Operator method in Quantum Computing

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N.D. Mermin, Quantum Computer Science An Introduction (Cambridge, 2007).

1. Definition of the operator $\hat{n}$

We define the operator $\hat{n}$ from the eigenvalue problem

$$\hat{n}|x\rangle = x|x\rangle,$$

with

$$\hat{n}|0\rangle = 0|0\rangle = 0, \qquad \hat{n}|1\rangle = 1|1\rangle = |1\rangle,$$

where x = 0 and 1. $\hat{n}$ is the projection operator and is defined by

$$\hat{n} = |1\rangle\langle 1|.$$

The matrix of $\hat{n}$ under the basis of $\{|0\rangle, |1\rangle\}$ is given by

$$\hat{n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operator $\hat{n}$ is the projection operator on the state .$|1\rangle$

2. Definition of the operator $\hat{m} = \hat{1} - \hat{n}$

We define the operator $\hat{m}$ as

$$\hat{m} = \hat{1} - \hat{n},$$

where $\hat{1}$ is the identity matrix of 2x2,

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For simplicity, here, we use $\hat{m}$ instead of $\tilde{n}$ (mermin used this notation). We note that

$$\hat{m}|x\rangle = (\hat{1} - \hat{n})|x\rangle = |x\rangle - x|x\rangle = (1-x)|x\rangle.$$

Thus we have

$$\hat{m}|0\rangle = |0\rangle, \qquad \hat{m}|1\rangle = 0|1\rangle = 0.$$

The matrix of $\hat{m}$ under the basis of $\{|0\rangle, |1\rangle\}$ is given by

$$\hat{m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The operator $\hat{n}$ is the projection operator on the state .$|0\rangle$

$$\hat{m} = |0\rangle\langle 0|.$$

3. Properties of $\hat{n}$ and $\hat{m}$

$$\hat{n}^2 = \hat{n},$$

$$\hat{m}^2 = \hat{m},$$

$$\hat{n}\hat{m} = \hat{m}\hat{n} = 0,$$

$$\hat{m} + \hat{n} = 1.$$

4. Pauli matrix

$$\hat{X} = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{Y} = \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{Z} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{n}\hat{X} = \hat{X}\hat{m}, \qquad \hat{m}\hat{X} = \hat{X}\hat{n},$$

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{1},$$

$$\hat{X}\hat{Z} = -\hat{Z}\hat{X},$$

$$\hat{n} = \frac{1}{2}(1 - \hat{Z}), \qquad \hat{m} = \frac{1}{2}(1 + \hat{Z}), \qquad \hat{Z} = \hat{n} - \hat{m},$$

$$\hat{Y} = i\hat{X}\hat{Z} = -i\hat{Z}\hat{X}, \qquad \hat{Z} = i\hat{Y}\hat{X} = -i\hat{X}\hat{Y}, \qquad \hat{X} = i\hat{Z}\hat{Y} = -i\hat{Y}\hat{Z}.$$

5. Hadamard operator

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{X} + \hat{Z}),$$

$$\hat{H}^2 = \frac{1}{2}(\hat{X} + \hat{Z})(\hat{X} + \hat{Z}) = \frac{1}{2}(\hat{X}^2 + \hat{X}\hat{Z} + \hat{Z}\hat{X} + \hat{Z}^2) = \hat{1},$$

$$\hat{H}\hat{X}\hat{H} = \hat{Z}, \qquad \hat{H}\hat{Z}\hat{H} = \hat{X}.$$

Note that

$$\begin{aligned}
\hat{H}\hat{X}\hat{H} &= \frac{1}{2}(\hat{X} + \hat{Z})\hat{X}(\hat{X} + \hat{Z}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z})(\ddot{1} + \hat{X}\hat{Z}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} + \hat{X}^2\hat{Z} + \hat{Z}\hat{X}Z) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} + \hat{X}^2\hat{Z} - \hat{Z}^2\hat{X}) \\
&= \hat{Z}
\end{aligned}$$

$$\begin{aligned}
\hat{H}\hat{Z}\hat{H} &= \frac{1}{2}(\hat{X} + \hat{Z})\hat{Z}(\hat{X} + \hat{Z}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z})(\hat{1} + \hat{Z}\hat{X}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} - \hat{X}^2\hat{Z} + \hat{Z}^2\hat{X}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} - \hat{Z} + \hat{X}) \\
&= \hat{X}
\end{aligned}$$

$$\hat{H} \otimes \hat{H} = \frac{1}{2}(\hat{X} + \hat{Z}) \otimes (\hat{X} + \hat{Z}) = \frac{1}{2}(\hat{X} \otimes \hat{X} + \hat{X} \otimes \hat{Z} + \hat{Z} \otimes \hat{X} + \hat{Z} \otimes \hat{Z}).$$

6. Calculation of matrices by using Mathematica

```
Clear["Global`*"]; I2 = IdentityMatrix[2];
```

n = $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; **m = I2 – n; X = PauliMatrix[1];**

Y = PauliMatrix[2]; Z = PauliMatrix[3]; $\phi 0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

$\phi 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

```
m // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

```
n // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

```
n.n – n // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
m.m – m // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
n.X - X.m // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
m.X - X.n // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
m - n // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
Z.X + X.Z // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\frac{1}{2}\,(\mathrm{I2} - \mathrm{Z})\ \text{// MatrixForm}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{2}\,(\mathrm{X} + \mathrm{Z})\ \text{// MatrixForm}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

H1 = $\frac{1}{\sqrt{2}}$ (X + Z); H1 // MatrixForm

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

X.X // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Z.Z // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

X.Z + Z.X // MatrixForm

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

H1.H1 // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

H1.X.H1 - Z // MatrixForm

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
H1.Z.H1 - X // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
H1.ϕ0 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
H1.ϕ1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
f11 = KroneckerProduct[m, I2] +
   KroneckerProduct[n, X];
f11 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
f12 =
 1/2 (KroneckerProduct[I2, I2 + X] +
    KroneckerProduct[Z, I2 - X]);
f12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
f13 =
 1/2 (KroneckerProduct[I2 + Z, I2] +
    KroneckerProduct[I2 - Z, X]);
f13 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
S11 = KroneckerProduct[n, n] +
  KroneckerProduct[m, m] +
  KroneckerProduct[X.n, X.m] +
  KroneckerProduct[X.m, X.n];
S11 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
C11 = KroneckerProduct[m, I2] +
  KroneckerProduct[n, X];
C11 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
Z + i X .Y // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
Y + i Z.X // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
X + i Y.Z // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
KroneckerProduct[H1, H1] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathtt{U1} = \begin{pmatrix} \mathtt{u11} & \mathtt{u12} \\ \mathtt{u21} & \mathtt{u22} \end{pmatrix};$$

```
KroneckerProduct[I2, U1] // MatrixForm
```

$$\begin{pmatrix} \mathtt{u11} & \mathtt{u12} & 0 & 0 \\ \mathtt{u21} & \mathtt{u22} & 0 & 0 \\ 0 & 0 & \mathtt{u11} & \mathtt{u12} \\ 0 & 0 & \mathtt{u21} & \mathtt{u22} \end{pmatrix}$$

```
h1 =
 KroneckerProduct[H1, H1].KroneckerProduct[X, X] //
  MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

7. The CNOT gate with $\hat{U}_{CNOT}$

The CNOT gate is defined by

$$
\begin{aligned}
\hat{U}_{CNOT} &= \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} \\
&= \frac{1}{2}(\hat{1} + \hat{Z}) \otimes \hat{1} + \frac{1}{2}(\hat{1} - \hat{Z}) \otimes \hat{X} \\
&= \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{Z} \otimes \hat{1} + \hat{1} \otimes \hat{X} - \hat{Z} \otimes \hat{X})
\end{aligned}
$$

with

$$
\hat{U}_{CNOT}{}^{2} = \hat{1},
$$

$$
\begin{aligned}
\hat{U}_{CNOT}(1-2) &= \hat{C}_{12} \otimes \hat{1} \\
&= \frac{1}{2}(\hat{1} \otimes \hat{1} \otimes \hat{1} + \hat{Z} \otimes \hat{1} \otimes \hat{1} + \hat{1} \otimes \hat{X} \otimes \hat{1} - \hat{Z} \otimes \hat{X} \otimes \hat{1})
\end{aligned}
$$

$$
\hat{U}_{CNOT}(1-3) = \frac{1}{2}(\hat{1} \otimes \hat{1} \otimes \hat{1} + \hat{Z} \otimes \hat{1} \otimes \hat{1} + \hat{1} \otimes \hat{1} \otimes \hat{X} - \hat{Z} \otimes \hat{1} \otimes \hat{X})
$$

((**Mathematica**))

```
Clear["Global`*"];
I2 = IdentityMatrix[2];
X = PauliMatrix[1]; Y = PauliMatrix[2];
Z = PauliMatrix[3];

UCNOT =
  1/2 (KroneckerProduct[I2, I2] +
     KroneckerProduct[Z, I2] +
     KroneckerProduct[I2, X] -
     KroneckerProduct[Z, X]);
UCNOT // MatrixForm
```

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

```
UCNOT12 =
 1/2 (KroneckerProduct[I2, I2, I2] +
    KroneckerProduct[Z, I2, I2] +
    KroneckerProduct[I2, X, I2] -
    KroneckerProduct[Z, X, I2]);
UCNOT12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

```
UCNOT13 =
 1/2 (KroneckerProduct[I2, I2, I2] +
     KroneckerProduct[Z, I2, I2] +
     KroneckerProduct[I2, I2, X] -
     KroneckerProduct[Z, I2, X]) //
  MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

8. Swap (exchange) operator

$\hat{G}_{swap}$ is called a swap (exchange) operator, which simply interchanges the states of qubits *1* and 2.

$$\begin{aligned}
\hat{G}_{SWAP} &= \hat{n}\otimes\hat{n} + \hat{m}\otimes\hat{m} + (\hat{X}\hat{n})\otimes(\hat{X}\hat{m}) + (\hat{X}\hat{m})\otimes(\hat{X}\hat{n}) \\
&= \frac{1}{4}(\hat{1}-\hat{Z})\otimes(\hat{1}-\hat{Z}) + \frac{1}{4}(\hat{1}+\hat{Z})\otimes(\hat{1}-\hat{Z}) + \frac{1}{4}[\hat{X}(\hat{1}-\hat{Z})]\otimes[X(\hat{1}+\hat{Z})] \\
&\quad + \frac{1}{4}[\hat{X}(\hat{1}+\hat{Z})]\otimes[X(\hat{1}-\hat{Z})] \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{Z}) + \frac{1}{4}(\hat{X} - \hat{X}\hat{Z})\otimes(X + \hat{X}\hat{Z}) + \frac{1}{4}(\hat{X} + \hat{X}\hat{Z})\otimes(X - \hat{X}\hat{Z}) \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{Z}) + \frac{1}{4}(\hat{X} + i\hat{Y})\otimes(X - i\hat{Y}) + \frac{1}{4}(\hat{X} - i\hat{Y})\otimes(X + i\hat{Y}) \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{X}\otimes X + \hat{Y}\otimes\hat{Y} + \hat{Z}\otimes\hat{Z})
\end{aligned}$$

Then the swap operator becomes equivalent to the Dirac exchange spin operator

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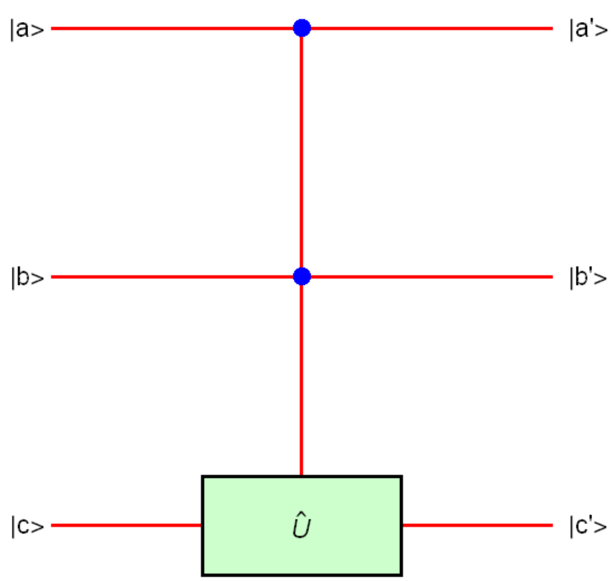
9. Toffoli gate



Only if $|a\cdot b\rangle = |1\rangle$,

$$|a'\rangle = |a\rangle, \qquad |b'\rangle = |b\rangle,$$

$$|c'\rangle = \hat{U}|c\rangle .$$

Other wise

$$|a'\rangle = |a\rangle , \qquad |b'\rangle = |b\rangle ,$$

$$|c'\rangle = |c\rangle .$$

((**Truth table**))

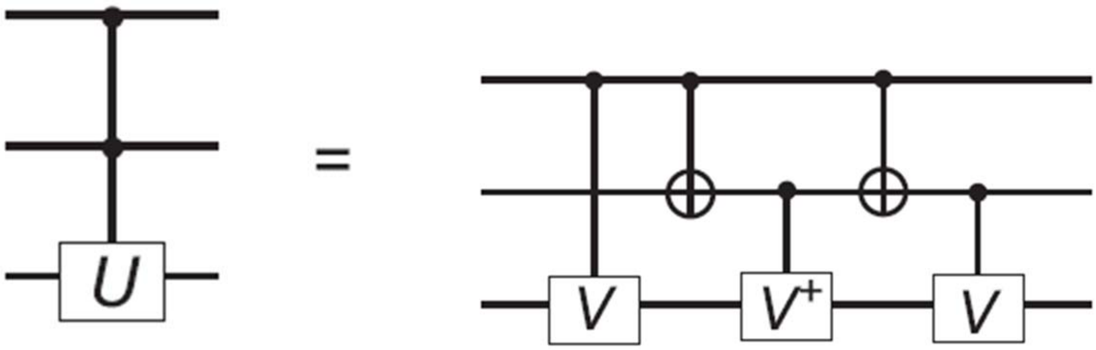
a	b	c	a’	b’	c’
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	$\hat{U}\|0\rangle = u_{11}\|0\rangle + u_{21}\|1\rangle$
1	1	1	1	1	$\hat{U}\|1\rangle = u_{12}\|0\rangle + u_{22}\|1\rangle$

$$\hat{G}_{Toffli}(U)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{11} & u_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{U} \end{pmatrix} .$$

10. Example: equivalent quantum circuits

We show that a two-qubit controlled gate canted using a combination of one qubit controlled gate as

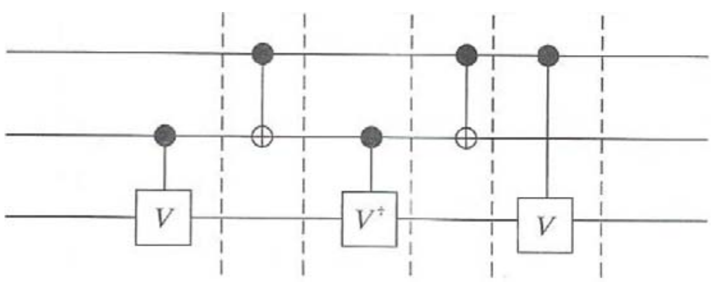
$$\hat{G}_{Toffoli}[U] = \hat{G}_{V23}(\hat{G}_{CNOT} \otimes \hat{1})(\hat{1} \otimes \hat{G}[V^{+}])(\hat{G}_{CNOT} \otimes \hat{1})\hat{G}_{V13} .$$



This can be also described by

$$\hat{G}_{Toffoli}[U] = \hat{G}_{V13}(\hat{G}_{CNOT} \otimes \hat{1})[\hat{1} \otimes \hat{G}(V^{+})](\hat{G}_{CNOT} \otimes \hat{1})\hat{G}_{V23},$$

which means that $\hat{G}_{Toffoli}[U]$ is a universal gate (reversible).



The left hand-side is the Toffoli gate with the matrix $\hat{U}$,

$$\hat{G}_{Toffoli}[U] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{21} & U_{22} \end{pmatrix}$$

where

$$\hat{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

Suppose that the matrix $\hat{U}$ is expressed by $\hat{U} = \hat{V}^2$, where $\hat{V}$ is the unitary operator.
The CNOT operator is given by

$$\hat{G}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix},$$

$\hat{G}_{CNOT} \otimes \hat{1}$ is obtained as

$$\hat{G}_{CNOT} \otimes \hat{1} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix} \otimes \hat{1} = \begin{pmatrix} \hat{I}_2 & 0 & 0 & 0 \\ 0 & \hat{I}_2 & 0 & 0 \\ 0 & 0 & 0 & \hat{I}_2 \\ 0 & 0 & \hat{I}_2 & 0 \end{pmatrix}.$$

The control V gate is given by

$$\hat{G}[V] = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{V} \end{pmatrix},. \qquad \hat{G}[V^+] = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{V}^+ \end{pmatrix}.$$

Then we have

$$\hat{G}_{V23} = \hat{1} \otimes \hat{U}[V] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{11} & V_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{V} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

$\hat{C}_{V13}$ is the matrix obtained from the matrix $\hat{1} \otimes \hat{G}[V]$ by the appropriate interchange of row and column,

$$\hat{G}_{V13}[V] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{V} & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}.$$

((**Mathematica**))

```
Clear["Global`*"]; I2 = IdentityMatrix[2]; CV = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, v11, v12}, {0, 0, v21, v22}};

CVP = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, v11c, v21c}, {0, 0, v12c, v22c}};

V1 = {{v11, v12}, {v21, v22}};

CUV13[A_] := Module[{A1, U, U1, U11, U12}, A1 = A;
  U = KroneckerProduct[I2, A1];
  U1 = {U[[All, 1]], U[[All, 2]], U[[All, 5]], U[[All, 6]],
    U[[All, 3]], U[[All, 4]], U[[All, 7]], U[[All, 8]] };
  U11 = Transpose[U1];
  U12 = {U11[[1]], U11[[2]], U11[[5]], U11[[6]], U11[[3]],
    U11[[4]], U11[[7]], U11[[8]]}];

CV13 = CUV13[CV]; CV13 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v11 & v12 & 0 & 0 \\ 0 & 0 & 0 & 0 & v21 & v22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v11 & v12 \\ 0 & 0 & 0 & 0 & 0 & 0 & v21 & v22 \end{pmatrix}$$

```
UCNOT = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}; UCNOTI2 = KroneckerProduct[UCNOT, I2];
```

$$\mathbf{UCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
UCNOTI2 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

```
CV23 = KroneckerProduct[I2, CV];
```

```
CV23 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v11 & v12 & 0 & 0 & 0 & 0 \\ 0 & 0 & v21 & v22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v11 & v12 \\ 0 & 0 & 0 & 0 & 0 & 0 & v21 & v22 \end{pmatrix}$$

```
CVP23 = KroneckerProduct[I2, CVP]; CVP23 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v11c & v21c & 0 & 0 & 0 & 0 \\ 0 & 0 & v12c & v22c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v11c & v21c \\ 0 & 0 & 0 & 0 & 0 & 0 & v12c & v22c \end{pmatrix}$$

```
K1 = CV23.UCNOTI2.CVP23.UCNOTI2.CV13 // FullSimplify;
```

$$vvc = \begin{pmatrix} v11 & v12 \\ v21 & v22 \end{pmatrix} . \begin{pmatrix} v11c & v21c \\ v12c & v22c \end{pmatrix};$$

$$vcv = \begin{pmatrix} v11c & v21c \\ v12c & v22c \end{pmatrix} . \begin{pmatrix} v11 & v12 \\ v21 & v22 \end{pmatrix};$$

```
U1 = V1.V1 // Simplify;
```

```
rule1 = {vvc[[1, 1]] → 1, vvc[[1, 2]] → 0, vvc[[2, 1]] → 0,
   vvc[[2, 2]] → 1, vcv[[1, 1]] → 1, vcv[[1, 2]] → 0, vcv[[2, 1]] → 0,
   vcv[[2, 2]] → 1 };
rule2 = {U1[[1, 1]] → U11, U1[[1, 2]] → U12, U1[[2, 1]] → U21,
   U1[[2, 2]] → U22};
```

```
K11 = K1 //. rule1; K12 = K11 //. rule2; K12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U11 & U12 \\ 0 & 0 & 0 & 0 & 0 & 0 & U21 & U22 \end{pmatrix}$$

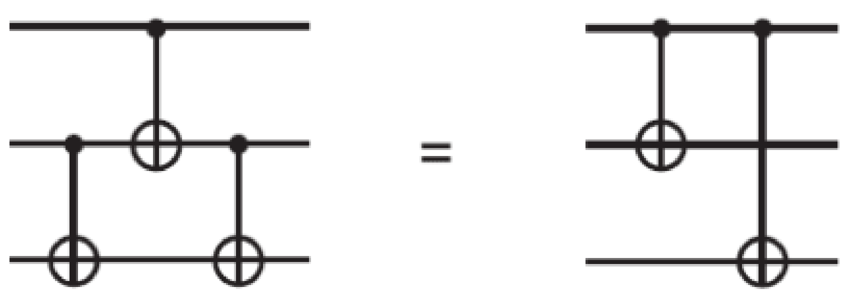
```
L1 = CV13.UCNOTI2.CVP23.UCNOTI2.CV23 // FullSimplify;

L11 = L1 //. rule1; L12 = L11 //. rule2; L12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U11 & U12 \\ 0 & 0 & 0 & 0 & 0 & 0 & U21 & U22 \end{pmatrix}$$

10. Equivalent circuits

We show the equivalence between two quantum circuits as shown below,



$$(\hat{1} \otimes \hat{G}_{CNOT})(\hat{G}_{CNOT} \otimes \hat{1})(\hat{1} \otimes \hat{G}_{CNOT}) = (\hat{G}_{CNOT} \otimes \hat{1})\hat{C}_{13}(CNOT)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\hat{G}_{CNOT} \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{1} \otimes \hat{G}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\hat{G}_{CNOT13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \hat{G}[V] &= \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{V} \\ &= \frac{1}{2}(\hat{1} + \hat{Z}) \otimes \hat{1} + \frac{1}{2}(\hat{1} - \hat{Z}) \otimes \hat{V} \\ &= \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{Z} \otimes \hat{1} + \hat{1} \otimes \hat{V} - \hat{Z} \otimes \hat{V}) \end{aligned}$$

$$\hat{G}_{12}[V]\otimes\hat{1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} \end{pmatrix}$$

$$\hat{G}_{23}[V]=\hat{1}\otimes\hat{G}[V]$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{11} & V_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix}=\begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{V} & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

$$\hat{G}_{13}[V]=\frac{1}{2}(\hat{1}\otimes\hat{1}\otimes\hat{1}+\hat{Z}\otimes\hat{1}\otimes\hat{1}+\hat{1}\otimes\hat{1}\otimes\hat{V}-\hat{Z}\otimes\hat{1}\otimes\hat{V})$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix}=\begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{I}_2 & 0 & 0 \\ 0 & 0 & \boldsymbol{V} & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"]; I2 = IdentityMatrix[2];
X = PauliMatrix[1]; Y = PauliMatrix[2];
Z = PauliMatrix[3];
V = ( V11  V12
      V21  V22 );

UV =
 1/2 (KroneckerProduct[I2, I2] +
    KroneckerProduct[Z, I2] +
    KroneckerProduct[I2, V] -
    KroneckerProduct[Z, V]);
UV // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & V11 & V12 \\ 0 & 0 & V21 & V22 \end{pmatrix}$$

```
C12V = KroneckerProduct[UV, I2];
C12V // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V11 & 0 & V12 & 0 \\ 0 & 0 & 0 & 0 & 0 & V11 & 0 & V12 \\ 0 & 0 & 0 & 0 & V21 & 0 & V22 & 0 \\ 0 & 0 & 0 & 0 & 0 & V21 & 0 & V22 \end{pmatrix}$$

```
C23V = KroneckerProduct[I2, UV];
C23V // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V11 & V12 & 0 & 0 & 0 & 0 \\ 0 & 0 & V21 & V22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V11 & V12 \\ 0 & 0 & 0 & 0 & 0 & 0 & V21 & V22 \end{pmatrix}$$

```
C13V =
 1/2 (KroneckerProduct[I2, I2, I2] +
    KroneckerProduct[Z, I2, I2] +
    KroneckerProduct[I2, I2, V] -
    KroneckerProduct[Z, I2, V]);
C13V // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V11 & V12 & 0 & 0 \\ 0 & 0 & 0 & 0 & V21 & V22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V11 & V12 \\ 0 & 0 & 0 & 0 & 0 & 0 & V21 & V22 \end{pmatrix}$$

11 Toffoli gate:

The Toffoli gate is given by

$$
\begin{aligned}
\hat{G}_{toffoli} &= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{G}_{CNOT} \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}) \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
\boldsymbol{I} & 0 & 0 & 0 \\
0 & \boldsymbol{I} & 0 & 0 \\
0 & 0 & \boldsymbol{I} & 0 \\
0 & 0 & 0 & \boldsymbol{X}
\end{pmatrix}
\end{aligned}
$$

where

$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X} \ .$$

((**Mathematica**))

```
Clear["Global`*"]; I2 = IdentityMatrix[2]; X = PauliMatrix[1];
Y = PauliMatrix[2]; Z = PauliMatrix[3]; n = (0 0
                                             0 1);
m = (1 0
     0 0);
UCNOT = (1 0 0 0
         0 1 0 0
         0 0 0 1
         0 0 1 0);

TOF1 = KroneckerProduct[m, I2, I2] +
  KroneckerProduct[n, UCNOT];
TOF1 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

12. Fredkin gate

The Fredkin gate is given by

$$\begin{aligned}\hat{G}_{Fredkin} &= \hat{1}\otimes\hat{1}\otimes\hat{m}+\hat{U}_{SWAP}\otimes\hat{n}\\ &= \hat{1}\otimes\hat{1}\otimes\hat{m}+\frac{1}{2}(\hat{1}\otimes\hat{1}+\hat{X}\otimes X+\hat{Y}\otimes\hat{Y}+\hat{Z}\otimes\hat{Z})\otimes\hat{n}\\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

where $\hat{G}_{SWAP}$ is the SWAP operator,

$$\hat{G}_{SWAP} = \frac{1}{2}(\hat{1}\otimes\hat{1}+\hat{X}\otimes X+\hat{Y}\otimes\hat{Y}+\hat{Z}\otimes\hat{Z}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

((**Mathematica**))

```
Clear["Global`*"]; X = PauliMatrix[1]; Y = PauliMatrix[2];
Z = PauliMatrix[3]; n = (0 0
                         0 1); m = (1 0
                                    0 0);
I2 = IdentityMatrix[2];
SWAP = (1 0 0 0
        0 0 1 0
        0 1 0 0
        0 0 0 1);

Fredkin =
 KroneckerProduct[I2, I2, m] +
   KroneckerProduct[SWAP, n] // Simplify;
Fredkin // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
SWAP =
 1/2 (KroneckerProduct[I2, I2] + KroneckerProduct[X, X] +
    KroneckerProduct[Y, Y] + KroneckerProduct[Z, Z]);
SWAP // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

13. Controlled V gate

$$\hat{C}[V] = \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{1} + \hat{1}\otimes\hat{V} - \hat{Z}\otimes\hat{V}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & V_{21} & V_{22} \end{pmatrix},$$

$$\hat{R}[V] = \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{1}\otimes\hat{Z} + \hat{V}\otimes\hat{1} - \hat{V}\otimes\hat{Z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 1 & 0 \\ 0 & V_{21} & 0 & V_{22} \end{pmatrix},$$

```
Clear["Global`*"]; I2 = IdentityMatrix[2];
X = PauliMatrix[1]; Y = PauliMatrix[2]; Z = PauliMatrix[3];
V = ( V11  V12
      V21  V22 );
```

```
UV =
  1/2 (KroneckerProduct[I2, I2] + KroneckerProduct[Z, I2] +
     KroneckerProduct[I2, V] - KroneckerProduct[Z, V]);
```

```
RUV =
  1/2 (KroneckerProduct[I2, I2] + KroneckerProduct[I2, Z] +
     KroneckerProduct[V, I2] - KroneckerProduct[V, Z]);
```

```
UV // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & V11 & V12 \\ 0 & 0 & V21 & V22 \end{pmatrix}$$

```
RUV // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V11 & 0 & V12 \\ 0 & 0 & 1 & 0 \\ 0 & V21 & 0 & V22 \end{pmatrix}$$

---

14. Controlled-*U* gate

$$\hat{C}[U] = \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{1} + \hat{1}\otimes\hat{U} - \hat{Z}\otimes\hat{U})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{U} \end{pmatrix}$$

$$\hat{R}[U] = \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{1} \otimes \hat{Z} + \hat{U} \otimes \hat{1} - \hat{U} \otimes \hat{Z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & 1 & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix}.$$

$\hat{R}(H)$

$$\hat{R}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

15. Controlled-CNOT

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix}.$$

16. Fredkin gate

$$\begin{aligned} \hat{G}_{Fredkin} &= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{G}_{SWAP} \otimes \hat{n} \\ &= \hat{1} \otimes \hat{1} \otimes \hat{m} + \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{X} \otimes X + \hat{Y} \otimes \hat{Y} + \hat{Z} \otimes \hat{Z}) \otimes \hat{n} \\ &== \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

17. Toffoli gate

$$
\begin{aligned}
\hat{G}_{toffoli} &= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{G}_{CNOT} \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}) \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X} \\
&== \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}&0&0&0 \\ 0&\boldsymbol{I}&0&0 \\ 0&0&\boldsymbol{I}&0 \\ 0&0&0&\boldsymbol{X} \end{pmatrix}
\end{aligned}
$$

18. R-CNOT

$$
\hat{R}_{CNOT} = \hat{1}\otimes\hat{m}+\hat{X}\otimes\hat{n} = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix}.
$$

19. Swap gate

$$
\begin{aligned}
\hat{G}_{SWAP} &= \hat{m}\otimes\hat{m}+\hat{n}\otimes\hat{n}+\hat{X}m\otimes\hat{X}\hat{n}+\hat{X}\hat{n}\otimes\hat{X}\hat{m} \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1}+\hat{X}\otimes X+\hat{Y}\otimes\hat{Y}+\hat{Z}\otimes\hat{Z}) \\
&= \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}
\end{aligned}
$$

$$
\hat{1} = \begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix},
$$

$$
\hat{Y} = \begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}, \qquad \hat{Z} = \begin{pmatrix} 1&0 \\ 0&-1 \end{pmatrix}.
$$

20. Hadamard gate:

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\hat{P}_\alpha = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix},$$

$$\hat{T}_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix},$$

$$\hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix},$$

$$\hat{G}_{V12}[V] = \hat{G}[V] \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} \end{pmatrix}$$

$$\hat{G}_{V23} = \hat{1} \otimes \hat{G}[V]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{V} & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

$$\hat{G}_{V13} = \frac{1}{2}(\hat{1}\otimes\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{1}\otimes\hat{1} + \hat{1}\otimes\hat{1}\otimes\hat{V} - \hat{Z}\otimes\hat{1}\otimes\hat{V})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{I}_2 & 0 & 0 \\ 0 & 0 & \boldsymbol{V} & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

---

10. Expression for $\hat{G}_{CNOT}$ in terms of Pauli operators $\hat{X}$ and $\hat{Z}$

We know that the controlled-CNOT gate can be expressed by

$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X}.$$

Here we consider another expressions for the controlled-CNOT gate. We start with

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

The projection operators are defined by

$$\hat{P}_+ = |+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\hat{P}_- = |-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We note that

$$\hat{P}_+ + \hat{P}_- = \hat{1}.$$

The operator $\hat{X}$ can be described as

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{P}_+ - \hat{P}_- .$$

The operator $\hat{Z}$ can be expressed by

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{m} - \hat{n} .$$

Note that

$$\hat{m} = \frac{1}{2}(\hat{1} + \hat{Z}), \qquad \hat{n} = \frac{1}{2}(\hat{1} - \hat{Z}),$$

since $\hat{m} + \hat{n} = \hat{1}$.

We now introduce the operator (controlled-CNOT gate), which can be generated from

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\hat{G}_{CNOT}$ is equivalent to the controlled-X gate

$$\hat{G}_{CNOT} = \hat{C}[X] = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix}.$$

This can be derived using the Kronecker product. Since

$$\hat{P}_+ + \hat{P}_- = \hat{1}, \qquad \hat{P}_+ - \hat{P}_- = \hat{X} ,$$

we get

$$\hat{P}_+ = \frac{1}{2}(\hat{1} + \hat{X}), \qquad \hat{P}_- = \frac{1}{2}(\hat{1} - \hat{X}).$$

Then the controlled-CNOT operator can be rewritten as

$$\begin{aligned}\hat{G}_{CNOT} &= \hat{m}\otimes(\hat{P}_+ + \hat{P}_-) + \hat{n}\otimes(\hat{P}_+ - \hat{P}_-) \\ &= \hat{m}\otimes\hat{P}_+ + \hat{m}\otimes\hat{P}_- + \hat{n}\otimes\hat{P}_+ - \hat{n}\otimes\hat{P}_- \\ &= (\hat{m}+\hat{n})\otimes\hat{P}_+ + (\hat{m}-\hat{n})\otimes\hat{P}_- \\ &= \hat{1}\otimes\hat{P}_+ + \hat{Z}\otimes\hat{P}_-\end{aligned}$$

or

$$\hat{G}_{CNOT} = \frac{1}{2}[\hat{1}\otimes(\hat{1}+\hat{X}) + \hat{Z}\otimes(\hat{1}-\hat{X})].$$

11. The expression of $\hat{G}_{CNOT}$ in terms of Hadamard gate $\hat{H}$

The Hadamard gate is defined by

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{Z}+\hat{X}).$$

Then we get

$$\hat{m}\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix},$$

$$\hat{H}\hat{m}\hat{H} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \hat{P}_+ = \frac{1}{2}(\hat{1}+\hat{X}).$$

Noting that $\hat{1}^2 = \hat{1}\,\hat{1}$ and using the property of the Kronecker product, we get

$$\hat{1}\otimes\hat{P}_+ = \hat{1}\otimes(\hat{H}\hat{m}\hat{H}) = (\hat{1}\otimes\hat{H})(\hat{1}\otimes\hat{m}\hat{H}) = (\hat{1}\otimes\hat{H})(\hat{1}\otimes\hat{m})(1\otimes\hat{H}) \qquad (1)$$

Similarly, we have

$$\hat{n}\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix},$$

$$\hat{H}\hat{n}\hat{H} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \hat{P}_- = \frac{1}{2}(\hat{1}-\hat{X}).$$

Noting that $\hat{Z} = \hat{1}\hat{Z}$ and using the property of the Kronecker product, we get

$$\hat{Z} \otimes \hat{P}_{-} = \hat{Z} \otimes (\hat{H}\hat{n}\hat{H}) = (\hat{Z} \otimes \hat{H})(\hat{1} \otimes \hat{n}\hat{H}) = (\hat{1} \otimes \hat{H})(\hat{Z} \otimes \hat{n})(\hat{1} \otimes \hat{H}) \quad (2)$$

From Eqs.(1) and (2), thus we have

$$\begin{aligned}\hat{G}_{CNOT} &= \hat{1} \otimes \hat{P}_{+} + \hat{Z} \otimes \hat{P}_{-} \\ &= (\hat{1} \otimes \hat{H})(\hat{1} \otimes \hat{m})(1 \otimes \hat{H}) + (\hat{1} \otimes \hat{H})(\hat{Z} \otimes \hat{n})(\hat{1} \otimes \hat{H}) \\ &= (\hat{1} \otimes \hat{H})(\hat{1} \otimes \hat{m} + \hat{Z} \otimes \hat{n})(\hat{1} \otimes \hat{H})\end{aligned}$$

which can be rewritten as

$$\hat{G}_{CNOT} = \hat{G}_{X} = (\hat{1} \otimes \hat{H})\hat{R}_{Z}(\hat{1} \otimes \hat{H}) = (\hat{1} \otimes \hat{H})\hat{G}_{Z}(\hat{1} \otimes \hat{H}).$$

since

$$\hat{R}_{Z} = \hat{G}_{Z}.$$

12. Matrix representation of Kronecker product for two qubits

$$\hat{m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{m} + \hat{n} = \hat{1}.$$

$$\hat{P}_{+} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \hat{P}_{-} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$\hat{m} \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{n} \otimes \hat{1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{1} \otimes \hat{m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{1} \otimes \hat{n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{m}\otimes\hat{m}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{m}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\hat{n}\otimes\hat{m}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},$$

$$\hat{m}\otimes\hat{X}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{X}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$\hat{X}\otimes\hat{m}=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{X}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},$$

$$\hat{m}\otimes\hat{U}=\begin{pmatrix}u_{11}&u_{12}&0&0\\u_{21}&u_{22}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{U}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&u_{11}&u_{12}\\0&0&u_{21}&u_{22}\end{pmatrix},$$

$$\hat{U}\otimes\hat{m}=\begin{pmatrix}u_{11}&0&u_{12}&0\\0&0&0&0\\u_{21}&0&u_{22}&0\\0&0&0&0\end{pmatrix},\qquad \hat{U}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&u_{11}&0&u_{12}\\0&0&0&0\\0&u_{21}&0&u_{22}\end{pmatrix},$$

$$\hat{X}\hat{m}\otimes\hat{X}\hat{n}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{X}\hat{n}\otimes\hat{X}\hat{m}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\hat{X}\hat{m}\otimes\hat{X}\hat{n}+\hat{X}\hat{n}\otimes\hat{X}\hat{m}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$\hat{m}\otimes\hat{P}_+=\frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{P}_+=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix},$$

$$\hat{m}\otimes\hat{P}_-=\frac{1}{2}\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{P}_-=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}.$$

13. Equivalence of quantum circuits between $\hat{G}_Z$ and $\hat{R}_Z$

We show that the controlled-*Z* gate $\hat{G}_Z$ is equivalent to the quantum circuit with $\hat{R}_Z$.

((Method-1)) The use of matrices

$$\begin{aligned}\hat{G}_Z&=\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z}\\&=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\\&=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
\hat{R}_Z &= \hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n}\\
&= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

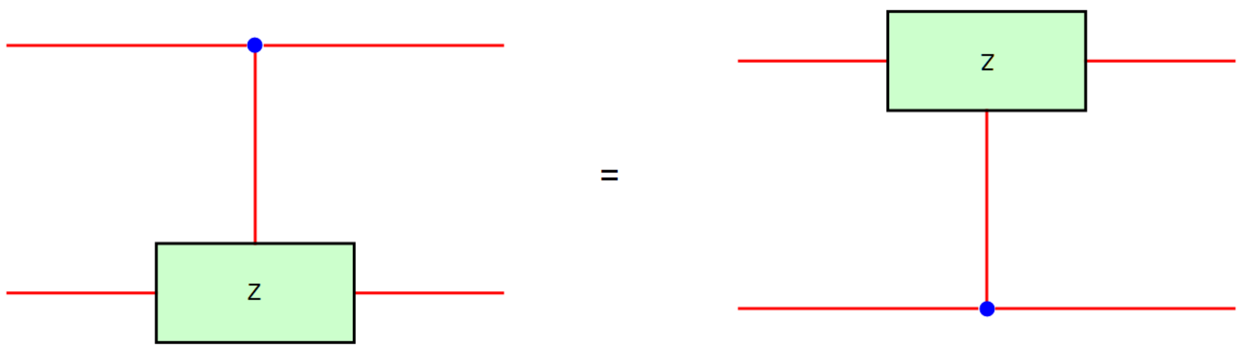


Fig. Quantum gates with $\hat{G}_Z$ and $\hat{R}_Z$.

((**Method-II**) Operation method

$$
\begin{aligned}
\hat{G}_Z{}^2 &= (\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z})(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z})\\
&= \hat{m}^2\otimes\hat{1}+\hat{m}\hat{n}\otimes\hat{Z}+\hat{n}\hat{m}\otimes\hat{Z}+\hat{n}^2\otimes\hat{Z}^2\\
&= \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{1}\\
&= (\hat{m}+\hat{n})\otimes\hat{1}\\
&= \hat{1}
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_Z{}^2 &= (\hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n})(\hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n})\\
&= \hat{1}\otimes\hat{m}^2+\hat{Z}\otimes\hat{m}\hat{n}+\hat{Z}\otimes\hat{n}\hat{m}+\hat{Z}^2\otimes\hat{n}^2\\
&= \hat{1}\otimes\hat{m}+\hat{1}\otimes\hat{n}\\
&= \hat{1}\otimes(\hat{m}+\hat{n})\\
&= \hat{1}
\end{aligned}
$$

$$
\begin{aligned}
\hat{G}_Z \hat{R}_Z &= (\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{Z})(\hat{1} \otimes \hat{m} + \hat{Z} \otimes \hat{n}) \\
&= \hat{m} \otimes \hat{m} + \hat{m}\hat{Z} \otimes \hat{n} + \hat{n} \otimes \hat{Z}\hat{m} + \hat{n}\hat{Z} \otimes \hat{Z}\hat{n} \\
&= \hat{m} \otimes \hat{m} + \hat{m} \otimes \hat{n} + \hat{n} \otimes \hat{m} + \hat{n} \otimes \hat{n} \\
&= (\hat{m} + \hat{n}) \otimes (\hat{m} + \hat{n}) \\
&= \hat{1}
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_Z \hat{G}_Z &= (\hat{1} \otimes \hat{m} + \hat{Z} \otimes \hat{n})(\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{Z}) \\
&= \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{m}\hat{Z} + \hat{Z}\hat{m} \otimes \hat{n} + \hat{Z}\hat{n} \otimes \hat{n}\hat{Z} \\
&= \hat{m} \otimes \hat{m} + \hat{m} \otimes \hat{n} + \hat{n} \otimes \hat{m} + \hat{n} \otimes \hat{n} \\
&= (\hat{m} + \hat{n}) \otimes (\hat{m} + \hat{n}) \\
&= \hat{1}
\end{aligned}
$$

where

$$
\hat{m}\hat{n} = 0, \qquad \hat{m} + \hat{n} = \hat{1},
$$

$$
\hat{m}\hat{Z} = \hat{Z}\hat{m} = \hat{m}, \qquad \hat{n}\hat{Z} = \hat{Z}n = -\hat{n} .
$$

Since

$$
\hat{G}_Z(\hat{G}_Z \hat{R}_Z) = \hat{G}_Z, \qquad \hat{R}(\hat{R}_Z \hat{G}_Z) = \hat{R},
$$

we get the relation

$$
\hat{G}_Z = \hat{R}_Z .
$$

The two quantum circuits are equivalent.

14. Quantum circuits related to the controlled *U*-gate

(a) Quantum circuit with $\hat{G}_U = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{U}$.

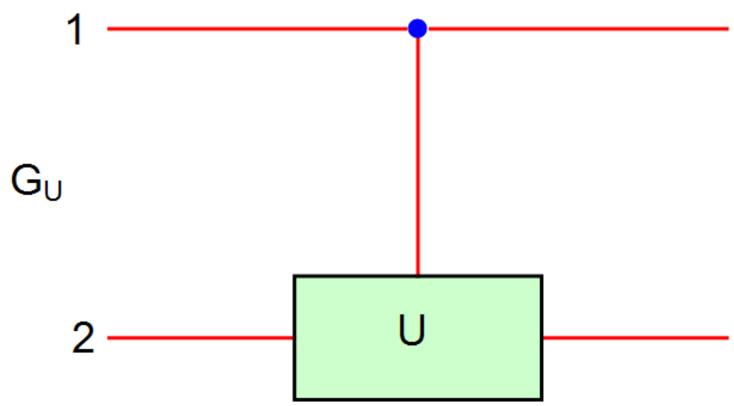


Fig. Controlled-U gate with $\hat{G}_U = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{U}$. $\hat{U}$ is the 2x2 matrix.

$$\hat{G}_U = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{U}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{pmatrix}$$

(b) Quantum circuit with $\hat{R}_U = \hat{1}\otimes\hat{m} + \hat{U}\otimes\hat{n}$

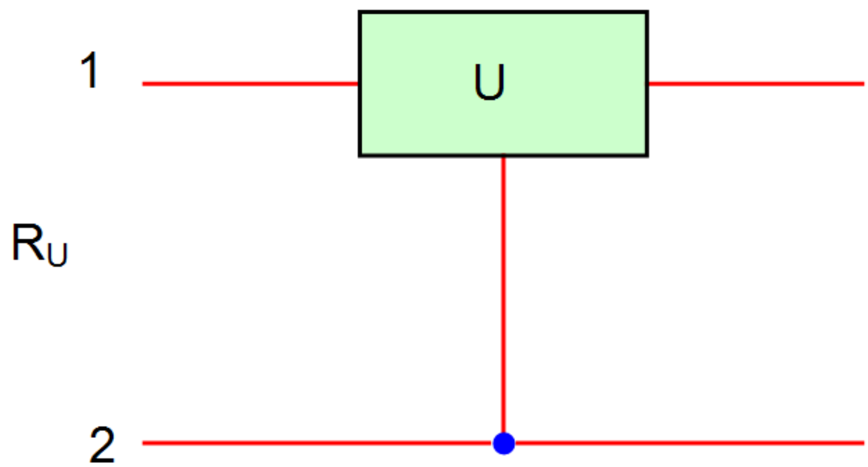


Fig. Quantum circuit with $\hat{R}_U = \hat{1}\otimes\hat{m} + \hat{U}\otimes\hat{n}$. $\hat{U}$ is the 2x2 matrix.

$$\hat{R}_U = \hat{1}\otimes\hat{m} + \hat{U}\otimes\hat{n}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix}$$

---

15. Qauntum circuits related to the controlled-CNOT

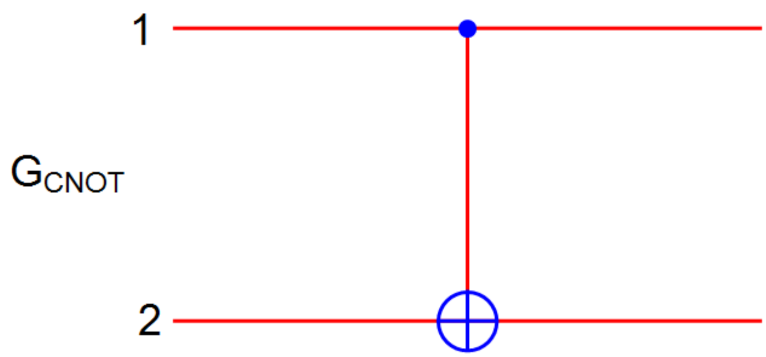


Fig. Controlled-CNOT with $\hat{G}_{CNOT} = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X}$

$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

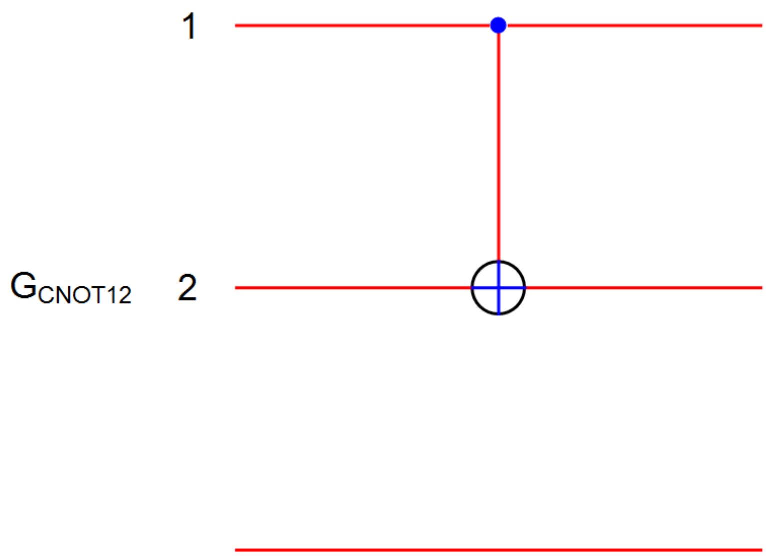


Fig. Quantum gate with Controlled-CNOT between 1 and 2. with $\hat{G}_{CNOT12} = \hat{G}_{CNOT} \otimes \hat{1} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{X} \otimes \hat{1}$.

$$
\begin{aligned}
\hat{G}_{CNOT12} &= \hat{G}_{CNOT} \otimes \hat{1} \\
&= \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{X} \otimes \hat{1} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

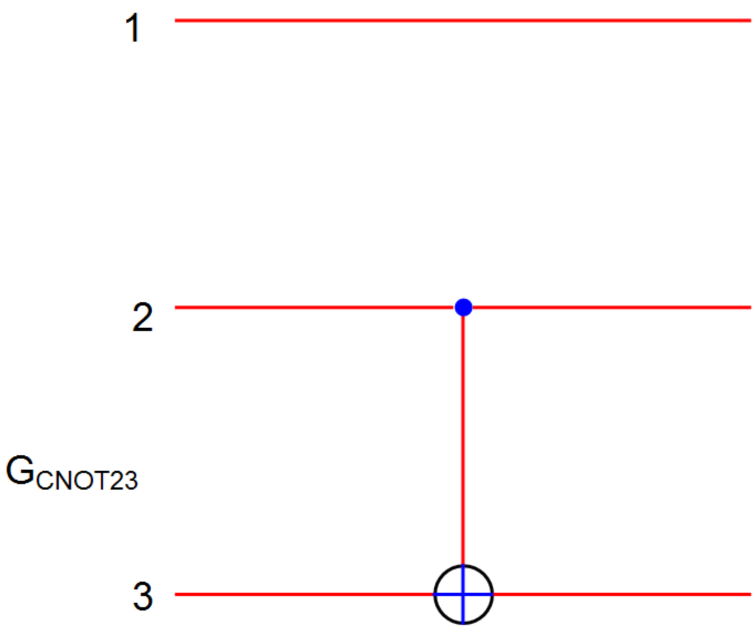


Fig. Quantum gate with Controlled-CNOT between 2 and 3. with $\hat{G}_{CNOT23} = \hat{1} \otimes \hat{G}_{CNOT} = \hat{1} \otimes \hat{m} \otimes \hat{1} + \hat{1} \otimes \hat{n} \otimes \hat{X}$.

$$
\begin{aligned}
\hat{G}_{CNOT23} &= \hat{1} \otimes \hat{G}_{CNOT} \\
&= \hat{1} \otimes \hat{m} \otimes \hat{1} + \hat{1} \otimes \hat{n} \otimes \hat{X} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{aligned}
$$

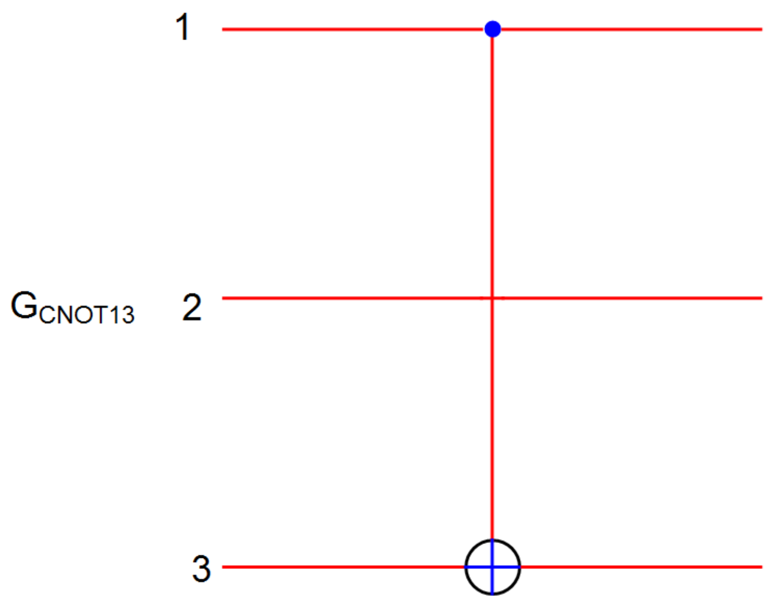


Fig. Quantum gate with Controlled-CNOT between 1 and 3. with $\hat{G}_{CNOT13} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{1} \otimes \hat{X}$.

$$\hat{G}_{CNOT13} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{1} \otimes \hat{X}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

16 R-CNOT gate with $\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}$

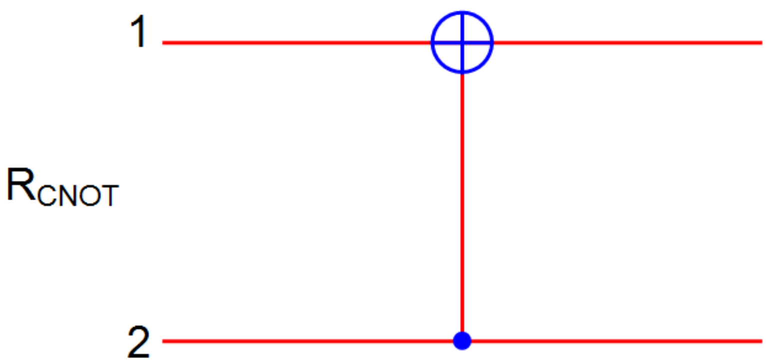


Fig. Quantum gate with $\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}$

$$\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

17. Quantrum circuits related to the SWAP gate

(a) Swap gate with $\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}$

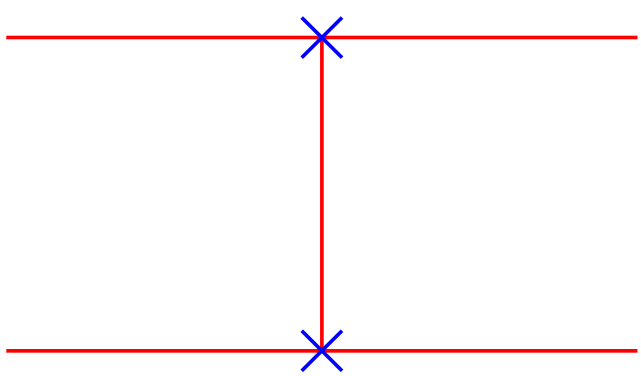


Fig. Swap gate with $\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}$

$$
\begin{aligned}
\hat{G}_{SWAP} &= \hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{X}\hat{m} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

(b) Quantum circuit $\hat{G}_{SWAP12} = \hat{G}_{SWAP} \otimes \hat{1}$

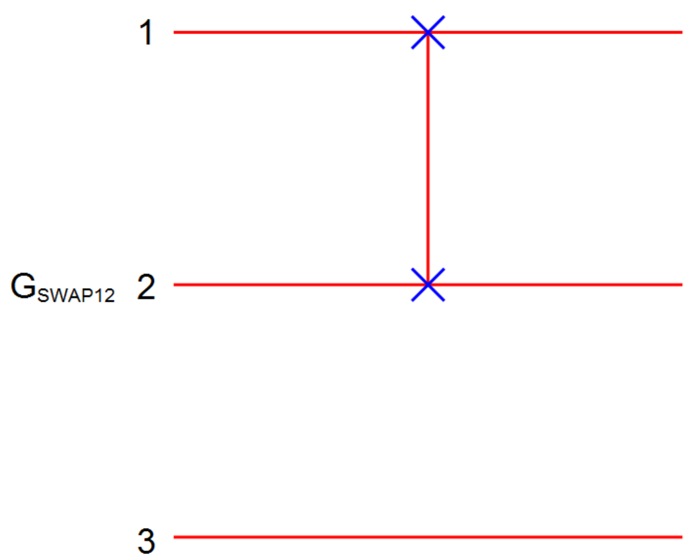


Fig. Quantum circuit including Swap gate between 1 and 2. $\hat{G}_{SWAP12} = \hat{G}_{SWAP} \otimes \hat{1}$

$$\hat{G}_{SWAP12} = \hat{G}_{SWAP} \otimes \hat{1}$$
$$= \hat{m} \otimes \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{n} \otimes \hat{1} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} \otimes \hat{1} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} \otimes \hat{1}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Quantum circuit with $\hat{G}_{SWAP23} = \hat{1} \otimes \hat{G}_{SWAP}$

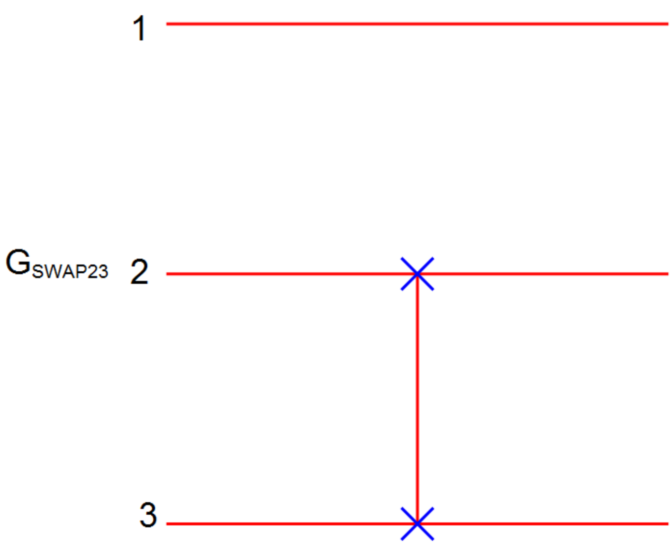


Fig. Quantum circuit including Swap gate between 2 and 3. $\hat{G}_{SWAP23} = \hat{1} \otimes \hat{G}_{SWAP}$

$$\hat{G}_{SWAP23} = \hat{1} \otimes \hat{G}_{SWAP}$$
$$= \hat{1} \otimes \hat{m} \otimes \hat{m} + \hat{1} \otimes \hat{n} \otimes \hat{n} + \hat{1} \otimes \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{1} \otimes \hat{X}\hat{n} \otimes \hat{X}\hat{m}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Quantum gate with $\hat{G}_{SWAP13}$

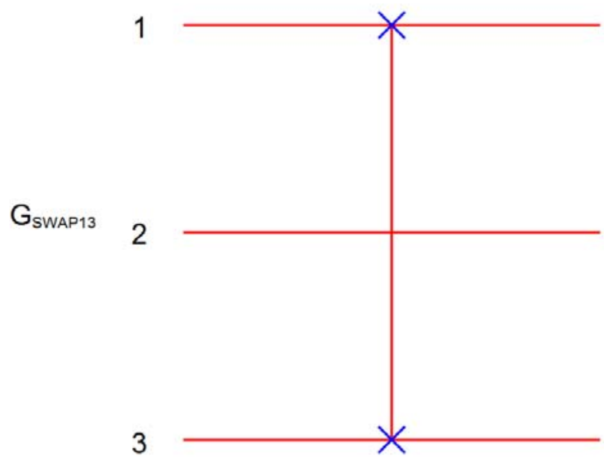


Fig. Quantum circuit including Swap gate between 1 and 3. $\hat{G}_{SWAP13}$

$$\hat{G}_{SWAP13} =$$

$$= \hat{m}\otimes\hat{1}\otimes\hat{m} + \hat{n}\otimes\hat{1}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{1}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{1}\otimes\hat{X}\hat{m}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

18. Matrix representation of typical tri-qubits

$$\hat{1}\otimes\hat{1}\otimes\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{m}\otimes\hat{1}\otimes\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{m}\otimes\hat{m}\otimes\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{m}\otimes\hat{m}\otimes\hat{m} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{1}\otimes\hat{1}\otimes\hat{m}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\end{pmatrix},\quad \hat{1}\otimes\hat{m}\otimes\hat{m}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

$$\hat{1}\otimes\hat{m}\otimes\hat{n}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

$$\hat{n}\otimes\hat{1}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},\quad \hat{n}\otimes\hat{n}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},$$

$$\hat{n}\otimes\hat{n}\otimes\hat{n}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\end{pmatrix}$$

$$\hat{m}\otimes\hat{n}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix},\quad \hat{n}\otimes\hat{m}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

19 Quantum gates related to the Toffoli gate

(a) Quantum gate $\hat{G}_{toffoli}$

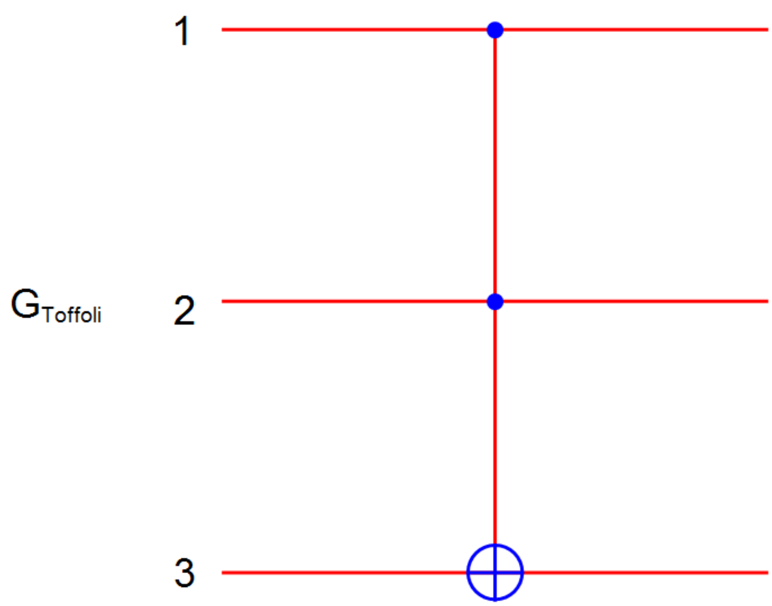


Fig. Toffoli gate with $\hat{G}_{toffoli} == \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X}$

$$\hat{G}_{toffoli} = \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{n}\otimes\hat{X}$$

$$= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}&0&0&0\\ 0&\boldsymbol{I}&0&0\\ 0&0&\boldsymbol{I}&0\\ 0&0&0&\boldsymbol{X} \end{pmatrix}$$

(b) Quantum gate with $\hat{R}_{toffoli}$

In the expression of

$$\hat{G}_{toffoli} = \hat{m}_1\otimes\hat{1}_2\otimes\hat{1}_3 + \hat{n}_1\otimes\hat{m}_2\otimes\hat{1}_3 + \hat{n}_1\otimes\hat{n}_2\otimes\hat{X}_3$$

we change the number of subscript as 1→3, 2→2, 3→1,

$$\hat{m}_3\otimes\hat{1}_2\otimes\hat{1}_1 + \hat{n}_3\otimes\hat{m}_2\otimes\hat{1}_1 + \hat{n}_3\otimes\hat{n}_2\otimes\hat{X}_1$$

This can be rewrirren as

$$\begin{aligned}\hat{R}_{toffoli} &= \hat{1}_1\otimes\hat{1}_2\otimes\hat{m}_3 + \hat{1}_1\otimes\hat{m}_2\otimes\hat{n}_3 + \hat{X}_1\otimes\hat{n}_2\otimes\hat{n}_3\\ &= \hat{1}\otimes\hat{1}\otimes\hat{m} + \hat{1}\otimes\hat{m}\otimes\hat{n} + \hat{X}\otimes\hat{n}\otimes\hat{n}\end{aligned}$$

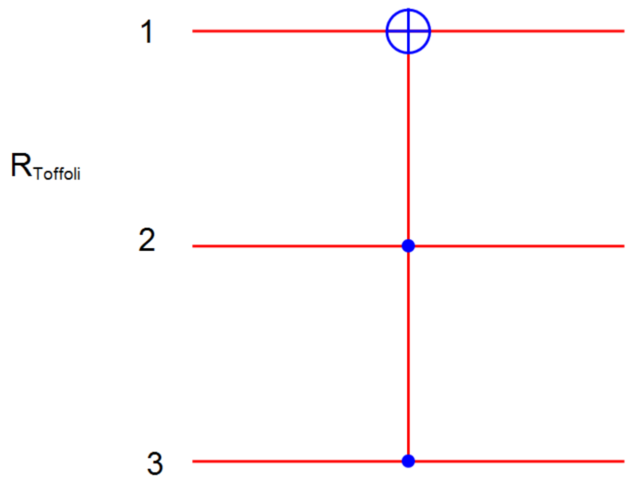


Fig. Quantum gate $\hat{R}_{toffoli} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{m} \otimes \hat{n} + \hat{X} \otimes \hat{n} \otimes \hat{n}$.

We note that

$$\hat{R}_{toffoli} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{m} \otimes \hat{n} + \hat{X} \otimes \hat{n} \otimes \hat{n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

20. Quantum gate related to the Fredkin gate

(a) Quantum gate with $\hat{G}_{Fredkin} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{G}_{SWAP}$

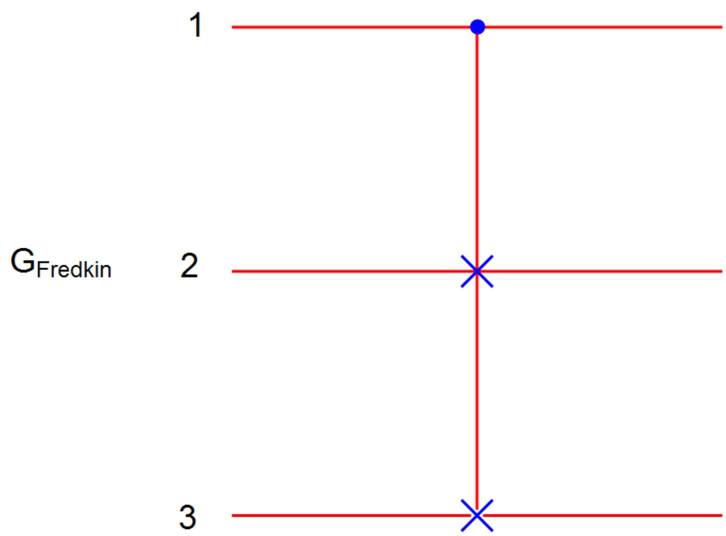


Fig. Fredkin gate with $\hat{G}_{Fredkin} = \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{G}_{SWAP}$.

$$
\begin{aligned}
\hat{G}_{Fredkin} &= \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{G}_{SWAP} \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes(\hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{X}\hat{m}) \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n}\otimes\hat{n} + \hat{n}\otimes\hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{n}\otimes\hat{X}\hat{n}\otimes\hat{X}\hat{m} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{G}_{SWAP} &= \hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{X}\hat{m} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

(b) Quantum gate with $\hat{R}_{Fredkin}$

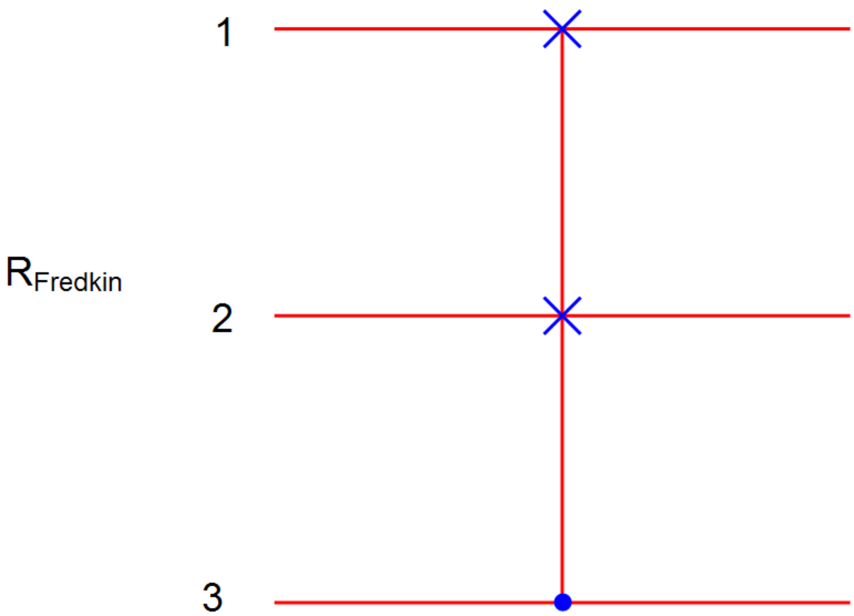
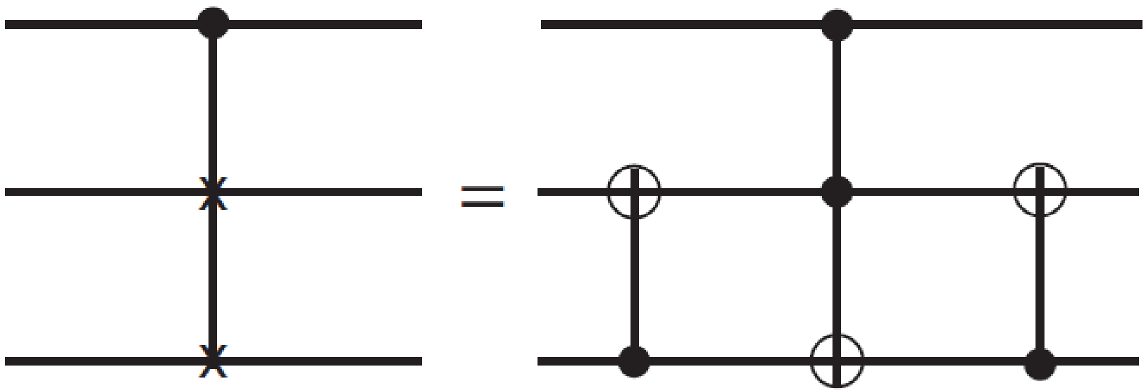


Fig. Modified Fredkin gate with $\hat{R}_{Fredkin} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{G}_{SWAP} \otimes \hat{n}$

$$
\begin{aligned}
\hat{R}_{Fredkin} &= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{G}_{SWAP} \otimes \hat{n} \\
&= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{m} \otimes \hat{m} \otimes \hat{n} + \hat{n} \otimes \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} \otimes \hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} \otimes \hat{n} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

21. Quantum circuit equivalence to the Fredkin gate



$$
\begin{aligned}
\hat{R}_{CNOT\,23} \cdot \hat{G}_{Toffoli} \cdot \hat{R}_{CNOT\,23} &= (\hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{X} \otimes \hat{n}) \cdot (\hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{n} \otimes \hat{X}) \\
&\cdot (\hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{X} \otimes \hat{n}) \\
&= (\hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{X} \otimes \hat{n}) \cdot (\hat{m} \otimes \hat{1} \otimes \hat{m} + \hat{m} \otimes \hat{X} \otimes \hat{n} \\
&+ \hat{n} \otimes \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{m}\hat{X} \otimes \hat{n} + \hat{n} \otimes \hat{n} \otimes \hat{X}\hat{m} + \hat{n} \otimes \hat{n}\hat{X} \otimes \hat{X}\hat{n}) \\
&= (\hat{m} \otimes \hat{1} \otimes \hat{m}^2 + \hat{m} \otimes \hat{X} \otimes \hat{m}\hat{n} + \hat{n} \otimes \hat{m} \otimes \hat{m}^2 + \hat{n} \otimes \hat{n} \otimes \hat{m}\hat{X}\hat{m} \\
&+ \hat{n} \otimes \hat{n}\hat{X} \otimes \hat{m}\hat{X}\hat{n} + \hat{m} \otimes \hat{X} \otimes \hat{n}\hat{m} + \hat{m} \otimes \hat{X}^2 \otimes \hat{n}^2 + n \otimes Xm \otimes \hat{n}\hat{m} \\
&+ \hat{n} \otimes \hat{X}\hat{n} \otimes \hat{n}\hat{X}\hat{m} + \hat{n} \otimes \hat{X}\hat{n}\hat{X} \otimes \hat{n}\hat{X}\hat{n}) \\
&= \hat{m} \otimes \hat{1} \otimes \hat{m} + +\hat{n} \otimes \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} \otimes \hat{m}\hat{n}\hat{X} \\
&+ \hat{n} \otimes \hat{n}\hat{X} \otimes \hat{m}^2\hat{X} + \hat{m} \otimes \hat{1} \otimes \hat{n} \\
&+ \hat{n} \otimes \hat{X}\hat{n} \otimes \hat{n}^2\hat{X} + \hat{n} \otimes \hat{X}\hat{n}\hat{X} \otimes \hat{n}\hat{m}\hat{X} \\
&= \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} \otimes \hat{n} + \hat{n} \otimes \hat{X}\hat{n} \otimes \hat{X}\hat{m} + +\hat{n} \otimes \hat{X}\hat{m} \otimes \hat{X}\hat{n}
\end{aligned}
$$

where

$$\hat{m}\hat{n} = 0, \qquad \hat{n}\hat{m} = 0, \qquad \hat{m}^2 = \hat{m}, \qquad \hat{n}^2 = \hat{n}, \qquad \hat{X}^2 = \hat{1},$$

$$\hat{m} + \hat{n} = \hat{1},$$

$$\hat{n}\hat{X} = \hat{X}\hat{m}, \qquad \hat{m}\hat{X} = \hat{X}\hat{n},$$

and

$$\hat{R}_{CNOT\,23} = \hat{1} \otimes \hat{R}_{CNOT}$$
$$= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{X} \otimes \hat{n}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\hat{G}_{toffoli} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{n} \otimes \hat{X}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus we have

$$\hat{R}_{CNOT\,23} \cdot \hat{G}_{Toffoli} \cdot \hat{R}_{CNOT\,23} = \hat{G}_{Fredkin},$$

where

$$\hat{G}_{Fredkin} = \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n}\otimes\hat{n} + \hat{n}\otimes\hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{n}\otimes\hat{X}\hat{n}\otimes\hat{X}\hat{m}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

22. Quantum circuit $\hat{G}_{CNOT}\hat{R}_{CNOT}\hat{G}_{CNOT}$ equivalent to $\hat{G}_{SWAP}$

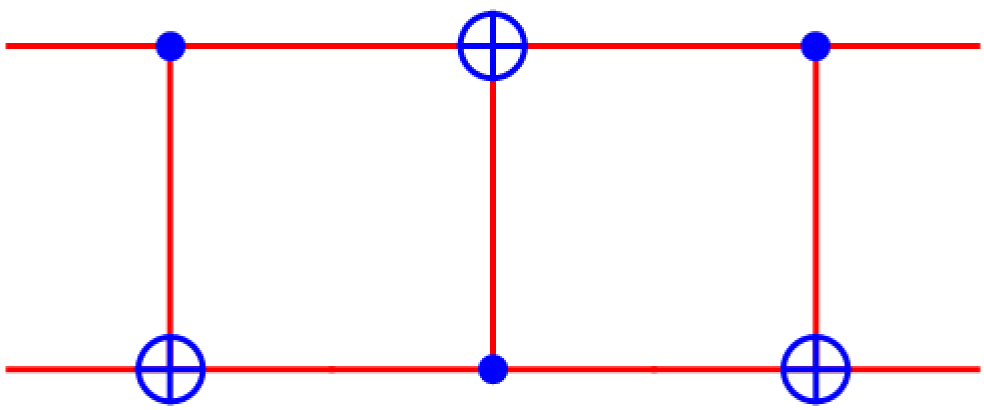


Fig.a Quantum circuit with $\hat{G}_{CNOT}\hat{R}_{CNOT}\hat{G}_{CNOT}$.

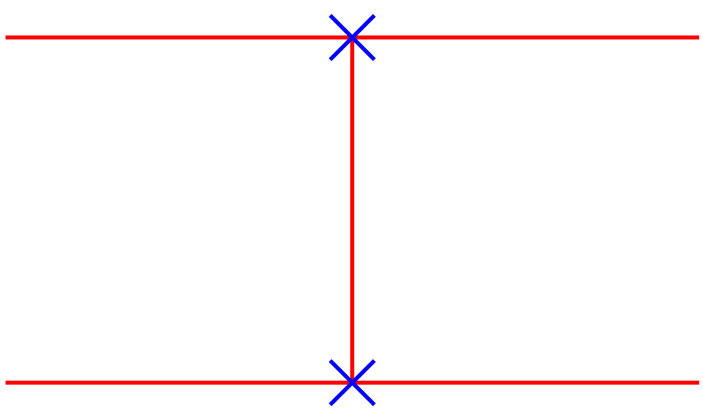


Fig.b Swap gate with $\hat{G}_{SWAP} = \hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{X}\hat{m}$.

We show that this quantum circuit is equivalent to the SWAP gate. We note that

$$\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n},$$

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} .$$

Then we get

$$\begin{aligned}\hat{G}_{CNOT}\hat{R}_{CNOT}\hat{G}_{CNOT} &= (\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X})(\hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n})(\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X}) \\ &= \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} + \hat{X}\hat{m} \otimes \hat{X}\hat{n}\end{aligned}$$

which is the same as

$$\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} .$$

23. Equivalent quantum circuits: $(\hat{X} \otimes \hat{1})\hat{G}_{CNOT}(\hat{X} \otimes \hat{1}) = \hat{n} \otimes \hat{1} + \hat{m} \otimes \hat{X}$

We show that these two quantum circuits are equivalent to each other.

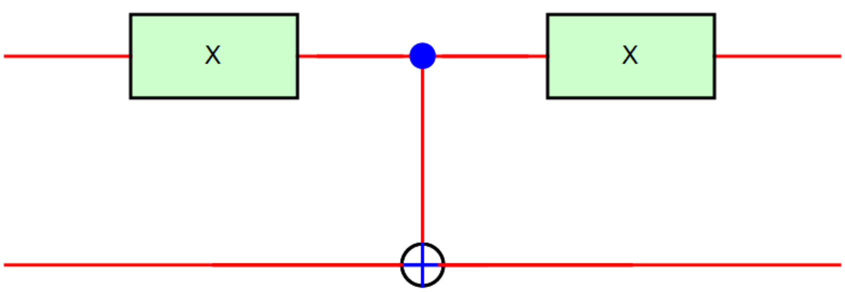


Fig.a Quantum circuit with $(\hat{X} \otimes \hat{1})\hat{G}_{CNOT}(\hat{X} \otimes \hat{1})$.

which is equivalent to a new type of quantum gate

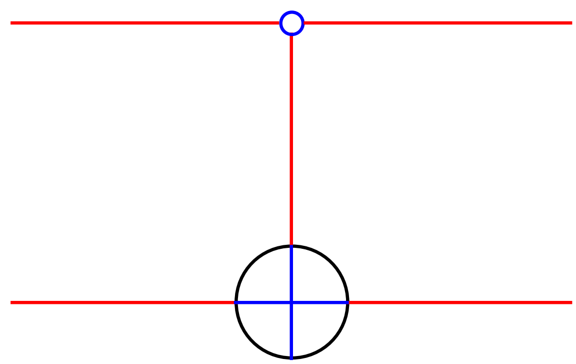


Fig.b Quantum circuit with $\hat{n}\otimes\hat{1}+\hat{m}\otimes\hat{X}$. Controlled operation with a NOT gate being performed on the second qubit, conditional on the first qubit being set to zero.

We note that

$$\hat{G}_{CNOT}=\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}\ ,$$

Then we have

$$\begin{aligned}(\hat{X}\otimes\hat{1})\hat{G}_{CNOT}(\hat{X}\otimes\hat{1}) &= (\hat{X}\otimes\hat{1})(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X})(\hat{X}\otimes\hat{1}) \\ &= (\hat{X}\otimes\hat{1})(\hat{m}\hat{X}\otimes\hat{1}+\hat{n}\hat{X}\otimes\hat{X}) \\ &= \hat{X}\hat{m}\hat{X}\otimes\hat{1}+\hat{X}\hat{n}\hat{X}\otimes\hat{X} \\ &= \hat{n}\hat{X}^2\otimes\hat{1}+\hat{m}\hat{X}^2\otimes\hat{X} \\ &= \hat{n}\otimes\hat{1}+\hat{m}\otimes\hat{X}\end{aligned}$$

In fact we obtain

$$(\hat{X}\otimes\hat{1})\hat{G}_{CNOT}(\hat{X}\otimes\hat{1})=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

24. Equivalence of two quantum circuits: $(\hat{1}\otimes\hat{H})\hat{G}_Z(\hat{1}\otimes\hat{H})=\hat{G}_X$

We show that these two quantum circuits are equivalent to each other.

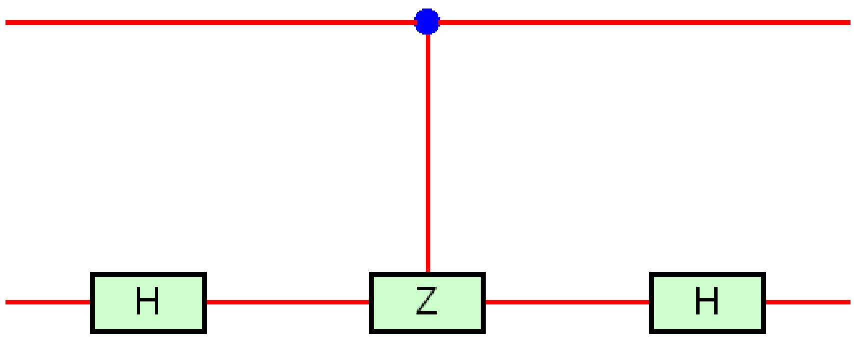


Fig.a Quantum circuit with $(\hat{1}\otimes\hat{H})\hat{G}_Z(\hat{1}\otimes\hat{H})$.

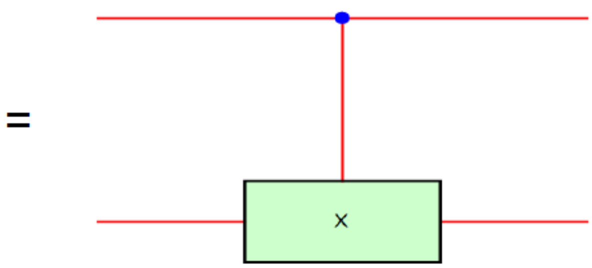


Fig.b Equivalent quantum circuit with $\hat{G}_X$

$$\hat{G}_Z = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{Z}, \qquad \hat{G}_X = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X}.$$

The quantum circuit can be represented by

$$\begin{aligned}
(\hat{1}\otimes\hat{H})\hat{G}_Z(\hat{1}\otimes\hat{H}) &= (\hat{1}\otimes\hat{H})(\hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{Z})(\hat{1}\otimes\hat{H}) \\
&= (\hat{1}\otimes\hat{H})(\hat{m}\otimes\hat{H} + \hat{n}\otimes\hat{Z}\hat{H}) \\
&= \hat{m}\otimes\hat{H}^2 + \hat{n}\otimes\hat{H}\hat{Z}\hat{H} \\
&= \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X} \\
&= \hat{G}_X
\end{aligned}$$

Note that

$$\hat{H}^2 = \frac{1}{2}(\hat{X}+\hat{Z})(\hat{X}+\hat{Z}) = \frac{1}{2}(\hat{X}^2+\hat{Z}^2+\hat{X}\hat{Z}+\hat{Z}\hat{X}) = \hat{1},$$

$$
\begin{aligned}
\hat{H}\hat{Z}\hat{H} &= \frac{1}{2}(\hat{X}+\hat{Z})\hat{Z}(\hat{X}+\hat{Z}) \\
&= \frac{1}{2}(\hat{X}+\hat{Z})(\hat{Z}\hat{X}+\hat{Z}^2) \\
&= \frac{1}{2}(\hat{X}+\hat{Z})(i\hat{Y}+\hat{1}) \\
&= \frac{1}{2}(i\hat{X}\hat{Y}+\hat{X}+i\hat{Z}\hat{Y}+\hat{Z}) \\
&= \hat{X}
\end{aligned}
$$

where

$$
\hat{X}\hat{Y} = -\hat{Y}\hat{X} = i\hat{Z}, \qquad \hat{Y}\hat{Z} = -\hat{Z}\hat{Y} = i\hat{X}, \qquad \hat{Z}\hat{X} = -\hat{X}\hat{Z} = i\hat{Y}.
$$

25. Equivalence of two quantum circuits; $(\hat{H}\otimes\hat{H})\hat{G}_{CNOT}(\hat{H}\otimes\hat{H}) = \hat{R}_{CNOT}$

We show that these two quantum circuits are equivalent to each other.

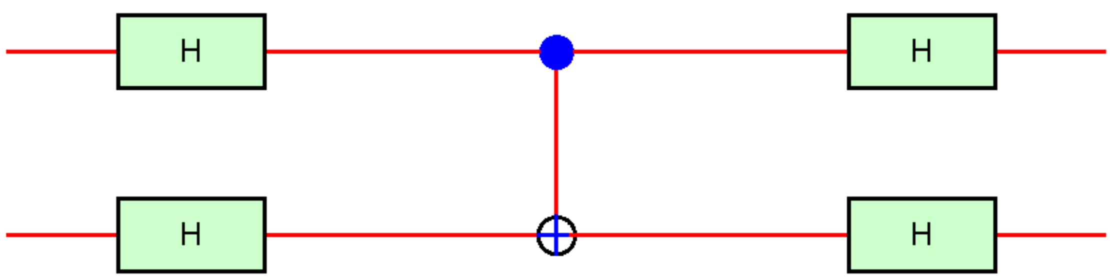


Fig.a Quantum circuit with $(\hat{H}\otimes\hat{H})\hat{G}_{CNOT}(\hat{H}\otimes\hat{H})$.

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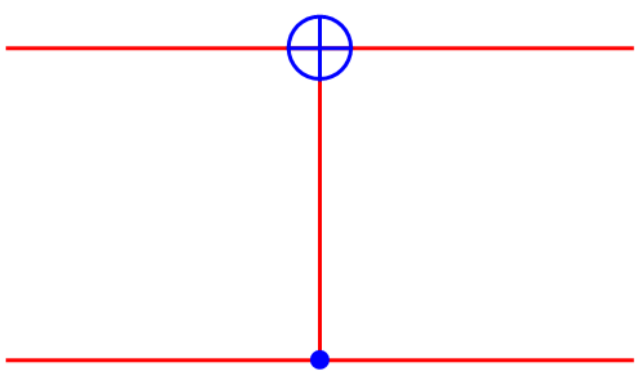


Fig.b Equivalent quantum circuit with $\hat{R}_{CNOT} = \hat{1}\otimes\hat{m} + \hat{X}\otimes\hat{n}$.

We note that

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} , \qquad \hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n} .$$

The quantum circuit (Fig.a) is expressed by

$$\begin{aligned}
(\hat{H} \otimes \hat{H})\hat{G}_{CNOT}(\hat{H} \otimes \hat{H}) &= (\hat{H} \otimes \hat{H})(\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X})(\hat{H} \otimes \hat{H}) \\
&= (\hat{H} \otimes \hat{H})(\hat{m}\hat{H} \otimes \hat{H} + \hat{n}\hat{H} \otimes \hat{X}\hat{H}) \\
&= \hat{H}\hat{m}\hat{H} \otimes \hat{H}^2 + \hat{H}\hat{n}\hat{H} \otimes \hat{H}\hat{X}\hat{H} \\
&= \hat{H}\hat{m}\hat{H} \otimes \hat{1} + \hat{H}\hat{n}\hat{H} \otimes \hat{H}\hat{X}\hat{H} \\
&= \hat{H}\hat{m}\hat{H} \otimes \hat{1} + \hat{H}\hat{n}\hat{H} \otimes \hat{Z}
\end{aligned}$$

or

$$\begin{aligned}
(\hat{H} \otimes \hat{H})\hat{G}_{CNOT}(\hat{H} \otimes \hat{H}) &= \frac{1}{2}(\hat{1} + \hat{X}) \otimes \hat{1} + \frac{1}{2}(\hat{1} - \hat{X}) \otimes \hat{Z} \\
&= \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{X} \otimes \hat{1} + \hat{1} \otimes \hat{Z} - \hat{X} \otimes \hat{Z}) \\
&= \hat{1} \otimes \frac{1}{2}(\hat{1} + \hat{Z}) + \hat{X} \otimes \frac{1}{2}(\hat{1} - \hat{Z}) \\
&= \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}
\end{aligned}$$

This agrees with

$$\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n} .$$

((**Note**))

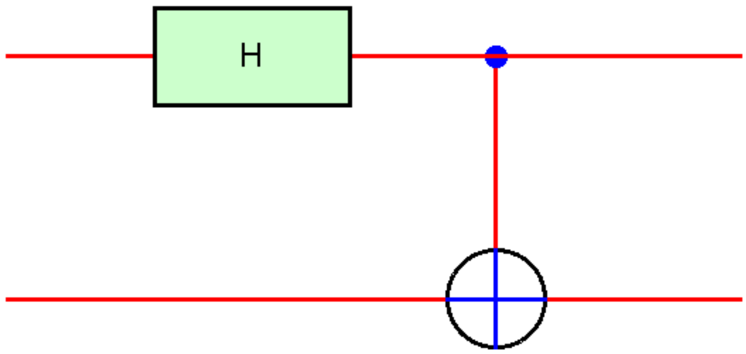
$$\hat{H}\hat{X}\hat{H} = \hat{Z} , \qquad \hat{H}\hat{Z}\hat{H} = \hat{X} ,$$

$$\hat{m} = \frac{1}{2}(\hat{1} + \hat{Z}) , \qquad \hat{n} = \frac{1}{2}(\hat{1} - \hat{Z}) .$$

$$\hat{H}\hat{m}\hat{H} = \frac{1}{2}\hat{H}(\hat{1} + \hat{Z})\hat{H} = \frac{1}{2}(\hat{H}^2 + \hat{H}\hat{Z}\hat{H}) = \frac{1}{2}(\hat{1} + \hat{X}) ,$$

$$\hat{H}\hat{n}\hat{H} = \frac{1}{2}\hat{H}(\hat{1} - \hat{Z})\hat{H} = \frac{1}{2}(\hat{H}^2 - \hat{H}\hat{Z}\hat{H}) = \frac{1}{2}(\hat{1} - \hat{X}) .$$

26. Construction of the Bell's states



Entangled qubits

$$\begin{aligned}\hat{G}_{CNOT}(\hat{H}\otimes\hat{1}) &= (\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X})(\hat{H}\otimes\hat{1})\\ &= \hat{m}\hat{H}\otimes\hat{1}+\hat{n}\hat{H}\otimes\hat{X}\\ &= \frac{1}{\sqrt{2}}[(\hat{1}+\hat{Z})\hat{H}\otimes\hat{1}+(\hat{1}-\hat{Z})\hat{H}\otimes\hat{X}]\end{aligned}$$

where

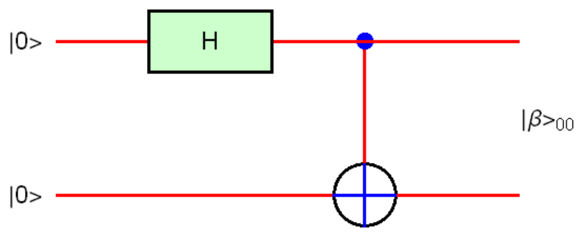
$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}\ ,$$

$$\hat{H} = \frac{1}{\sqrt{2}}(\hat{X}+\hat{Z}), \qquad \hat{Z}\hat{X} = i\hat{Y}\ ,$$

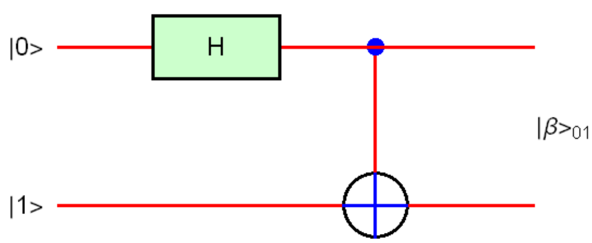
$$\hat{m}\hat{H} = \frac{1}{\sqrt{2}}(\hat{1}+\hat{Z})\hat{H}\ , \qquad \hat{H}\hat{n} = \frac{1}{\sqrt{2}}\hat{H}(\hat{1}-\hat{Z})\ .$$

The matrix of $\hat{G}_{CNOT}(\hat{H}\otimes\hat{1})$ is given by

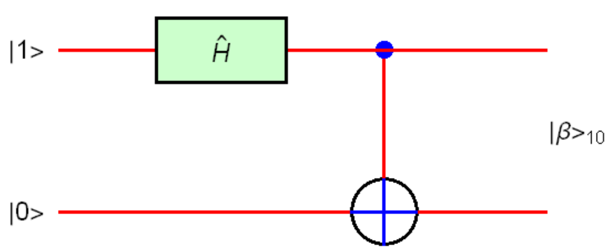
$$\hat{G}_{CNOT}(\hat{H}\otimes\hat{1}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & -1\\ 1 & 0 & -1 & 0\end{pmatrix}$$



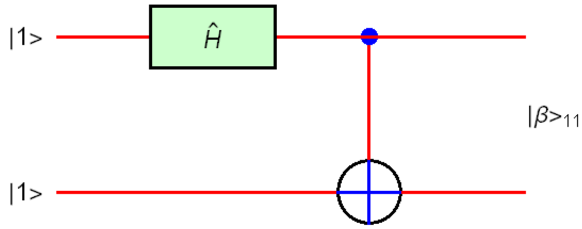
$$|\beta\rangle_{00} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle).$$



$$|\beta\rangle_{01} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle).$$



$$|\beta\rangle_{10} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle).$$



$$|\beta\rangle_{11} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle).$$

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Jacques Salomon Hadamard;(8 December 1865 – 17 October 1963) was a French mathematician who made major contributions in number theory, complex function theory, differential geometry and partial differential equations.

Tommaso Toffoli is a professor of electrical and computer engineering at Boston University where he joined the faculty in 1995. He has worked on cellular automata and the theory of artificial life (with Edward Fredkin and others), and is known for the invention of the Toffoli gate.

Edward Fredkin (born 1934) is a distinguished career professor at Carnegie Mellon University (CMU) and an early pioneer of digital physics. His primary contributions include his work on reversible computing and cellular automata. While Konrad Zuse's book, *Calculating Space* (1969), mentioned the importance of reversible computation, the Fredkin gate represented the essential breakthrough. In recent work, he uses the term digital philosophy (DP). During his career Fredkin also served on the faculties of MIT in Computer Science, was a Fairchild Distinguished Scholar at Caltech, and Research Professor of Physics at Boston University.

David Elieser Deutsch, FRS (born 18 May 1953) is a British physicist at the University of Oxford. He is a non-stipendiary Visiting Professor in the Department of Atomic and Laser Physics at the Centre for Quantum Computation (CQC) in the Clarendon Laboratory of the University of Oxford. He pioneered the field of quantum computation by formulating a description for a quantum Turing machine, as well as specifying an algorithm designed to run on a quantum computer. He is a proponent of the many-worlds interpretation of quantum mechanics.
