Quantum teleportation

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Quantum teleportation is a process by which quantum information (e.g. the exact state of an atom or photon) can be transmitted (exactly in principle) from one location to another, with the help of classical communication and previously shared quantum entanglement between the sending and receiving location. Because it depends on classical communication, which can proceed no faster than the speed of light, it cannot be used for superluminal transport or communication. And because it disrupts the quantum system at the sending location, it cannot be used to violate the no-cloning theorem by producing two copies of the system. Quantum teleportation is unrelated to the kind of teleportation commonly used in fiction, as it does not transport the system itself, does not function instantaneously, and does not concern rearranging particles to copy the form of an object. Thus, despite the provocative name, it is best thought of as a kind of communication, rather than a kind of transportation. The seminal paper first expounding the idea was published by C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters in 1993. Since then, quantum teleportation has been realized in various physical systems. Presently, the record distance for quantum teleportation is 143 km (89 mi) with photons, and 21m with material systems. On September 11th, 2013, the "Furusawa group at the University of Tokyo has succeeded in demonstrating complete quantum teleportation of photonic quantum bits by a hybrid technique for the first time worldwide."

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1. Quantum teleportation

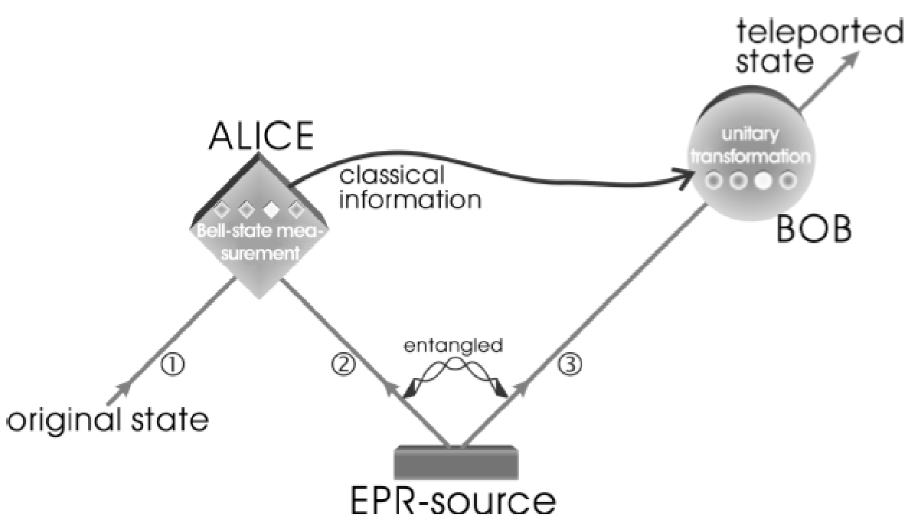


Fig. Schematic diagram of quantum teleportation. When Alice and Bob each receive one particle of an entangled pair from an EPR source (for Einstein-Podolsky-Rosen), the quantum state from particle 1 can be transferred (teleported) to particle 3.

[J. Audretsch (Editor) Entangled World (Wiley-VCH Verlag GmbH & Co, KGaA, Weinheim, 2002).] p.155 Fig.6.3.

We consider the four Bell-states given by

$$\left|\psi_{12}^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|-z\right\rangle_2 \pm \left|-z\right\rangle_1\left|+z\right\rangle_2],$$

$$\left|\Phi_{12}^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|+z\right\rangle_2 \pm \left|-z\right\rangle_1\left|-z\right\rangle_2].$$

The state of the particle 1, the particle Alice wishes to teleport, is simply

$$\left|\psi_1\right\rangle = a\left|+\right\rangle_1 + b\left|-\right\rangle_1.$$

Before Alice makes a measurement, the state of the three particles is given by

$$\begin{aligned}\left|\psi_{123}\right\rangle &= (a\left|+z\right\rangle_1 + b\left|-z\right\rangle_1)\left|\psi_{23}^{(-)}\right\rangle \\ &= (a\left|+z\right\rangle_1 + b\left|-z\right\rangle_1)\frac{1}{\sqrt{2}}[\left|+z\right\rangle_2\left|-z\right\rangle_3 - \left|-z\right\rangle_2\left|+z\right\rangle_3] \\ &= \frac{1}{\sqrt{2}}a(\left|+z\right\rangle_1\left|+z\right\rangle_2\left|-z\right\rangle_3 - \left|+z\right\rangle_1\left|-z\right\rangle_2\left|+z\right\rangle_3) \\ &\quad + \frac{1}{\sqrt{2}}b(\left|-z\right\rangle_1\left|+z\right\rangle_2\left|-z\right\rangle_3 - \left|-z\right\rangle_1\left|-z\right\rangle_2\left|+z\right\rangle_3)\end{aligned}$$

where

$$\left|\psi_{23}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_2\left|-z\right\rangle_3 - \left|-z\right\rangle_2\left|+z\right\rangle_3].$$

Alice makes a special type of measurement called a Bell-state measurement.

(a) The state of Bell basis:

$$\left|\psi_{12}^{(+)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|-z\right\rangle_2 + \left|-z\right\rangle_1\left|+z\right\rangle_2] = \left|1,0\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$

$$\left|\psi_{12}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|-z\right\rangle_2 - \left|-z\right\rangle_1\left|+z\right\rangle_2] = \left|0,0\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix},$$

where $\left|1,0\right\rangle = \left|j=1, m=0\right\rangle$, $\left|0,0\right\rangle = \left|j=0, m=0\right\rangle$.

(b) The additional basis (it is still called Bell basis)

$$\left|\Phi_{12}^{(+)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|+z\right\rangle_2 + \left|-z\right\rangle_1\left|-z\right\rangle_2] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

$$\left|\Phi_{12}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|+z\right\rangle_2 - \left|-z\right\rangle_1\left|-z\right\rangle_2] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}.$$

Then we have

$$\left|+z\right\rangle_1\left|-z\right\rangle_2 = \frac{1}{\sqrt{2}}(\left|\psi_{12}^{(+)}\right\rangle + \left|\psi_{12}^{(-)}\right\rangle)$$
$$\left|-z\right\rangle_1\left|+z\right\rangle_2 = \frac{1}{\sqrt{2}}(\left|\psi_{12}^{(+)}\right\rangle - \left|\psi_{12}^{(-)}\right\rangle)$$

$$\left|+z\right\rangle_1\left|+z\right\rangle_2 = \frac{1}{\sqrt{2}}(\left|\Phi_{12}^{(+)}\right\rangle + \left|\Phi_{12}^{(-)}\right\rangle) = \left|1,1\right\rangle$$
$$\left|-z\right\rangle_1\left|-z\right\rangle_2 = \frac{1}{\sqrt{2}}(\left|\Phi_{12}^{(+)}\right\rangle - \left|\Phi_{12}^{(-)}\right\rangle) = \left|1,-1\right\rangle$$

where

$$|1,1\rangle = |\, j=1, m=1\rangle\,, \qquad |1,-1\rangle = |\, j=1, m=-1\rangle\,.$$

Using the above Bell basis, the state $|\psi_{123}\rangle$

$$|\psi_{123}\rangle = \frac{1}{\sqrt{2}}a(|+z\rangle_1|+z\rangle_2|-z\rangle_3 - |+z\rangle_1|-z\rangle_2|+z\rangle_3)$$
$$+\frac{1}{\sqrt{2}}b(|-z\rangle_1|+z\rangle_2|-z\rangle_3 - |-z\rangle_1|-z\rangle_2|+z\rangle_3)$$

can be expressed as follows.

$$\begin{aligned}|\psi_{123}\rangle &= \frac{1}{2}a((|\Phi_{12}^{(+)}\rangle + |\Phi_{12}^{(-)}\rangle)|-z\rangle_3 - \frac{1}{2}a(|\psi_{12}^{(+)}\rangle + |\psi_{12}^{(-)}\rangle)|+z\rangle_3) \\ &\quad + \frac{1}{2}b((|\psi_{12}^{(+)}\rangle - |\psi_{12}^{(-)}\rangle)|-z\rangle_3 - \frac{1}{2}b(|\Phi_{12}^{(+)}\rangle - |\Phi_{12}^{(-)}\rangle)|+z\rangle_3) \\ &= \frac{1}{2}a((|\Phi_{12}^{(+)}\rangle + |\Phi_{12}^{(-)}\rangle)|-z\rangle_3 - \frac{1}{2}a(|\psi_{12}^{(+)}\rangle + |\psi_{12}^{(-)}\rangle)|+z\rangle_3) \\ &\quad + \frac{1}{2}b((|\psi_{12}^{(+)}\rangle - |\psi_{12}^{(-)}\rangle)|-z\rangle_3 - \frac{1}{2}b(|\Phi_{12}^{(+)}\rangle - |\Phi_{12}^{(-)}\rangle)|+z\rangle_3) \\ &= \frac{1}{2}|\psi_{12}^{(-)}\rangle(-a|+z\rangle_3 - b)|-z\rangle_3) + \frac{1}{2}|\psi_{12}^{(+)}\rangle(-a|+z\rangle_3 + b|-z\rangle_3) \\ &\quad + \frac{1}{2}|\Phi_{12}^{(-)}\rangle(a|-z\rangle_3 + b|+z\rangle_3) + \frac{1}{2}|\Phi_{12}^{(+)}\rangle(a|-z\rangle_3 - b|+z\rangle_3)\end{aligned}$$

or

$$\begin{aligned}|\psi_{123}\rangle &= \frac{1}{2}|\psi_{12}^{(-)}\rangle(-a|+z\rangle_3 - b)|-z\rangle_3) \\ &\quad + \frac{1}{2}|\psi_{12}^{(+)}\rangle(-a|+z\rangle_3 + b|-z\rangle_3) \\ &\quad + \frac{1}{2}|\Phi_{12}^{(-)}\rangle(b|+z\rangle_3 + a|-z\rangle_3) \\ &\quad + \frac{1}{2}|\Phi_{12}^{(+)}\rangle(-b|+z\rangle_3 + a|-z\rangle_3)\end{aligned}$$

2. The use of $|\psi_{12}^{(-)}\rangle$

If Alice's Bell-state measurement on particles 1 and 2 collapses the two particle state to the state $\left|\psi_{12}^{(-)}\right\rangle$,

$$\left|\psi_{12}^{(-)}\right\rangle(-a\left|+z\right\rangle_3 - b\left|-z\right\rangle_3),$$

then the particle 3, Bob's particle, is forced to be in the state

$$\left|\psi_3\right\rangle = -a\left|+z\right\rangle_3 - b\left|-z\right\rangle_3 = -(a\left|+z\right\rangle_3 + b\left|-z\right\rangle_3),$$

which is exactly the state, up to an overall phase, of the particle before the measurement.

This is a dramatic illustration of the spooky action at a distance of entangled states that so troubled Einstein.

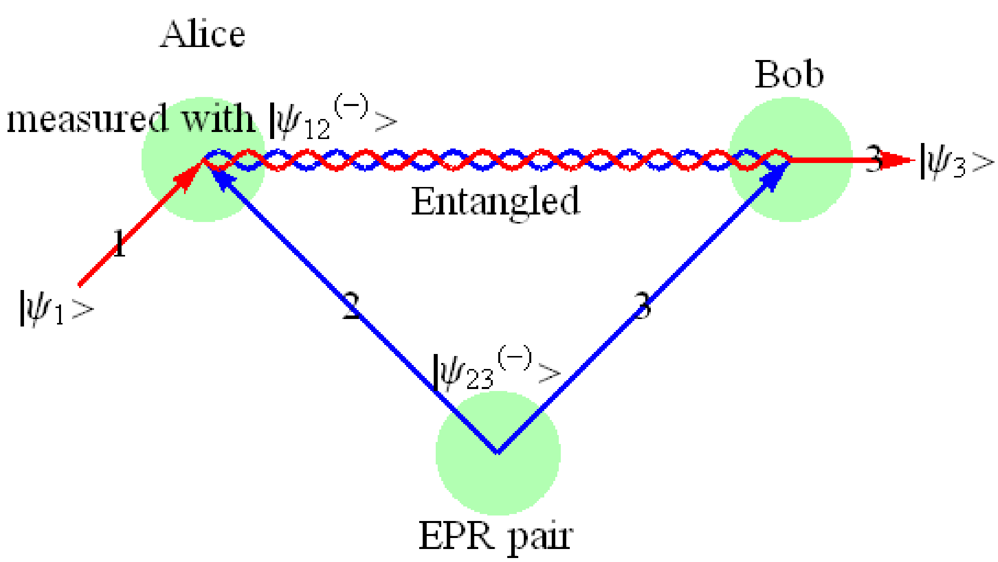


Fig. Alice measures with the Bell's state $\left|\psi_{12}^{(-)}\right\rangle$. After Bob measures, we get the state $\left|\psi_3\right\rangle = -(a\left|+z\right\rangle_3 + b\left|-z\right\rangle_3)$, which is the same as $\left|\psi_1\right\rangle$, except for the phase factor (-1). The entangled line should not be confused with the classical information line. After Alice measures with the Bell's state, Bob instantly get a copy state $\left|\psi_3\right\rangle$.

3. The use of the Bell's state $\left|\Phi_{12}^{(+)}\right\rangle$ and the rotation operator $\hat{R}_y(\pi)$

If Alice's Bell-state measurement on particles 1 and 2 collapses the two particle state to the state $\left|\Phi_{12}^{(+)}\right\rangle$,

$$\left|\Phi_{12}^{(+)}\right\rangle(-b\left|+z\right\rangle_3+a\left|-z\right\rangle_3),$$

then the particle 3, Bob's particle, is forced to be in the state

$$\left|\psi_3\right\rangle=(-b\left|+z\right\rangle_3+a\left|-z\right\rangle_3)=\begin{pmatrix}-b\\a\end{pmatrix},$$

which is exactly the state, up to an overall phase, of the particle before the measurement. We consider the rotation operator around the *x* axis.

$$\hat{R}_y(\theta)=\cos\frac{\theta}{2}\hat{1}-i\sin\frac{\theta}{2}\hat{\sigma}_y=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix},$$

and

$$\hat{R}_y(\pi)=\begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Then we have

$$\hat{R}_y(\pi)\left|\psi_3\right\rangle=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-b\\a\end{pmatrix}=-\begin{pmatrix}a\\b\end{pmatrix}=-(a\left|+z\right\rangle_3+b\left|-z\right\rangle_3).$$

which is the same as $\left|\psi_1\right\rangle$, except for the phase factor (-1).

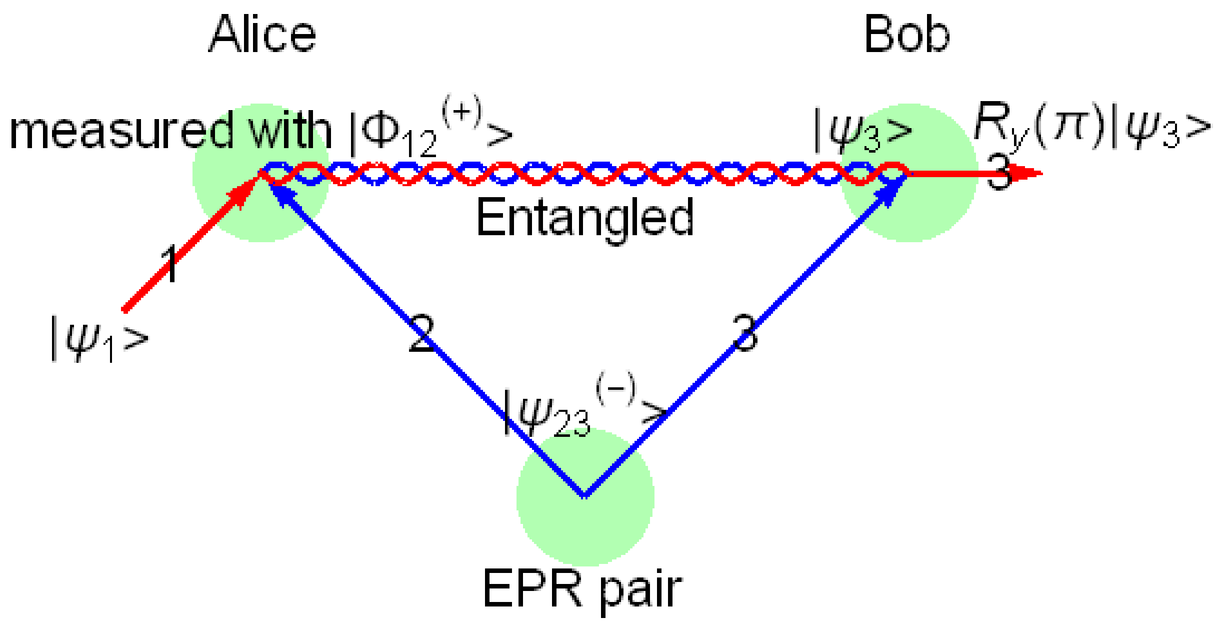


Fig. Alice measures with the Bell's state $\left|\Phi_{12}^{(+)}\right\rangle$. After Bob measures, we get the state $-b\left|+z\right\rangle_3+a\left|-z\right\rangle_3$. Through the unitary operator $\hat{R}_y(\pi)$, we get the state $-(a\left|+z\right\rangle_3+b\left|-z\right\rangle_3)$, which is the same as $\left|\psi_1\right\rangle$, except for the phase factor (-1).

4. The use of $\left|\Phi_{12}^{(-)}\right\rangle$ and $\hat{R}_x(\pi)$

If Alice's Bell state measurement results in her tangling particles 1 and 2 in the state $\left|\Phi_{12}{}^{(-)}\right\rangle$,

$$\left|\Phi_{12}^{(-)}\right\rangle[b\left|+z\right\rangle_3+a\left|-z\right\rangle_3],$$

then the particle 3, Bob's particle, is forced to be in the state

$$\left|\psi_3\right\rangle=b\left|+z\right\rangle_3+a\left|-z\right\rangle_3=\begin{pmatrix} b \\ a \end{pmatrix},$$

which is not the same as $\left|\psi_1\right\rangle$. Here we use the rotation operator

$$\hat{R}_x(\theta)=\exp(-\frac{i}{\hbar}\hat{S}_x\theta)=\cos\frac{\theta}{2}\hat{1}-i\hat{\sigma}_x\sin\frac{\theta}{2},$$

and

$$\hat{R}_x(\pi) = \cos\frac{\pi}{2}\hat{1} - i\hat{\sigma}_x \sin\frac{\pi}{2} = -i\hat{\sigma}_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Then we have

$$\hat{R}_x(\pi)|\psi_3\rangle = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} = -i\begin{pmatrix} a \\ b \end{pmatrix} = -i(a|+z\rangle_3 + b|-z\rangle_3).$$

which is the same as $|\psi_1\rangle$, except for the phase factor (-*i*).

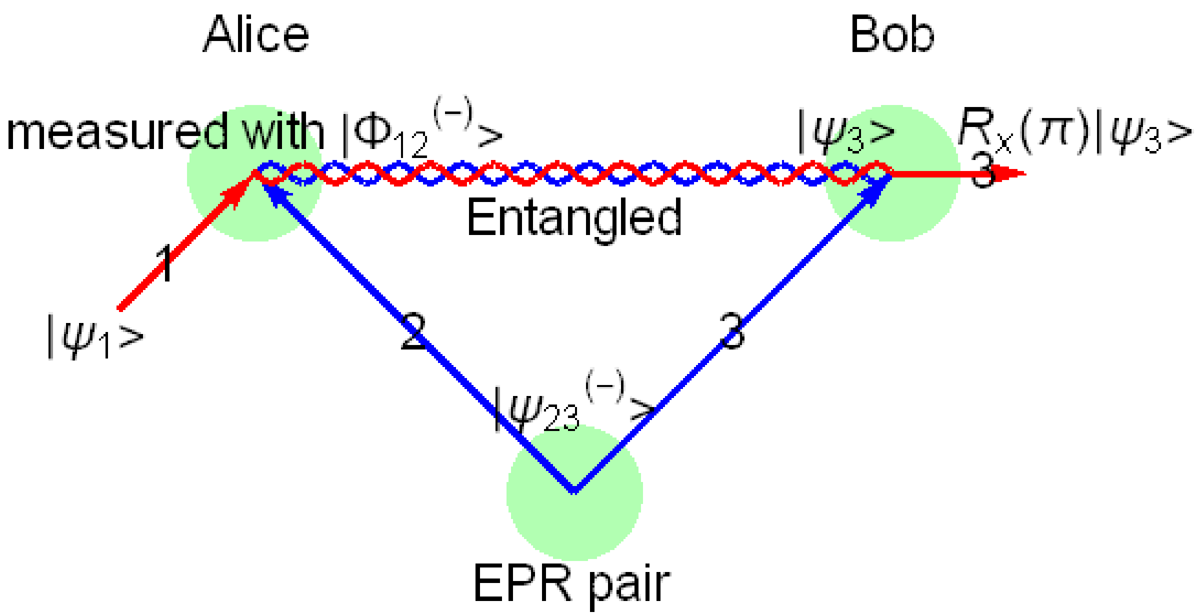


Fig. Alice measures with the Bell's state $|\Phi_{12}^{(-)}\rangle$. After Bob measures, we get the state $b|+z\rangle_3 + a|-z\rangle_3$. Through the unitary operator $\hat{R}_x(\pi)$, we get the state $-i(a|+z\rangle_3 + b|-z\rangle_3)$, which is the same as $|\psi_1\rangle$.

5. The use of $|\psi_{12}^{(+)}\rangle$ and $\hat{R}_z(\pi)$

If Alice's Bell state measurement results in her tangling particles 1 and 2 in the state $|\Phi_{12}^{(-)}\rangle$,

$$|\psi_{12}^{(+)}\rangle[-a|+z\rangle_3 + b|-z\rangle_3],$$

then the particle 3, Bob's particle, is forced to be in the state

$$-a|+z\rangle_3 + b|-z\rangle_3 = \begin{pmatrix} -a \\ b \end{pmatrix}.$$

which is not the same as $|\psi_1\rangle$. Here we use the rotation operator

$$\hat{R}_z(\phi) = \cos\frac{\phi}{2}\hat{1} - i\hat{\sigma}_z \sin\frac{\phi}{2} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix},$$

or

$$\hat{R}_z(\pi) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Thus using matrix mechanics, we have

$$\hat{R}_z(\pi)[-a|+z\rangle_3 + b|-z\rangle_3] = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} -a \\ b \end{pmatrix} = i\begin{pmatrix} a \\ b \end{pmatrix},$$

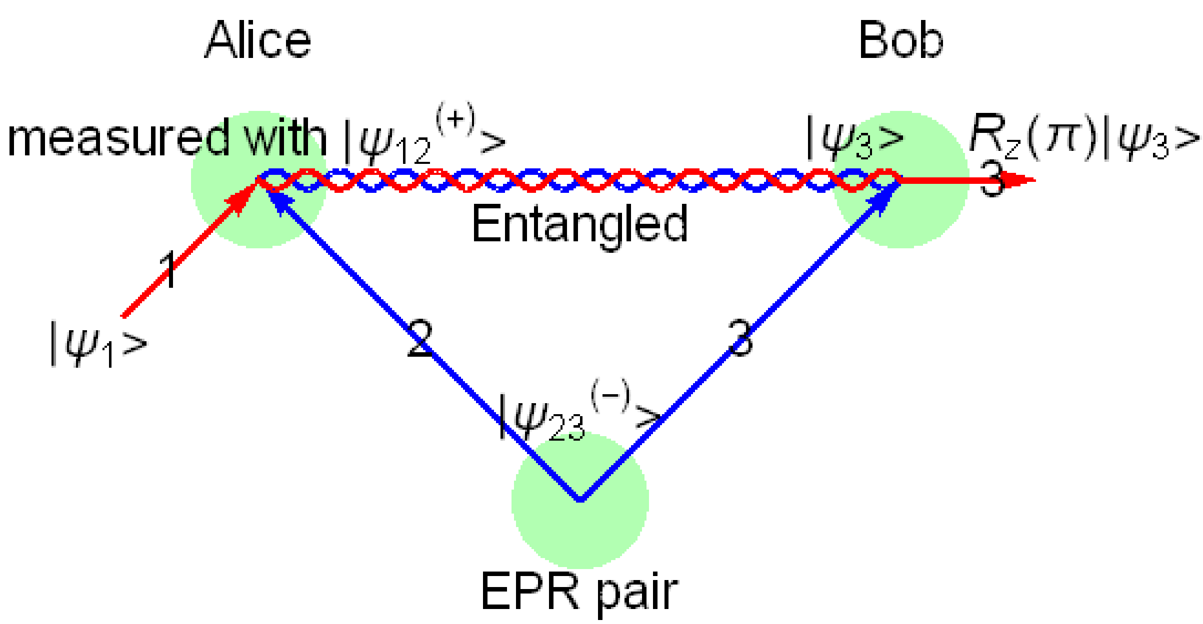
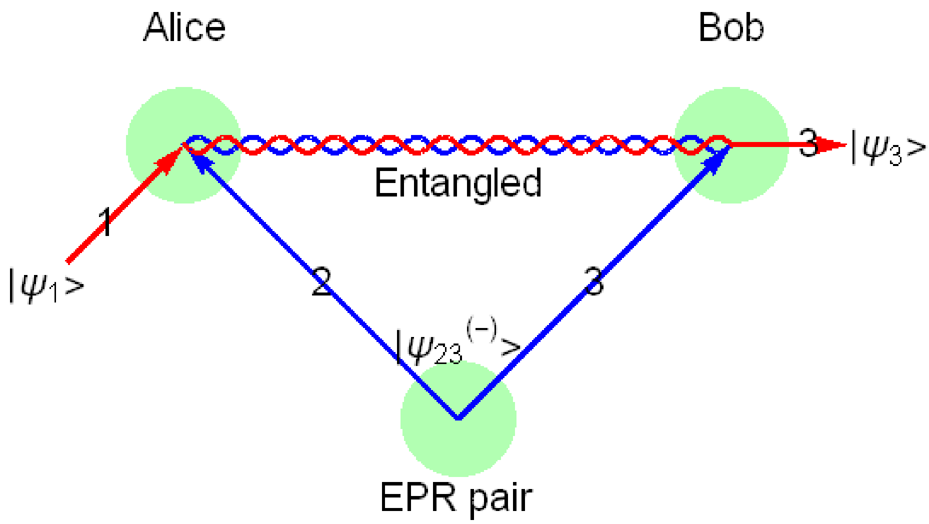


Fig. Alice measures with the Bell's state $|\psi_{12}^{(+)}\rangle$. After Bob measures, we get the state $-a|+z\rangle_3 + b|-z\rangle_3$. Through the unitary operator $\hat{R}_z(\pi)$, we get the state $i(a|+z\rangle_3 + b|-z\rangle_3)$, which is the same as $|\psi_1\rangle$, except for the phase factor *i*.

5. Approach from the reduced density operator



We consider the pure particle state $|\psi_{123}\rangle$ which is related to the quantum teleportation. The density operator for this pure state is given by

$$\hat{\rho} = |\psi_{123}\rangle\langle\psi_{123}|,$$

where

$$|\psi_{123}\rangle = (a|+z\rangle_1 + b|-z\rangle_1)|\psi_{23}^{(-)}\rangle$$
$$= \frac{1}{2}|\psi_{12}^{(-)}\rangle[-a|+z\rangle_3 - b|-z\rangle_3] + \frac{1}{2}|\psi_{12}^{(+)}\rangle[-a|+z\rangle_3 + b|-z\rangle_3]$$
$$+ \frac{1}{2}|\Phi_{12}^{(-)}\rangle[a|-z\rangle_3 + b|+z\rangle_3] + \frac{1}{2}|\Phi_{12}^{(+)}\rangle[a(|-z\rangle_3 - b|+z\rangle_3)$$

and

$$|\psi_{23}^{(-)}\rangle = \frac{1}{\sqrt{2}}[|+z,2\rangle|-z,3\rangle + |-z,2\rangle|+z,3\rangle].$$

Note that there are four Bell's states for particles 1 and 2, which are defined by

$$\left|\psi_{12}^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|-z\right\rangle_2 \pm \left|-z\right\rangle_1\left|+z\right\rangle_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm 1 \\ 0 \end{pmatrix},$$

$$\left|\Phi_{12}^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|+z\right\rangle_2 \pm \left|-z\right\rangle_1\left|-z\right\rangle_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix}.$$

Note that

$$|a|^2 + |b|^2 = 1. \qquad \text{(normalization)}$$

The density operator $\hat{\rho}$ can be obtained as

$$\hat{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |a|^2/2 & -|a|^2/2 & 0 & 0 & (ab^*)/2 & -(ab^*)/2 & 0 \\ 0 & -|a|^2/2 & |a|^2/2 & 0 & 0 & -(ab^*)/2 & (ab^*)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (a^*b)/2 & -(a^*b)/2 & 0 & 0 & |b|^2/2 & -|b|^2/2 & 0 \\ 0 & -(a^*b)/2 & (a^*b)/2 & 0 & 0 & -|b|^2/2 & |b|^2/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Tracing out particle 1, the reduced density operators are obtained as

$$\hat{\rho}_{23} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & -|a|^2 & 0 \\ 0 & -|a|^2 & |a|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |b|^2 & -|b|^2 & 0 \\ 0 & -|b|^2 & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2+|b|^2 & -|a|^2-|b|^2 & 0 \\ 0 & -|a|^2-|b|^2 & |a|^2+|b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This reduced operator is the same as that of the density operator for the Bell's state. Note that for the Bell's two-particle entangled state,

$$\left|\Psi_{12}{}^{(-)}\right\rangle = \frac{1}{\sqrt{2}}[\left|+z;1\right\rangle\left|-z;2\right\rangle - \left|-z;1\right\rangle\left|+z;2\right\rangle],$$

we have the density operator given by

$$\hat{\rho} = \left|\Psi_{12}{}^{(-)}\right\rangle\left\langle\Psi_{12}{}^{(-)}\right| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Tracing over particle 2 furthermore, we have

$$\hat{\rho}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is equivalent to a completely un-polarized state. So Bob (particle 3) has no information about the state of the particle Alice is attempting to teleport. On the other hand, if Bob waits until he receives the result of Alice's Bell state measurement, Bob can then maneuver his particle into the state $|\psi\rangle$ that Alice's particle was in initially.

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$$\psi 1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}; \ \psi 2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix};$$

$$\phi 1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix};$$

$$\phi 2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

$$\chi 1 = \begin{pmatrix} -a \\ -b \end{pmatrix}; \ \chi 2 = \begin{pmatrix} -a \\ b \end{pmatrix}; \ \chi 3 = \begin{pmatrix} b \\ a \end{pmatrix}; \ \chi 4 = \begin{pmatrix} -b \\ a \end{pmatrix};$$

```
ψ123 = 1/2 KroneckerProduct[ψ1, χ1] + 1/2 KroneckerProduct[ψ2, χ2] +
    1/2 KroneckerProduct[ϕ1, χ3] + 1/2 KroneckerProduct[ϕ2, χ4] //
   Simplify;

K1 = Transpose[ψ123][[1]];

K2 = Transpose[ψ123] //. {a → a1, b → b1};

ρ = Outer[Times, K1, K2[[1]]] // FullSimplify;
ρ // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{a\,a1}{2} & -\frac{a\,a1}{2} & 0 & 0 & \frac{a\,b1}{2} & -\frac{a\,b1}{2} & 0 \\ 0 & -\frac{a\,a1}{2} & \frac{a\,a1}{2} & 0 & 0 & -\frac{a\,b1}{2} & \frac{a\,b1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{a1\,b}{2} & -\frac{a1\,b}{2} & 0 & 0 & \frac{b\,b1}{2} & -\frac{b\,b1}{2} & 0 \\ 0 & -\frac{a1\,b}{2} & \frac{a1\,b}{2} & 0 & 0 & -\frac{b\,b1}{2} & \frac{b\,b1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6. Approach from the quantum qubits

Suppose Alice and Bob share a pair of qubits in the entangled state

$$\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle .$$

Alice needs to communicate to Bob one qubit of information

$$\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle .$$

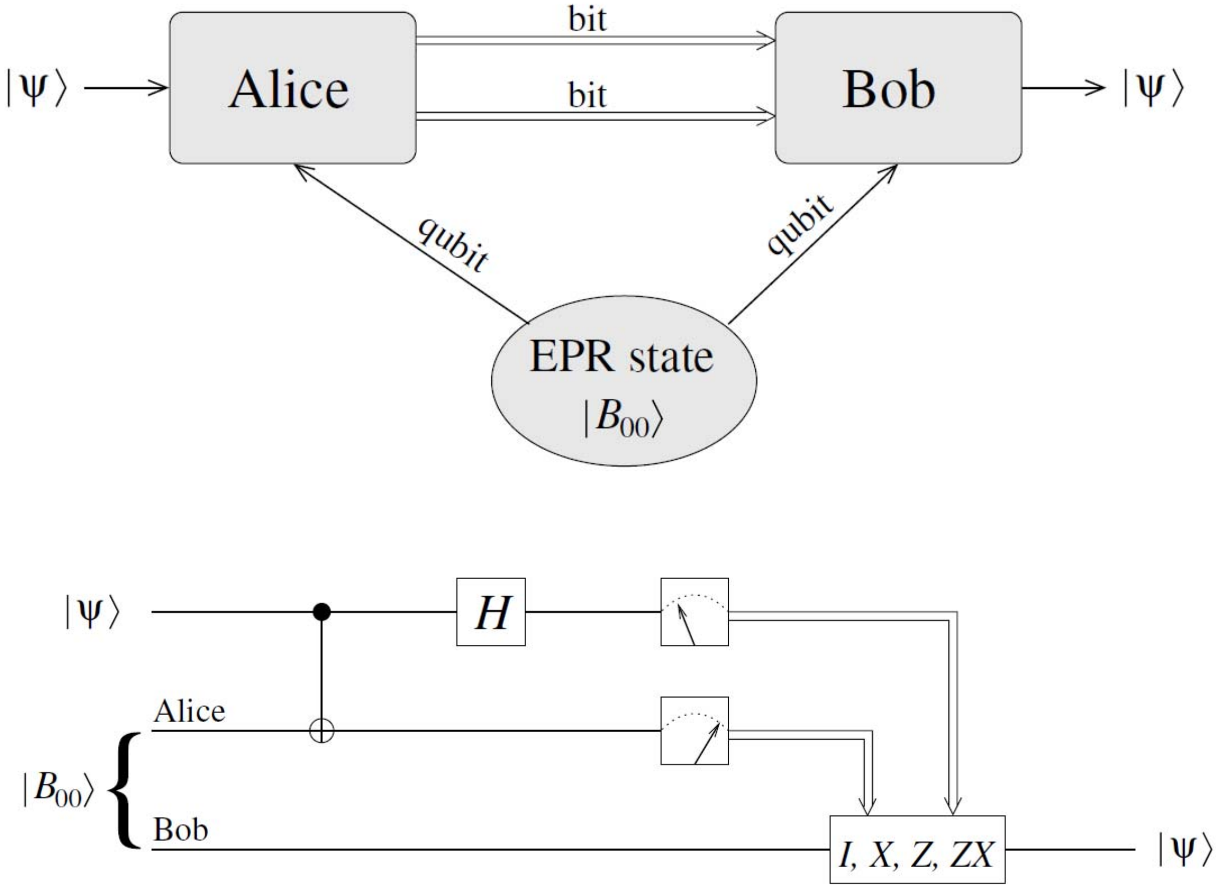


Fig. Quantum teleportation scheme and corresponding circuit.
(P. Lambropoulos and D. Petrosyan, Fundamentals of Quantum Optics and Quantum Information, Springer-Verlag, 2007). p.228.

The initial state of the system of three qubits is given by

$$\left|\psi_3^{(0)}\right\rangle = |\psi\rangle|B_{00}\rangle$$
$$= (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle$$
$$= \frac{1}{\sqrt{2}}[\alpha|0\rangle(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle + \beta|1\rangle(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle]$$

The first two qubits are at the Alice's location and the last bit is at the Bob's location. Alice applies the CNOT transformation to her two qubits, with the control qubit being the quibit to be teleported to Bob.

$$\left|\psi_3^{(1)}\right\rangle = \frac{1}{\sqrt{2}}[\alpha|0\rangle\left|0^A\right\rangle\left|0^B\right\rangle + \alpha|0\rangle\left|1^A\right\rangle\left|1^B\right\rangle + \beta|1\rangle\left|1^A\right\rangle\left|0^B\right\rangle + \beta|1\rangle\left|0^A\right\rangle\left|1^B\right\rangle]$$
$$= \frac{1}{\sqrt{2}}[\alpha|0\rangle(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle) + \beta|1\rangle(\left|1^A\right\rangle\left|0^B\right\rangle + \left|0^A\right\rangle\left|1^B\right\rangle)]$$

where

$$\hat{U}_{CNOT}|0\rangle \otimes \left|0^A\right\rangle = |0\rangle \otimes \left|0^A\right\rangle,$$

$$\hat{U}_{CNOT}|0\rangle \otimes \left|1^A\right\rangle = |0\rangle \otimes \left|1^A\right\rangle,$$

$$\hat{U}_{CNOT}|1\rangle \otimes \left|0^A\right\rangle = |1\rangle \otimes \left|1^A\right\rangle,$$

$$\hat{U}_{CNOT}|1\rangle \otimes \left|1^A\right\rangle = |1\rangle \otimes \left|0^A\right\rangle.$$

She then applies the Hadamard transformation to the first qubit.

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad \hat{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Then we get

$$\left|\psi_3{}^{(1)}\right\rangle = \frac{1}{2}[\alpha(\left|0\right\rangle + \left|1\right\rangle)(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle) + \beta(\left|0\right\rangle - \left|1\right\rangle)(\left|1^A\right\rangle\left|0^B\right\rangle + \left|0^A\right\rangle\left|1^B\right\rangle)]$$

$$= \frac{1}{2}[\alpha[(\left|0\right\rangle + \left|1\right\rangle)(\left|0^A\right\rangle\left|0^B\right\rangle + \left|1^A\right\rangle\left|1^B\right\rangle)] + \beta(\left|0\right\rangle - \left|1\right\rangle)(\left|1^A\right\rangle\left|0^B\right\rangle + \left|0^A\right\rangle\left|1^B\right\rangle)]$$

$$= \frac{1}{2}\alpha[\left|00^A\right\rangle\left|0^B\right\rangle + \left|01^A\right\rangle\left|1^B\right\rangle + \left|10^A\right\rangle\left|0^B\right\rangle + \left|11^A\right\rangle\left|1^B\right\rangle]$$

$$+ \frac{1}{2}\beta[\left|01^A\right\rangle\left|0^B\right\rangle + \left|00^A\right\rangle\left|1^B\right\rangle - \left|11^A\right\rangle\left|0^B\right\rangle - \left|10^A\right\rangle\left|1^B\right\rangle]$$

$$= \frac{1}{2}[\left|00^A\right\rangle(\alpha\left|0^B\right\rangle + \beta\left|1^B\right\rangle) + \left|01^A\right\rangle(\alpha\left|1^B\right\rangle + \beta\left|0^B\right\rangle)$$

$$+ \left|10^A\right\rangle(\alpha\left|0^B\right\rangle - \beta\left|1^B\right\rangle) + \left|11^A\right\rangle(\alpha\left|1^B\right\rangle - \beta\left|0^B\right\rangle)$$

Finally, Alice measures the two quibits in her possession. The measurement outcome. For the measurement of $\left|00^A\right\rangle$ Alice, the state of Bob's qubit is equivalent to the original state

$$\left|\psi_1\right\rangle = \alpha\left|0^B\right\rangle + \beta\left|1^B\right\rangle .$$

So Bob does not change, which is indicated by the identity operator $\hat{I}$,

$$\hat{I}\left|\psi_1\right\rangle = \alpha\left|0^B\right\rangle + \beta\left|1^B\right\rangle .$$

For the measurement of $\left|01^A\right\rangle$ Alice, the state of Bob's qubit is given by

$$\left|\psi_2\right\rangle = \alpha\left|1^B\right\rangle + \beta\left|0^B\right\rangle = \begin{pmatrix}\beta\\ \alpha\end{pmatrix}.$$

If Bob applies the $\hat{X}$ transformation to his qubit, the state becomes

$$\hat{X}\left|\psi_2\right\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \left|\psi_1\right\rangle .$$

For the measurement of $\left|10^A\right\rangle$ Alice, the state of Bob's qubit is given by

$$\left|\psi_3\right\rangle = \alpha\left|0^B\right\rangle - \beta\left|1^B\right\rangle = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}.$$

If Bob applies the $\hat{Z}$ transformation to his qubit, the state becomes

$$\hat{Z}|\psi_3\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\psi_1\rangle .$$

For the measurement of $|11^A\rangle$ Alice, the state of Bob's qubit is given by

$$|\psi_4\rangle = (\alpha|1^B\rangle - \beta|0^B\rangle) = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}.$$

If Bob applies the $\hat{Z}\hat{X}$ transformation to his qubit, the state becomes

$$\hat{Z}\hat{X}|\psi_4\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\psi_1\rangle .$$

Here note that we use the operators $\hat{I}$, $\hat{X}$, $\hat{Z}$, and $\hat{Z}\hat{X}$, where

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{Z}\hat{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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APPENDIX
((**Mathematica**))
Bell's states

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :→ Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);

ψB1 =
 1/√2 (KroneckerProduct[ψ1, ψ1] +
     KroneckerProduct[ψ2, ψ2]) // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψB2 =
 1/√2 (KroneckerProduct[ψ1, ψ2] +
     KroneckerProduct[ψ2, ψ1]) // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
ψB3 =
  1/√2 (KroneckerProduct[ψ1, ψ1] -
      KroneckerProduct[ψ2, ψ2]) // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψB4 =
  1/√2 (KroneckerProduct[ψ1, ψ2] -
      KroneckerProduct[ψ2, ψ1]) // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$
