Neutron Interferometry Experiment

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It is known that the operator for rotation through 2π radians for a fermion causes a reversal of the sign of the wave function. Only after two full rotations through 4π does the system come back to its original value. In the early days of quantum mechanics, people reassured themselves that in the experimentally accessible probabilities $\langle\psi|\psi\rangle$ the phase factor cancels, hence there seemed to be no reason to worry about this strange property. Aharonov and Suskind (1967) and independently, Berstein (1967) predicted the possibility of observation of the 4p symmetry for interferometric experiments, and Eder and Zeilinger (1967) gave the theoretical framework for the neutron interferometric realization.

It can be confirmed from the experiment using the precession of 2π radians in a magnetic field using a neutron interferometer (Werner et al. and Rauch et al. (1975)]. Here we discuss the detail of the experiment, a 2π rotation of spin 1/2 system, based on the textbook of Sakurai and Napolitano, Modern Quantum Mechanics (2011, revised version).

1. Spin and magnetic moment of neutron

The neutron spin and the magnetic moment. The neutron is a fermion with spin 1/2. The spin operators are given by

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x, \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y, \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z .$$

The neutron wave function can be expressed by

$$|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle .$$

The neutron has a magnetic moment

$$\hat{\boldsymbol{\mu}}_n = \gamma_n \mu_N \hat{\boldsymbol{\sigma}} = \gamma_n \mu_N \frac{2}{\hbar}\hat{\boldsymbol{S}} ,$$

where

$$\gamma_n = -1.9130427(5).$$

and μ_N is the nuclear magneton.

$$\mu_N = \frac{e\hbar}{2m_p c} = 5.0507832413 \times 10^{-24} \text{ emu},$$

where m_p is the rest mass of proton. So the magnitude of the magnetic moment is

$$|\mu_n| = |\gamma_n|\mu_N = 9.662347055(71)\times 10^{-24}$$ emu.

where emu = erg/Oe. Here we use the expression for the nuclear magnetic moment as

$$\hat{\boldsymbol{\mu}}_n = -|\mu_n|\frac{2}{\hbar}\hat{\boldsymbol{S}}.$$

2. Rotation operator

We now consider the rotation operator around the u axis by α for the spin 1/2 system (neutron), which is given by

$$\hat{R}_u(\alpha) = e^{-i\frac{\alpha(\hat{\boldsymbol{\sigma}}\cdot\mathbf{u})}{2}} = \hat{1}\cos\frac{\alpha}{2} - i(\hat{\boldsymbol{\sigma}}\cdot\mathbf{u})\sin\frac{\alpha}{2}.$$

When $\alpha = 2\pi$,

$$\hat{R}_u(2\pi) = e^{-i\pi(\hat{\sigma}\cdot\boldsymbol{u})} = \hat{1}\cos\pi - i(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u})\sin\pi = -\hat{1}.$$

For any ket $|\psi\rangle$, we have

$$\hat{R}_u(2\pi)|\psi\rangle = -|\psi\rangle.$$

The ket for the 2π rotated state differs from the original ket by a minus sign. When $|\psi\rangle = |\pm z\rangle$, we have

$\hat{R}_u(2\pi)|+z\rangle = -|+z\rangle$, and $\hat{R}_u(2\pi)|-z\rangle = -|-z\rangle$.

We also note that

$\hat{R}_u(4\pi)|+z\rangle = |+z\rangle$, and $\hat{R}_u(4\pi)|-z\rangle = |-z\rangle$.

3. Magnetic moment of neutron

Here we have some comments on the magnetic moment of neutron.

Spin of neutron: $I = \frac{1}{2}$.

Gyromagnetic ratio:

$$\gamma_n = \frac{\mu_n}{\hbar I} = -1.83247171(43)\times 10^4 \ (\text{s}^{-1}\text{G}^{-1}).$$

Nuclear magneton is defined as

$$\mu_N = \frac{e\hbar}{2m_p c} = 5.05079 \times 10^{-24} \text{ emu.}$$

where emu=erg/G, m_p is the mass of proton and emu=erg/G. The magnetic moment of neutron is

$$\mu_n = \hbar I \gamma_n = \frac{1}{2}\hbar\gamma_n = -9.66236 \text{ x } 10^{-24} \text{ emu}$$

or

$$\mu_n = -1.91304\mu_N = \frac{1}{2} g \mu_N$$

The g-factor is

$$g = -3.82608545(90) .$$

4 Neutron interferometry experiment

A beam of thermal neutrons is split into two parts, path A-B-D and path A-C-D. That path A-B-D goes through a magnetic-field-free region. In contrast, the path A-C-D enters a small region where a static magnetic field is present. A variable phase shift is obtained by having the neutrons of the path A-C-D pass through a uniform magnetic field over a distance l.

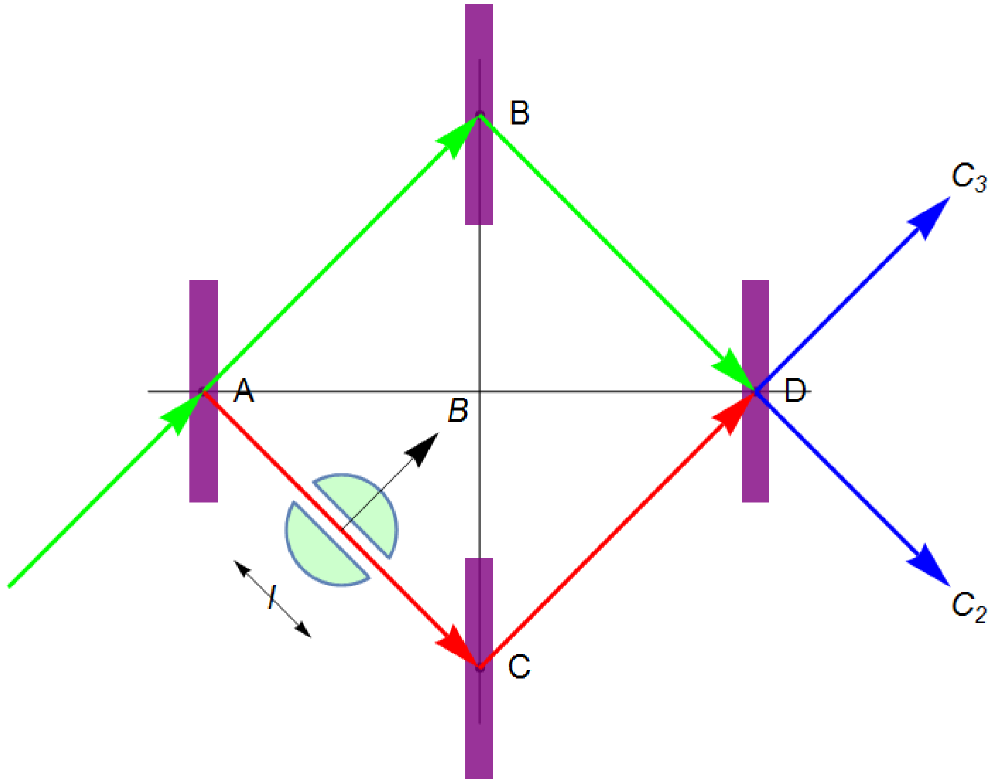


Fig. A schematic diagram of the neutron interferometer. On the path AC the neutrons are in a magnetic field B (0 to 500 Oe) for a distance l (= 2 cm).

The spin Hamiltonian $\hat{H}$ of the neutron in the presence of a magnetic field $\boldsymbol{B}$ (//z)

$$\hat{H} = -\hat{\boldsymbol{\mu}}_n \cdot \mathbf{B} = -\frac{\hbar}{2}\gamma_n B\hat{\sigma}_z = 1.91304\mu_N B\hat{\sigma}_z .$$

We now consider the time dependence of the ket vector $|\psi(t)\rangle$,

$$|\psi(t)\rangle = \hat{U}(t)|\psi(t=0)\rangle ,$$

where the time evolution operator $\hat{U}(t)$ is given by

$$\hat{U}(\Delta t) = e^{-\frac{i}{\hbar}\hat{H}\Delta t} = e^{-\frac{i}{\hbar}1.91304\mu_N B\Delta t\hat{\sigma}_z} .$$

We define the angle ϕ as

$$\phi = \frac{2}{\hbar}1.91304\mu_N B\Delta t = \frac{1}{\hbar}|g|\mu_N B\Delta t = 18324.7 B\Delta t$$

where $|g| = 3.82608545(90)$, and

$$\frac{1}{\hbar}|g|\mu_N = 18324.7 \ \ (\mathrm{G}^{-1}\mathrm{s}^{-1}).$$

The time Δt during which the neutron passes the magnetic-field region, is given by

$$\Delta t = \frac{l}{v}.$$

Note that the momentum of the neutron is given from the de Broglie relation,

$$p = m_n v = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda},$$

When $t = \Delta t$, the unitary operator $\hat{U}(t)$ is equivalent to the rotation operator $\hat{R}_z(\phi)$,

$$\hat{U}(t) = \hat{R}_z(\phi) = e^{-i\frac{\varphi\hat{\sigma}_z}{2}} = \hat{1}\cos\frac{\phi}{2} - i\hat{\sigma}_z\sin\frac{\phi}{2}.$$

since

$$\begin{aligned}
e^{-i\frac{\phi\hat{\sigma}_z}{2}} &= e^{-i\frac{\phi\hat{\sigma}_z}{2}}(|+z\rangle\langle+z| + |-z\rangle\langle-z|) \\
&= e^{-i\frac{\phi}{2}}|+z\rangle\langle+z| + e^{i\frac{\phi}{2}}|-z\rangle\langle-z| \\
&= \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \\
&= \cos\frac{\phi}{2}\hat{1} - i\sin\frac{\phi}{2}\hat{\sigma}_z
\end{aligned}$$

Then we have

$$\hat{R}_z(\phi)|+z\rangle = e^{-i\frac{\phi\hat{\sigma}_z}{2}}|+z\rangle = \cos\frac{\phi}{2}|+z\rangle - i\sin\frac{\phi}{2}\hat{\sigma}_z|+z\rangle = e^{-i\frac{\phi}{2}}|+z\rangle,$$

and

$$\hat{R}_z(\phi)\left|-z\right\rangle = e^{-i\frac{\phi\hat{\sigma}_z}{2}}\left|-z\right\rangle = \cos\frac{\phi}{2}\left|-z\right\rangle - i\sin\frac{\phi}{2}\hat{\sigma}_z\left|-z\right\rangle = e^{i\frac{\phi}{2}}\left|+\right\rangle.$$

For any ket given by

$$\left|\psi\right\rangle = C_1\left|+z\right\rangle + C_2\left|-z\right\rangle,$$

we have

$$\hat{R}_z(\phi)\left|\psi\right\rangle = e^{-i\frac{\phi\sigma_z}{2}}\left|\psi\right\rangle = C_1 e^{-i\frac{\phi}{2}}\left|+z\right\rangle + C_2 e^{i\frac{\phi}{2}}\left|-z\right\rangle.$$

The appearance of the half-angle $\phi/2$ has an extremely interesting consequence.

The state $\left|\chi\right\rangle$ in the region D is given by

$$\left|\chi\right\rangle = \left|\psi\right\rangle + \hat{R}_z(\phi)\left|\psi\right\rangle.$$

The intensity is proportional to $\left\langle\chi\middle|\chi\right\rangle$,

$$\left\langle\chi\middle|\chi\right\rangle = (\left\langle\psi\right| + \left\langle\psi\right|\hat{R}_z^{+}(\phi))(\left|\psi\right\rangle + \hat{R}_z(\phi)\left|\psi\right\rangle) = 2 + \left\langle\psi\right|\hat{R}_z^{+}(\phi) + \hat{R}_z(\phi)\left|\psi\right\rangle,$$

where $\left\langle\psi\middle|\psi\right\rangle = 1$.

$$\hat{R}_z(\phi) = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}, \qquad \hat{R}_z^{+}(\phi) = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix},$$

$$\hat{R}_z(\phi) + \hat{R}_z^{+}(\phi) = \begin{pmatrix} 2\cos\frac{\phi}{2} & 0 \\ 0 & 2\cos\frac{\phi}{2} \end{pmatrix} = 2\cos\frac{\phi}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the formula, we have

$$\hat{R}_z(\phi) = \hat{1}\cos\frac{\phi}{2} - i\hat{\sigma}_z\sin\frac{\phi}{2}, \qquad \hat{R}_z^{+}(\phi) = \hat{1}\cos\frac{\phi}{2} + i\hat{\sigma}_z\sin\frac{\phi}{2}$$

we have directly get

$$\hat{R}_z(\phi)+\hat{R}_z^{+}(\phi)=\hat{1}(2\cos\frac{\phi}{2})$$

Then we have

$$\langle\chi|\chi\rangle=2+2\cos\frac{\phi}{2}=4\cos^2\frac{\phi}{4}.$$

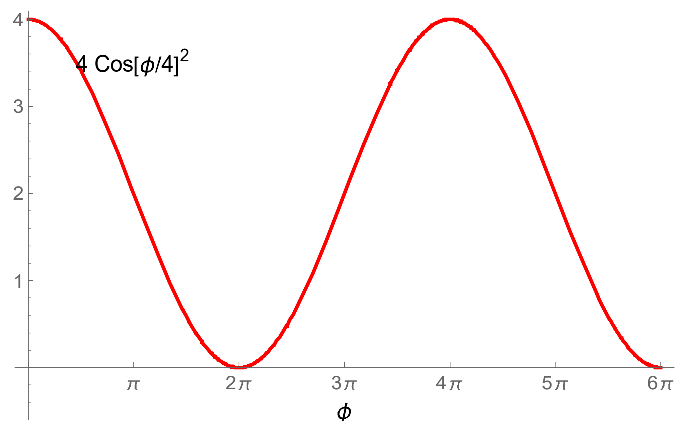


Fig. Plot of $\langle\chi|\chi\rangle=2+2\cos\frac{\phi}{2}=4\cos^2\frac{\phi}{4}$ as a function of the phase difference ϕ.

The period of ϕ is

$$\phi=4\pi\ .$$

We now calculate the value of the magnetic field when $\phi=4\pi$ from the relation

$$\phi=\frac{2}{\hbar}1.91304\mu_N B\Delta t=\frac{1}{\hbar}|g|\mu_N B\Delta t\ ,$$

Note that

$$\frac{1}{\hbar}|g|\mu_N=18324.7\ \ (\mathrm{G^{-1}s^{-1}})$$

((Experimental result)) Werner et al. (1975)

Here we need to know the value of l (length of the msgnrtic field region) and velocity of neutron. Since the wavelength of neutron is given by $\lambda = 1.445$ Å (Rausch and Werner), using the de Broglie relation ($p = mv = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$, we get the velocity as

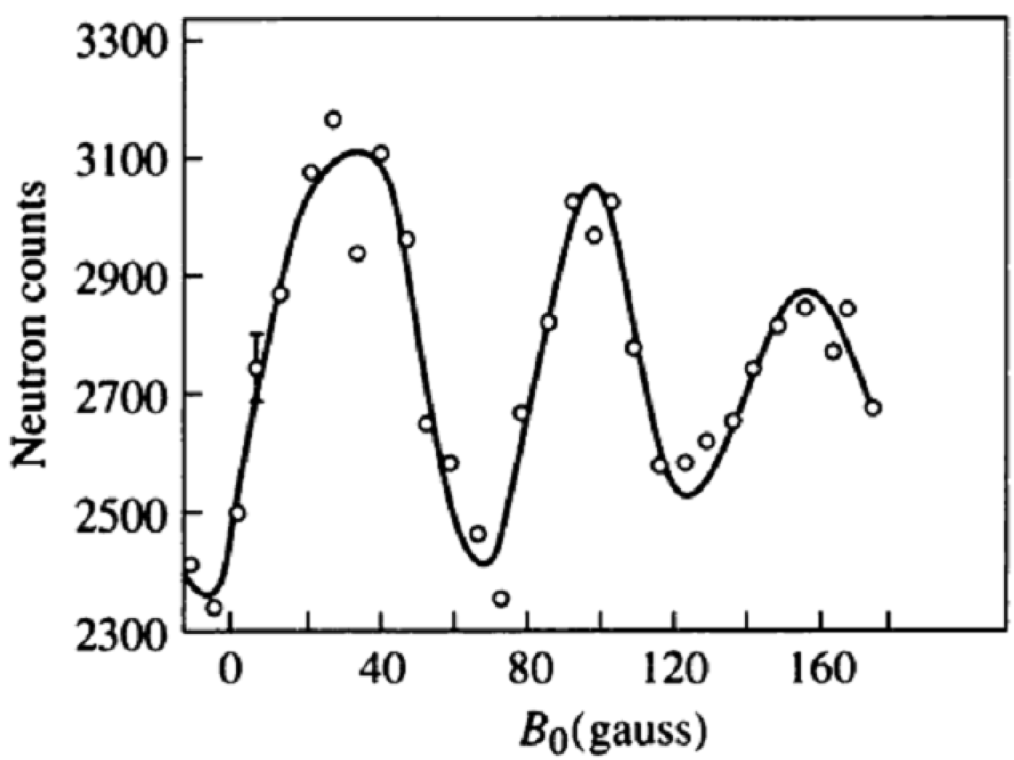
$$v = \frac{h}{m_n \lambda} = 2.73774 \text{ km/s.}$$

where m_n is the mass of neutron. Note that in thermal equilibrium at 290 K, the most probable velocity of neutron can be evaluated as

$$v_{mp} = \sqrt{\frac{2k_B T}{m_n}} = 2.1865 \text{ km/s.}$$

When $l = 2.0$ cm, the time Δt for the neutron passes through the region where the magnetic field is applied, is

$$\Delta t = \frac{l}{v} = 7.305\, \mu\text{s.}$$



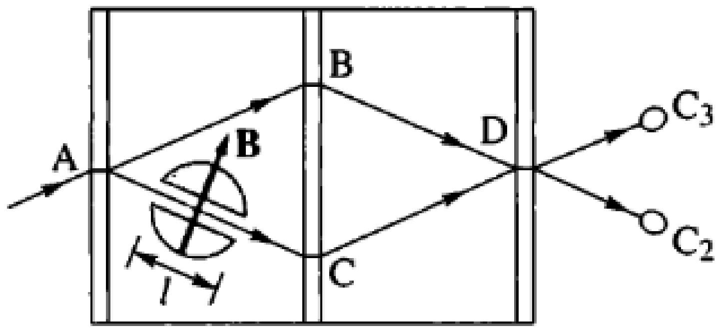


Fig. The difference in counts between the counters C_3 and C_2 as a function of B (Werner et al., 1975) From the textbook of Townsend (p.121), J.S. Townsend, A Modern Approach to Quantum Mechanics, 2nd edition (University Science Books. 2012).

5. Numerical calculation

When $\phi = 4\pi$

$$B\Delta t = \frac{4\pi}{18324.7} = 6.85762 \times 10^{-4} \text{ (G s)} \tag{1}$$

which is universal constant.

When $\Delta t = 7.305 \mu s$ ($\lambda = 1.445$ Å, $l = 2.0$ cm), the value of B can be evaluated as

$B = 93.6$ Oe,

This evaluated value of B is relatively larger than the experimental value

$B = 62 \pm 2$ Oe.

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   gn → 1.9130427, mn → 1.674927211 × 10^-24,
   qe → 4.8032068 × 10^-10, Å → 10^-8,  μn → 9.66237055 × 10^-24,
   neV → 1.602176487 × 10^-12 10^-9};

ω1 = (2 μn / ℏ) B /. rule1

18324.7 B

f1 = ω1 / (2 π) /. rule1

2916.47 B

ϵ1 = (ℏ ω1 / 2) / neV /. rule1

0.00603078 B

a = (4 π c ℏ) / (qe gn (Å / 2 π)) /. rule1

27.5252

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f1 /. B → 165 // ScientificForm

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ϵ1 /. B → 165

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