Projection operator

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The outer product $|\psi_1\rangle\langle\psi_2|$ is not a number. It is an operator. A special case is the outer product of a ket with its corresponding bra vector, $|\psi\rangle\langle\psi|$. Assuming that $|\psi\rangle$ is normalized, this operator is called a projection operator. The projection operator can be said to project a vector onto the direction defined by $|\psi\rangle$. We discuss the properties of the projection operator.

1. Adjoint operator

The adjoint operator of

$$\hat{A} = |\psi_1\rangle\langle\psi_2|,$$

is given by

$$\hat{A}^{+} = |\psi_2\rangle\langle\psi_1|.$$

((**Proof**))
From the definition, we have

$$\langle\beta|\hat{A}^{+}|\alpha\rangle = \langle\alpha|\hat{A}|\beta\rangle^{*}.$$

Here we note that

$$\langle\alpha|\hat{A}|\beta\rangle^{*} = (\langle\alpha|\psi_1\rangle\langle\psi_2|\beta\rangle)^{*} = \langle\psi_2|\beta\rangle^{*}\langle\alpha|\psi_1\rangle^{*} = \langle\beta|\psi_2\rangle\langle\psi_1|\alpha\rangle.$$

Then we have

$$\hat{A}^{+} = |\psi_2\rangle\langle\psi_1|.$$

2. Properties of projection operator

The projection operator is defined as

$$\hat{P}_1 = |\psi_1\rangle\langle\psi_1|,$$

where $|\psi_1\rangle$ is normalized; $\langle\psi_1|\psi_1\rangle = 1$

(a)

$$\hat{P}_1^+ = |\psi_1\rangle\langle\psi_1| = \hat{P}_1,$$

which means that $\hat{P}_1$ is the Hermitian operator.

$$\hat{P}_1|\psi_1\rangle = |\psi_1\rangle\langle\psi_1|\psi_1\rangle = |\psi_1\rangle .$$

(b)

$$\hat{P}_1^2 = \hat{P}_1|\psi_1\rangle\langle\psi_1| = |\psi_1\rangle\langle\psi_1| = \hat{P}_1 ,$$

or

$$\hat{P}_1^2 = \hat{P}_1 .$$

In general, we have

$$\hat{P}_1 = |\psi_1\rangle\langle\psi_1| , \qquad \hat{P}_1|\psi_1\rangle = |\psi_1\rangle .$$

It follows that for any vector $|\psi\rangle$

$$\hat{P}_1^2|\psi\rangle = \hat{P}_1|\psi_1\rangle\langle\psi_1|\psi\rangle = |\psi_1\rangle\langle\psi_1|\psi\rangle = \hat{P}_1|\psi\rangle ,$$

or

$$\hat{P}_1^2 = \hat{P}_1 .$$

The identity operator is a simple example of a projection operator, since

$$\hat{I}^+ = \hat{I} , \qquad \hat{I}^2 = \hat{I} .$$

(c)

$$\hat{P}_i = |\psi_i\rangle\langle\psi_i|, \qquad \hat{P}_j = |\psi_j\rangle\langle\psi_j|.$$

where $i \neq j$. Hence

$$\hat{P}_i\hat{P}_j|\psi\rangle = \hat{P}_i|\psi_j\rangle\langle\psi_j|\psi\rangle = 0,$$

$$\hat{P}_j\hat{P}_i|\psi\rangle = \hat{P}_j|\psi_i\rangle\langle\psi_i|\psi\rangle = 0.$$

(d) The product of two commuting projection operators $\hat{P}_1$ and $\hat{P}_2$, is also a projection operator, since

$$(\hat{P}_1\hat{P}_2)^+ = \hat{P}_2^{\,+}\hat{P}_1^{\,+} = \hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_2,$$

$$(\hat{P}_1\hat{P}_2)^2 = \hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2 = \hat{P}_1^{\,2}\hat{P}_2^{\,2} = \hat{P}_1\hat{P}_2.$$

(e)

$$|\psi\rangle = \sum_{i=1}^{n} c_i|\phi_i\rangle,$$

$$\sum_{i=1}^{n}\hat{P}_i|\psi\rangle = \sum_{i=1}^{n}|\phi_i\rangle\langle\phi_i|\psi\rangle = \sum_{i=1}^{n}c_i|\phi_i\rangle = |\psi\rangle,$$

where

$$\hat{P}_i = |\phi_i\rangle\langle\phi_i|.$$

Hence

$$\sum_{i=1}^{n}\hat{P}_i = \hat{1}. \qquad \text{(closure relation)}$$

(f)
$|a_n\rangle$ is an eigenket of $\hat{A}$ with the eigenvalue a_n,

$$\hat{A}|a_n\rangle = a_n|a_n\rangle .$$

Then we have

$$\hat{A} = \sum_n \hat{A}|a_n\rangle\langle a_n| = \sum_n a_n|a_n\rangle\langle a_n| .$$

3. Measurement

Suppose we measure the physical quantity *A*. The eigenstate of the operator $\hat{A}$ is given by

$$\hat{A}|\psi_n\rangle = \lambda_n|\psi_n\rangle .$$

The initial state is given by $|\psi\rangle$. After the measurement we have

$$\hat{P}_n|\psi\rangle = |\psi_n\rangle\langle\psi_n|\psi\rangle ,$$

with the probability as

$$|\langle\psi_n|\psi\rangle|^2 .$$

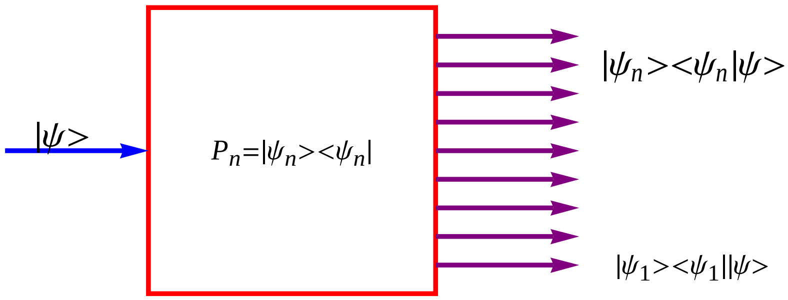

Fig. Projection operator. $|\psi_n\rangle$ is the eigenket of the operator $\hat{A}$ corresponding to the measurement of physical quantity *A*.

4. Examples of projection operator (2x2 matrix)

$$\hat{P}_+ = |+z\rangle\langle +z| , \qquad \hat{P}_- = |-z\rangle\langle -z| ,$$

where

$$\hat{P}_+ + \hat{P}_- = \hat{1}. \qquad \text{(Closure relation)}$$

The matrix representation of $\hat{P}_+$,

$$\hat{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix representation of $\hat{P}_-$,

$$\hat{P}_- = 1 - \hat{P}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

5. Property of the projection operator:

$$\hat{P}_+|+z\rangle = |+z\rangle\langle +z|+z\rangle = |+z\rangle, \qquad \hat{P}_+|-z\rangle = |+z\rangle\langle +z|-z\rangle = 0.$$

$$\hat{P}_-|+z\rangle = |-z\rangle\langle -z|+z\rangle = 0, \qquad .\hat{P}_-|-z\rangle = |-z\rangle\langle -z|-z\rangle = |-z\rangle$$

This means that

(i) $|+z\rangle$ is the eigenket of $\hat{P}_+$ with the eigenvalue 1, $|-z\rangle$ is the eigenket of $\hat{P}_+$ with the eigenvalue 0.

(ii) $|+z\rangle$ is the eigenket of $\hat{P}_-$ with the eigenvalue 0, $|-z\rangle$ is the eigenket of $\hat{P}_+$ with the eigenvalue 1.

$$\hat{P}_+^2 = \hat{P}_+|+z\rangle\langle +z| = |+z\rangle\langle +z| = \hat{P}_+,$$

$$\hat{P}_-^2 = \hat{P}_-|-z\rangle\langle -z| = |-z\rangle\langle -z| = \hat{P}_-,$$

$$\hat{P}_+\hat{P}_- = \hat{P}_+|-z\rangle\langle -z| = 0,$$

$$\hat{P}_-\hat{P}_+ = \hat{P}_-|+z\rangle\langle +z| = 0,$$

6. Eigenvalue problems for the projection operator

((**Mathematica**))

```
Clear["Global`*"]; P1 = (1 0
                         0 0); P2 = (0 0
                                     0 1);
eq1 = Eigensystem[P1]

{{1, 0}, {{1, 0}, {0, 1}}}

ψ1 = Normalize[eq1[[2, 1]]]

{1, 0}

ψ0 = Normalize[eq1[[2, 2]]]

{0, 1}

eq2 = Eigensystem[P2]

{{1, 0}, {{0, 1}, {1, 0}}}

ϕ1 = Normalize[eq2[[2, 1]]]

{0, 1}

ϕ0 = Normalize[eq2[[2, 2]]]

{1, 0}
```

7. Definition of Outer[Times,…] in the Mathematica

We assume that

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_n \end{pmatrix}, \qquad \langle\psi| = \begin{pmatrix} a_1^* & a_2^* & a_3^* & . & . & . & a_n^* \end{pmatrix}.$$

Then we have the matrix of the projection operator as

$$|\psi\rangle\langle\psi| = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_n \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* & a_3^* & . & . & . & a_n^* \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_1^* & a_1 a_2^* & a_1 a_3^* . & . & . & . & a_1 a_n^* \\ a_2 a_1^* & a_2 a_2^* & a_2 a_3^* . & . & . & . & a_2 a_n^* \\ a_3 a_1^* & a_3 a_2^* & a_3 a_3^* & . & . & . & a_3 a_n^* \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_n a_1^* & a_n a_2^* & a_n a_3^* . & . & . & . & a_n a_n^* \end{pmatrix}$$

We note that the matrix of the projection operator can be calculated using the Mathematica as
(a)

$$|\psi\rangle\langle\psi| \rightarrow \qquad |\psi\rangle . \langle\psi| ,$$

where we use the dot mark "." between $|\psi\rangle$ and $\langle\psi|$, we use the standard form of $|\psi\rangle$ and $\langle\psi|$ as

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ . \\ a_n \end{pmatrix}, \qquad \langle\psi| = (a_1^* \quad a_2^* \quad a_3^* \quad . \quad . \quad . \quad a_n^*) .$$

(b)

$$|\psi\rangle\langle\psi| \rightarrow \qquad \text{Outer[Times, } |\psi\rangle, \ \langle\psi| \text{]}$$

where we use

$$|\psi\rangle \rightarrow \{a_1 \quad a_2 \quad a_3 \quad . \quad . \quad . \quad a_n\},$$

$$\langle\psi| \to \{a_1^* \quad a_2^* \quad a_3^* \quad . \quad . \quad . \quad a_n^*\}.$$

((**Mathematica**))

```
Clear["Global`*"];

ψ1 = {a1, a2, a3, a4, a5}; ψ2 = {b1, b2, b3, b4, b5};

A1 = Outer[Times, ψ1, ψ2] // Simplify // MatrixForm
```

$$\begin{pmatrix} a1\,b1 & a1\,b2 & a1\,b3 & a1\,b4 & a1\,b5 \\ a2\,b1 & a2\,b2 & a2\,b3 & a2\,b4 & a2\,b5 \\ a3\,b1 & a3\,b2 & a3\,b3 & a3\,b4 & a3\,b5 \\ a4\,b1 & a4\,b2 & a4\,b3 & a4\,b4 & a4\,b5 \\ a5\,b1 & a5\,b2 & a5\,b3 & a5\,b4 & a5\,b5 \end{pmatrix}$$

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8. Examples: Projection operators

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-x\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The projection operator

(a)

$$|+z\rangle\langle+z| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(10) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad |-z\rangle\langle-z| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(01) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$|+z\rangle\langle+z| + |-z\rangle\langle-z| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b)

$$|+x\rangle\langle+x| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \qquad |-x\rangle\langle-x| = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},$$

$$|+x\rangle\langle+x| + |-x\rangle\langle-x| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

(c)

$$|+y\rangle\langle+y| = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}, \qquad |-y\rangle\langle-y| = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix},$$

$$|+y\rangle\langle+y| + |-y\rangle\langle-y| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

((Mathematica))

```
Clear["Global`*"];
expr_* := expr /. Complex[a_, b_] :→ Complex[a, -b];

ψxp = 1/√2 {1, 1}; ψxn = 1/√2 {1, -1}; ψyp = 1/√2 {1, ⅈ};
ψyn = 1/√2 {1, -ⅈ}; ψzp = {1, 0};
ψzn = {0, 1};

Pxp = Outer[Times, ψxp, ψxp*] // Simplify;
Pxn = Outer[Times, ψxn, ψxn*] // Simplify;
Pyp = Outer[Times, ψyp, ψyp*] // Simplify;
Pyn = Outer[Times, ψyn, ψyn*] // Simplify;
Pzp = Outer[Times, ψzp, ψzp*] // Simplify;
Pzn = Outer[Times, ψzn, ψzn*] // Simplify;
```

```
Pzp // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

```
Pzn // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Pzp + Pzn // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Pxp // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

```
Pxn // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

```
Pxp + Pxn // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Pyp // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

```
Pyn // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

```
Pyp + Pyn // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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9. Spin opertors

$$\begin{aligned}\hat{S}_z &= \hat{S}_z(|+z\rangle\langle +z| + |-z\rangle\langle -z|) \\ &= \frac{\hbar}{2}[|+z\rangle\langle +z| - |-z\rangle\langle -z|] \\ &= \frac{\hbar}{2}(\hat{P}_{+z} - \hat{P}_{-z})\end{aligned}$$

$$\begin{aligned}\hat{S}_x &= \hat{S}_x(|+x\rangle\langle +x| + |-x\rangle\langle -x|) \\ &= \frac{\hbar}{2}[|+x\rangle\langle +x| - |-x\rangle\langle -x|] \\ &= \frac{\hbar}{2}(\hat{P}_{+x} - \hat{P}_{-x})\end{aligned}$$

$$\begin{aligned}\hat{S}_y &= \hat{S}_y(|+y\rangle\langle +y| + |-y\rangle\langle -y|) \\ &= \frac{\hbar}{2}[|+y\rangle\langle +y| - |-y\rangle\langle -y|] \\ &= \frac{\hbar}{2}(\hat{P}_{+y} - \hat{P}_{-y})\end{aligned}$$

$$
\begin{aligned}
\hat{\boldsymbol{S}} \cdot \boldsymbol{n} &= \frac{\hbar}{2}(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}) \\
&= \frac{\hbar}{2}(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n})[|+\boldsymbol{n}\rangle\langle+\boldsymbol{n}| + |-\boldsymbol{n}\rangle\langle-\boldsymbol{n}|] \\
&= \frac{\hbar}{2}[|+\boldsymbol{n}\rangle\langle+\boldsymbol{n}| - |-\boldsymbol{n}\rangle\langle-\boldsymbol{n}|] \\
&= \frac{\hbar}{2}(\hat{P}_{+\boldsymbol{n}} - \hat{P}_{+\boldsymbol{n}})
\end{aligned}
$$