Local Realism and GHZ states

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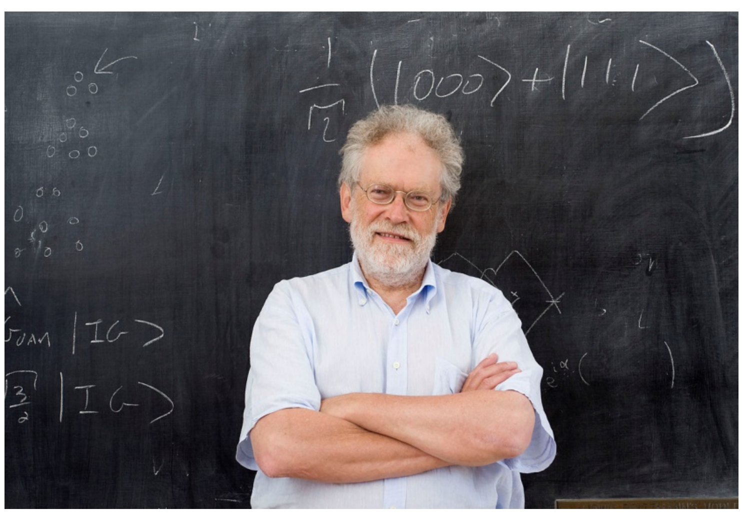
For many years, everyone thought that Bell had basically exhausted the subject by considering all really interesting situations, and two-spin systems provides the most spectacular quantum violations of local realism. It therefore came as a surprise to many when in 1989 Greenberger, Hone, and Zeilinger (GHZ) showed that systems containing more than two correlated particles may actually exhibit even more dramatic violations of local realism. They involve a sign contradiction (100 % violation) for perfect correlations, while the CHSH inequalities are violated about 40 % (Tsirelson bound = 2 for the CHSH inequality) and deal with situations where the results of measurements are not completely correlated. (F. Laloë, Do we really understand Quantum Mechanics?, Cambridge, 2012).

Anton Zeilinger (born on 20 May 1945) is an Austrian quantum physicist who in 2008 received the Inaugural Isaac Newton Medal of the Institute of Physics (UK) for "his pioneering conceptual and experimental contributions to the foundations of quantum physics, which have become the cornerstone for the rapidly-evolving field of quantum information". Zeilinger is professor of physics at the University of Vienna and Senior Scientist at the Institute for Quantum Optics and Quantum Information IQOQI at the Austrian Academy of Sciences. Most of his research concerns the fundamental aspects and applications of quantum entanglement.



http://en.wikipedia.org/wiki/Anton_Zeilinger

1. Element of reality

We consider the decay of a simple system into a pair of spin 1/2 particles such as

$$\pi^0 \to e^+ + e^- ,$$

where e^+ is a positron and e^- is an electron. After the decay products have separated and are very far apart, we measure a component of the spin of one of them. This is the entangled state.

Suppose that $\hat{S}_z$ of the electron is measured by Alice using the SGz device with *B*//*z* and is found to be equal to $\hbar/2$. Then Alice can be sure that $\hat{S}_z$ of positron will turn out equal to $-\hbar/2$, if Bob measures it, since the positron and electron form the entangled state.

Next we consider the different situation. Alice measures the eigenvalue of $\hat{S}_z$ for the electron by using her SGz device with *B*//*z*. She finds that the eigenvalue of $\hat{S}_z$ for electron is equal to $\hbar/2$. Suppose that Bob measure the eigenvalue of $\hat{S}_x$ for the positron by using his SGx with *B*//*x*, instead of measuring with the SGz device. What is the eigenvalue of $\hat{S}_x$ for the positron measured by Bob?

According to quantum mechanics, the probability of finding the state $|+x\rangle$ is the same as that of finding the state $|-x\rangle$. The probability is equal to 1/2 for each case, since

$$P = |\langle +x|-z\rangle|^2 = |\langle -x|-z\rangle|^2 = \frac{1}{2},$$

with

$$|+x\rangle = \frac{1}{\sqrt{2}}[|+z\rangle + |-z\rangle], \qquad |-x\rangle = \frac{1}{\sqrt{2}}[|+z\rangle - |-z\rangle].$$

We note that the spin operators $\hat{S}_z$ and $\hat{S}_x$ for the positron are not commutable; $[\hat{S}_z, \hat{S}_x] \neq 0$. Thus the eigenvalue of $\hat{S}_x$ cannot be determined definitely, even if the eigenvalue of $\hat{S}_z$ for the positron can be determined uniquely as $-\hbar/2$ because of the entangled state.

In the element of reality as defined by EPR theory (local theory), it is assume that all the spin operators are commutable. So all three components of the spin for the positron can be predictable with certainty, if we measure the corresponding spin component of the positron. This claim, however, is incompatible with quantum mechanics, which asserts that at most one spin component of each particle may be definite.

2. Local realism and quantum mechanics at odds with the use of mathematics

We consider two spin 1/2 particles, far apart from each other, in a singlet state. The Bell's state (singlet, spin zero) is given by

$$\left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|-z\right\rangle_2 - \left|-z\right\rangle_1\left|+z\right\rangle_2).$$

We know that measurements of $\hat{\sigma}_{1x}$ and $\hat{\sigma}_{2x}$, if performed, shall yield opposite values, that we denote by m_{1x} and m_{2x}, respectively.

$$m_{1x} = -m_{2x},$$

where m_{1x} and m_{2x} are either 1 or -1. Likewise, measurements of $\hat{\sigma}_{1y}$ and $\hat{\sigma}_{2y}$, if performed, shall yield opposite values, that we denote by m_{1y} and m_{2y}, respectively.

$$m_{1y} = -m_{2y}.$$

where m_{1y} and m_{2y} are either 1 or -1. Furthermore, since $\hat{\sigma}_{1x}$ and $\hat{\sigma}_{2y}$ commute, and both correspond to elements of reality, their product $\hat{\sigma}_{1x}\hat{\sigma}_{2y}$ also corresponds to an element of reality. The numerical value assigned to the product $\hat{\sigma}_{1x}\hat{\sigma}_{2y}$ is the product of the individual numerical values, $m_{1x}m_{2y}$. Likewise, the numerical value assigned to the product $\hat{\sigma}_{1y}\hat{\sigma}_{2x}$ is the product of the individual numerical values, $m_{1y}m_{2x}$.These two products must be equal, since

$$m_{1x}m_{2y} = (-m_{2x})(-m_{2y}) = m_{2x}m_{2y},$$

We note that the quantum mechanics asserts that the singlet state (Bell's state) satisfies

$$(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \otimes \hat{\sigma}_{2x})\left|\Phi^{(-)}\right\rangle_{12} = 0.$$

The proof of this equation will be given later. From this equation, we can predict with certainty that if we measure $(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \otimes \hat{\sigma}_{2x})$, we have

$$m_{1x}m_{2y} + m_{1y}m_{2x} = 0,$$

where each operator corresponds to an EPR element of reality.

We note that this equation is totally inconsistent with the equation derived above based on EPR element of reality; $m_{1x}m_{2y} = m_{1y}m_{2x}$

3. Mathematica

Here, using the Mathematica, we show that the singlet state (Bell's state) satisfies the following relations

$$(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \otimes \hat{\sigma}_{2x}) \left| \Phi^{(-)} \right\rangle_{12} = 0,$$

$$(\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2z} + \hat{\sigma}_{1z} \otimes \hat{\sigma}_{2y}) \left| \Phi^{(-)} \right\rangle_{12} = 0,$$

$$\left(\hat{\sigma}_{1z} \otimes \hat{\sigma}_{2x} + \hat{\sigma}_{1x} \otimes \hat{\sigma}_{2z}\right) \left| \Phi^{(-)} \right\rangle_{12} = 0,$$

and

$$\left(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x}\right) \left| \Phi^{(-)} \right\rangle_{12} = -\left| \Phi^{(-)} \right\rangle_{12},$$

$$(\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2y}) \left| \Phi^{(-)} \right\rangle_{12} = -\left| \Phi^{(-)} \right\rangle_{12},$$

$$\left(\hat{\sigma}_{z1} \otimes \hat{\sigma}_{z2}\right) \left| \Phi^{(-)} \right\rangle_{12} = -\left| \Phi^{(-)} \right\rangle_{12}.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :→ Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
σx = PauliMatrix[1]; σy = PauliMatrix[2];
σz = PauliMatrix[3];
χ =
  1/√2
   (KroneckerProduct[ψ1, ψ2] -
     KroneckerProduct[ψ2, ψ1]);

A12 =
 (KroneckerProduct[σx, σy] +
   KroneckerProduct[σy, σx]);
A12.χ

{{0}, {0}, {0}, {0}}
```

```
A23 =
 (KroneckerProduct[σy, σz] +
   KroneckerProduct[σz, σy]);
A23.χ
```

```
{{0}, {0}, {0}, {0}}
```

```
A31 =
 (KroneckerProduct[σx, σz] +
   KroneckerProduct[σz, σx]);
A31.χ
```

```
{{0}, {0}, {0}, {0}}
```

```
(KroneckerProduct[σx, σx]).χ + χ
```

```
{{0}, {0}, {0}, {0}}
```

```
(KroneckerProduct[σy, σy]).χ + χ
```

```
{{0}, {0}, {0}, {0}}
```

```
(KroneckerProduct[σz, σz]).χ + χ
```

```
{{0}, {0}, {0}, {0}}
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4. GHZ state $\left|\psi_{GHZ}^{-}\right\rangle$ for the spin 1/2 systems

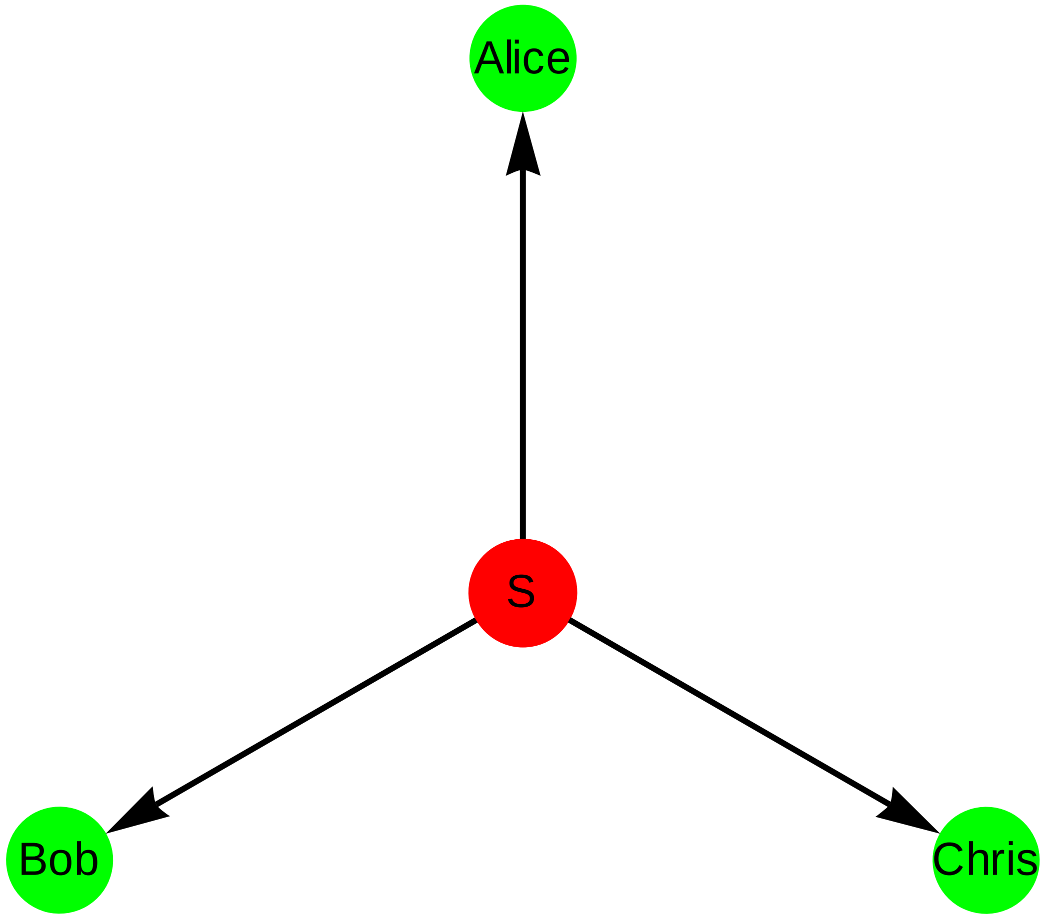

Fig. A source (S) of particle triples produces three identical particles which then move towards three equidistant magnetic orientation detectors. Alice, Bob and Chris set up detectors to measure the magnetization along the direction of the particle's motion (the z-direction) or along two other mutually perpendicular directions (*x* and *y*-directions).

In 1989, a striking extension of Bell's theorem to the case of three particles was taken by Greenberger, Horne, and Zeilinger. In contrast to Bell's theorem, which concerns statistical averages, this so-called GHZ theorem shows that a conflict between quantum mechanics and local realism can be obtained with a single measurements. GHZ consider three observers, Alice (1), Bob (2), and Chris(3). The GHZ experiments are a class of physics experiments that may be used to generate starkly contrasting predictions from local hidden variable theory and quantum mechanics, and permits immediate comparison with actual experimental results.

Using the Mathematica, here we show that the singlet state (GHZ state) are the eigenkets of the following operators

$$\hat{A}_{xyy} = \hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3},$$

$$\hat{A}_{yxy} = \hat{\sigma}_{y1} \otimes \hat{\sigma}_{x2} \otimes \hat{\sigma}_{y3},$$

$$\hat{A}_{yyx} = \hat{\sigma}_{y1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{x3},$$

$$\hat{A}_{xxx} = \hat{\sigma}_{x1} \otimes \hat{\sigma}_{x2} \otimes \hat{\sigma}_{x3}.$$

First we consider the GHZ state $\left|\psi_{GHZ}{}^{-}\right\rangle$ given by

$$\left|\psi_{GHZ}{}^{-}\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|+z\right\rangle_2\left|+z\right\rangle_3 - \left|-z\right\rangle_1\left|-z\right\rangle_2\left|-z\right\rangle_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

It is seen that $\left|\psi_{GHS}{}^{-}\right\rangle$ is an eigenstate of several operators with the eigenvalue +1,

$$\hat{A}_{xyy}\left|\psi_{GHS}{}^{-}\right\rangle = \left|\psi_{GHS}{}^{-}\right\rangle, \tag{1a}$$

$$\hat{A}_{yxy}\left|\psi_{GHS}{}^{-}\right\rangle = \left|\psi_{GHZ}{}^{-}\right\rangle, \tag{1b}$$

$$\hat{A}_{yyx}\left|\psi_{GHS}{}^{-}\right\rangle = \left|\psi_{GHS}{}^{-}\right\rangle. \tag{1c}$$

(1d)

Then we obtain

$$\hat{A}_{xyy}\hat{A}_{yxy}\hat{A}_{yyx}\left|\psi_{GHZ}{}^{-}\right\rangle = \left|\psi_{GHZ}{}^{-}\right\rangle,$$

We can also show that $\left|\psi_{GHS}{}^{-}\right\rangle$ is an eigenstate of the operator $\hat{A}_{xxx}$ with the eigenvalue -1,

$$\hat{A}_{xxx}\left|\psi_{GHS}{}^{-}\right\rangle = -\left|\psi_{GHS}{}^{-}\right\rangle .$$

Once three particles are sufficiently far apart, each spin of them possesses its own physical characteristics. We use A_x to denote the result of measuring the x component of the spin of particle 1 by Alice, B_y the result of measuring the y component of the spin of particle 2 by Bob, and C_y the result of measuring the y component of the spin of particle 3 by Chris, and so on, with $A_x = \pm 1$ …, $C_y = \pm 1$. When the x component is measured in connection with two measurements of the y component, we see that the product is +1:

$$A_x B_y C_y = +1 ,$$

Similarly, we have

$$A_y B_x C_y = +1, \qquad A_y B_y C_x = +1.$$

However, when the particles are in flight, two of the three experimentalists can decide to modify the direction of their analyzer axes, orienting them in the x axis direction. Then the product of the three spin components will be -1:

$$A_x B_x C_x = -1. \tag{2}$$

However, we note that

$$A_x B_x C_x = (A_x B_y C_y)(A_y B_x C_y)(A_y B_y C_x) = 1, \tag{3}$$

because $A_y{}^2 = B_y{}^2 = C_y{}^2 = 1$. Thus Eqs.(2) and (3) are incompatible.

Local realism would mean that $\hat{\sigma}_{x1}$ has a physical reality in the EPR sense, since it can be measured without disturbing $\hat{\sigma}_{y2}$ and $\hat{\sigma}_{y3}$,

$$A_x = B_y C_y .$$

However, it is also possible to obtain A_x by measuring $\hat{\sigma}_{x2}$ and $\hat{\sigma}_{x3}$:

$$A_x = -B_x C_x$$

Local realism implies that it is the same A_x, but this is not the case in quantum mechanics. The value of A_x is contextual. It depends on physical properties incompatible with each other which are measured simultaneously.

((**Mathematica**))

```
Clear["Global`*"]; ψ1 = (1
                         0); ψ2 = (0
                                   1);
σx = PauliMatrix[1]; σy = PauliMatrix[2];
σz = PauliMatrix[3];
χ =
  1/√2 (KroneckerProduct[ψ1, ψ1, ψ1] -
     KroneckerProduct[ψ2, ψ2, ψ2]);

Axyy = KroneckerProduct[σx, σy, σy]; Axyy.χ - χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Ayxy = KroneckerProduct[σy, σx, σy]; Ayxy.χ - χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Ayyx = KroneckerProduct[σy, σy, σx]; Ayyx.χ - χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Axxx = KroneckerProduct[σx, σx, σx]; Axxx.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

A123 = Axyy.Ayxy.Ayyx; A123.χ + Axxx.χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}
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5. $\left|\psi_{GHZ}{}^{+}\right\rangle$ state for spin 1/2 systems

We consider the GHZ state $\left|\psi_{GHZ}{}^{+}\right\rangle$ defined by

$$\left|\psi_{GHZ}{}^{+}\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|+z\right\rangle_2\left|+z\right\rangle_3 + \left|-z\right\rangle_1\left|-z\right\rangle_2\left|-z\right\rangle_3) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}.$$

It is seen that $\left|\psi_{GHS}{}^{+}\right\rangle$ is an eigenstate of several operators with the eigenvalue -1,

$$\hat{A}_{xyy}\left|\psi_{GHS}{}^{+}\right\rangle = -\left|\psi_{GHS}{}^{+}\right\rangle, \quad (1a)$$

$$\hat{A}_{yxy}\left|\psi_{GHS}\right\rangle = -\left|\psi_{GHZ}\right\rangle, \quad (1b)$$

$$\hat{A}_{yyx}\left|\psi_{GHS}\right\rangle = -\left|\psi_{GHS}\right\rangle. \quad (1c)$$

Then we obtain

$$\hat{A}_{xyy}\hat{A}_{yxy}\hat{A}_{yyx}\left|\psi_{GHZ}{}^{-}\right\rangle = -\hat{A}_{xyy}\hat{A}_{yxy}\left|\psi_{GHZ}{}^{-}\right\rangle = \hat{A}_{xyy}\left|\psi_{GHZ}{}^{-}\right\rangle = -\left|\psi_{GHZ}{}^{-}\right\rangle, \quad (2)$$

We can also show that $\left|\psi_{GHS}{}^{+}\right\rangle$ is an eigenstate of $\hat{A}_{xxx}$ with the eigenvalue +1,

$$\hat{A}_{xxx}\left|\psi_{GHS}{}^{+}\right\rangle = \left|\psi_{GHS}{}^{-}\right\rangle.$$

Once the three particles are sufficiently far apart, each spin of them possesses its own physical characteristics. We use A_x to denote the result of measuring the *x* component of the spin of particle 1 by Alice, …, C_y the result of measuring the *y* component of the spin of particle 3 by Charlotte, and so on, with $A_x = \pm 1$ …, $C_y = \pm 1$. When the *x* component is measured in connection with two measurements of the *y* component, we see that the products is +1:

$$A_x B_y C_y = -1, \quad A_y B_x C_y = -1, \qquad A_y B_y C_x = -1.$$

However, when the particles are in flight, two of the three experimentalists can decide to modify the direction of their analyzer axes, orienting them in the *x* axis direction. Then the product of the three spin components will be -1:

$$A_x B_x C_x = 1. \qquad (2)$$

However, we note that

$$A_x B_x C_x = (A_x B_y C_y)(A_y B_x C_y)(A_y B_y C_x) = -1, \qquad (3)$$

because $A_y^{\ 2} = B_y^{\ 2} = C_y^{\ 2} = 1$. Thus Eqs.(2) and (3) are incompatible. Local realism would mean that $\hat{\sigma}_{x1}$ has a physical reality in the EPR sense, since it can be measured without disturbing $\hat{\sigma}_{y2}$ and $\hat{\sigma}_{y3}$,

$$A_x = -B_y C_y .$$

However, it is also possible to obtain A_x by measuring $\hat{\sigma}_{x2}$ and $\hat{\sigma}_{x3}$:

$$A_x = B_x C_x .$$

Local realism implies that it is the same A_x, but this is not the case in quantum mechanics. The value of A_x is contextual. It depends on physical properties incompatible with each other which are measured simultaneously

((Mathematica))

```
Clear["Global`*"]; ψ1 = (1
                        0); ψ2 = (0
                                  1); σx = PauliMatrix[1];
σy = PauliMatrix[2]; σz = PauliMatrix[3];
χ = 1/√2 (KroneckerProduct[ψ1, ψ1, ψ1] + KroneckerProduct[ψ2, ψ2, ψ2]);

Axyy = KroneckerProduct[σx, σy, σy]; Axyy.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Ayxy = KroneckerProduct[σy, σx, σy]; Ayxy.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Ayyx = KroneckerProduct[σy, σy, σx]; Ayyx.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Axxx = KroneckerProduct[σx, σx, σx]; Axxx.χ - χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

A123 = Axyy.Ayxy.Ayyx; A123.χ + Axxx.χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Axyy.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Ayxy.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Ayyx.χ + χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

Axxx.χ - χ

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}
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6. GHZ state for photon systems

$$\left|45^\circ\right\rangle = \left|H'\right\rangle = \frac{1}{\sqrt{2}}(\left|x\right\rangle + \left|y\right\rangle), \qquad \left|+x\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle + \left|-z\right\rangle),$$

$$\left|-45°\right\rangle = \left|V'\right\rangle = \frac{1}{\sqrt{2}}(\left|x\right\rangle - \left|y\right\rangle), \qquad \left|-x\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle - \left|-z\right\rangle),$$

$$\left|R\right\rangle = \frac{1}{\sqrt{2}}(\left|x\right\rangle + i\left|y\right\rangle), \qquad \left|+y\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle + i\left|-z\right\rangle),$$

$$\left|L\right\rangle = \frac{1}{\sqrt{2}}(\left|x\right\rangle - i\left|y\right\rangle), \qquad \left|-y\right\rangle = \frac{1}{\sqrt{2}}(\left|+z\right\rangle - i\left|-z\right\rangle).$$

(a) Basis {$\left|H'\right\rangle$, $\left|V'\right\rangle$}

$$\left|\psi_{GHZ}{}^{XXY(+)}\right\rangle = \frac{1}{\sqrt{2}}(\left|x\right\rangle_1\left|x\right\rangle_2\left|y\right\rangle_3 + \left|y\right\rangle_1\left|y\right\rangle_2\left|x\right\rangle_3),$$

where $\left|x\right\rangle$ and $\left|y\right\rangle$ denote horizontal and vertical polarizations, respectively. This state indicates that the three photons are in a quantum superposition of the state $\left|x\right\rangle_1\left|x\right\rangle_2\left|y\right\rangle_3$ and $\left|y\right\rangle_1\left|y\right\rangle_2\left|x\right\rangle_3$. Note that

$$\left|x\right\rangle = \frac{1}{\sqrt{2}}[\left|H'\right\rangle + \left|V'\right\rangle], \qquad \left|y\right\rangle = \frac{1}{\sqrt{2}}[\left|H'\right\rangle - \left|V'\right\rangle].$$

Then we have the form

$$\left|\psi_{GHZ}{}^{XXY(+)}\right\rangle = \frac{1}{2}(\left|H'\right\rangle_1\left|H'\right\rangle_2\left|H'\right\rangle_3 - \left|H'\right\rangle_1\left|V'\right\rangle_2\left|V'\right\rangle_3 - \left|V'\right\rangle_1\left|H'\right\rangle_2\left|V'\right\rangle_3 + \left|V'\right\rangle_1\left|V'\right\rangle_2\left|H'\right\rangle_3).$$

((Mathematica))

```
Clear["Global`*"]; x1 = (1
                         0); y1 = (0
                                   1);
χ =
  1/√2 (KroneckerProduct[x1, x1, y1] +
     KroneckerProduct[y1, y1, x1]);

H1 = 1/√2 (1
           1); V1 = 1/√2 (1
                          -1);
HHH1 = KroneckerProduct[H1, H1, H1] // Simplify;
HHV1 = KroneckerProduct[H1, H1, V1] // Simplify;
HVH1 = KroneckerProduct[H1, V1, H1] // Simplify;
HVV1 = KroneckerProduct[H1, V1, V1] // Simplify;
VHH1 = KroneckerProduct[V1, H1, H1] // Simplify;
VHV1 = KroneckerProduct[V1, H1, V1] // Simplify;
VVH1 = KroneckerProduct[V1, V1, H1] // Simplify;
VVV1 = KroneckerProduct[V1, V1, V1] // Simplify;

f1 = a1 HHH1 + a2 HHV1 + a3 HVH1 + a4 HVV1 +
     a5 VHH1 + a6 VHV1 + a7 VVH1 + a8 VVV1 // Simplify;

eq1 = Solve[f1 == χ, {a1, a2, a3, a4, a5, a6, a7, a8}];

rule1 = {b1 → HHH, b2 → HHV, b3 → HVH, b4 → HVV,
   b5 → VHH, b6 → VHV, b7 → VVH, b8 → VVV};
P1 = a1 b1 + a2 b2 + a3 b3 + a4 b4 + a5 b5 +
   a6 b6 + a7 b7 + a8 b8;
P1 /. rule1 /. eq1[[1]]
```

$$\frac{\text{HHH}}{2} - \frac{\text{HVV}}{2} - \frac{\text{VHV}}{2} + \frac{\text{VVH}}{2}$$

(b) **Basis** $\{|H'\rangle, |V'\rangle\}$

$$|\psi_{GHZ}{}^{XXX(+)}\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|x\rangle_2|x\rangle_3 + |y\rangle_1|y\rangle_2|y\rangle_3)$$

where $|x\rangle$ and $|y\rangle$ denote horizontal and vertical polarizations, respectively. This state indicates that the three photons are in a quantum superposition of the state $|x\rangle_1|x\rangle_2|x\rangle_3$ and $|y\rangle_1|y\rangle_2|y\rangle_3$. Using the basis of $\{|H'\rangle, |V'\rangle\}$, $|\psi_{GHZ}{}^{XXX(+)}\rangle$ can be rewritten as

$$|\psi_{GHZ}{}^{XXX(+)}\rangle = \frac{1}{2}(|H'\rangle_1|H'\rangle_2|H'\rangle_3 + |H'\rangle_1|V'\rangle_2|V'\rangle_3 + |V'\rangle_1|H'\rangle_2|V'\rangle_3 + |V'\rangle_1|V'\rangle_2|H'\rangle_3)$$

((**Mathematica**))

```
Clear["Global`*"]; x1 = (1
                         0); y1 = (0
                                   1);
χ =
 1/√2 (KroneckerProduct[x1, x1, x1] +
     KroneckerProduct[y1, y1, y1]);

H1 = 1/√2 (1
           1); V1 = 1/√2 (1
                          -1);
HHH1 = KroneckerProduct[H1, H1, H1] // Simplify;
HHV1 = KroneckerProduct[H1, H1, V1] // Simplify;
HVH1 = KroneckerProduct[H1, V1, H1] // Simplify;
HVV1 = KroneckerProduct[H1, V1, V1] // Simplify;
VHH1 = KroneckerProduct[V1, H1, H1] // Simplify;
VHV1 = KroneckerProduct[V1, H1, V1] // Simplify;
VVH1 = KroneckerProduct[V1, V1, H1] // Simplify;
VVV1 = KroneckerProduct[V1, V1, V1] // Simplify;

f1 = a1 HHH1 + a2 HHV1 + a3 HVH1 + a4 HVV1 +
     a5 VHH1 + a6 VHV1 + a7 VVH1 + a8 VVV1 // Simplify;

eq1 = Solve[f1 == χ, {a1, a2, a3, a4, a5, a6, a7, a8}];

rule1 = {b1 → HHH, b2 → HHV, b3 → HVH, b4 → HVV,
   b5 → VHH, b6 → VHV, b7 → VVH, b8 → VVV};
P1 = a1 b1 + a2 b2 + a3 b3 + a4 b4 + a5 b5 +
   a6 b6 + a7 b7 + a8 b8;
P1 /. rule1 /. eq1[[1]]
```

$$\frac{\text{HHH}}{2} + \frac{\text{HVV}}{2} + \frac{\text{VHV}}{2} + \frac{\text{VVH}}{2}$$

(c) Basis $\{|R\rangle, |L\rangle\}$

$$\left|\psi_{GHZ}{}^{XXX(-)}\right\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|x\rangle_2|x\rangle_3 - |y\rangle_1|y\rangle_2|x\rangle_3)$$

where $|x\rangle$ and $|y\rangle$ denote horizontal and vertical polarizations, respectively. This state indicates that the three photons are in a quantum superposition of the state $|x\rangle_1|x\rangle_2|x\rangle_3$ and $|y\rangle_1|y\rangle_2|x\rangle_3$. Using the basis of $\{|R\rangle,|L\rangle\}$, $\left|\psi_{GHZ}{}^{XXX(-)}\right\rangle$ can be rewritten as

$$\left|\psi_{GHZ}{}^{XXX(-)}\right\rangle = \frac{1}{2}(|R\rangle_1|R\rangle_2|R\rangle_3 + |R\rangle_1|R\rangle_2|L\rangle_3 + |L\rangle_1|L\rangle_2|R\rangle_3 + |L\rangle_1|L\rangle_2|L\rangle_3)$$

((**Mathematica**))

```
Clear["Global`*"]; x1 = (1
                         0); y1 = (0
                                   1);
χ =
  1/√2 (KroneckerProduct[x1, x1, x1] -
     KroneckerProduct[y1, y1, x1]);

R1 = 1/√2 (1
           ⅈ); L1 = 1/√2 (1
                          -ⅈ);
RRR1 = KroneckerProduct[R1, R1, R1] // Simplify;
RRL1 = KroneckerProduct[R1, R1, L1] // Simplify;
RLR1 = KroneckerProduct[R1, L1, R1] // Simplify;
RLL1 = KroneckerProduct[R1, L1, L1] // Simplify;
LRR1 = KroneckerProduct[L1, R1, R1] // Simplify;
LRL1 = KroneckerProduct[L1, R1, L1] // Simplify;
LLR1 = KroneckerProduct[L1, L1, R1] // Simplify;
LLL1 = KroneckerProduct[L1, L1, L1] // Simplify;

f1 = a1 RRR1 + a2 RRL1 + a3 RLR1 + a4 RLL1 +
     a5 LRR1 + a6 LRL1 + a7 LLR1 + a8 LLL1 // Simplify;

eq1 = Solve[f1 == χ, {a1, a2, a3, a4, a5, a6, a7, a8}];

rule1 = {b1 → RRR, b2 → RRL, b3 → RLR, b4 → RLL,
   b5 → LRR, b6 → LRL, b7 → LLR, b8 → LLL};
P1 = a1 b1 + a2 b2 + a3 b3 + a4 b4 + a5 b5 +
   a6 b6 + a7 b7 + a8 b8;
P1 /. rule1 /. eq1[[1]]
```

$$\frac{\text{LLL}}{2} + \frac{\text{LLR}}{2} + \frac{\text{RRL}}{2} + \frac{\text{RRR}}{2}$$

7. General case

$$\left|\psi_{GHZ}^{XXX(+)}\right\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|x\rangle_2|x\rangle_3 + |y\rangle_1|y\rangle_2|y\rangle_3),$$

$$|H'\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle,$$

$$|V'\rangle = -\sin\theta|x\rangle + \cos\theta|y\rangle,$$

$$\begin{aligned}|\psi_{GHZ}{}^{XXX(+)}\rangle &= \frac{1}{\sqrt{2}}(\cos^3\theta + \sin^3\theta)|H'\rangle_1|H'\rangle_2|H'\rangle_3 \\ &+ \frac{1}{\sqrt{2}}(\cos^3\theta - \sin^3\theta)|V'\rangle_1|V'\rangle_2|V'\rangle_3 \\ &+ \frac{1}{\sqrt{2}}\sin\theta(\sin\theta - \cos\theta)(|H'\rangle_1|H'\rangle_2|V'\rangle_3 + |H'\rangle_1|V'\rangle_2|H'\rangle_3 + |V'\rangle_1|H'\rangle_2|H'\rangle_3) \\ &+ \frac{1}{\sqrt{2}}\sin\theta(\sin\theta + \cos\theta)(|H'\rangle_1|V'\rangle_2|V'\rangle_3 + |V'\rangle_1|V'\rangle_2|H'\rangle_3 + |V'\rangle_1|H'\rangle_2|V'\rangle_3)\end{aligned}$$

We make a plot of the probabilities for the eight states $\{|H'\rangle_1|H'\rangle_2|H'\rangle_3,\ |V'\rangle_1|V'\rangle_2|V'\rangle_3,\ ....\}$ as a function of θ.

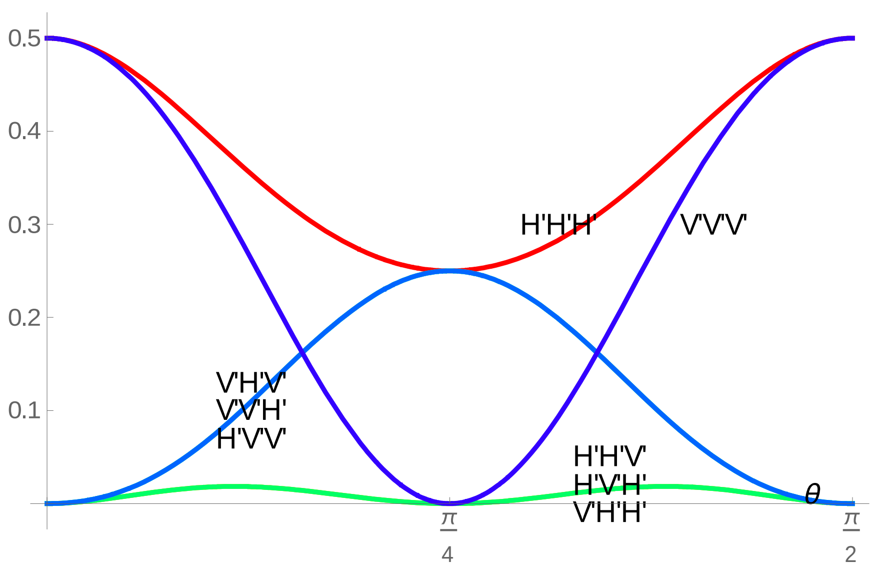


Fig. plot of the probabilities for the eight states $\{|H'\rangle_1|H'\rangle_2|H'\rangle_3,\ |V'\rangle_1|V'\rangle_2|V'\rangle_3,\ ....\}$ as a function of θ.

8. Reality

We have three particles, and we can choose to measure each on arbitrary basis. We designate a chosen set of observation on these particles by a sequence of symbol *X* and *Y*. If we may choose to measure particles 1, 2, and 3 in only the basis *X*. This measurement is denoted by *XXX*.

$$\left|\psi_{GHZ}{}^{XXX(+)}\right\rangle = \frac{1}{\sqrt{2}}(\left|x\right\rangle_1\left|x\right\rangle_2\left|x\right\rangle_3 + \left|y\right\rangle_1\left|y\right\rangle_2\left|y\right\rangle_3)$$
$$= \frac{1}{2}(\left|H_1'\right\rangle\left|H_2'\right\rangle\left|H_3'\right\rangle + \left|H_1'\right\rangle\left|V_2'\right\rangle\left|V_3'\right\rangle + \left|V_1'\right\rangle\left|H_2'\right\rangle\left|V_3'\right\rangle + \left|V_1'\right\rangle\left|V_2'\right\rangle\left|H_3'\right\rangle)$$

We may also choose to measure particles 1 and 2 in basis Y (right-left basis) and particle 3 in basis X (horizontal-vertical basis). This measurement is denoted by YYX. The state $\left|\psi_{GHZ}{}^{YYX}\right\rangle$ expressed in the corresponding basis set becomes

$$\left|\psi_{GHZ}{}^{YYX(+)}\right\rangle = \frac{1}{\sqrt{2}}(\left|y\right\rangle_1\left|y\right\rangle_2\left|x\right\rangle_3 + \left|x\right\rangle_1\left|x\right\rangle_2\left|y\right\rangle_3)$$
$$= \frac{1}{2}[\left|R_1\right\rangle\otimes\left|L_2\right\rangle\otimes\left|H_3'\right\rangle + \left|L_1\right\rangle\otimes\left|R_2\right\rangle\otimes\left|H_3'\right\rangle$$
$$+ \left|R_1\right\rangle\otimes\left|R_2\right\rangle\otimes\left|V_3'\right\rangle + \left|L_1\right\rangle\otimes\left|L_2\right\rangle\otimes\left|V_3'\right\rangle]$$

((**Mathematica**))

```
Clear["Global`*"]; x1 = (1, 0)ᵀ; y1 = (0, 1)ᵀ;
```

$$\mathbf{H11} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix};\ \mathbf{V11} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix};\ \mathbf{R1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix};$$

$$\mathbf{L1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$\left\{\left\{\frac{1}{\sqrt{2}}\right\}, \left\{-\frac{i}{\sqrt{2}}\right\}\right\}$$

```
χ =
  1/2 (KroneckerProduct[R1, L1, H11] +
    KroneckerProduct[L1, R1, H11] +
    KroneckerProduct[R1, R1, V11] +
    KroneckerProduct[L1, L1, V11]);

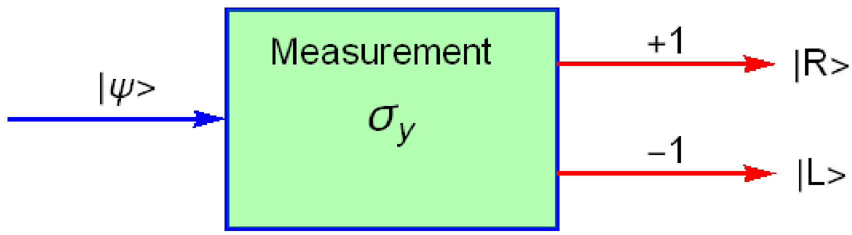
χ // Simplify
```

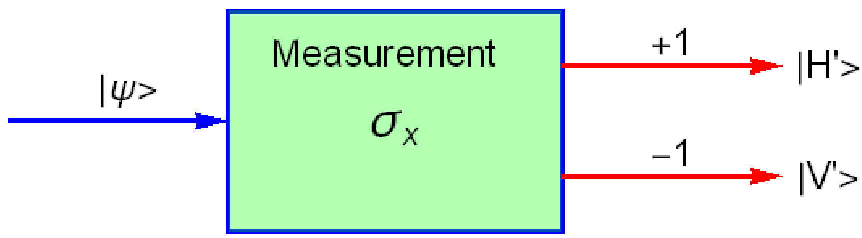
$$\left\{\left\{\frac{1}{\sqrt{2}}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{\frac{1}{\sqrt{2}}\right\}\right\}$$

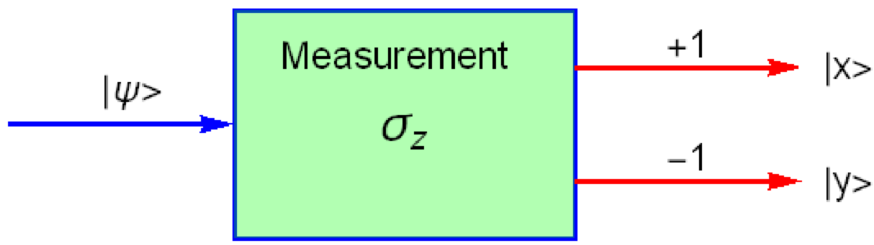
9. Pauli matrices for photon polarization

For convenience we use the Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$







(a) The Pauli operator $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$|H'\rangle$ and $|V'\rangle$ are the eigenkets of $\hat{\sigma}_x$ with the eigenvalues +1, and -1, respectively.

$$\hat{\sigma}_x |H'\rangle = |H'\rangle, \qquad \hat{\sigma}_x |V'\rangle = -|V'\rangle$$

(b) The Pauli operator $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$:

$|R\rangle$ and $|L\rangle$ are the eigenkets of $\hat{\sigma}_y$ with the eigenvalues +1, and -1, respectively.

$$\hat{\sigma}_y|R\rangle = |R\rangle, \qquad \hat{\sigma}_y|L\rangle = -|L\rangle.$$

(c) The Pauli operator $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$|x\rangle$ and $|y\rangle$ are the eigenkets of $\hat{\sigma}_z$ with the eigenvalues +1, and -1, respectively.

10. *YYX*-, *YXY*-, *XYY*-, and *XXX*-type measurement

Suppose that the particle 3 is in the state $|V_3'\rangle$,

$$\hat{\sigma}_{3x}|V_3'\rangle = -|V_3'\rangle$$

The relation

$$\begin{aligned}
|\psi_{GHZ}{}^{YYX(+)}\rangle &= \frac{1}{\sqrt{2}}(|y_1\rangle|y_2\rangle|x_3\rangle + |x_1\rangle|x_2\rangle|y_3\rangle \\
&= \frac{1}{2}[|R_1\rangle\otimes|L_2\rangle\otimes|H_3'\rangle + |L_1\rangle\otimes|R_2\rangle\otimes|H_3'\rangle \\
&+ |R_1\rangle\otimes|R_2\rangle\otimes|V_3'\rangle + |L_1\rangle\otimes|L_2\rangle\otimes|V_3'\rangle]
\end{aligned}$$

states that the particles 1 and 2 must be in either of the states $|R_1\rangle\otimes|R_2\rangle$ or $|L_1\rangle\otimes|L_2\rangle$.

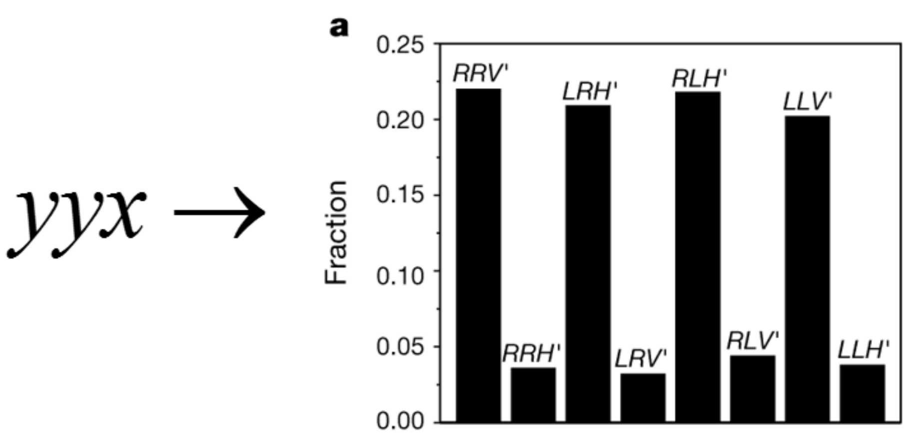


Fig. from the paper of **Pan et.al.**

If the system 2 is also in the $|V_2'\rangle$ state, then the relation

$$\left|\psi_{GHZ}{}^{YXY(+)}\right\rangle = \frac{1}{2}[\left|L_1\right\rangle \otimes \left|H_2'\right\rangle \otimes \left|R_3\right\rangle + \left|R_1\right\rangle \otimes \left|H_2'\right\rangle \otimes \left|L_3\right\rangle$$
$$+ \left|R_1\right\rangle \otimes \left|V_2'\right\rangle \otimes \left|R_3\right\rangle + \left|L_1\right\rangle \otimes \left|V_2'\right\rangle \otimes \left|L_3\right\rangle]$$

similarly implies that the particles 2 and 3 are in the state $\left|R_1\right\rangle \otimes \left|R_3\right\rangle$ or $\left|L_1\right\rangle \otimes \left|L_3\right\rangle$. In this case these states represent reality, and the conclusions above described combine to state that particles 2 and 3 are in the state $\left|R_2\right\rangle \otimes \left|R_3\right\rangle$ or $\left|L_2\right\rangle \otimes \left|L_3\right\rangle$.

$yxy \rightarrow$

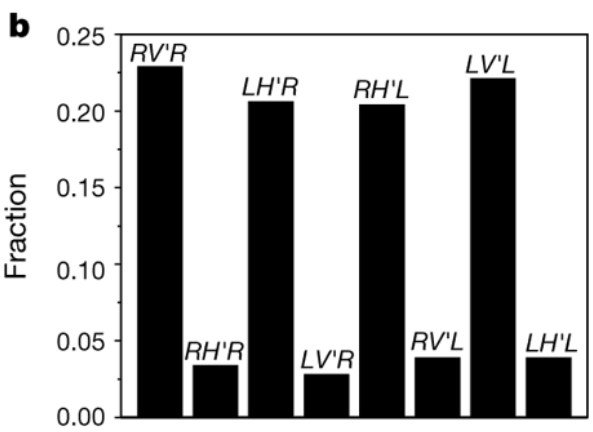


Fig. from the paper of **Pan et.al.**

From the relation

$$\left|\psi_{GHZ}{}^{XYY(+)}\right\rangle = \frac{1}{2}[\left|H_1'\right\rangle \otimes \left|R_2\right\rangle \otimes \left|L_3\right\rangle + \left|H_1'\right\rangle \otimes \left|L_2\right\rangle \otimes \left|R_3\right\rangle$$
$$+ \left|V_1'\right\rangle \otimes \left|R_2\right\rangle \otimes \left|R_3\right\rangle + \left|V_1'\right\rangle \otimes \left|L_2\right\rangle \otimes \left|L_3\right\rangle]$$

we can conclude that the particle 1 is in the state $\left|V_1'\right\rangle$, if the particle 2 and 3 are in the state $\left|V_2'\right\rangle \otimes \left|V_3'\right\rangle$.

$xyy \rightarrow$

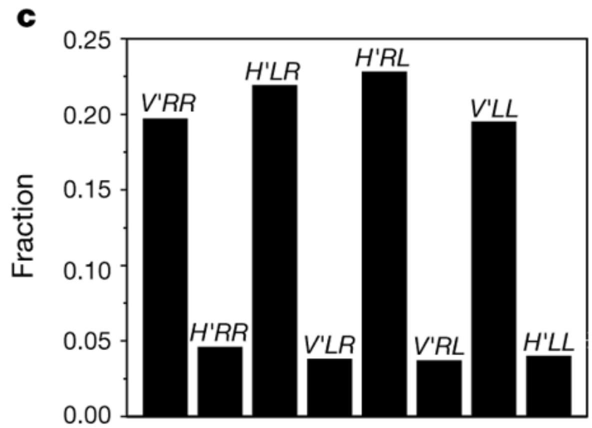


Fig. from the paper of **Pan et.al.**

After going through analogous arguments four times, we have the following states allowed,

$$\{ |V_1'\rangle \otimes |V_2'\rangle \otimes |V_3'\rangle,\ |H_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle,\ |H_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle,\ |V_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle \}.$$

LHV:

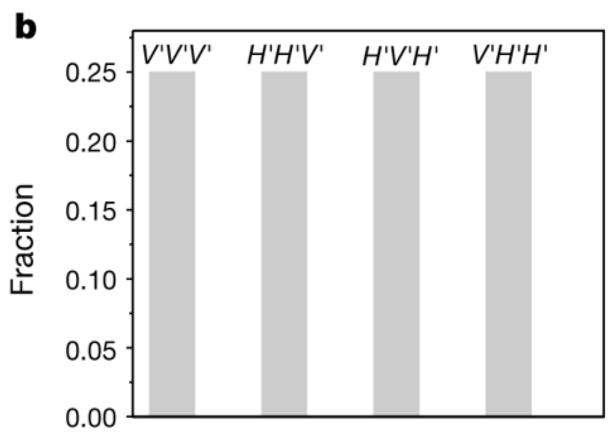


Fig. Prediction from the local hidden theory. There are four states denoted by $|V_1'\rangle \otimes |V_2'\rangle \otimes |V_3'\rangle$, $|H_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle$, $|H_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle$, and $|V_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle$. (**Pan et.al.)**

In quantum mechanics, $\left|\psi_{GHZ}{}^{XXX(+)}\right\rangle$ is the eigenket of $\hat{\sigma}_{1x}\otimes\hat{\sigma}_{2x}\otimes\hat{\sigma}_{3x}$ with the eigenvalue of +1, since

$$\hat{\sigma}_{1x}\otimes\hat{\sigma}_{2x}\otimes\hat{\sigma}_{3x}\left|\psi_{GHZ}{}^{XXX(+)}\right\rangle=\left|\psi_{GHZ}{}^{XXX(+)}\right\rangle$$

where

$$\left|\psi_{GHZ}{}^{XXX(+)}\right\rangle=\frac{1}{2}(\left|H_1'\right\rangle\left|H_2'\right\rangle\left|H_3'\right\rangle+\left|H_1'\right\rangle\left|V_2'\right\rangle\left|V_3'\right\rangle+\left|V_1'\right\rangle\left|H_2'\right\rangle\left|V_3'\right\rangle+\left|V_1'\right\rangle\left|V_2'\right\rangle\left|H_3'\right\rangle).$$

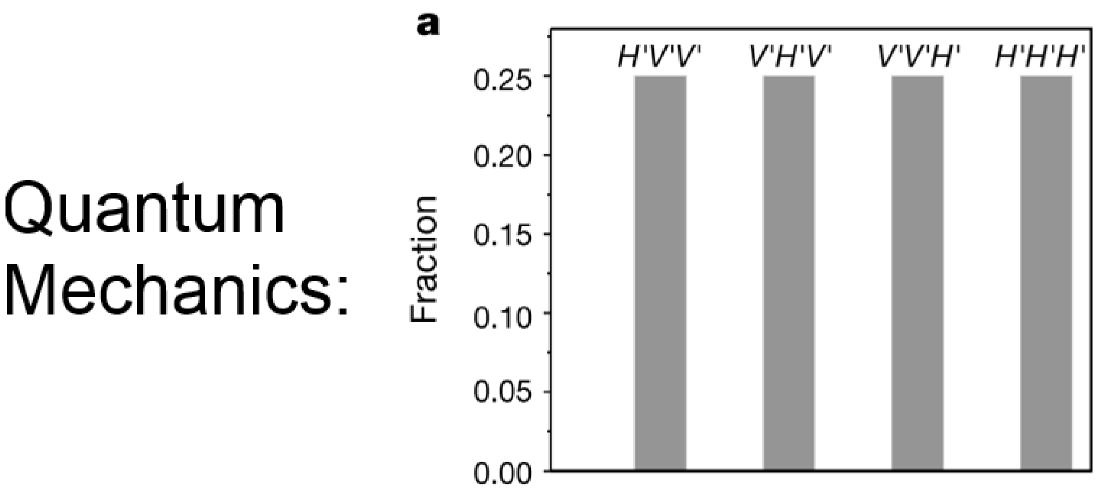


Fig. Prediction from the quantum mechanics. There are four states denoted by $\left|H_1'\right\rangle\left|H_2'\right\rangle\left|H_3'\right\rangle$, $\left|H_1'\right\rangle\left|V_2'\right\rangle\left|V_3'\right\rangle$, $\left|V_1'\right\rangle\left|H_2'\right\rangle\left|V_3'\right\rangle$, and $\left|V_1'\right\rangle\left|V_2'\right\rangle\left|H_3'\right\rangle$, which are different from the states predicted from the local hidden theory. (**Pan et.al.)**

11. Detail of the GHZ experiment

We consider the GHZ state given by

$$\left|\psi_{GHZ}{}^{YYX(+)}\right\rangle=\frac{1}{2}[\left|R_1\right\rangle\otimes\left|L_2\right\rangle\otimes\left|H_3'\right\rangle+\left|L_1\right\rangle\otimes\left|R_2\right\rangle\otimes\left|H_3'\right\rangle$$
$$+\left|R_1\right\rangle\otimes\left|R_2\right\rangle\otimes\left|V_3'\right\rangle+\left|L_1\right\rangle\otimes\left|L_2\right\rangle\otimes\left|V_3'\right\rangle]$$

For polarization measurements in the L/R basis, the photons in modes 1 and 2 have equal probability for the combinations $|R_1\rangle \otimes |L_2\rangle$, $|L_1\rangle \otimes |R_2\rangle$, $|R_1\rangle \otimes |R_2\rangle$, and $|L_1\rangle \otimes |L_2\rangle$. If $|R_1\rangle \otimes |R_2\rangle$ is obtained, the photon in mode 3 has to be in the state $|V_3'\rangle$.

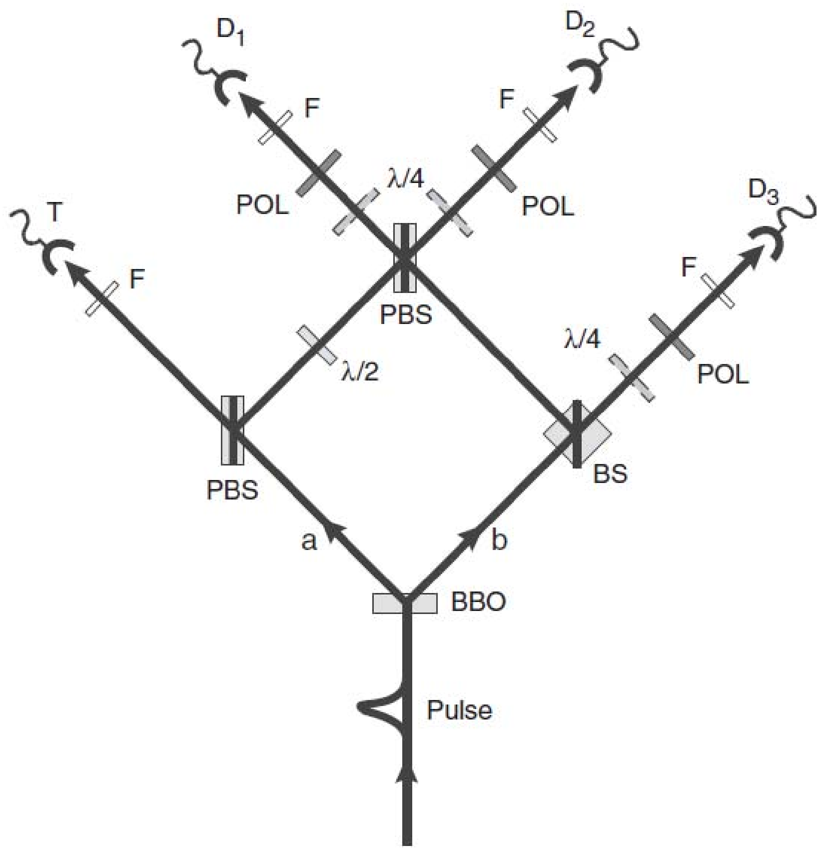


Fig. Set up for the creation of a GHZ state using two pairs of polarization entangled photon (**Pan et al.**). BS: half beam splitter. PBS: polarized beam splitter. POL: polarizer. $\lambda/4$ quarter wave plate. F: narrow bandwidth filter. $\lambda/2$: half wave plate. The half wave plate switches $|y\rangle$ to $|H'\rangle = \frac{1}{\sqrt{2}}[|x\rangle + |y\rangle$. Quarter wave plates and polarizer just before the detectors are used for correlation measurements.

This figure shows the experimental results for this correlation measurement. Quarter wave plates and polarizers just before detectors D_1, D_2, and D_3 in Fig. are set to $|R_1\rangle \otimes |R_2\rangle \otimes |V_3'\rangle$ or

$|R_1\rangle \otimes |R_2\rangle \otimes |H_3'\rangle$. The results clearly confirms the strong correlations of $|R_1\rangle \otimes |R_2\rangle \otimes |V_3'\rangle$ in comparison to $|R_1\rangle \otimes |R_2\rangle \otimes |H_3'\rangle$.

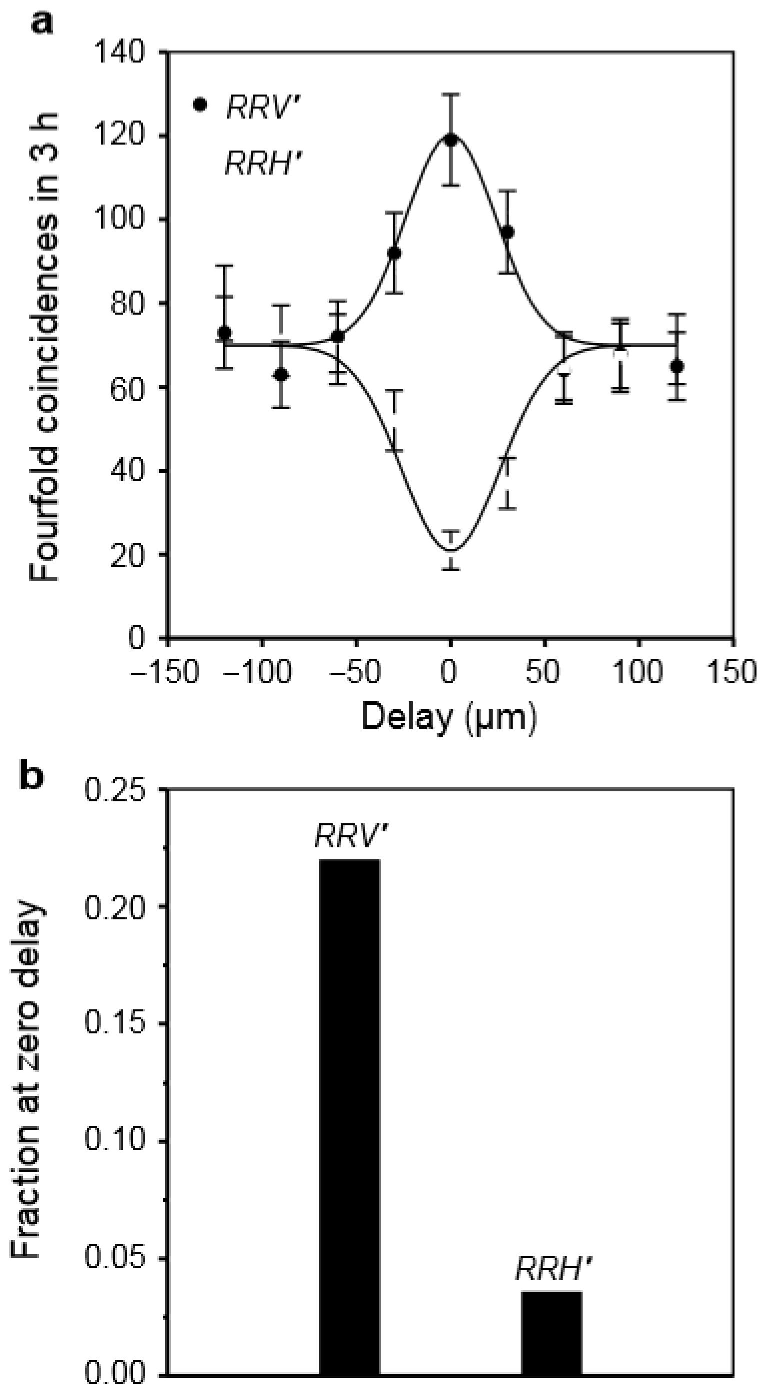


Fig. A typical experimental result used in the GHZ argument.This is the *yyx* experiment measuring circular polarization on photons 1 and 2 and linear polarization on the third. (**Pan et.al.)**

For photon *i* we call these elements of reality X_i with values +1 (-1) for $|H'\rangle$ ($|V'\rangle$') polarizations and Y_i with values +1 (-1) for $|R\rangle$ ($|L\rangle$); we thus obtain the relations

(a) The relation: $Y_1Y_2X_3 = -1$,

form

	Each eigenvlaues			Resultant eigenvalue
$\lvert R_1\rangle\otimes\lvert L_2\rangle\otimes\lvert H_3'\rangle$	1	-1	1	-1
$\lvert L_1\rangle\otimes\lvert R_2\rangle\otimes\lvert H_3'\rangle$	-1	1	1	-1
$\lvert R_1\rangle\otimes\lvert R_2\rangle\otimes\lvert V_3'\rangle$	1	1	-1	-1

in order to be able to reproduce the quantum predictions of equation

$$\left|\psi_{GHZ}{}^{YYX(+)}\right\rangle = \frac{1}{2}[|R_1\rangle\otimes|L_2\rangle\otimes|H_3'\rangle + |L_1\rangle\otimes|R_2\rangle\otimes|H_3'\rangle \\ + |R_1\rangle\otimes|R_2\rangle\otimes|V_3'\rangle + |L_1\rangle\otimes|L_2\rangle\otimes|V_3'\rangle]$$

(b) The relation $Y_1X_2Y_3 = -1$,

from

	Each eigenvlaues			Resultant eigenvalue
$\lvert L_1\rangle\otimes\lvert H_2'\rangle\otimes\lvert R_3\rangle$	-1	1	1	-1 (eigenvalue)
$\lvert R_1\rangle\otimes\lvert H_2'\rangle\otimes\lvert L_3\rangle$	1	1	-1	-1
$\lvert R_1\rangle\otimes\lvert V_2'\rangle\otimes\lvert R_3\rangle$	1	-1	1	-1
$\lvert L_1\rangle\otimes\lvert V_2'\rangle\otimes\lvert L_3\rangle]$	-1	-1	-1	-1

in order to be able to reproduce the quantum predictions of equation

$$\left|\psi_{GHZ}{}^{YXY(+)}\right\rangle = \frac{1}{2}[|L_1\rangle\otimes|H_2'\rangle\otimes|R_3\rangle + |R_1\rangle\otimes|H_2'\rangle\otimes|L_3\rangle + |R_1\rangle\otimes|V_2'\rangle\otimes|R_3\rangle \\ + |L_1\rangle\otimes|V_2'\rangle\otimes|L_3\rangle]$$

(c) The relation $X_1Y_2Y_3$ = -1,

from

	Each eigenvlaues			Resultant eigenvalue
$\lvert R_1\rangle \otimes \lvert L_2\rangle \otimes \lvert H_3'\rangle$	1	-1	1	-1
$\lvert L_1\rangle \otimes \lvert R_2\rangle \otimes \lvert H_3'\rangle$	-1	1	1	-1
$\lvert R_1\rangle \otimes \lvert R_2\rangle \otimes \lvert V_3'\rangle$	1	1	-1	-1
$\lvert L_1\rangle \otimes \lvert L_2\rangle \otimes \lvert V_3'\rangle$	-1	-1	-1	-1

in order to be able to reproduce the quantum predictions of equation

$$\left|\psi_{GHZ}{}^{YYX(+)}\right\rangle = \frac{1}{2}[\left|R_1\right\rangle \otimes \left|L_2\right\rangle \otimes \left|H_3'\right\rangle + \left|L_1\right\rangle \otimes \left|R_2\right\rangle \otimes \left|H_3'\right\rangle \\ + \left|R_1\right\rangle \otimes \left|R_2\right\rangle \otimes \left|V_3'\right\rangle + \left|L_1\right\rangle \otimes \left|L_2\right\rangle \otimes \left|V_3'\right\rangle]$$

Because of Einstein locality any specific measurement for X must be independent of whether an X or Y measurement is performed on the other photon. As $Y_1Y_1 = Y_2Y_2 = Y_3Y_3 = 1$, we can write

$$X_1X_2X_3 = (X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3) = (-1)(-1)(-1) = -1 \qquad (1)$$

We now consider a fourth experiment measuring linear $\left|H'\right\rangle / \left|V'\right\rangle$ polarization on all three photons, that is, an XXX experiment. We investigate the possible outcomes that will be predicted by local realism based on the elements of reality introduced to explain the XXX experiments. We obtain

$$X_1X_2X_3 = 1 \qquad (2)$$

from

	Each eigenvlaue			Resultant eigenvalue
$\lvert H_1'\rangle \lvert H_2'\rangle \lvert H_3'\rangle$	1	1	1	1
$\lvert H_1'\rangle \lvert V_2'\rangle \lvert V_3'\rangle$	1	-1	-1	1
$\lvert V_1'\rangle \lvert H_2'\rangle \lvert V_3'\rangle$	-1	1	-1	1
$\lvert V_1'\rangle \lvert V_2'\rangle \lvert H_3'\rangle$	-1	-1	1	1

in order to be able to reproduce the quantum predictions of equation

$$\left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle = \frac{1}{2}(\left|H_1'\right\rangle\left|H_2'\right\rangle\left|H_3'\right\rangle + \left|H_1'\right\rangle\left|V_2'\right\rangle\left|V_3'\right\rangle + \left|V_1'\right\rangle\left|H_2'\right\rangle\left|V_3'\right\rangle + \left|V_1'\right\rangle\left|V_2'\right\rangle\left|H_3'\right\rangle).$$

with

$$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \left|\Psi_{GHZ}{}^{(+)}\right\rangle = +\left|\Psi_{GHZ}{}^{(+)}\right\rangle.$$

Then the value $X_1 X_2 X_3 = -1$ from Eq.(1) is incompatible with the value $X_1 X_2 X_3 = 1$ from Eq.(2), predicted from the quantum mechanics.

---

12. Eigenstates of $\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$

$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are eight eigenstates. The eigenvalues of the four states is -1 and the eigenvalue of the four states is (+1).

(i) Four states are the eigenkets of $\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$ with the eigenvalue (-1). One of these states is the GHZ state

$$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle = -\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle,$$

$$\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle|x_2\rangle|x_3\rangle - |y_1\rangle|y_2\rangle|y_3\rangle$$
$$= \frac{1}{2}(|H_1'\rangle|H_2'\rangle|V_3'\rangle + |H_1'\rangle|V_2'\rangle|H_3'\rangle + |V_1'\rangle|H_2'\rangle|H_3'\rangle + |V_1'\rangle|V_2'\rangle|V_3'\rangle)$$

The other three states are not the GHZ state,

$$\frac{1}{2}(-|H_1'\rangle|H_2'\rangle|V_3'\rangle + |H_1'\rangle|V_2'\rangle|H_3'\rangle + |V_1'\rangle|H_2'\rangle|H_3'\rangle - |V_1'\rangle|V_2'\rangle|V_3'\rangle),$$

$$\frac{1}{2}(|H_1'\rangle|H_2'\rangle|V_3'\rangle - |H_1'\rangle|V_2'\rangle|H_3'\rangle + |V_1'\rangle|H_2'\rangle|H_3'\rangle - |V_1'\rangle|V_2'\rangle|V_3'\rangle),$$

$$\frac{1}{2}(-|H_1'\rangle|H_2'\rangle|V_3'\rangle - |H_1'\rangle|V_2'\rangle|H_3'\rangle + |V_1'\rangle|H_2'\rangle|H_3'\rangle + |V_1'\rangle|V_2'\rangle|V_3'\rangle).$$

(ii) Four states are the eigenkets of $\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$ with the eigenvalue (+1). One of these states is the GHZ state

$$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle = \left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle,$$

$$\left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle|x_2\rangle|x_3\rangle + |y_1\rangle|y_2\rangle|y_3\rangle$$
$$= \frac{1}{2}(|H_1'\rangle|H_2'\rangle|H_3'\rangle + |H_1'\rangle|V_2'\rangle|V_3'\rangle + |V_1'\rangle|H_2'\rangle|V_3'\rangle + |V_1'\rangle|V_2'\rangle|H_3'\rangle)$$

The other three states are not the GHZ state.

$$\frac{1}{2}(|H_1'\rangle|H_2'\rangle|H_3'\rangle - |H_1'\rangle|V_2'\rangle|V_3'\rangle - |V_1'\rangle|H_2'\rangle|V_3'\rangle + |V_1'\rangle|V_2'\rangle|H_3'\rangle),$$

$$\frac{1}{2}(|H_1'\rangle|H_2'\rangle|H_3'\rangle - |H_1'\rangle|V_2'\rangle|V_3'\rangle + |V_1'\rangle|H_2'\rangle|V_3'\rangle - |V_1'\rangle|V_2'\rangle|H_3'\rangle),$$

$$\frac{1}{2}(|H_1'\rangle|H_2'\rangle|H_3'\rangle + |H_1'\rangle|V_2'\rangle|V_3'\rangle - |V_1'\rangle|H_2'\rangle|V_3'\rangle - |V_1'\rangle|V_2'\rangle|H_3'\rangle).$$

12. Eigenstates of $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$

$\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are eight eigenstates. The eigenvalues of the four states is -1 and the eigenvalue of the four states is (+1).

(i) Four states are the eigenkets of $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$ with the eigenvalue (-1). These states are not the GHZ state.

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|L_3\rangle + |L_1\rangle|R_2\rangle|R_3\rangle + |R_1\rangle|L_2\rangle|R_3\rangle + |R_1\rangle|R_2\rangle|L_3\rangle)$$

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|L_3\rangle + |L_1\rangle|R_2\rangle|R_3\rangle + |R_1\rangle|L_2\rangle|R_3\rangle - |R_1\rangle|R_2\rangle|L_3\rangle)$$

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|L_3\rangle + |L_1\rangle|R_2\rangle|R_3\rangle - |R_1\rangle|L_2\rangle|R_3\rangle + |R_1\rangle|R_2\rangle|L_3\rangle)$$

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|L_3\rangle - |L_1\rangle|R_2\rangle|R_3\rangle + |R_1\rangle|L_2\rangle|R_3\rangle + |R_1\rangle|R_2\rangle|L_3\rangle)$$

(ii) Four states are the eigenkets of $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$ with the eigenvalue (+1). These states are not the GHZ state.

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|R_3\rangle+|L_1\rangle|R_2\rangle|L_3\rangle+|R_1\rangle|L_2\rangle|L_3\rangle+|R_1\rangle|R_2\rangle|R_3\rangle)$$

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|R_3\rangle-|L_1\rangle|R_2\rangle|L_3\rangle-|R_1\rangle|L_2\rangle|L_3\rangle+|R_1\rangle|R_2\rangle|R_3\rangle)$$

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|R_3\rangle+|L_1\rangle|R_2\rangle|L_3\rangle-|R_1\rangle|L_2\rangle|L_3\rangle+|R_1\rangle|R_2\rangle|R_3\rangle)$$

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|R_3\rangle+|L_1\rangle|R_2\rangle|L_3\rangle-|R_1\rangle|L_2\rangle|L_3\rangle-|R_1\rangle|R_2\rangle|R_3\rangle)$$

In conclusion, no state corresponding to the GHZ state is obtained in this configuration.

13. Eigenstates of $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x$

$\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x$ =

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are eight eigenstates. The eigenvalues of the four states is -1 and the eigenvalue of the four states is (+1).

(i) Four states are the eigenkets of $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x$ with the eigenvalue (-1). One of these states is the GHZ state $|\Psi_{GHZ}{}^{XXX(+)}\rangle$, given by

$$\left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle = \frac{1}{\sqrt{2}}(|y_1\rangle|y_2\rangle|x_3\rangle + |x_1\rangle|x_2\rangle|y_3\rangle$$
$$= \frac{1}{2}(|L_1\rangle|L_2\rangle|V_3'\rangle + |L_1\rangle|R_2\rangle|H_3'\rangle + |R_1\rangle|L_2\rangle|H_3'\rangle + |R_1\rangle|R_2\rangle|V_3'\rangle)$$

The other three states are not the GHZ state;

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|V_3'\rangle - |L_1\rangle|R_2\rangle|H_3'\rangle - |R_1\rangle|L_2\rangle|H_3'\rangle + |R_1\rangle|R_2\rangle|V_3'\rangle),$$

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|V_3'\rangle + |L_1\rangle|R_2\rangle|H_3'\rangle - |R_1\rangle|L_2\rangle|H_3'\rangle + |R_1\rangle|R_2\rangle|V_3'\rangle),$$

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|V_3'\rangle + |L_1\rangle|R_2\rangle|H_3'\rangle - |R_1\rangle|L_2\rangle|H_3'\rangle - |R_1\rangle|R_2\rangle|V_3'\rangle).$$

(ii) Four states are the eigenkets of $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x$ with the eigenvalue (+1). One of these state is the GHZ state $\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle$,

$$\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle|x_2\rangle|x_3\rangle - |y_1\rangle|y_2\rangle|y_3\rangle$$
$$= \frac{1}{2}(|L_1\rangle|L_2\rangle|H_3'\rangle + |L_1\rangle|R_2\rangle|V_3'\rangle + |R_1\rangle|L_2\rangle|V_3'\rangle + |R_1\rangle|R_2\rangle|H_3'\rangle)$$

The other three states are not the GHZ state,

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|H_3'\rangle + |L_1\rangle|R_2\rangle|V_3'\rangle + |R_1\rangle|L_2\rangle|V_3'\rangle - |R_1\rangle|R_2\rangle|H_3'\rangle),$$

$$\frac{1}{2}(-|L_1\rangle|L_2\rangle|H_3'\rangle + |L_1\rangle|R_2\rangle|V_3'\rangle - |R_1\rangle|L_2\rangle|V_3'\rangle + |R_1\rangle|R_2\rangle|H_3'\rangle),$$

$$\frac{1}{2}(|L_1\rangle|L_2\rangle|H_3'\rangle + |L_1\rangle|R_2\rangle|V_3'\rangle - |R_1\rangle|L_2\rangle|V_3'\rangle - |R_1\rangle|R_2\rangle|H_3'\rangle).$$

14. Eigenvalue problem $\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$

$\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$ =

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are eight eigenstates. The eigenvalues of the four states is -1 and the eigenvalue of the four states is (+1).

(i) Four states are the eigenkets of $\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$ with the eigenvalue (-1). One state is the GHZ state $\left|\Psi_{GHZ}{}^{(+)}\right\rangle$,

$$\begin{aligned} \left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle &= \frac{1}{\sqrt{2}}(|x_1\rangle|x_2\rangle|x_3\rangle + |y_1\rangle|y_2\rangle|y_3\rangle \\ &= \frac{1}{2}(|H_1'\rangle|L_2\rangle|R_3\rangle + |H_1'\rangle|R_2\rangle|L_3\rangle + |V_1'\rangle|L_2\rangle|L_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|R_3\rangle) \end{aligned}$$

The other three states are not the GHZ state;

$$\frac{1}{2}(|H_1'\rangle|L_2\rangle|R_3\rangle - |H_1'\rangle|R_2\rangle|L_3\rangle - |V_1'\rangle|L_2\rangle|L_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|R_3\rangle),$$

$$\frac{1}{2}(-|H_1'\rangle|L_2\rangle|R_3\rangle + |H_1'\rangle|R_2\rangle|L_3\rangle - |V_1'\rangle|L_2\rangle|L_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|R_3\rangle),$$

$$\frac{1}{2}(-|H_1'\rangle|L_2\rangle|R_3\rangle - |H_1'\rangle|R_2\rangle|L_3\rangle + |V_1'\rangle|L_2\rangle|L_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|R_3\rangle).$$

(ii) The four states are the eigenkets of $\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$ with the eigenvalue (+1). One state is the GHZ state denoted by $\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle$,

$$\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle|x_2\rangle|x_3\rangle - |y_1\rangle|y_2\rangle|y_3\rangle$$
$$= \frac{1}{2}(|H_1'\rangle|L_2\rangle|L_3\rangle + |H_1'\rangle|R_2\rangle|R_3\rangle + |V_1'\rangle|L_2\rangle|R_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|L_3\rangle)$$

The other three states are not the GHZ state;

$$\frac{1}{2}(-|H_1'\rangle|L_2\rangle|L_3\rangle + |H_1'\rangle|R_2\rangle|R_3\rangle + |V_1'\rangle|L_2\rangle|R_3\rangle - \left|V_1^{'}\right\rangle|R_2\rangle|L_3\rangle),$$

$$\frac{1}{2}(-|H_1'\rangle|L_2\rangle|L_3\rangle + |H_1'\rangle|R_2\rangle|R_3\rangle - |V_1'\rangle|L_2\rangle|R_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|L_3\rangle),$$

$$\frac{1}{2}(-|H_1'\rangle|L_2\rangle|L_3\rangle - |H_1'\rangle|R_2\rangle|R_3\rangle + |V_1'\rangle|L_2\rangle|R_3\rangle + \left|V_1^{'}\right\rangle|R_2\rangle|L_3\rangle),$$

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15. Eigenvalue problem $\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y$

$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are eight eigenstates. The eigenvalues of the four states is -1 and the eigenvalue of the four states is (+1). There is no eigenkets corresponding to the GHZ state.

16. The GHS state in the configuration *YYX*, *YXY*, and *XYY*

(i) $\left|\Psi_{GHZ}{}^{YYX(+)}\right\rangle$ state

$$\left|\psi_{GHZ}{}^{YYX(+)}\right\rangle = \frac{1}{2}[\left|R_1\right\rangle \otimes \left|L_2\right\rangle \otimes \left|H_3'\right\rangle + \left|L_1\right\rangle \otimes \left|R_2\right\rangle \otimes \left|H_3'\right\rangle + \left|R_1\right\rangle \otimes \left|R_2\right\rangle \otimes \left|V_3'\right\rangle + \left|L_1\right\rangle \otimes \left|L_2\right\rangle \otimes \left|V_3'\right\rangle]$$

$$\left|\psi_{GHZ}{}^{YXY(+)}\right\rangle = \frac{1}{2}[\left|L_1\right\rangle \otimes \left|H_2'\right\rangle \otimes \left|R_3\right\rangle + \left|R_1\right\rangle \otimes \left|H_2'\right\rangle \otimes \left|L_3\right\rangle + \left|R_1\right\rangle \otimes \left|V_2'\right\rangle \otimes \left|R_3\right\rangle + \left|L_1\right\rangle \otimes \left|V_2'\right\rangle \otimes \left|L_3\right\rangle]$$

$$\left|\psi_{GHZ}{}^{XYY(+)}\right\rangle = \frac{1}{2}[\left|H_1'\right\rangle \otimes \left|R_2\right\rangle \otimes \left|L_3\right\rangle + \left|H_1'\right\rangle \otimes \left|L_2\right\rangle \left|R_3\right\rangle + \left|V_1'\right\rangle \otimes \left|R_2\right\rangle \otimes \left|R_3\right\rangle + \left|V_1'\right\rangle \otimes \left|L_2\right\rangle \otimes \left|L_3\right\rangle]$$

where

$$\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x \left|\psi_{GHZ}{}^{YYX(+)}\right\rangle = -\left|\psi_{GHZ}{}^{YYX(+)}\right\rangle,$$

$$\hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y \left|\psi_{GHZ}{}^{YXY(+)}\right\rangle = -\left|\psi_{GHZ}{}^{YXY(+)}\right\rangle,$$

$$\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y \left|\psi_{GHZ}{}^{XYY(+)}\right\rangle = -\left|\psi_{GHZ}{}^{XYY(+)}\right\rangle.$$

(ii) $\left|\Psi_{GHZ}{}^{YYX(-)}\right\rangle$ state

$$\left|\Psi_{GHZ}{}^{YYX(-)}\right\rangle = \frac{1}{2}(\left|L_1\right\rangle\left|L_2\right\rangle\left|H_3'\right\rangle + \left|R_1\right\rangle\left|R_2\right\rangle\left|H_3'\right\rangle + \left|L_1\right\rangle\left|R_2\right\rangle\left|V_3'\right\rangle + \left|R_1\right\rangle\left|L_2\right\rangle\left|V_3'\right\rangle),$$

$$\left|\Psi_{GHZ}{}^{YXY(-)}\right\rangle = \frac{1}{2}(\left|L_1\right\rangle\left|H_2'\right\rangle\left|L_3\right\rangle + \left|R_1\right\rangle\left|H_2'\right\rangle\left|R_3\right\rangle + \left|L_1\right\rangle\left|V_2'\right\rangle\left|R_3\right\rangle + \left|R_1\right\rangle\left|V_2'\right\rangle\left|L_3\right\rangle),$$

$$\left|\Psi_{GHZ}{}^{XYY(-)}\right\rangle = \frac{1}{2}(\left|H_1'\right\rangle\left|L_2\right\rangle\left|L_3\right\rangle + \left|H_1'\right\rangle\left|R_2\right\rangle\left|R_3\right\rangle + \left|V_1'\right\rangle\left|L_2\right\rangle\left|R_3\right\rangle + \left|V_1'\right\rangle\left|R_2\right\rangle\left|L_3\right\rangle),$$

where

$$\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x \left|\psi_{GHZ}{}^{YYX(-)}\right\rangle = +\left|\psi_{GHZ}{}^{YYX(-)}\right\rangle,$$

$$\hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y \left|\psi_{GHZ}{}^{YXY(-)}\right\rangle = +\left|\psi_{GHZ}{}^{YXY(-)}\right\rangle,$$

$$\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y \left|\psi_{GHZ}{}^{XYY(-)}\right\rangle = +\left|\psi_{GHZ}{}^{XYY(-)}\right\rangle.$$

((**Note**))

(i) The GHZ state $\left|\Psi_{GHZ}{}^{(+)}\right\rangle$ is an eigenket of three operators

$$\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y, \; \hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y, \; \hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x,$$

with corresponding eigenvalue -1

(ii) The GHZ state $\left|\Psi_{GHZ}{}^{(+)}\right\rangle$ is an eigenket of $\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$ with corresponding eigenvalue +1.

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APPENDIX **Comment on the eigenstates of $\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x}$**

We consider the eigenstates of $\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x}$. In quantum mechanics, we get

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} \left|V_1'\right\rangle \otimes \left|V_2'\right\rangle \otimes \left|V_3'\right\rangle = -\left|V_1'\right\rangle \otimes \left|V_2'\right\rangle \otimes \left|V_3'\right\rangle,$$

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |H_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle = -|H_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle,$$

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |H_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle = -|H_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle,$$

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |V_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle = -|V_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle.$$

Then the four states

$|V_1'\rangle \otimes |V_2'\rangle \otimes |V_3'\rangle$, $|H_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle$, $|H_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle$, and $|V_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle$,

have the same eigenvalue (-1). Thus any superposition of these states has the eigenvalue (-1). However, there is only one GHZ state, among these states, as

$$\left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle = \frac{1}{2}(|H_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle + |H_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle$$
$$+ |V_1'\rangle|H_2'\rangle|H_3'\rangle + |V_1'\rangle|V_2'\rangle|V_3'\rangle)$$

where

$$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \left|\Psi_{GHZ}{}^{XXX(-)}\right\rangle = -\left|\Psi_{GHZ}{}^{(-)}\right\rangle.$$

Similarly, in quantum mechanics we get

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |H_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle = |H_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle,$$

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |V_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle = |V_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle,$$

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |V_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle = |V_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle,$$

$$\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x} \otimes \hat{\sigma}_{3x} |H_1'\rangle \otimes |V_2'\rangle \otimes |V_3'\rangle = |H_1'\rangle \otimes |V_2'\rangle \otimes |V_3'\rangle.$$

Then the four states

$|H_1'\rangle \otimes |H_2'\rangle \otimes |H_3'\rangle$, $|V_1'\rangle \otimes |V_2'\rangle \otimes |H_3'\rangle$, $|V_1'\rangle \otimes |H_2'\rangle \otimes |V_3'\rangle$, and $|H_1'\rangle \otimes |V_2'\rangle \otimes |V_3'\rangle$

have the same eigenvalue (+1). Thus any superposition of these states has the eigenvalue (+1). However, there is only one GHZ state, among these states, as

$$\left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle = \frac{1}{2}(\left|H_1'\right\rangle\left|H_2'\right\rangle\left|H_3'\right\rangle + \left|H_1'\right\rangle\left|V_2'\right\rangle\left|V_3'\right\rangle + \left|V_1'\right\rangle\left|H_2'\right\rangle\left|V_3'\right\rangle + \left|V_1'\right\rangle\left|V_2'\right\rangle\left|H_3'\right\rangle),$$

where

$$\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle = +\left|\Psi_{GHZ}{}^{XXX(+)}\right\rangle.$$