**Quantum dense coding**

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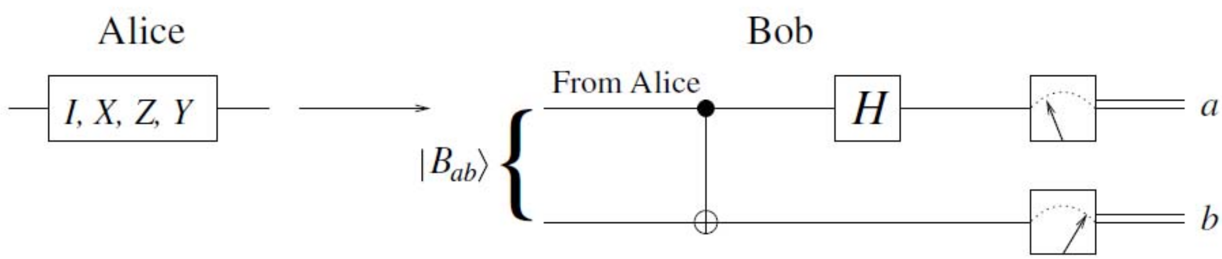
**(Date: August 27, 2014)**

In quantum information theory, **superdense coding** is a technique used to send two bits of classical information using only one qubit, with the aid of entanglement.

## 1. Quantum dense coding




Alice and Bob possess two qubits in the entangled $|B_{00}\rangle$ state,

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|0^A 0^B\rangle + |1^A 1^B\rangle,$$

but can communicate with each other using a reliable quantum communication channel, rather than classical channel. First of all, note that a reliable quantum channel allows one to transmit reliably classical information as well, since the two possible states 0 and 1 of a classical bit can be represented by a qubit being in state $|0\rangle|$ and $|1\rangle$, respectively. Here we show that Alice, applying local transformations to her qubit and then sending it to Bob, can communicate two bits of classical information, as shown in Fig. Depending on which of the four possible two-bit sequences 00, 01, 10, or 11 Alice wants to encode, she acts on her qubits with $\hat{I}_A$, $\hat{X}_A$, $\hat{Z}_A$, and $\hat{Z}_A \hat{X}_A$. The resulting two qubits are

$$\hat{I}_A |B_{00}\rangle = \frac{1}{\sqrt{2}}(|0^A 0^B\rangle + |1^A 1^B\rangle,$$

$$\hat{X}_A |B_{00}\rangle = \frac{1}{\sqrt{2}}(|1^A 0^B\rangle + |0^A 1^B\rangle,$$

$$\hat{Z}_A|B_{00}\rangle = \frac{1}{\sqrt{2}}(|0^A0^B\rangle - |1^A1^B\rangle,$$

$$\hat{Z}_A\hat{X}_A|B_{00}\rangle = \frac{1}{\sqrt{2}}(|0^A1^B\rangle - |1^A0^B\rangle).$$

Alice then sends her Qubits to Bob. He sends the pair through the controlled-NOT gate $\hat{U}^{AB}_{CNOT}$ using the Quibits he received from Alice as control, to get

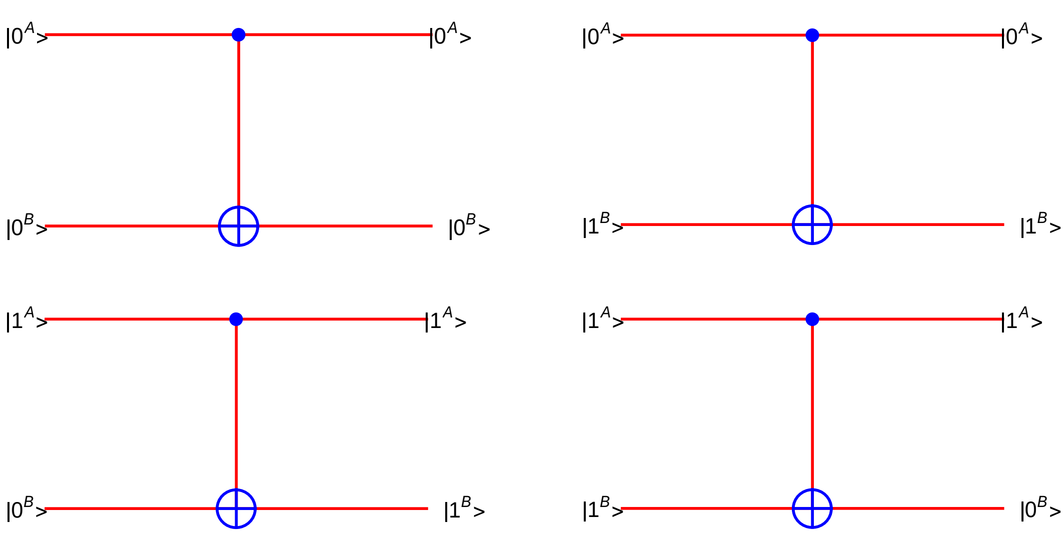
$$\hat{U}^{AB}_{CNOT}\hat{I}_A|B_{00}\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(|0^A0^B\rangle + |1^A0^B\rangle = \frac{1}{\sqrt{2}}(|0^A\rangle + |1^A\rangle)|0^B\rangle,$$

$$\hat{U}^{AB}_{CNOT}\hat{X}_A|B_{00}\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(|1^A0^B\rangle + |0^A1^B\rangle = \frac{1}{\sqrt{2}}(|0^A\rangle + |1^A\rangle)|1^B\rangle,$$

$$\hat{U}^{AB}_{CNOT}\hat{Z}_A|B_{00}\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(|0^A0^B\rangle - |1^A1^B\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(|0^A\rangle - |1^A\rangle)|0^B\rangle,$$

$$\hat{U}^{AB}_{CNOT}\hat{Z}_A\hat{X}_A|B_{00}\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(|0^A1^B\rangle - |1^A0^B\rangle) = \frac{1}{\sqrt{2}}(|0^A\rangle - |1^A\rangle)|1^B\rangle,$$

where




Then Bob applies a Hadamard transform to get

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{I}_A \left| B_{00} \right\rangle = \frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle + \left| 1^A \right\rangle) \left| 0^B \right\rangle = \left| 0^A \right\rangle \left| 0^B \right\rangle ,$$

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{X}_A \left| B_{00} \right\rangle = \frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle + \left| 1^A \right\rangle) \left| 1^B \right\rangle = \left| 0^A \right\rangle \left| 1^B \right\rangle ,$$

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{Z}_A \left| B_{00} \right\rangle = \frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle - \left| 1^A \right\rangle) \left| 0^B \right\rangle = \left| 1^A \right\rangle \left| 0^B \right\rangle ,$$

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{Z}_A \hat{X}_A \left| B_{00} \right\rangle = \frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle - \left| 1^A \right\rangle) \left| 1^B \right\rangle = \left| 1^A \right\rangle \left| 1^B \right\rangle ,$$

where

$$\hat{U}_H^A \left| 0^A \right\rangle = \frac{1}{\sqrt{2}} (\left| 0^A \right\rangle + \left| 1^A \right\rangle) ,$$

$$\hat{U}_H^A \left| 1^A \right\rangle = \frac{1}{\sqrt{2}} (\left| 0^A \right\rangle - \left| 1^A \right\rangle) ,$$

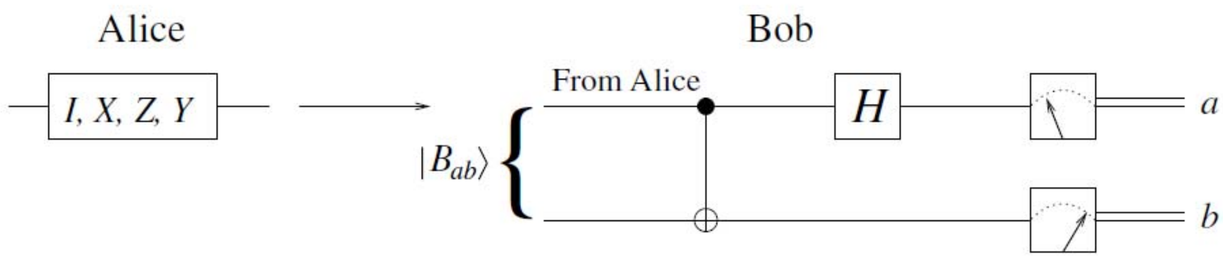
and

$$\frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle + \left| 1^A \right\rangle) = \left| 0^A \right\rangle ,$$

$$\frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle - \left| 1^A \right\rangle) = \left| 1^A \right\rangle .$$

Measuring the two qubits then gives him 00, 01, 10, or 11, - precisely the two-bit message Alice wished to send.

## 2. Example




Alice and Bob share a pair of entangled qubits A and B in state $|\psi\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle].$$

Alice wishes to send Bob two classical bits of information *i* and *j*, *j* = 0, 1, while using a single qubit. She transforms the state of her qubit by applying on it the operator $\hat{A}_{ij}$ acting on the qubit *A*

$$\hat{A}_{ij} = (\hat{\sigma}_x)^i(\hat{\sigma}_z)^j,$$

where *i* and *j* are exponents. She then sends her qubit to Bob, who gets the pair in the state $\hat{A}_{ij}|\psi\rangle$.

1. Give the explicit expression of $\hat{A}_{00}|\psi\rangle$, $\hat{A}_{01}|\psi\rangle$, $\hat{A}_{10}|\psi\rangle$, $\hat{A}_{11}|\psi\rangle$ in terms of the states $|0^A0^B\rangle$, $|0^A0^B\rangle$, $|0^A0^B\rangle$, and $|0^A0^B\rangle$.
2. Bob uses the logic circuit of Fig. with a CNOT gate and a Hadamard gate *H*. Examine the four possibilities for $\hat{A}_{ij}|\psi\rangle$, show that the CNOT gate transforms $\hat{A}_{ij}|\psi\rangle$ into a tensor product and the measurement of qubit *B* gives the value of *i*. Show finally that measurement of qubit *A* gives the value of *j*. Thus Alice transmits two bits of information while sending only one qubit.

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|0^A0^B\rangle + |1^A1^B\rangle.$$

$$\hat{A}_{ij} = (\hat{\sigma}_x)^i(\hat{\sigma}_z)^j.$$

where *i* and *j* are exponents.

$$\hat{\sigma}_x\left|0^A\right\rangle = \left|1^A\right\rangle, \qquad \hat{\sigma}_x\left|1^A\right\rangle = \left|0^A\right\rangle,$$

$$\hat{\sigma}_z\left|0^A\right\rangle = \left|0^A\right\rangle, \qquad \hat{\sigma}_z\left|1^A\right\rangle = -\left|1^A\right\rangle,$$

$$\hat{A}_{00}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A 0^B\right\rangle + \left|1^A 1^B\right\rangle,$$

$$\hat{A}_{01}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}\hat{\sigma}_z(\left|0^A 0^B\right\rangle + \left|1^A 1^B\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A 0^B\right\rangle - \left|1^A 1^B\right\rangle,$$

$$\hat{A}_{10}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}\hat{\sigma}_x(\left|0^A 0^B\right\rangle + \left|1^A 1^B\right\rangle = \frac{1}{\sqrt{2}}(\left|1^A 0^B\right\rangle + \left|0^A 1^B\right\rangle,$$

$$\hat{A}_{11}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}\hat{\sigma}_x\hat{\sigma}_z(\left|0^A 0^B\right\rangle + \left|1^A 1^B\right\rangle = \frac{1}{\sqrt{2}}(\left|1^A 0^B\right\rangle - \left|0^A 1^B\right\rangle.$$

Let us first examine the action of the CNOT-gate on the four states.

$$\hat{U}^{AB}_{CNOT}(\hat{A}_{00}\left|B_{00}\right\rangle) = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(\left|0^A 0^B\right\rangle + \left|1^A 1^B\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A\right\rangle + \left|1^A\right\rangle)\left|0^B\right\rangle,$$

$$\hat{U}^{AB}_{CNOT}\hat{A}_{01}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(\left|0^A 0^B\right\rangle - \left|1^A 1^B\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A\right\rangle - \left|1^A\right\rangle)\left|0^B\right\rangle,$$

$$\hat{U}^{AB}_{CNOT}\hat{A}_{10}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(\left|1^A 0^B\right\rangle + \left|0^A 1^B\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A\right\rangle + \left|1^A\right\rangle)\left|1^B\right\rangle,$$

$$\hat{U}^{AB}_{CNOT}\hat{A}_{11}\left|B_{00}\right\rangle = \frac{1}{\sqrt{2}}\hat{U}^{AB}_{CNOT}(\left|1^A 0^B\right\rangle - \left|0^A 1^B\right\rangle = -\frac{1}{\sqrt{2}}(\left|0^A\right\rangle - \left|1^A\right\rangle)\left|1^B\right\rangle.$$

Then Bob applies a Hadamard transform to get

$$\hat{U}^A_H\hat{U}^{AB}_{CNOT}(\hat{A}_{00}\left|B_{00}\right\rangle) = \frac{1}{\sqrt{2}}\hat{U}^A_H(\left|0^A\right\rangle + \left|1^A\right\rangle)\left|0^B\right\rangle = \left|0^A\right\rangle \otimes \left|0^B\right\rangle,$$

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{A}_{01} \left| B_{00} \right\rangle = \frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle - \left| 1^A \right\rangle) \left| 0^B \right\rangle = \left| 1^A \right\rangle \otimes \left| 0^B \right\rangle,$$

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{A}_{10} \left| B_{00} \right\rangle = \frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle + \left| 1^A \right\rangle) \left| 1^B \right\rangle = \left| 0^A \right\rangle \otimes \left| 1^B \right\rangle,$$

$$\hat{U}_H^A \hat{U}_{CNOT}^{AB} \hat{A}_{11} \left| B_{00} \right\rangle = -\frac{1}{\sqrt{2}} \hat{U}_H^A (\left| 0^A \right\rangle - \left| 1^A \right\rangle) \left| 1^B \right\rangle = -\left| 1^A \right\rangle \otimes \left| 1^B \right\rangle.$$

The measurement qubit B gives the value of $i$ and the measurement of qubit A gives the value of $j$. Thus Alice transmits two bits of information while sending only one qubit.