{J}_y{}^5 = \hat{J}_y{}^4\hat{J}_y = \hbar^2\hat{J}_y{}^2\hat{J}_y = \hbar^2\hat{J}_y{}^3 = \hbar^4\hat{J}_y$$

Therefore

$$\begin{aligned}
\exp(-\frac{i}{\hbar}\theta\hat{J}_y) &= \hat{1} + \frac{\hat{J}_y}{\hbar}[(-\theta) + \frac{1}{3!}(-i\theta)^3 + \frac{1}{5!}(-i\theta)^5 + ..] \\
&\quad + \frac{\hat{J}_y{}^2}{\hbar^2}[\frac{1}{2!}(-i\theta)^2 + \frac{1}{4!}(-i\theta)^4 + ..] \\
&= \hat{1} - \frac{\hat{J}_y}{\hbar}(i\sin\theta) + \frac{\hat{J}_y{}^2}{\hbar^2}(\cos\theta - 1) \\
&= \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}
\end{aligned}$$

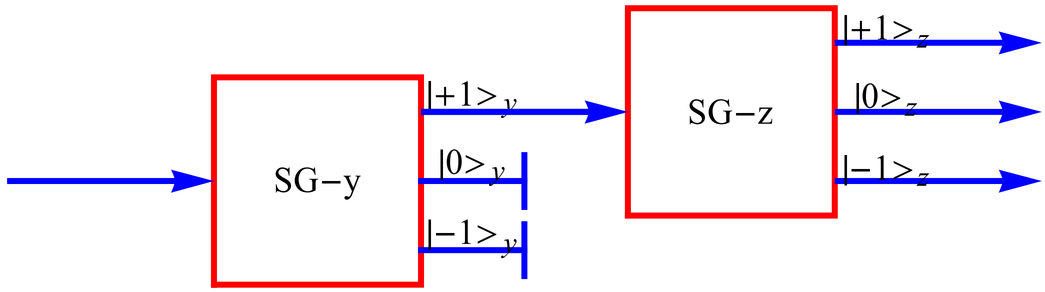
We also get

$$\exp(-\frac{i}{\hbar}\phi\hat{J}_z) = \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}.$$

which is a diagonal matrix.

4. Examples of SG experiments with $J = 1$

(1) A spin-1 particle exists an SG$_y$ device in a state with $S_y = \hbar$. The beam then enters an SG$_z$ device. What is the probability that the measurement of S_z yields the value 0, $+\hbar$, and $-\hbar$?



$$|1,y\rangle = \begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix} = \frac{1}{2}|1,z\rangle + \frac{i}{\sqrt{2}}|0,z\rangle + \frac{-1}{2}|-1,z\rangle .$$

The probability for finding the state $|1,z\rangle$ is

$$P_1 = |\langle 1,y|1,z\rangle|^2 = |\langle 1,z|1,y\rangle|^2 = \frac{1}{4}.$$

The probability for finding the state $|0,z\rangle$ is

$$P_1 = |\langle 1,y|0,z\rangle|^2 = |\langle 0,z|1,y\rangle|^2 = \frac{1}{2}.$$

The probability for finding the state $|-1,z\rangle$ is

$$P_{-1} = |\langle 1,y|-1,z\rangle|^2 = |\langle -1,z|1,y\rangle|^2 = \frac{1}{4}.$$

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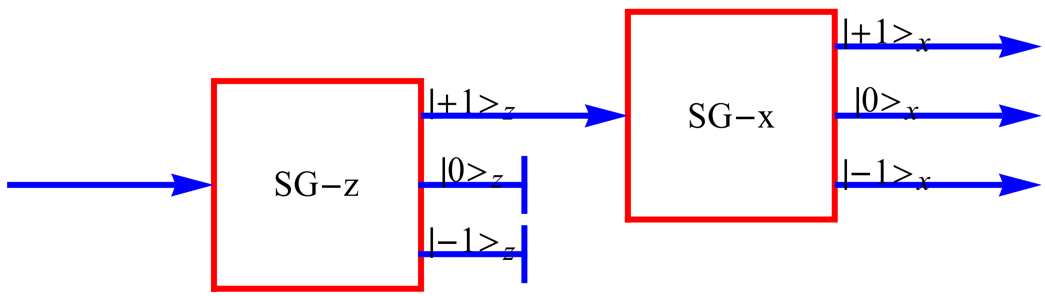
(2) ((Townsend 3.16)) A spin-1 particle exists an SG$_z$ device in a state with $S_z = \hbar$. The beam then enters an SG$_x$ device. What is the probability that the measurement of S_x yields the value 0, $+\hbar$, and $-\hbar$?



$$\left|0, x\right\rangle = -\frac{1}{\sqrt{2}}\left|1, z\right\rangle + \frac{1}{\sqrt{2}}\left|-1, z\right\rangle .$$

The probability for finding the state $\left|1, x\right\rangle$ is

$$P_1 = \left|\left\langle 0, x \middle| 1, z\right\rangle\right|^2 = \left|\left\langle 1, z \middle| 0, x\right\rangle\right|^2 = \frac{1}{2} .$$

The probability for finding the state $\left|0, x\right\rangle$ is

$$P_1 = \left|\left\langle 0, x \middle| 0, z\right\rangle\right|^2 = \left|\left\langle 0, z \middle| 0, x\right\rangle\right|^2 = 0 .$$

The probability for finding the state $\left|-1, x\right\rangle$ is

$$P_{-1} = \left|\left\langle 0, x \middle| -1, z\right\rangle\right|^2 = \left|\left\langle -1, z \middle| 0, x\right\rangle\right|^2 = \frac{1}{2} .$$

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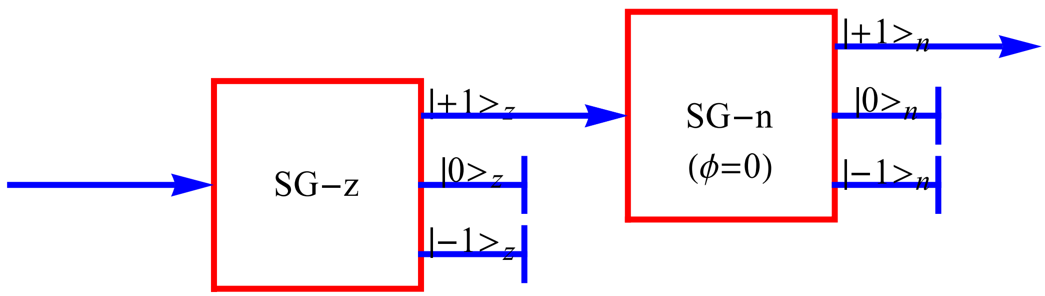
(3) ((Shankhar)) A beam of spin 1 particles, moving along the *y* axis, is incident on two collinear SG apparatuses, the first with *B* along the *z* axis and the second with *B* along the z' axis, which lies in the *x*-*z* plane at an angle θ relative to the *z* axis. Both apparatuses transmit only the uppermost beams. What fraction leaving the first will pass the second?

The intial state after passing the first SG_z, is $\left|1, z\right\rangle$



We note that

$$\left|1,\boldsymbol{n}\right\rangle = \frac{1+\cos\theta}{2}\left|1,z\right\rangle + \frac{\sin\theta}{\sqrt{2}}\left|0,z\right\rangle + \frac{1-\cos\theta}{2}\left|-1,z\right\rangle,$$

$$\left|0,\boldsymbol{n}\right\rangle = -\frac{\sin\theta}{\sqrt{2}}\left|1,z\right\rangle + \cos\theta\left|0,z\right\rangle + \frac{\sin\theta}{\sqrt{2}}\left|-1,z\right\rangle,$$

$$\left|-1,\boldsymbol{n}\right\rangle = \frac{1-\cos\theta}{2}\left|1,z\right\rangle - \frac{\sin\theta}{\sqrt{2}}\left|0,z\right\rangle + \frac{1+\cos\theta}{2}\left|-1,z\right\rangle,$$

The probability for finding the state $\left|1,\boldsymbol{n}\right\rangle$ is

$$P_1 = \left|\left\langle 1,\boldsymbol{n}\middle|1,z\right\rangle\right|^2 = \left|\left\langle 1,z\middle|1,\boldsymbol{n}\right\rangle\right|^2 = \frac{1}{4}(1+\cos\theta)^2.$$

The probability for finding the state $\left|0\right\rangle_n$ is

$$P_0 = \left|\left\langle 0,\boldsymbol{n}\middle|1,z\right\rangle\right|^2 = \left|\left\langle 1,z\middle|0,\boldsymbol{n}\right\rangle\right|^2 = \frac{1}{2}\sin^2\theta.$$

The probability for finding the state $\left|-1,\boldsymbol{n}\right\rangle$ is

$$P_{-1} = \left|\left\langle -1,\boldsymbol{n}\middle|1,z\right\rangle\right|^2 = \left|\left\langle 1,z\middle|-1,\boldsymbol{n}\right\rangle\right|^2 = \frac{1}{4}(1-\cos\theta)^2.$$

The total probability is

$$P_1 + P_0 + P_{-1} = 1.$$

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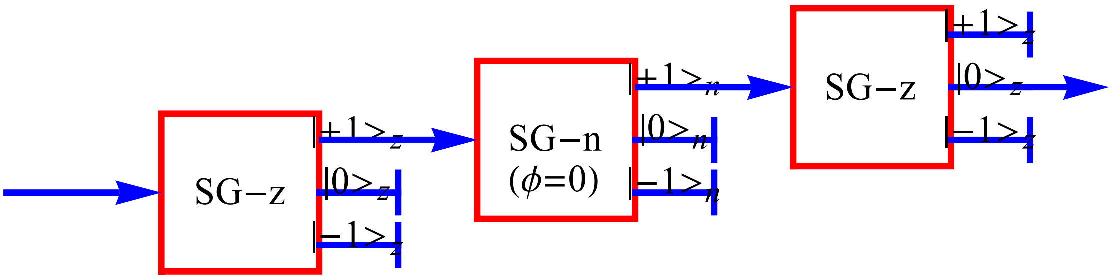
(4) ((Townsend 3.20)) A beam of spin-1 particle is sent through a series of three Stern-Gerlach measuring devices. The first SG$_z$ device transmits particles with $S_z = \hbar$ and

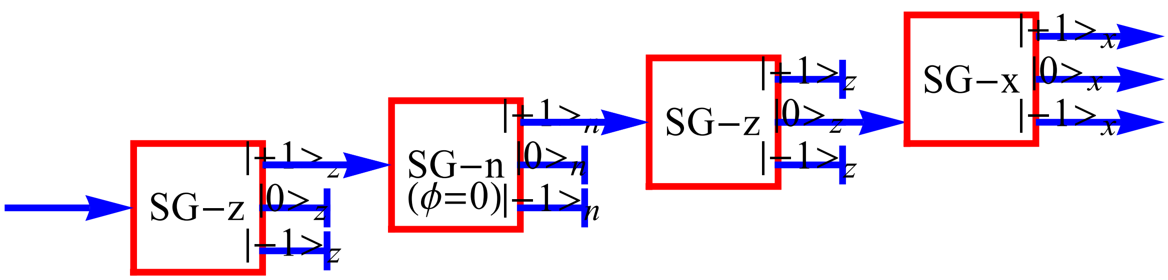
filters out particles with S_z = 0 and S_z = -$\hbar$. The second device, an SG_n device, transmits particles with S_n = $\hbar$ and filters out particles with S_n = 0 and S_n = - $\hbar$, where the axis *n* makes an angle θ ($0 \le \theta \le \pi/2$) in the *x*-*z* plane with respect to the *z* axis. A last SG_z device transmits particles with S_z = 0 and filters out particles with S_z = $\hbar$ and S_z = -$\hbar$.



(a) **What fraction of the particles transmitted by the first SG_z device will survive the third measurement?**

(b) **How must the angle θ of the SG_n device be oriented so as to maximize the number of particles that are transmitted by the final SG_z device? What fraction of the particles survive the third measurements for this value of θ?**

(c) **What fractions of the particles with S_x = $\hbar$, S_x =0, and S_x = -$\hbar$, respectively, survive after the fourth device, SG_x which device transmits particles with S_x = $\hbar$, S_x =0, and S_x = -$\hbar$?**



We note that

$$\left|1,\boldsymbol{n}\right\rangle = \frac{1+\cos\theta}{2}\left|1,z\right\rangle + \frac{\sin\theta}{\sqrt{2}}\left|0,z\right\rangle + \frac{1-\cos\theta}{2}\left|-1,z\right\rangle,$$

$$\left|0,\boldsymbol{n}\right\rangle = -\frac{\sin\theta}{\sqrt{2}}\left|1,z\right\rangle + \cos\theta\left|0,z\right\rangle + \frac{\sin\theta}{\sqrt{2}}\left|-1,z\right\rangle,$$

$$\left|-1,\boldsymbol{n}\right\rangle = \frac{1-\cos\theta}{2}\left|1,z\right\rangle - \frac{\sin\theta}{\sqrt{2}}\left|0,z\right\rangle + \frac{1+\cos\theta}{2}\left|-1,z\right\rangle,$$

$$\left|1,x\right\rangle = \frac{1}{2}\left|1,z\right\rangle + \frac{1}{\sqrt{2}}\left|0,z\right\rangle + \frac{1}{2}\left|-1,z\right\rangle,$$

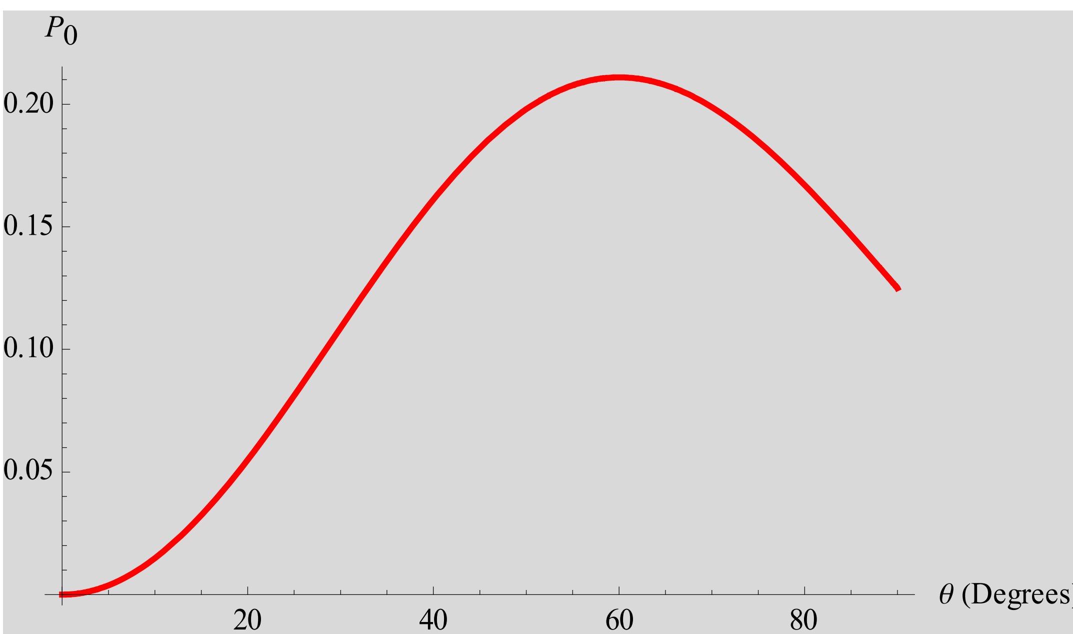
$$\left|0,x\right\rangle = -\frac{1}{\sqrt{2}}\left|1,z\right\rangle + \frac{1}{\sqrt{2}}\left|-1,z\right\rangle,$$

$$\left|-1,x\right\rangle = \frac{1}{2}\left|1,z\right\rangle - \frac{1}{\sqrt{2}}\left|0,z\right\rangle + \frac{1}{2}\left|-1,z\right\rangle.$$

(a)
The fraction of the particles transmitted by the first SG_z device will survive the third measurement is

$$P_0 = \left|\left\langle 1,z \middle| 1,\boldsymbol{n}\right\rangle\right|^2 \left|\left\langle 1,\boldsymbol{n} \middle| 0,z\right\rangle\right|^2 = \left|\left\langle 0,z \middle| 1,\boldsymbol{n}\right\rangle\right|^2 = \frac{\sin^2\theta(1+\cos\theta)^2}{8}.$$

(b)



$$\frac{dP_0}{d\theta} = 2\cos^5\frac{\theta}{2}\sin\frac{\theta}{2}(2\cos\theta - 1).$$

P_0 takes a maximum (= 0.210938) at $\theta = 60°$

(c)

The fractions of the particles with $S_x = \hbar$,

$$P_0 |\langle 0,z|1,x\rangle|^2 = \frac{P_0}{2}.$$

The fractions of the particles with $S_x = 0$,

$$P_0 |\langle 0,z|0,x\rangle|^2 = 0.$$

The fractions of the particles with $S_x = -\hbar$,

$$P_0 |\langle 0,z|-1,x\rangle|^2 = \frac{P_0}{2}.$$

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(5)
A spin-1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} = \frac{1}{\sqrt{14}}(|1,z\rangle + 2|0,z\rangle + 3i|-1,z\rangle).$$

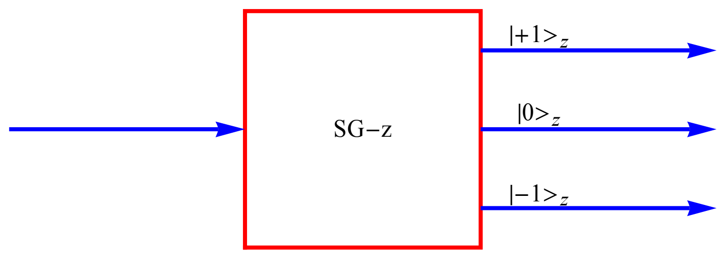
(a) What are the probabilities that a measurement of S_z will yield the values $\hbar$, 0, or - $\hbar$ for this state?
(b) What is $\langle S_z \rangle$?
(c) What is the probability that a measurement of S_x will yield the values $\hbar$, 0, or - $\hbar$ for this state?
(d) What is $\langle S_x \rangle$ for this state?

$$|1,x\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2}|1,z\rangle + \frac{1}{\sqrt{2}}|0,z\rangle + \frac{1}{2}|-1,z\rangle,$$

$$|0,x\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}|1,z\rangle + \frac{1}{\sqrt{2}}|-1,z\rangle,$$

$$\left|-1,x\right\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1.2 \end{pmatrix} = \frac{1}{2}\left|1,z\right\rangle - \frac{1}{\sqrt{2}}\left|0,z\right\rangle + \frac{1}{2}\left|-1,z\right\rangle .$$

(a) and (b)



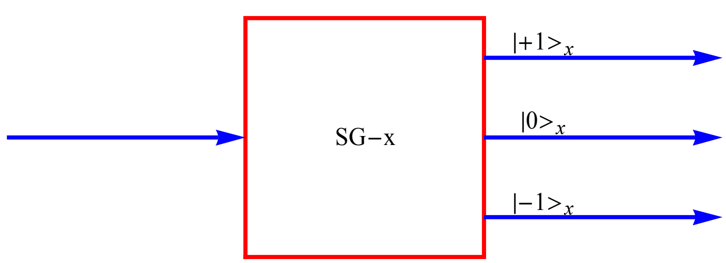
$$P(\left|1,z\right\rangle) = \left|\left\langle 1,z|\psi\right\rangle\right|^2 = \frac{1}{14},$$

$$P(\left|0,z\right\rangle) = \left|\left\langle 0,z|\psi\right\rangle\right|^2 = \frac{4}{14} = \frac{2}{7},$$

$$P(\left|-1,z\right\rangle) = \left|\left\langle -1,z|\psi\right\rangle\right|^2 = \frac{9}{14},$$

$$\begin{aligned}\left\langle S_z\right\rangle &= \hbar P(\left|1,z\right\rangle) + 0\hbar P(\left|0,z\right\rangle) - \hbar P(\left|-1,z\right\rangle) \\ &= \frac{\hbar}{14} - \frac{9\hbar}{14} = -\frac{8\hbar}{14} = -\frac{4\hbar}{7} = -0.57143\hbar\end{aligned}$$

(c) and (d)



$$P(\left|1,x\right\rangle) = \left|\left\langle 1,x|\psi\right\rangle\right|^2 = \frac{9+2\sqrt{2}}{28},$$

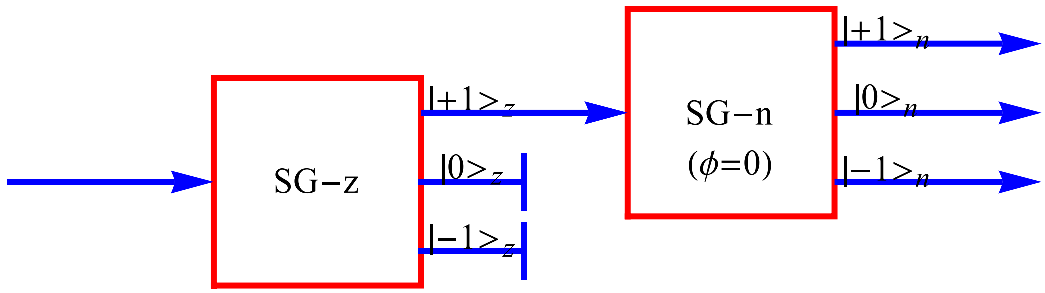
$$P(\left|0,x\right\rangle) = \left|\left\langle 0,x|\psi\right\rangle\right|^2 = \frac{5}{14},$$

$$P(|-1,x\rangle) = |\langle -1,x|\psi\rangle|^2 = \frac{9-2\sqrt{2}}{28},$$

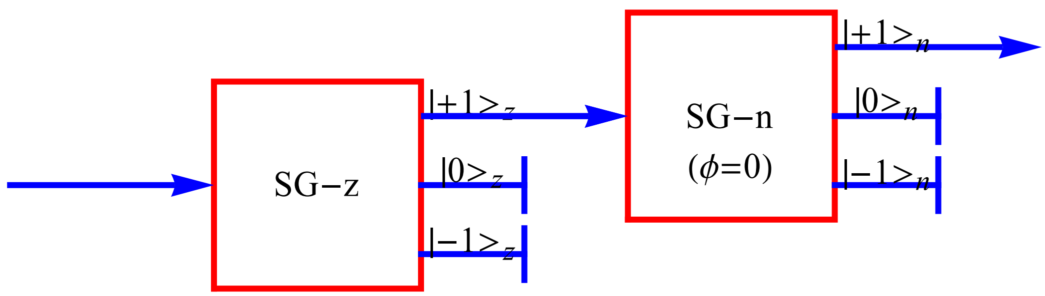
$$\begin{aligned}\langle S_x\rangle &= \hbar P(|1,x\rangle) + 0\hbar P(|0,x\rangle) - \hbar P(|-1,x\rangle) \\ &= \hbar[P(|1,x\rangle) - P(|-1,x\rangle)] \\ &= \frac{\sqrt{2}}{7}\hbar = 0.20231\hbar\end{aligned}$$

5. Feynman's thinking SG experiment

(1) Experiment-1

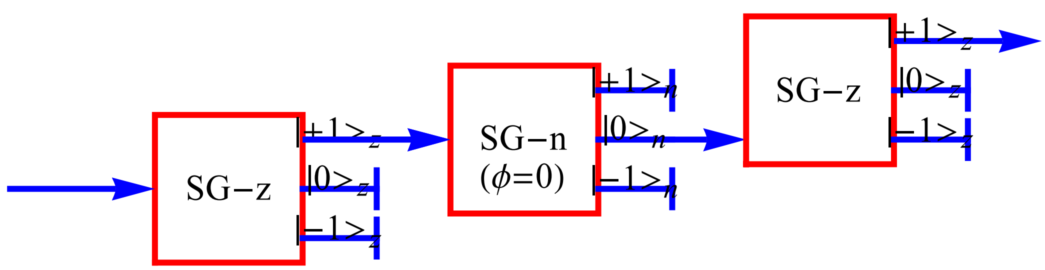


(2) Experiment-2

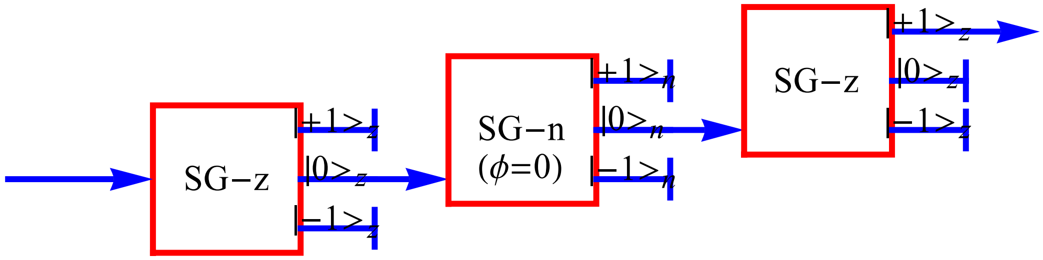


Two Stern-Gerlach type filters in series; the second is tilted at the angle θ from the z axis in the x-z plane.

(3) Experiment-3



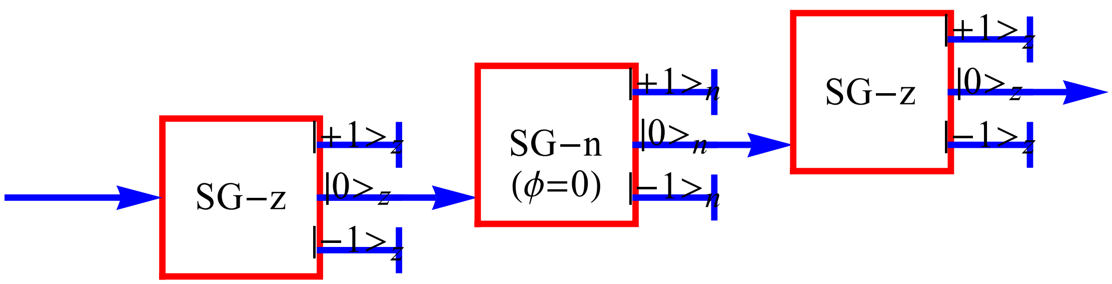
(4) Experiment 4



The probability that an atom that comes out of SG_z (the first) will also go through both $SG_{\mathbf{n}}$ and SGz (the second) is

$$P_4 = \left|\langle 1,z|0,\boldsymbol{n}\rangle\right|^2 \left|\langle 0,\boldsymbol{n}|0,z\rangle\right|^2 .$$

(5) Experiment-5



The probability that an atom that comes out of SG_z (the first) will also go through both $SG_{\mathbf{n}}$ and SG_z (the second) is

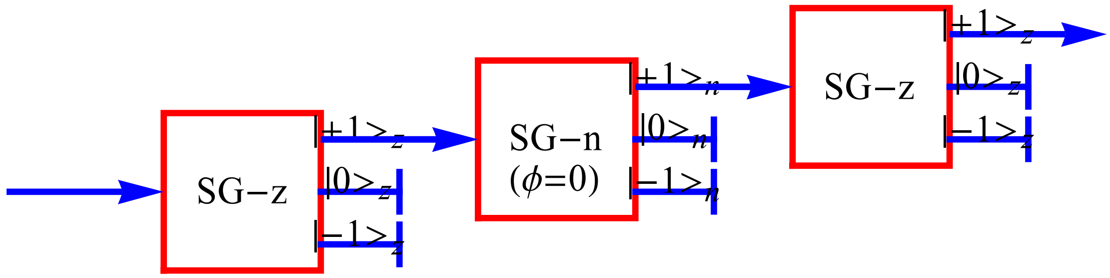
$$P_5 = \left|\langle 0,z|0,\boldsymbol{n}\rangle\right|^2 \left|\langle 0,\boldsymbol{n}|0,z\rangle\right|^2 .$$

Then we have

$$\frac{P_4}{P_5} = \frac{\left|\langle 1,z|0,\boldsymbol{n}\rangle\right|^2}{\left|\langle 0,z|0,\boldsymbol{n}\rangle\right|^2} = \frac{\left|\langle 1,z|0,\boldsymbol{n}\rangle\right|^2}{\left|\langle 0,z|0,\boldsymbol{n}\rangle\right|^2} = \frac{1}{2}\tan^2\theta .$$

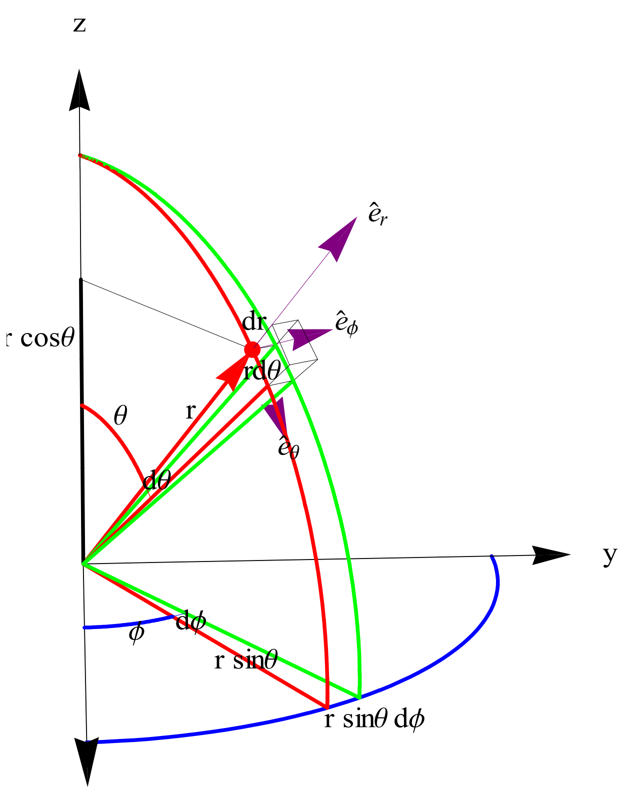
This ratio does not depend on which state is selected by the first SG_z.

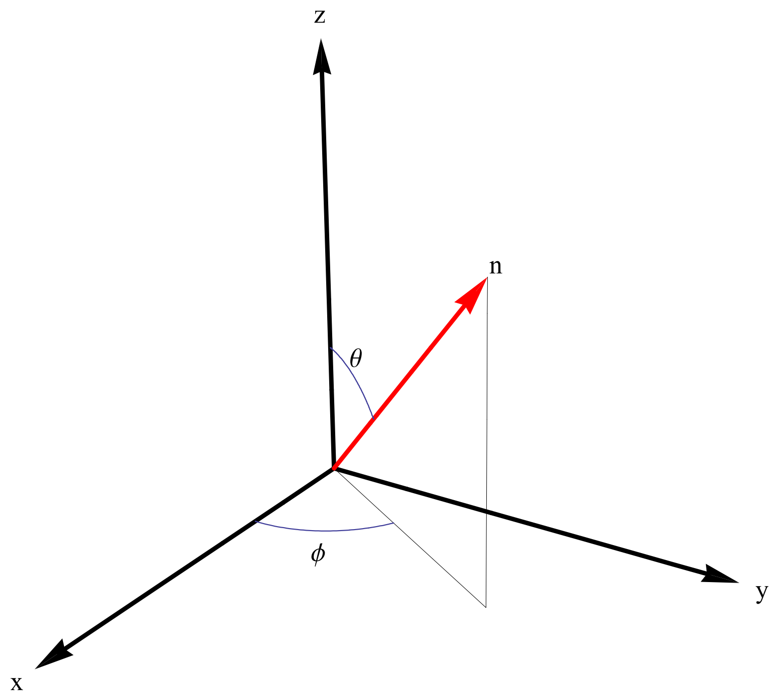
(6) Experiment-6



6. Rotation operator with $j = 1$

$$\boldsymbol{n} = (\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$





The rotation operator with $J = 1$ is given by

$$\hat{R} = D^{(1)}(\theta,\phi) = \begin{pmatrix} e^{-i\phi}(\frac{1+\cos\theta}{2}) & -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}} & e^{-i\phi}(\frac{1-\cos\theta}{2}) \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ e^{i\phi}(\frac{1-\cos\theta}{2}) & e^{i\phi}\frac{\sin\theta}{\sqrt{2}} & e^{i\phi}(\frac{1+\cos\theta}{2}) \end{pmatrix},$$

The eigenkets $\left|1,\boldsymbol{n}\right\rangle$, $\left|0,\boldsymbol{n}\right\rangle$, and $\left|-1,\boldsymbol{n}\right\rangle$ are obtained as

$$\left|1,\boldsymbol{n}\right\rangle = \hat{R}\left|1,z\right\rangle = \begin{pmatrix} \frac{1+\cos\theta}{2}e^{-i\phi} \\ \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2}e^{i\phi} \end{pmatrix},$$

$$\left|0,\boldsymbol{n}\right\rangle = \hat{R}\left|0,z\right\rangle = \begin{pmatrix} -\dfrac{\sin\theta}{\sqrt{2}}e^{-i\phi} \\ \cos\theta \\ \dfrac{\sin\theta}{\sqrt{2}}e^{i\phi} \end{pmatrix},$$

$$\left|-1,\boldsymbol{n}\right\rangle = \hat{R}\left|-1,z\right\rangle = \begin{pmatrix} \dfrac{1-\cos\theta}{2}e^{-i\phi} \\ -\dfrac{\sin\theta}{\sqrt{2}} \\ \dfrac{1+\cos\theta}{2}e^{i\phi} \end{pmatrix}.$$

For $\phi = 0$, the rotation operator is given by

$$\hat{R} = D^{(1)}(\theta,\phi) = \begin{pmatrix} \dfrac{1+\cos\theta}{2} & -\dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1-\cos\theta}{2} \\ \dfrac{\sin\theta}{\sqrt{2}} & \cos\theta & -\dfrac{\sin\theta}{\sqrt{2}} \\ \dfrac{1-\cos\theta}{2} & \dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1+\cos\theta}{2} \end{pmatrix},$$

and

$$\left|1,\boldsymbol{n}\right\rangle = \hat{R}\left|1,z\right\rangle = \begin{pmatrix} \dfrac{1+\cos\theta}{2} \\ \dfrac{\sin\theta}{\sqrt{2}} \\ \dfrac{1-\cos\theta}{2} \end{pmatrix},$$

$$\left|0,\boldsymbol{n}\right\rangle_n = \hat{R}\left|0,z\right\rangle = \begin{pmatrix} -\dfrac{\sin\theta}{\sqrt{2}} \\ \cos\theta \\ \dfrac{\sin\theta}{\sqrt{2}} \end{pmatrix},$$

$$\left|-1,\boldsymbol{n}\right\rangle = \hat{R}\left|-1,z\right\rangle = \begin{pmatrix} \dfrac{1-\cos\theta}{2} \\ -\dfrac{\sin\theta}{\sqrt{2}} \\ \dfrac{1+\cos\theta}{2} \end{pmatrix}.$$

((**Mathematica**))

Matrices j = 1

```
Clear["Global`*"];

Jx[ℓ_, n_, m_] := 1/2 Sqrt[(ℓ - m) (ℓ + m + 1)] KroneckerDelta[n, m + 1] +
  1/2 Sqrt[(ℓ + m) (ℓ - m + 1)] KroneckerDelta[n, m - 1]

Jy[ℓ_, n_, m_] := -1/2 ⅈ Sqrt[(ℓ - m) (ℓ + m + 1)] KroneckerDelta[n, m + 1] +
  1/2 ⅈ Sqrt[(ℓ + m) (ℓ - m + 1)] KroneckerDelta[n, m - 1]

Jz[ℓ_, n_, m_] := m KroneckerDelta[n, m]


Jx = Table[Jx[1, n, m], {n, 1, -1, -1}, {m, 1, -1, -1}];

Jy = Table[Jy[1, n, m], {n, 1, -1, -1}, {m, 1, -1, -1}];

Jz = Table[Jz[1, n, m], {n, 1, -1, -1}, {m, 1, -1, -1}];

Ry[θ_] := MatrixExp[-ⅈ Jy θ] // Simplify

Rz[ϕ_] := MatrixExp[-ⅈ Jz ϕ] // Simplify

Rz[ϕ].Ry[θ] // MatrixForm
```

$$\begin{pmatrix} e^{-i\phi}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{e^{-i\phi}\,\mathrm{Sin}[\theta]}{\sqrt{2}} & e^{-i\phi}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2 \\ \frac{\mathrm{Sin}[\theta]}{\sqrt{2}} & \mathrm{Cos}[\theta] & -\frac{\mathrm{Sin}[\theta]}{\sqrt{2}} \\ e^{i\phi}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{e^{i\phi}\,\mathrm{Sin}[\theta]}{\sqrt{2}} & e^{i\phi}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

```
u1 = Rz[ϕ].Ry[θ].{1, 0, 0} // Simplify
```

$$\left\{e^{-i\phi}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]^2,\ \frac{\mathrm{Sin}[\theta]}{\sqrt{2}},\ e^{i\phi}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2\right\}$$

```
u2 = Rz[ϕ].Ry[θ].{0, 1, 0} // Simplify
```

$$\left\{-\frac{e^{-i\phi}\,\mathrm{Sin}[\theta]}{\sqrt{2}},\ \mathrm{Cos}[\theta],\ \frac{e^{i\phi}\,\mathrm{Sin}[\theta]}{\sqrt{2}}\right\}$$

```
u3 = Rz[ϕ].Ry[θ].{0, 0, 1} // Simplify
```

$$\left\{e^{-i\phi}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2,\ -\frac{\mathrm{Sin}[\theta]}{\sqrt{2}},\ e^{i\phi}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]^2\right\}$$

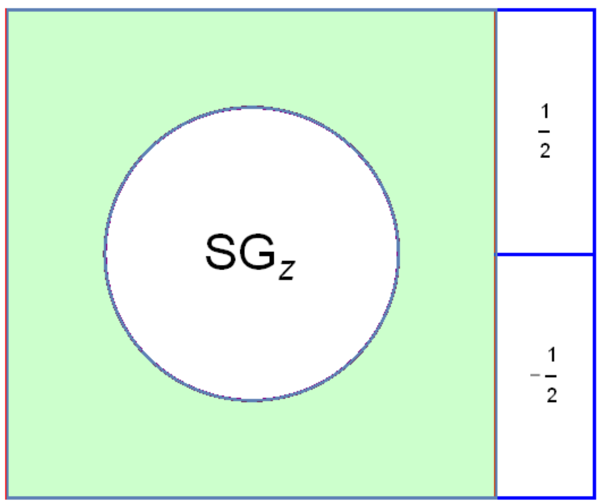
REFERENCES

1. R.P. Feynman, R.,B. Leighton, and M. Sands, *The Feynman Lectures in Physics*, 6th edition (Addison Wesley, Reading Massachusetts, 1977).
2. J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics 2nd Edition, (Pearson, 2011).
3. J.S. Townsend, A Modern Approach to Quantum Mechanics, 2nd edition (University Science Books, 2012).
4. D.H. McInTyre, Quantum mechanics: A Paradigms Approach (Pearson, 2012).

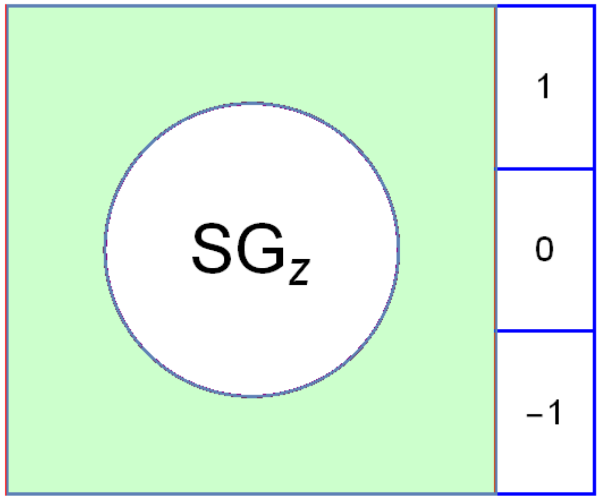
APPENDIX

Schematic diagram of the Stern-Gerlach experiments with spin 1/2, 1, 3/2, 2, and 5/2.

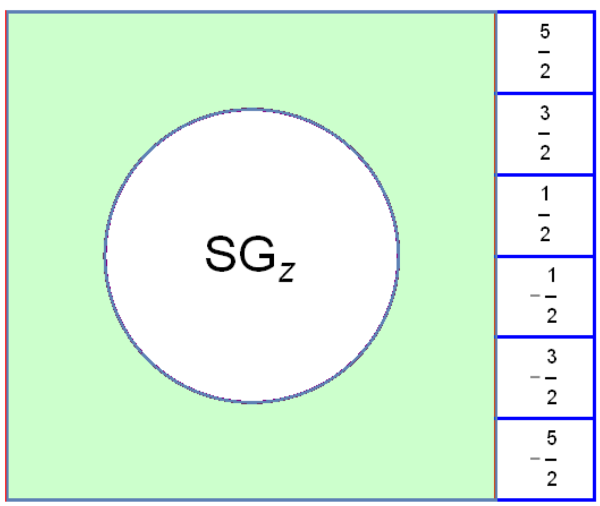
S=1/2



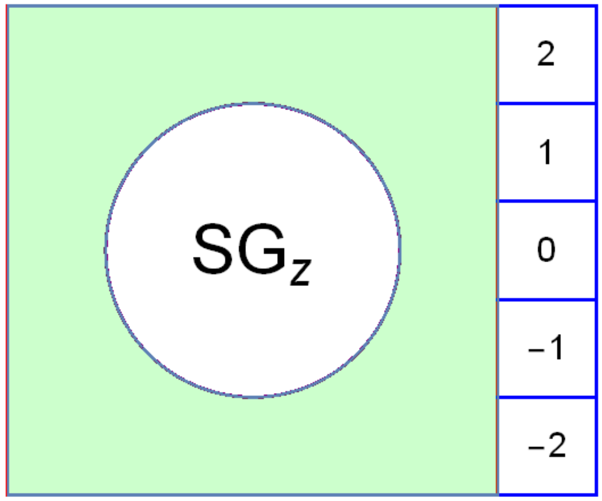
$S = 1$



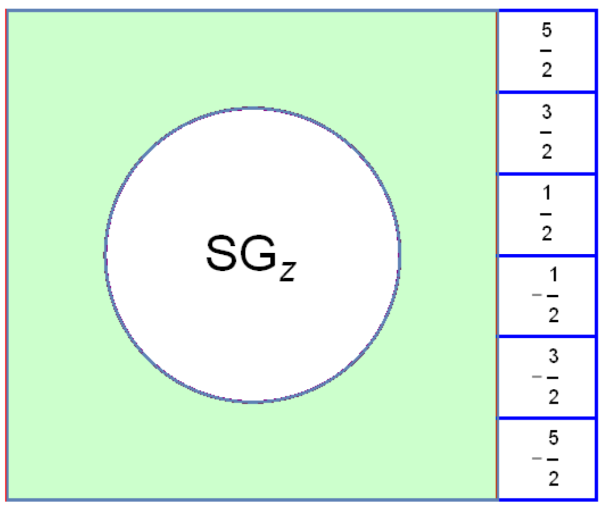
S=3/2



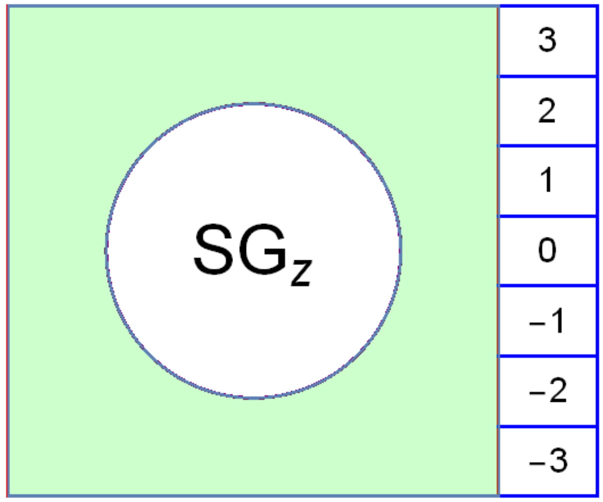
S = 2



$S = 5/2$



$S = 3$



((**Mathematica**)) Schematic diagram for $S = 4$. The number of states is $N = 2S + 1 = 9$.

```
Clear["Gobal`"]; N1 = 9;
f0 = Graphics[{Text[Style["SGz", Black, 40], {0.5, 0.5}]}];
f1 =
 Graphics[
  {Red, Thick, Line[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}],
   Blue, Line[{{1, 0}, {1.2, 0}, {1.2, 1}, {1, 1}}],
   Table[Line[{{1, k / N1}, {1.2, k / N1}}], {k, 1, N1 - 1}],
   Table[Text[Style["" <> ToString[StandardForm[k]],
      Black, 20], {1.1, (1/(2 N1) + k/N1) + (-1 + N1)/(2 N1)}],
    {k, -(N1 - 1)/2, (N1 - 1)/2, 1}]}];
f2 = ParametricPlot[{0.5 + 0.3 Cos[θ], 0.5 + 0.3 Sin[θ]},
  {θ, 0, 2 π}, PlotStyle → {Purple, Thick}];
f3 =
 RegionPlot[0 < x < 1 && 0 < y < 1 && (x - 0.5)^2 + (y - 0.5)^2 > 0.3^2,
  {x, 0, 1}, {y, 1, 0}, PlotStyle → {Green, Opacity[0.2]}];
Show[f0, f1, f2, f3]
```