Quantum computing
Equivalent circuits of quantum teleportation and swap

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Quantum circuit is a sort of electric circuits (such as Wheatstone bridge and ladder circuit), which one may study in the class of circuit analysis (electricity and magnetism). For complicated circuits (such as network), it is essential to make use of the theorems of circuit analysis (such as Thevinin theorem and Norton theorem). So, the circuit becomes much more simplified by using the corresponding equivalent circuits. It may be true for quantum computing. We can also apply various kinds of techniques (based on the quantum mechanics) to the quantum circuits (such as the quantum teleportation and SWAP). The equivalent circuits can be used for the simplification of quantum circuits.

In a Website, we find a very interesting article on the discussion on the equivalence of quantum computer circuit between quantum teleportation and the SWAP circuit. It is surprising for one that the SWAP circuit is literally equivalent to the quantum teleportation. The title: From Swapping to teleporting with Simple Circuit Moves; https://algassert.com/post/1628. In the introduction of this article, we found the following exciting statements. "We are going to prove that quantum teleportation works. Not by carefully considering how it affects input states, but by starting with a circuit that *obviously* moves a qubit from one place to another and then applying simple obviously-correct transformations until we end up with the quantum teleportation circuit."

We also had an excellent opportunity to listen to a series of lectures on the quantum computer, in a web site (in Japanese). In the second lecture (Quantum teleportation, done by Eisuke Abe, Keio University on November 15, 2009), the quantum circuit for the quantum teleportation circuit is discussed in the association with the SWAP circuit. We were very impressed with a possible equivalence of the quantum circuits between the quantum teleportation and the SWAP. Note that unfortunately, these lectures were done in Japanese.

Here we will show that the quantum circuit of the SWAP circuit is essentially equivalent to that of the quantum teleportation. This lecture note is mainly written in order to reproduce the content of lectures given by Eisuke Abe. In other words, there is nothing new in this note. Nevertheless, we think that the content of this lecture note may be very useful to undergraduate students and graduate students who want to know about the principle of the quantum entanglement [quantum teleportation among three people Alice (A), Bob (B), and Charlie (C)]. The information of Charlie (or state) is delivered to Bob, immediately after the Bell state shared with Alice and Charlie is observed by Alice.

Here we discuss the quantum circuits of the quantum teleportation and the swap based on the lecture. We show the similarity of the quantum circuits between the quantum teleportation and SAP. In this note, we first discuss the fundamental circuits in quantum computer, in particular, various kinds of equivalent circuits.

1. Quantum bits

In quantum computing, a qubit or quantum bit is the basic unit of quantum information—the quantum version of the classical binary bit physically realized with a two-state device. A qubit is a two-state (or two-level) quantum-mechanical system, one of the simplest quantum systems displaying the peculiarity of quantum mechanics. Examples include: the spin of the electron in which the two levels can be taken as spin up $\left|+z\right\rangle = \left|0\right\rangle$ and spin down $\left|-z\right\rangle = \left|1\right\rangle$; or the polarization of a single photon in which the two states can be taken to be the and the horizontal polarization $\left|x\right\rangle$ and the vertical polarization $\left|y\right\rangle$. In a classical system, a bit would have to be in one state or the other. However, quantum mechanics allows the qubit to be in a coherent superposition of both states simultaneously, a property which is fundamental to quantum mechanics and quantum computing. Here we use the Dirac notation, the eigenkets $\left|0\right\rangle$ and $\left|1\right\rangle$;

$$\left|0\right\rangle = \left|+z\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \left|1\right\rangle = \left|-z\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

https://en.wikipedia.org/wiki/Qubit

The quantum circuits consist of various kinds of quantum gate (unitary operators, Pauli operators, and so on). These gates are reversible in the processes. Here we discuss the quantum circuits of the quantum teleportation and swap, as typical examples.

2. Quantum NOT: Pauli spin 1/2 operators

X corresponds to the Pauli matrix; $X = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We note that

$$X\left|a\right\rangle = \left|\bar{a}\right\rangle,$$

$$X\left|0\right\rangle = \left|\bar{0}\right\rangle = \left|1\right\rangle, \qquad X\left|1\right\rangle = \left|\bar{1}\right\rangle = \left|0\right\rangle.$$

Y corresponds to the Pauli matrix $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

$$Y\left|a\right\rangle = (-1)^a i\left|\bar{a}\right\rangle.$$

Z corresponds to the Pauli matrix $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$Z|a\rangle = (-1)^a |a\rangle,$$

Note that

$$X^2 = Y^2 = Z^2 = 1.$$

Commutation relations: $[X,Y] = 2iZ$, $[Y,Z] = 2iX$, $[Z,X] = 2iY$

3. Hadamard gate

Hadamard gate is expressed by H in the quantum circuit.

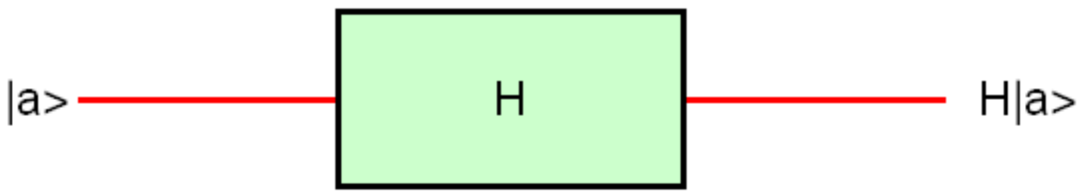

Fig1. Hadamard gate H.

$$\hat{U}_x = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{\sigma}_z + \hat{\sigma}_x);$$

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

In general, we have

$$H|a\rangle = \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}|b\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^a|1\rangle],$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\sum_{b=0}^{1}|b\rangle, \qquad H|1\rangle = \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^b|b\rangle.$$

Note that

$$H^2 = 1$$

$$HXH = Z, \qquad HYH = -Y, \qquad HZH = X,$$

$$X|a\rangle = |\bar{a}\rangle = |a \oplus 1\rangle, \qquad Y|a\rangle = (-1)^a i|\bar{a}\rangle, \qquad Z|a\rangle = (-1)^a|a\rangle.$$

X, Y, and Z are the Pauli matrices of 2x2. The detail of the properties is presented in the APPENDIX.

3.1. Proof of $HXH = Z$

$$
\begin{aligned}
HXH|a\rangle &= HX\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}|b\rangle \\
&= H\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}X|b\rangle \\
&= H\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}|\overline{b}\rangle \\
&= H\frac{1}{\sqrt{2}}[|\overline{0}\rangle+(-1)^{a}|\overline{1}\rangle] \\
&= H\frac{1}{\sqrt{2}}[|1\rangle+(-1)^{a}|0\rangle] \\
&= (-1)^{a}H\frac{1}{\sqrt{2}}[|0\rangle+(-1)^{a}|1\rangle] \\
&= (-1)^{a}H^{2}|a\rangle \\
&= (-1)^{a}|a\rangle
\end{aligned}
$$

or

$$HXH|a\rangle = (-1)^{a}|a\rangle = Z|a\rangle .$$

3.2 Proof of $HYH = -Y$

$$
\begin{aligned}
HYH\left|a\right\rangle &= HY\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}\left|b\right\rangle \\
&= H\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}Y\left|b\right\rangle \\
&= \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{(a+1)b}iH\left|\bar{b}\right\rangle \\
&= \frac{1}{\sqrt{2}}i\sum_{b=0}^{1}(-1)^{(a+1)b}\frac{1}{\sqrt{2}}\sum_{c=0}^{1}(-1)^{\bar{b}c}\left|c\right\rangle \\
&= \frac{1}{2}i\sum_{b,c}(-1)^{(a+1)b+\bar{b}c}\left|c\right\rangle \\
&= \frac{1}{2}i[1-(-1)^{a}]\left|0\right\rangle+\frac{1}{2}[1+(-1)^{a}]\left|1\right\rangle \\
&= \frac{1}{2}[\left|0\right\rangle-\left|1\right\rangle]-\frac{1}{2}(-1)^{a}[\left|0\right\rangle+\left|1\right\rangle]
\end{aligned}
$$

3.3 Proof of $HZH = X$

$$
\begin{aligned}
HZH\left|a\right\rangle &= HZ\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}\left|b\right\rangle \\
&= H\frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}Z\left|b\right\rangle \\
&= \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{(a+1)b}H\left|b\right\rangle \\
&= \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{(a+1)b}\frac{1}{\sqrt{2}}\sum_{c=0}^{1}(-1)^{bc}\left|c\right\rangle \\
&= \frac{1}{2}\sum_{b,c}(-1)^{(a+c+1)b}\left|c\right\rangle \\
&= \frac{1}{2}[1-(-1)^{a}]\left|0\right\rangle+\frac{1}{2}[1+(-1)^{a}]\left|1\right\rangle \\
&= \frac{1}{2}[\left|0\right\rangle+\left|1\right\rangle]+\frac{1}{2}(-1)^{a}[\left|0\right\rangle-\left|1\right\rangle] \\
&= \left|\bar{a}\right\rangle
\end{aligned}
$$

or

$$HZH\left|a\right\rangle=\left|\bar{a}\right\rangle=X\left|a\right\rangle.$$

4. Controlled-*U* gate

The controlled-U gate is defined by

$$|a\rangle \otimes |b\rangle \rightarrow |a\rangle \otimes U^{a}|b\rangle,$$

with

Control bit: $|a\rangle$,

Target bit: $|b\rangle$,

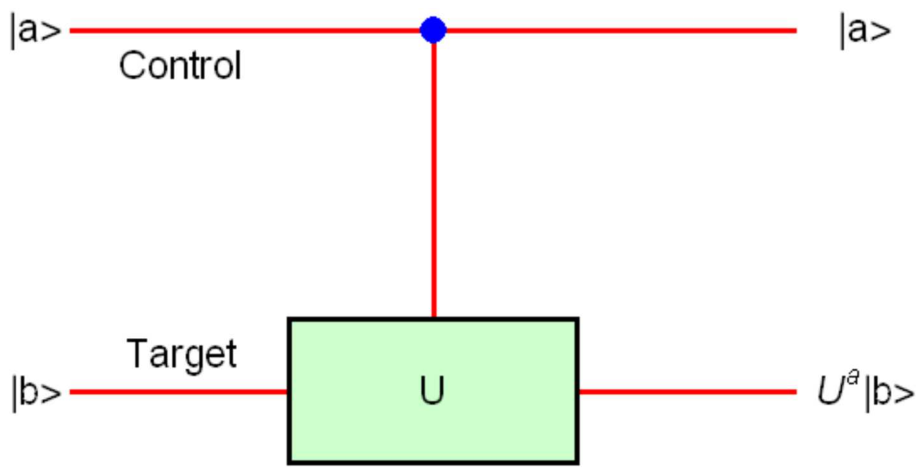

Fig.2 Controlled U gate. Control ($|a\rangle$). Target ($|b\rangle$).

We have another type of control-U gate where the target and the control are different from that in the above control-U gate.

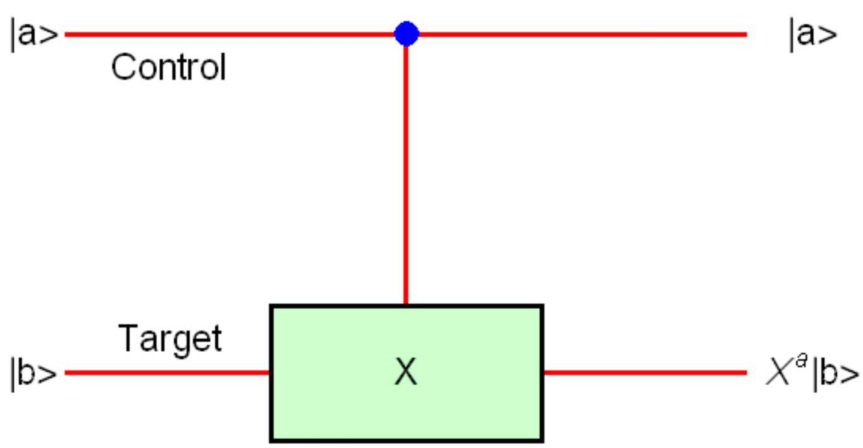

Fig.3 Controlled X-gate. $|a\rangle$: control. $|b\rangle$: target. $X^{a}|b\rangle$.

5. CNOT gate

The **controlled NOT gate** (**CNOT**) is a quantum logic gate that is an essential component in the construction of a gate-based quantum computer. It can be used to entangle and disentangle EPR states. Any quantum circuit can be simulated to an arbitrary degree of accuracy using a combination of CNOT gates and single qubit rotations.
https://en.wikipedia.org/wiki/Controlled_NOT_gate

Here we use C_{12} as the CNOT gate.

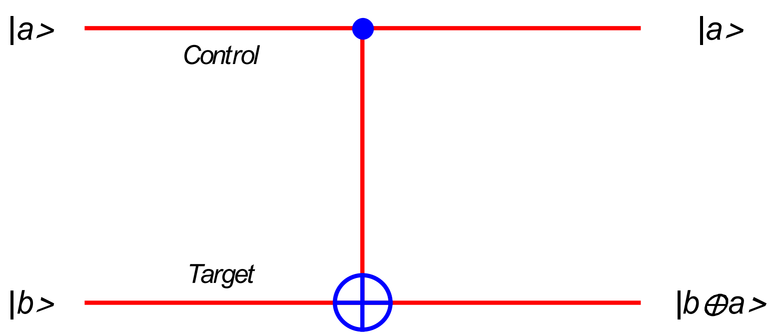

Fig.4 Controlled Not gate (CNOT). $|a\rangle$: control. $|b\rangle$: target.

$$|a\rangle \otimes |b\rangle \rightarrow |a\rangle \otimes X^a |b\rangle = |a\rangle \otimes |b \oplus a\rangle$$

$$C_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $|a\rangle$: control. $|b\rangle$: target.

C_{21}

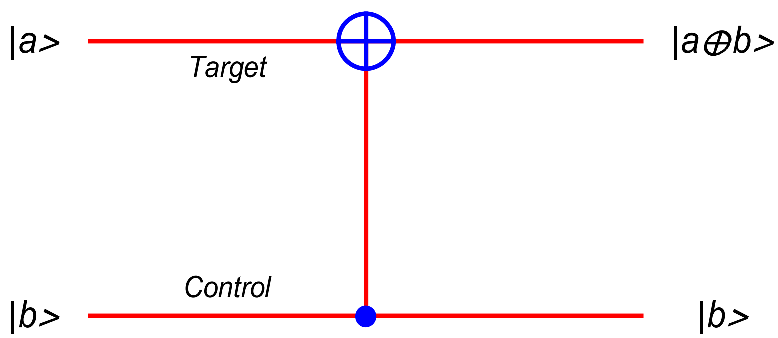

Fig.5 CNOT gate: $|b\rangle$: control. $|a\rangle$: target.

$$|a\rangle \otimes |b\rangle \rightarrow |a \oplus b\rangle \otimes |b\rangle .$$

where $|b\rangle$: control. $|a\rangle$: target.

The matrix of C_{12} is

$$C_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

((**Note**))

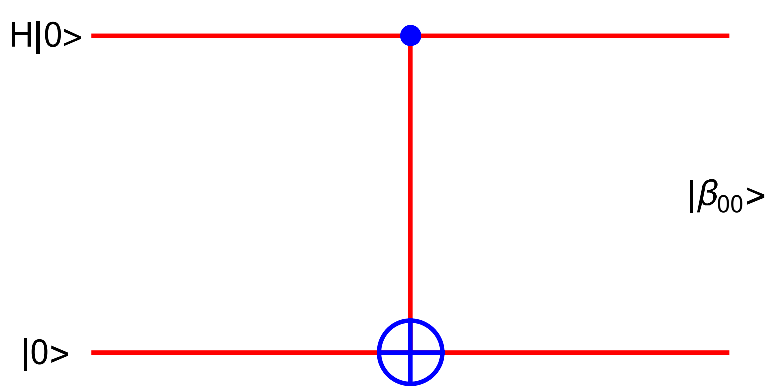

Fig.6 Encode of the Bell state $|\beta_{00}\rangle$ using CNOT gate with control ($H|0\rangle$) and target ($|0\rangle$.

$H|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$. $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$ (Bell state).

Using the CNOT the input state of two unentangled qubits can be changed into an entangled state.

Input:

$$(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle)\otimes|0\rangle .$$

Output:

$$C_{12}[\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle)\otimes|0\rangle]=|\beta_{00}\rangle=\frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\otimes|1\rangle .$$

6. SWAP: Dirac exchange operator

$$\begin{aligned}|a\rangle\otimes|b\rangle &\to |a\rangle\otimes|b\oplus a\rangle\\ &\to |a\oplus b\oplus a\rangle\otimes|b\oplus a\rangle=|b\rangle\otimes|b\oplus a\rangle\\ &\to |b\rangle\otimes|b\oplus a\oplus b\rangle=|b\rangle\otimes|a\rangle\end{aligned}$$

Note

$$a\oplus a=0 .$$

Dirac exchange operator is defined by

$$\hat{P}=\frac{1}{2}(\hat{1}+\hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{\sigma}}_2) ,$$

in terms of the Pauli operators.

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

The SWAP circuit is expressed by a circuit

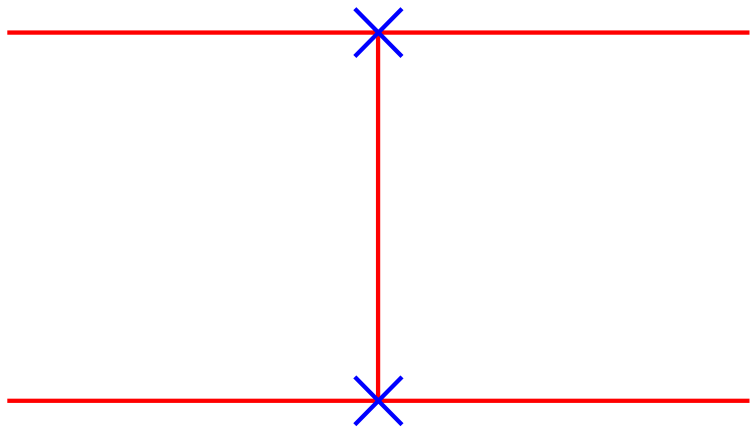

Fig.7(a) SWAP circuit.

which is equivalent to

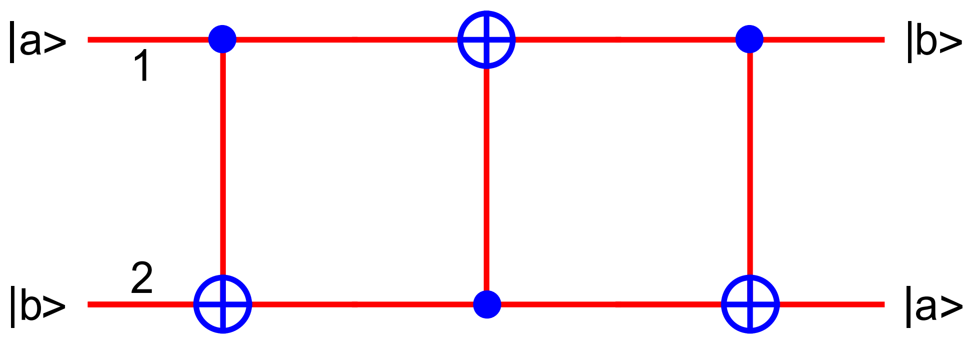

Fig.7(b) Circuit equivalent to SWAP circuit.

using the CNOT circuits. The proof for this is given as follows.

$$
\begin{aligned}
C_{12}C_{21}C_{12}|a\rangle\otimes|b\rangle &= C_{12}C_{21}|a\rangle\otimes|b\oplus a\rangle \\
&= C_{12}|a\oplus b\oplus a\rangle\otimes|b\oplus a\rangle \\
&= C_{12}|b\rangle\otimes|b\oplus a\rangle \\
&= |b\rangle\otimes|b\oplus a\oplus b\rangle \\
&= |b\rangle\otimes|a\rangle
\end{aligned}
$$

where $b\oplus b=0$.

7. Controlled Z gate

The controlled Z-gate is expressed by a circuit,

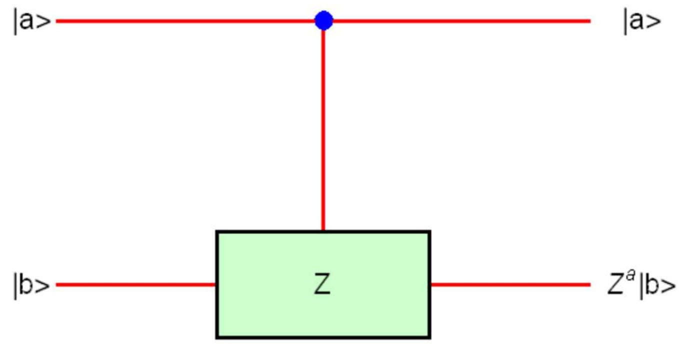

Fig.8(a) Controlled Z gate with control ($|a\rangle$) and target ($|b\rangle$. $Z^a|b\rangle = (-1)^{ab}|b\rangle$.

$$|a\rangle_C \otimes |b\rangle_T \to |a\rangle \otimes Z^a|b\rangle = (-1)^{ab}|a\rangle \otimes |b\rangle,$$

which is equivalent to the upside-down circuit, where C is control and T is target.

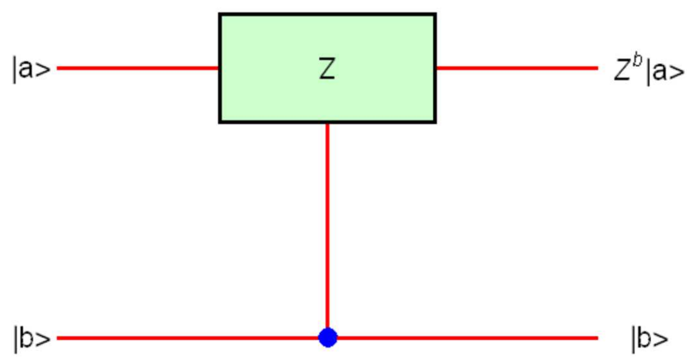

Fig.8(b) Controlled Z gate with control ($|b\rangle$) and target ($|a\rangle$, which is equivalent to Fig.8(a)

since

$$|a\rangle_T \otimes |b\rangle_C \to (Z^b|a\rangle) \otimes |b\rangle = (-1)^{ab}|a\rangle \otimes |b\rangle.$$

Here we note that

$$Z^a|b\rangle = (-1)^{ab}|b\rangle, \qquad Z|b\rangle = (-1)^b|b\rangle.$$

8. Base transformation for the Bell states

8.1 Encode of Bell state: $|x\rangle \otimes |y\rangle \to |\beta_{xy}\rangle$

Encode of Bell circuit is obtained using the combination (H-CNOT) circuits.

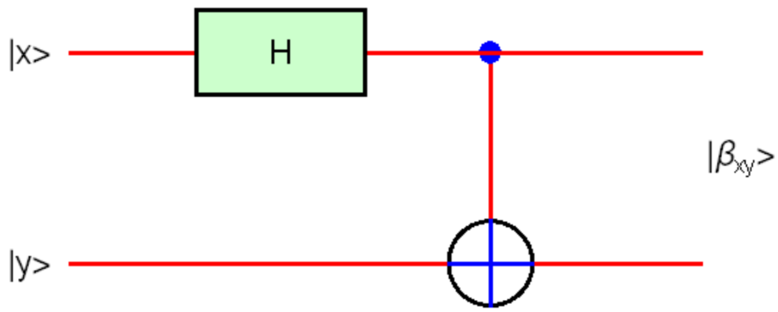

Fig.9 Encode of Bell state $|\beta_{xy}\rangle$. Base transformation ($|x\rangle \otimes |y\rangle \rightarrow |\beta_{xy}\rangle$ by H-CNOT gate.

The transformation

$$|x\rangle \otimes |y\rangle \rightarrow |\beta_{xy}\rangle$$

can be proved as follows.

Step A: **Application of H-gate**

$$\begin{aligned} |x\rangle \otimes |y\rangle &\xrightarrow{H} (H|x\rangle) \otimes |y\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{b=0}^{1} (-1)^{xb} |b\rangle \otimes |y\rangle \\ &= \frac{1}{\sqrt{2}} [|0\rangle \otimes |y\rangle + (-1)^{x} |1\rangle \otimes |y\rangle \end{aligned}$$

where

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{b=0}^{1} (-1)^{xb} |b\rangle = \frac{1}{\sqrt{2}} [|0\rangle + (-1)^{x} |1\rangle]$$

Step B: **Application of CNOT gate**

$$\frac{1}{\sqrt{2}}[|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y\rangle \xrightarrow{C_{12}} \frac{1}{\sqrt{2}}[|0\rangle \otimes |y \oplus 0\rangle + (-1)^x |1\rangle \otimes |y \oplus 1\rangle$$
$$= \frac{1}{\sqrt{2}}[|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle$$
$$= |\beta_{xy}\rangle$$

or

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle .$$

where

$$y \oplus 0 = y,$$
$$y \oplus 1 = \overline{y}$$

or more directly, we have

$$C_{12}H_1|x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{xb} C_{12}|b\rangle \otimes |y\rangle$$
$$= \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{xb}|b\rangle \otimes |y \oplus b\rangle$$
$$= \frac{1}{\sqrt{2}}[|0\rangle \otimes |y \oplus 0\rangle + (-1)^x |1\rangle \otimes |y \oplus 1\rangle]$$
$$= \frac{1}{\sqrt{2}}[|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle]$$
$$= |\beta_{xy}\rangle$$

Note that the Bell states (4 states) are defined by

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle ,$$

where

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

8.2 Decode of Bell states: $\left|\beta_{xy}\right\rangle \rightarrow |x\rangle \otimes |y\rangle$

The decode of the Bell state is obtained from the combination CNOT-*H*;

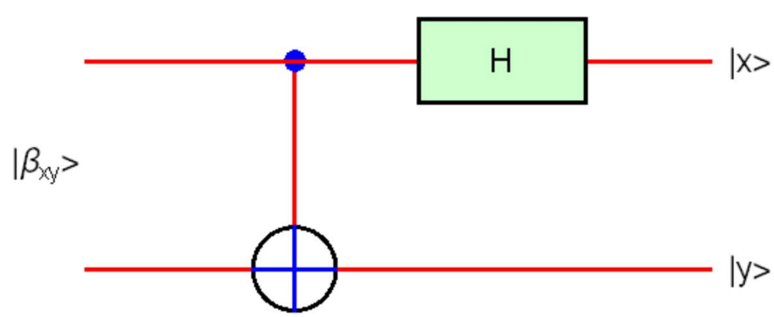

Fig.10 Decode of Bell state $\left|\beta_{xy}\right\rangle$. Base transformation ($|x\rangle \otimes |y\rangle \rightarrow \left|\beta_{xy}\right\rangle$ by H-CNOT gate.

$$\left|\beta_{xy}\right\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle$$

Step A. **CNOT gate (C_{12})**

$$\left|\beta_{xy}\right\rangle \xrightarrow{C_{12}} \frac{1}{\sqrt{2}}[\left|0\right\rangle \otimes \left|y \oplus 0\right\rangle + (-1)^x \left|1\right\rangle \otimes \left|\bar{y} \oplus 1\right\rangle]$$
$$= \frac{1}{\sqrt{2}}[\left|0\right\rangle \otimes \left|y\right\rangle + (-1)^x \left|1\right\rangle \otimes \left|y\right\rangle]$$
$$= \frac{1}{\sqrt{2}}[\left|0\right\rangle + (-1)^x \left|1\right\rangle] \otimes \left|y\right\rangle]$$
$$= (H\left|x\right\rangle) \otimes \left|y\right\rangle$$

Step B. ***H* gate**

$$(H\left|x\right\rangle) \otimes \left|y\right\rangle \xrightarrow{H} (H^2\left|x\right\rangle) \otimes \left|y\right\rangle = \left|x\right\rangle \otimes \left|y\right\rangle$$

since $H^2 = 1$.

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9. Equivalent quantum circuits

Here, we discuss several equivalent-circuits. These equivalent circuits are essential to the simplification of quantum circuits for the quantum teleportation and the swap circuit.

9.1 Equivalent quantum circuits-1: Controlled *X* - CNOT

The following two circuits are equivalent.

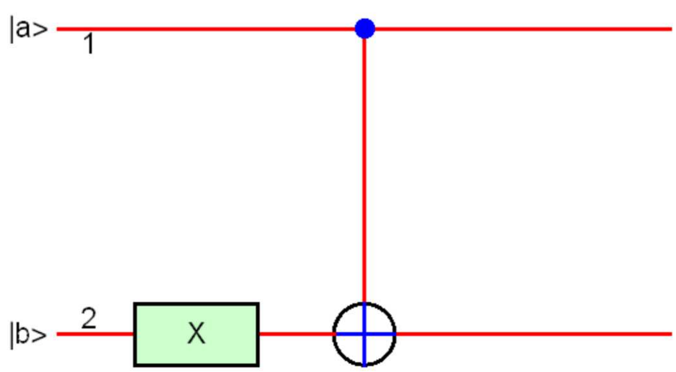

Fig.11(a) Quantum circuit-**1**

$$C_{12}X_2\left|a\right\rangle \otimes \left|b\right\rangle = C_{12}\left|a\right\rangle \otimes \left|b \oplus 1\right\rangle$$
$$= \left|a\right\rangle \otimes \left|a \oplus b \oplus 1\right\rangle$$

where

$$\bar{b} = b \oplus 1$$

This circuit is equivalent to

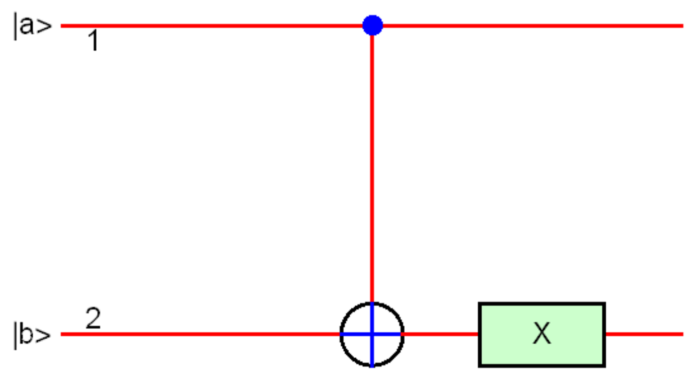

Fig.11(b) Quantum circuit equivalent to Fig.11(a).

$$\begin{aligned} X_2 C_{12} \left| a \right\rangle \otimes \left| b \right\rangle &= X_2 \left| a \right\rangle \otimes \left| a \oplus b \right\rangle \\ &= \left| a \right\rangle \otimes \left| a \oplus b \oplus 1 \right\rangle \end{aligned}$$

9.2 Equivalent quantum circuits-2

The following two circuits are equivalent.

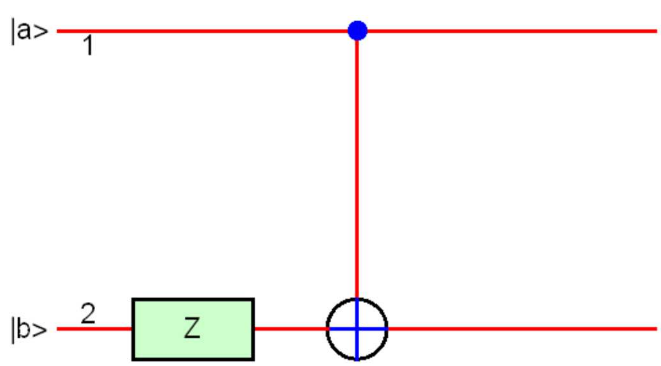

Fig.12(a) Quantum circuit-2

$$\begin{aligned} C_{12} Z_2 \left| a \right\rangle \otimes \left| b \right\rangle &= (-1)^b C_{12} \left| a \right\rangle \otimes \left| b \right\rangle \\ &= (-1)^b \left| a \right\rangle \otimes \left| a \oplus b \right\rangle \end{aligned}$$

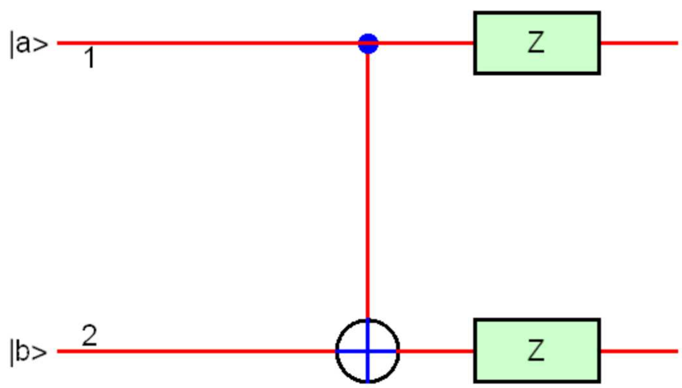

Fig.12(b) Quantum circuit equivalent to Fig.12(a).

$$
\begin{aligned}
Z_1 Z_2 C_{12} \left|a\right\rangle \otimes \left|b\right\rangle &= Z_1 Z_2 \left|a\right\rangle \otimes \left|a \oplus b\right\rangle \\
&= (-1)^{a+a+b} \left|a\right\rangle \otimes \left|a \oplus b\right\rangle \\
&= (-1)^{b} \left|a\right\rangle \otimes \left|a \oplus b\right\rangle
\end{aligned}
$$

9.3 Equivalent quantum circuits-3

The following two circuits are equivalent.

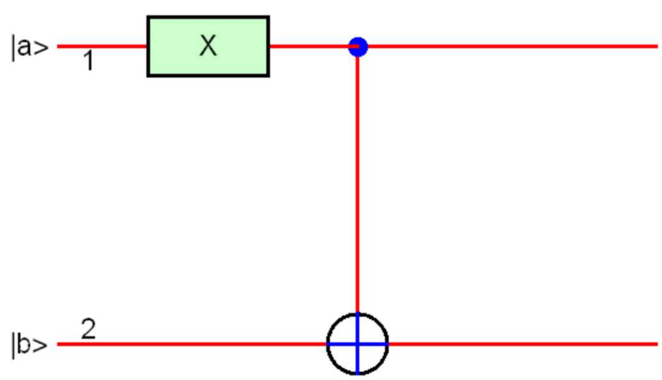

Fig.13(a) Quantum circuit-3.

$$
\begin{aligned}
C_{12} X_1 \left|a\right\rangle \otimes \left|b\right\rangle &= C_{12} \left|a \oplus 1\right\rangle \otimes \left|b\right\rangle \\
&= \left|a \oplus 1\right\rangle \otimes \left|a \oplus b \oplus 1\right\rangle
\end{aligned}
$$

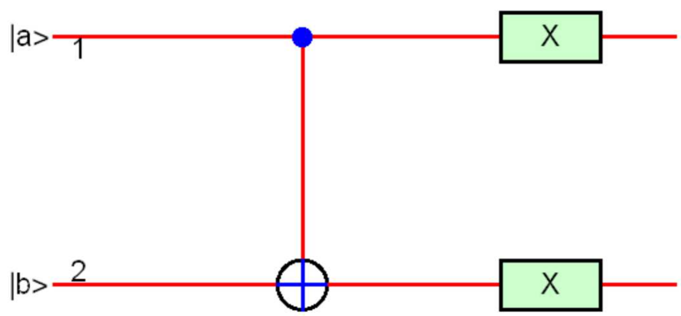

Fig.13(b) Quantum circuit equivalent to Fig.13(a).

$$X_1 X_2 C_{12} |a\rangle \otimes |b\rangle = X_1 X_2 |a\rangle \otimes |a \oplus b\rangle$$
$$= |a \oplus 1\rangle \otimes |a \oplus b \oplus 1\rangle$$

9.4 Equivalent quantum circuits-4

The following two circuits are equivalent.

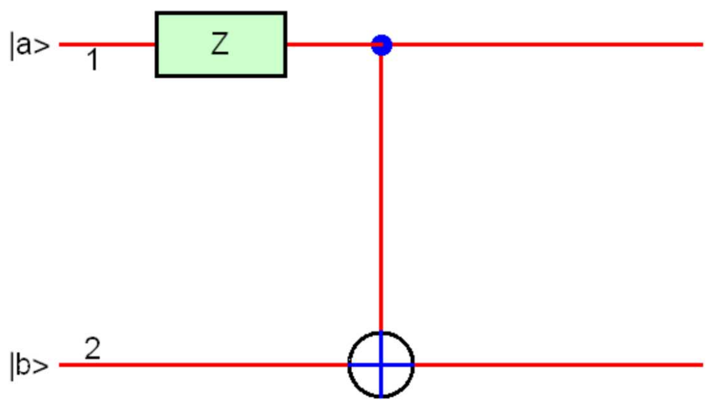

Fig.14(a) Z-gate and CNOT. Quantum circuit-4. 1: control. 2: target. Input: $|a\rangle \otimes |b\rangle$. Output: $(-1)^a |a\rangle \otimes |a \oplus b\rangle$.

$$Z_1 C_{12} (|a\rangle \otimes |b\rangle) = Z_1 |a\rangle \otimes |a \oplus b\rangle$$
$$= (-1)^a |a\rangle \otimes |a \oplus b\rangle$$

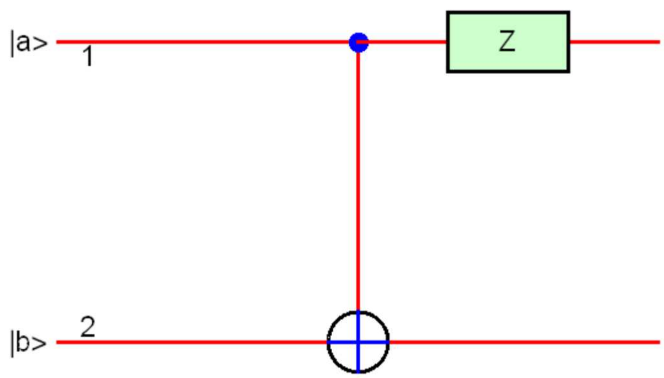

Fig.14(b) CNOT and Z-gate. Quantum circuit equivalent to the quantum circuit -4 of Fig.14(a). 1: control. 2: target. Input: $|a\rangle \otimes |b\rangle$. Output: $(-1)^a |a\rangle \otimes |a \oplus b\rangle$. Figs.14(a) and (b) are equivalent.

$$
\begin{aligned}
C_{12} Z_1 (|a\rangle \otimes |b\rangle) &= (-1)^a C_{12} |a\rangle \otimes |a\rangle \\
&= (-1)^a |a\rangle \otimes |a \oplus b\rangle
\end{aligned}
$$

9.5 Equivalent quantum circuits-5

The following two circuits are equivalent.

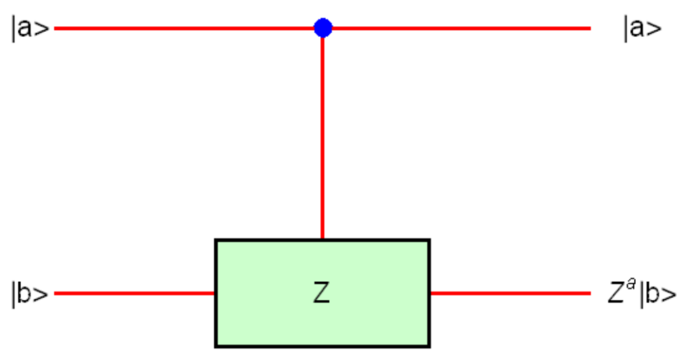

Fig.15(a) Controlled Z gate. Quantum circuit-5. Control gate ($|a\rangle$) and target gate ($|b\rangle$). 1: control ($|a\rangle$. 2: target ($|b\rangle$). Input: $|a\rangle \otimes |b\rangle$. Output: $(-1)^{ab} |a\rangle \otimes |b\rangle$.

$$
\begin{aligned}
|a\rangle \otimes Z^a |b\rangle &= |a\rangle \otimes (-1)^{ab} |b\rangle \\
&= (-1)^{ab} |a\rangle \otimes |b\rangle
\end{aligned}
$$

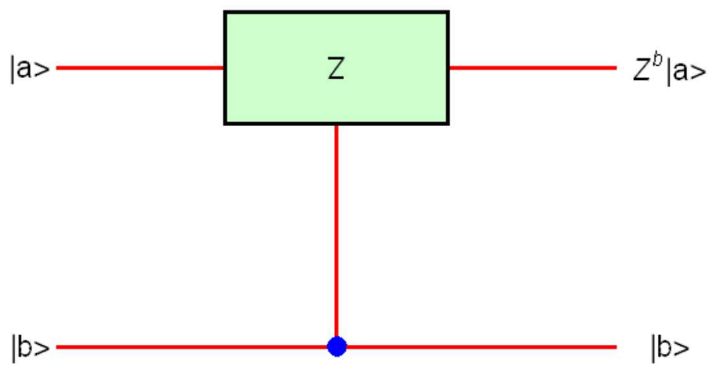

Fig.15(b) Controlled Z gate. Control gate ($|b\rangle$) and target gate ($|a\rangle$). $Z^b|a\rangle\otimes|b\rangle=(-1)^{ab}|a\rangle\otimes|b\rangle$.Figs.15(a) and 15(b) are equivalent. CZ gate is non-local, independent of the choice of control and target.

9.6 Equivalent quantum circuit-6

The following two circuits are equivalent.

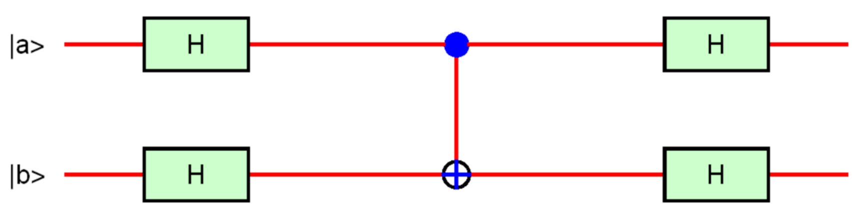

Fig.16(a) Quantum circuit-6, with 4 H-gates and 1 CZ-gate. Control gate (a) and target gate ($|b\rangle$). Input: $|a\rangle\otimes|b\rangle$. Output:$|a\oplus b\rangle\otimes|b\rangle$.

This quantum circuit is equivalent to the following circuit.

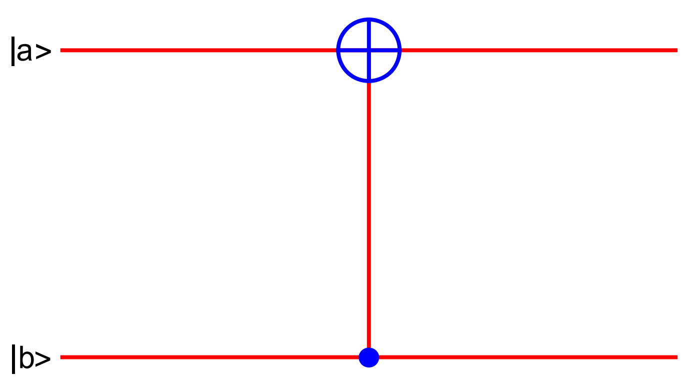

Fig.16(b) *CNOT* gate. Quantum circuit equivalent to Fig.16(a). Control gate ($|b\rangle$) and target gate ($|a\rangle$). The input: Input: $|a\rangle \otimes |b\rangle$. Output: $|a \oplus b\rangle \otimes |b\rangle$.

Step-1:

$$H|a\rangle = \frac{1}{\sqrt{2}} \sum_{a'=0}^{1} (-1)^{aa'} |a'\rangle, \qquad H|b\rangle = \frac{1}{\sqrt{2}} \sum_{b'=0}^{1} (-1)^{bb'} |b'\rangle$$

$$H|a\rangle \otimes H|b\rangle = \frac{1}{2} \sum_{a',b'=0}^{1} (-1)^{aa'+bb'} |a'\rangle \otimes |b'\rangle$$

Step-2:

$$\begin{aligned} C_{12}(H|a\rangle \otimes H|b\rangle) &= \frac{1}{2} \sum_{a',b'=0}^{1} (-1)^{aa'+bb'} C_{12} |a'\rangle \otimes |b'\rangle \\ &= \frac{1}{2} \sum_{a',b'=0}^{1} (-1)^{aa'+bb'} |a'\rangle \otimes |b' \oplus a'\rangle \\ &= \frac{1}{2} [|0\rangle \otimes |0 \oplus 0\rangle + (-1)^{b} |0\rangle \otimes |1 \oplus 0\rangle \\ &\quad + (-1)^{a} |1\rangle \otimes |0 \oplus 1\rangle + (-1)^{a+b} |1\rangle \otimes |1 \oplus 1\rangle] \\ &= \frac{1}{2} [|0\rangle \otimes |0\rangle + (-1)^{b} |0\rangle \otimes |1\rangle \\ &\quad + (-1)^{a} |1\rangle \otimes |1\rangle + (-1)^{a+b} |1\rangle \otimes |0\rangle] \end{aligned}$$

or

$$\begin{aligned} C_{12}(H|a\rangle \otimes H|b\rangle) &= \frac{1}{2} \{|0\rangle \otimes [|0\rangle + (-1)^{b} |1\rangle] \\ &\quad + (-1)^{a+b} |1\rangle [|0\rangle + (-1)^{-b} |1\rangle]\} \\ &= \frac{1}{2} \{|0\rangle \otimes [|0\rangle + (-1)^{b} |1\rangle] \\ &\quad + (-1)^{a+b} |1\rangle \otimes [|0\rangle + (-1)^{b} |1\rangle]\} \\ &= \frac{1}{2} [|0\rangle + (-1)^{a+b} |1\rangle] \otimes [|0\rangle + (-1)^{b} |1\rangle] \\ &= H|a \oplus b\rangle \otimes H|b\rangle \end{aligned}$$

Step-3:

$$H^2\left|a\oplus b\right\rangle\otimes H^2\left|b\right\rangle=\left|a\oplus b\right\rangle\otimes\left|b\right\rangle$$

since $H^2=1$.

10. Quantum teleportation

10.1 Principle of quantum teleportation

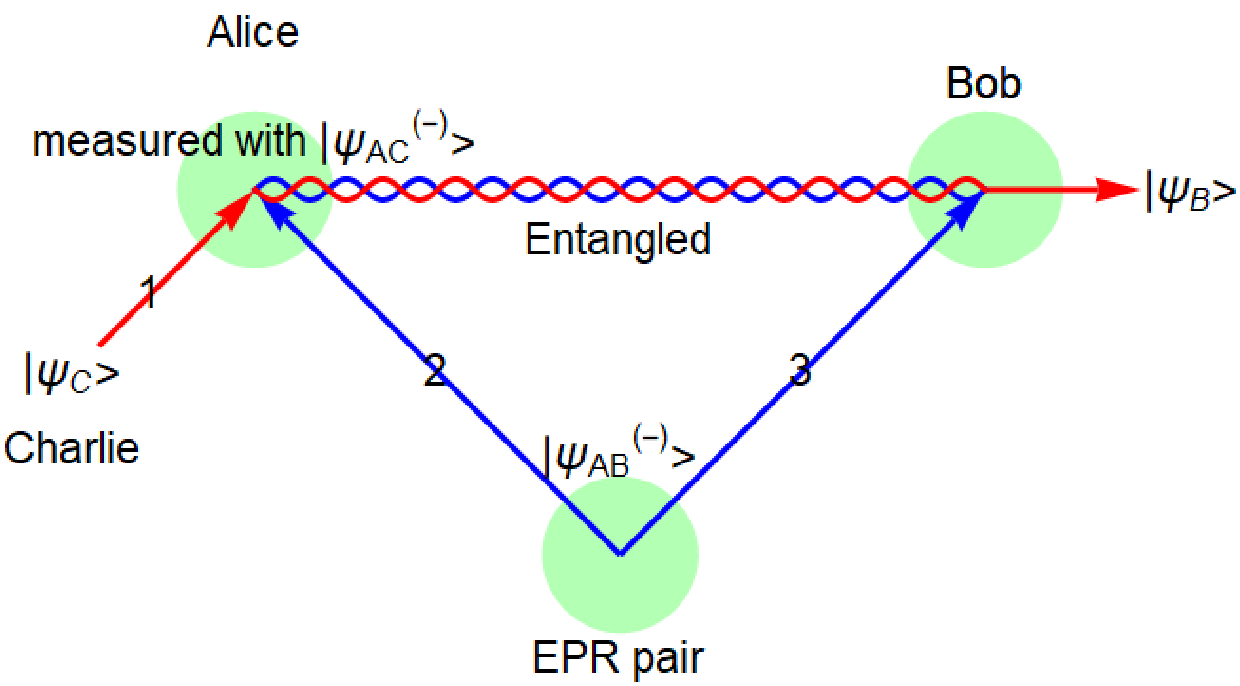

Fig.17 Quantum teleportation. The detail of this Figure is given in the section of Quantum teleportation> http://bingweb.binghamton.edu/~suzuki/QuantumMechanicsFiles/10-3_Quantum_teleportation.pdf

$$\left|\psi\right\rangle_C=\alpha\left|0\right\rangle+\beta\left|1\right\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}:$$ the state of Charlie

$$Z\left|\psi\right\rangle_C=\alpha Z\left|0\right\rangle+\beta Z\left|1\right\rangle=\alpha\left|0\right\rangle-\beta\left|1\right\rangle$$

$$X\left|\psi\right\rangle_C=\alpha X\left|0\right\rangle+\beta X\left|1\right\rangle=\alpha\left|1\right\rangle+\beta\left|0\right\rangle=\beta\left|0\right\rangle+\alpha\left|1\right\rangle$$

$$XZ|\psi\rangle_C = \alpha X|0\rangle - \beta X|1\rangle = \alpha|1\rangle - \beta|0\rangle$$

$|\beta_{xy}\rangle$: Bell state (EP pair; Alice and Bob)

$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(the Bell state shared by Alice and Bob)

It is shown that

$$\begin{aligned} |\psi\rangle_C \otimes |\beta_{00}\rangle_{AB} &= \frac{1}{2}|\beta_{00}\rangle_{CA} \otimes |\psi\rangle_B + \frac{1}{2}|\beta_{01}\rangle_{CA} \otimes X|\psi\rangle_B \\ &\quad + \frac{1}{2}|\beta_{10}\rangle_{CA} \otimes Z|\psi\rangle_B + \frac{1}{2}|\beta_{11}\rangle_{CA} \otimes XZ|\psi\rangle_B \\ &= \frac{1}{2}\sum_{x,y=0}^{1} |\beta_{xy}\rangle_{CA} \otimes X^y Z^x |\psi\rangle_B \end{aligned}$$

where

$$|\psi\rangle_C \otimes |\beta_{00}\rangle_{AB} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{pmatrix}$$

which is equal to

$$
\begin{array}{cccc}
\alpha & 0 & \alpha & 0 \\
\beta & 0 & -\beta & 0 \\
0 & \beta & 0 & -\beta \\
0 & \alpha & 0 & \alpha \\
0 & \beta & 0 & \beta \\
0 & \alpha & 0 & -\alpha \\
\alpha & 0 & -\alpha & 0 \\
\beta & 0 & \beta & 0
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
2\alpha \\ 0 \\ 0 \\ 2\alpha \\ 2\beta \\ 0 \\ 0 \\ 2\beta
\end{array}
$$

where

$$
\left|\beta_{00}\right\rangle_{CA} \otimes \left|\psi\right\rangle_{B} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\\beta\\0\\0\\0\\0\\\alpha\\\beta\end{pmatrix}
$$

$$
\left|\beta_{01}\right\rangle_{CA} \otimes X\left|\psi\right\rangle_{B} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \otimes \begin{pmatrix}\beta\\\alpha\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\\beta\\\alpha\\\beta\\\alpha\\0\\0\end{pmatrix}
$$

$$
\left|\beta_{10}\right\rangle_{CA} \otimes Z\left|\psi\right\rangle_{B} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \otimes \begin{pmatrix}\alpha\\-\beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\-\beta\\0\\0\\0\\0\\-\alpha\\\beta\end{pmatrix}
$$

$$\left|\beta_{11}\right\rangle_{CA}\otimes XZ\left|\psi\right\rangle_B=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\otimes\begin{pmatrix}-\beta\\\alpha\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\-\beta\\\alpha\\\beta\\-\alpha\\0\\0\end{pmatrix}$$

$$\begin{aligned}\left|\psi\right\rangle_C\otimes\left|\beta_{00}\right\rangle_{AB}&=\frac{1}{2}\left|\beta_{00}\right\rangle_{CA}\otimes\left|\psi\right\rangle_B+\frac{1}{2}\left|\beta_{01}\right\rangle_{CA}\otimes X\left|\psi\right\rangle_B\\&+\frac{1}{2}\left|\beta_{10}\right\rangle_{CA}\otimes Z\left|\psi\right\rangle_B+\frac{1}{2}\left|\beta_{11}\right\rangle_{CA}\otimes XZ\left|\psi\right\rangle_B\\&=\frac{1}{2}\sum_{x,y=0}^{1}\left|\beta_{xy}\right\rangle_{CA}\otimes X^yZ^x\left|\psi\right\rangle_B\end{aligned}$$

Suppose that Alice measures the state $\left|\beta_{x'y'}\right\rangle_{CA}$ shared with Charlie. Immediately, the system collapses, and Bob can measure the state

$$X^yZ^x\left|\psi\right\rangle_B,$$

$$\begin{aligned}\left|\psi\right\rangle_C\otimes\left|\beta_{00}\right\rangle_{AB}&\rightarrow\frac{1}{2}\sum_{x,y=0}^{1}\left|xy\right\rangle_{CA}\otimes(Z^xX^y)(X^yZ^x)\left|\psi\right\rangle_B\\&=\frac{1}{2}\sum_{x,y=0}^{1}\left|xy\right\rangle_{CA}\otimes\left|\psi\right\rangle_B\end{aligned}$$

Immediately after Alice measures the Bell state $\left|xy\right\rangle_{CA}$, the system collapses into $\left|xy\right\rangle_{CA}\otimes\left|\psi\right\rangle_B$. So, Bob gets the state $\left|\psi\right\rangle_B$.

10.2 Quantum circuit of quantum teleportation (I)

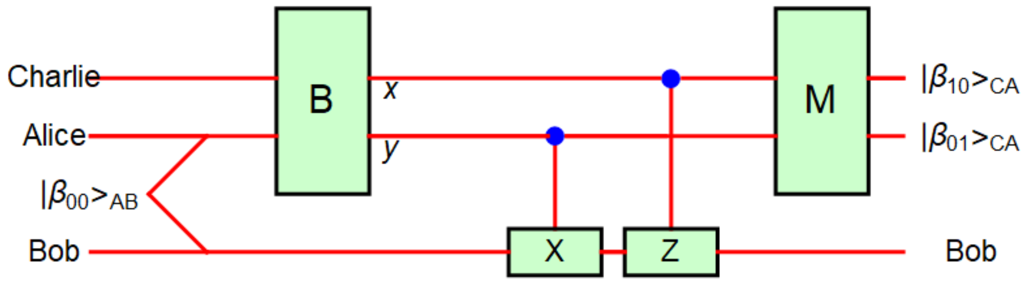

Fig.18 Quantum teleportation circuit (Alice, Bob, and Charlie). The box B denotes the generation of the Bell state. The box M denotes the measurement

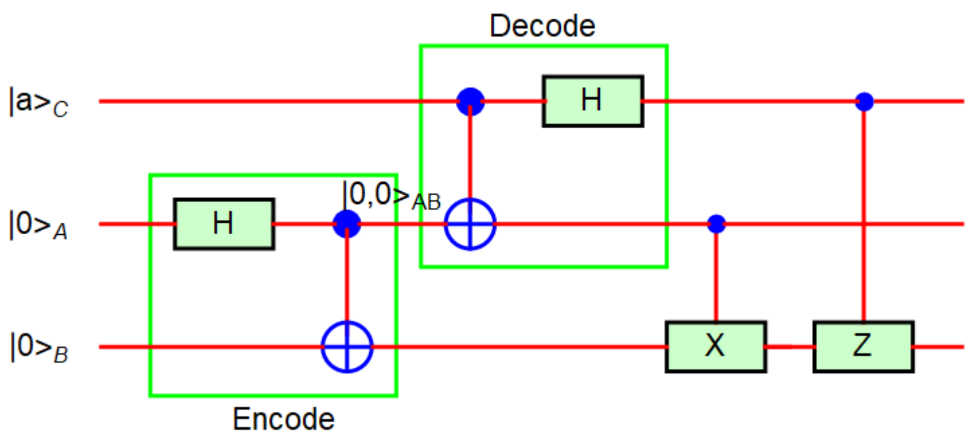

Fig.19 Equivalent quantum circuit of quantum teleportation, with encode of Bell state $|\beta_{00}\rangle$ and decode of the Bell state.

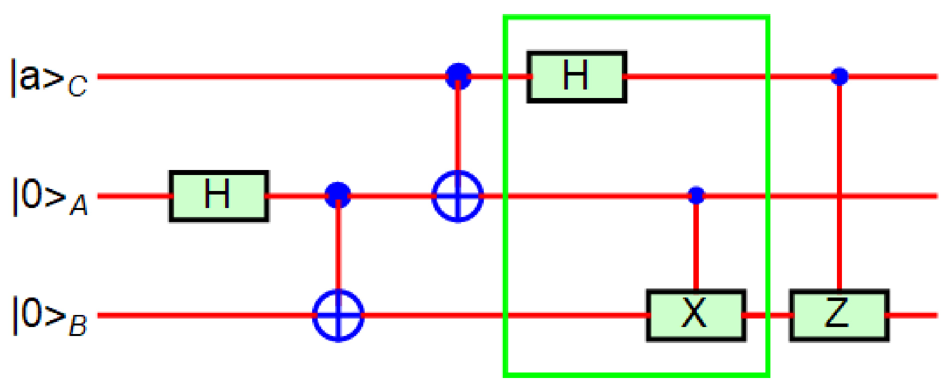

Fig.20 The quantum circuit of quantum teleportation. Within the green box, the *H*-gate can be shifted to the right, while the *X*-gate can be shifted to the left. The output is not influenced by these shifts.

After shifting the Hadamard gate to the right in Charlie-channel, equivalently, we have

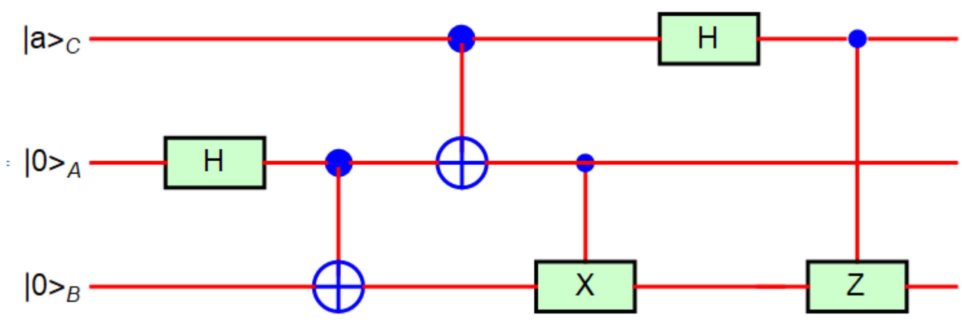

Fig.21 The circuit as input with the states $|a\rangle$ (Charlie), $|0\rangle$ (Alice), and $|0\rangle$ (Bob).

We now discuss the response of this circuit when the states $|a\rangle$ (Charlie), $|0\rangle$ (Alice), and $|0\rangle$ (Bob) are given as input from the right.

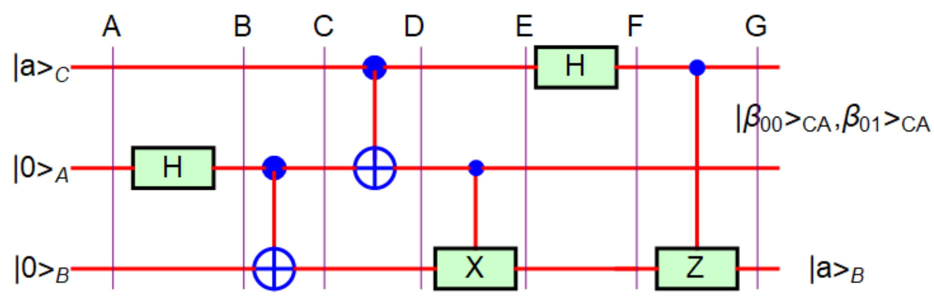

Fig.22 Quantum teleportation circuit. The output after step-*G* is $\frac{1}{\sqrt{2}}(|\beta_{00}\rangle_{CA} + |\beta_{01}\rangle_{CA}) \otimes |a\rangle_B$. Immediately after Alice measures one of the Bell states ($|\beta_{00}\rangle_{CA}$, $|\beta_{01}\rangle_{CA}$), the state of the system collapses, leading the state of Bob into the state $|a\rangle_B$. In other words, the state $|a\rangle_C$ of Charlie is transferred to Bob as the state $|a\rangle_B$.

We will show that this quantum circuit is similar to the circuit of SWAP, by each step.

Step-*A*:

$$|a\rangle_C \otimes |0\rangle_A \otimes |0\rangle_B \qquad \text{(as input)}$$

Step-*B*:

$$|a\rangle \otimes (H|0\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}|a\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}[|a\rangle \otimes |0\rangle \otimes |0\rangle + |a\rangle \otimes |1\rangle \otimes |0\rangle]$$

Step *C*:

$$C_{23}\frac{1}{\sqrt{2}}[|a\rangle \otimes (|0\rangle \otimes |0\rangle) + |a\rangle \otimes (|1\rangle \otimes |0\rangle)] = \frac{1}{\sqrt{2}}[|a\rangle \otimes |0\rangle \otimes |0\rangle + |a\rangle \otimes |1\rangle \otimes |1\rangle]$$

Step-*D*:

$$\frac{1}{\sqrt{2}}C_{12}[(|a\rangle \otimes |0\rangle) \otimes |0\rangle + (|a\rangle \otimes |1\rangle) \otimes |1\rangle] = \frac{1}{\sqrt{2}}[|a\rangle \otimes |a\rangle \otimes |0\rangle + |a\rangle \otimes |a \oplus 1\rangle \otimes |1\rangle]$$

Step-*E:*

$$\frac{1}{\sqrt{2}}[|a\rangle \otimes [|a\rangle \otimes X^a|0\rangle + |a \oplus 1\rangle \otimes X^{a+1}|1\rangle] = \frac{1}{\sqrt{2}}[|a\rangle \otimes [|a\rangle \otimes |a\rangle + |a \oplus 1\rangle \otimes |a \oplus 1 \oplus 1\rangle]$$
$$= |a\rangle \otimes H|0\rangle \otimes |a\rangle]$$

where

$$|a\rangle + |a \oplus 1\rangle = X^a|0\rangle + X^a|1\rangle$$
$$= \sqrt{2}X^a H|0\rangle$$
$$= \sqrt{2}H|0\rangle$$

$$X^a H|0\rangle = HZ^a|0\rangle = H|0\rangle$$

Step-*F*:

$$H|a\rangle \otimes H|0\rangle \otimes |a\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^a|1\rangle] \otimes H|0\rangle \otimes |a\rangle$$

where

$$H|a\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^a |1\rangle]$$

Step-*G*:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}[|0\rangle \otimes H|0\rangle \otimes Z^0 |a\rangle + (-1)^a |1\rangle \otimes H|0\rangle \otimes Z^1 |a\rangle \\
&= \frac{1}{\sqrt{2}}[|0\rangle \otimes H|0\rangle \otimes |a\rangle + (-1)^a |1\rangle \otimes H|0\rangle \otimes (-1)^a |a\rangle \\
&= \frac{1}{\sqrt{2}}[|0\rangle \otimes H|0\rangle \otimes |a\rangle + |1\rangle \otimes H|0\rangle \otimes |a\rangle \\
&= \frac{1}{2}[|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \otimes |a\rangle \\
&= \frac{1}{\sqrt{2}} \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}) \otimes |a\rangle + \frac{1}{\sqrt{2}} \frac{|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle}{\sqrt{2}} \otimes |a\rangle \\
&= \frac{1}{\sqrt{2}} |\beta_{00}\rangle \otimes |a\rangle + \frac{1}{\sqrt{2}} |\beta_{01}\rangle \otimes |a\rangle
\end{aligned}$$

or

$$\text{output} = \frac{1}{\sqrt{2}}(|\beta_{00}\rangle + |\beta_{01}\rangle) \otimes |a\rangle .$$

where $(-1)^{2a} = 1$, and the Bell states are defined as

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

Finally, Alice can measure either the state $|\beta_{00}\rangle_{CA}$ or $|\beta_{01}\rangle_{CA}$ with the probability of 50 %. After the measurement by Alice, the state collapses. As a result, Bob measures the state $|a\rangle$, independent of her choice of the state.

11. More simplified circuit of quantum teleportation (I)

We start with the quantum circuit of quantum teleportation which is previous derived.

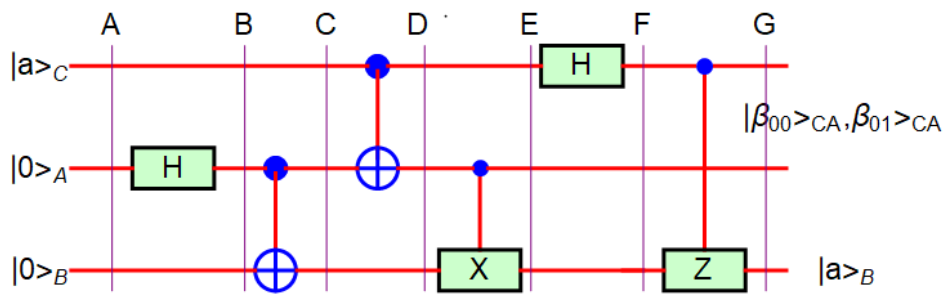

Fig.23(a) The same figure as Fig.22

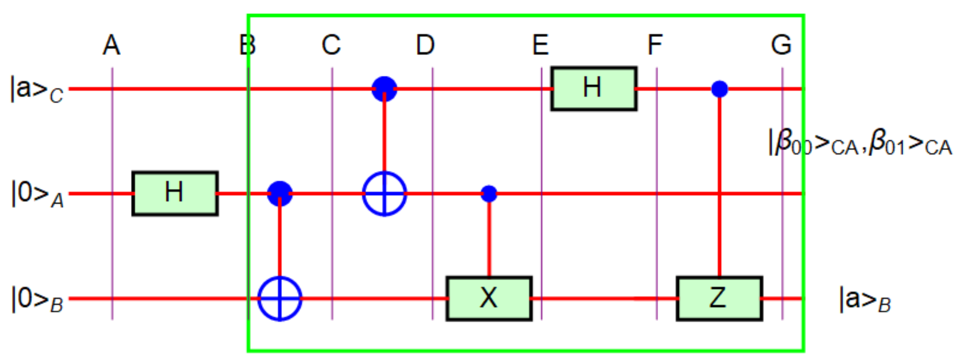

Fig.23(b) The change of the location of the input from the position A to B. The input at the position A is $|a\rangle_C \otimes H|0\rangle_A \otimes |0\rangle_B$.

The input is expressed by the state, $|a\rangle_C \otimes |0\rangle_A \otimes |0\rangle_B$. The state at the line B is given by $|a\rangle_C \otimes H|0\rangle_A \otimes |0\rangle_B$. The circuit after the line B can be rewritten as

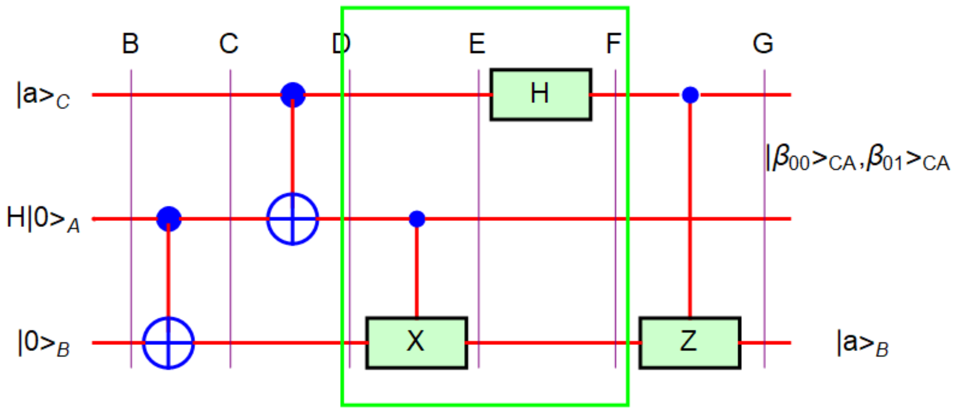

Fig.23(c) The change of location between *H* and *X* withing the green box. The element *H* shifts to the left, while the element *X* shifts to the right, without any effect on the output.

There are H and CX within enclosed by the green box. It is clear that the output at the line F is not influenced by the shift of H to the left and the shift of CX to the right within the green box. The output at the line G is expressed by the state

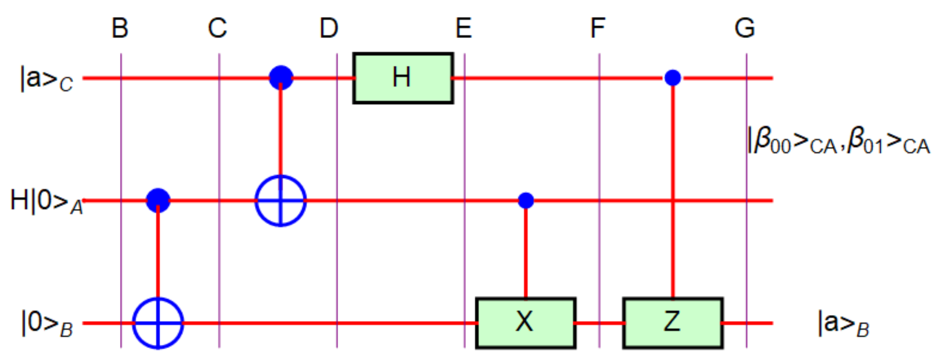

Fig.23 (d) Quantum circuit of quantum teleportation. The output at the line G is given by $\frac{1}{\sqrt{2}}(\left|\beta_{00}\right\rangle_{CA}+\left|\beta_{01}\right\rangle_{CA})\otimes\left|a\right\rangle_{B}$. Immediately after Alice measures the Bell states (either $\left|\beta_{00}\right\rangle_{CA}$ or $\left|\beta_{01}\right\rangle_{CA}$), the state of Bob collapses to the state $\left|a\right\rangle_{B}$.

12. Swap circuit

12.1 Original swap circuit

We consider the following SWAP circuit. Is this another quantum teleportation? In the input, the states of Bob and Charlie are $\left|0\right\rangle_{B}$ and $\left|\psi\right\rangle_{C}$, respectively. Alice is not involved in the process. There is also no classical communication, unlike the quantum teleportation.

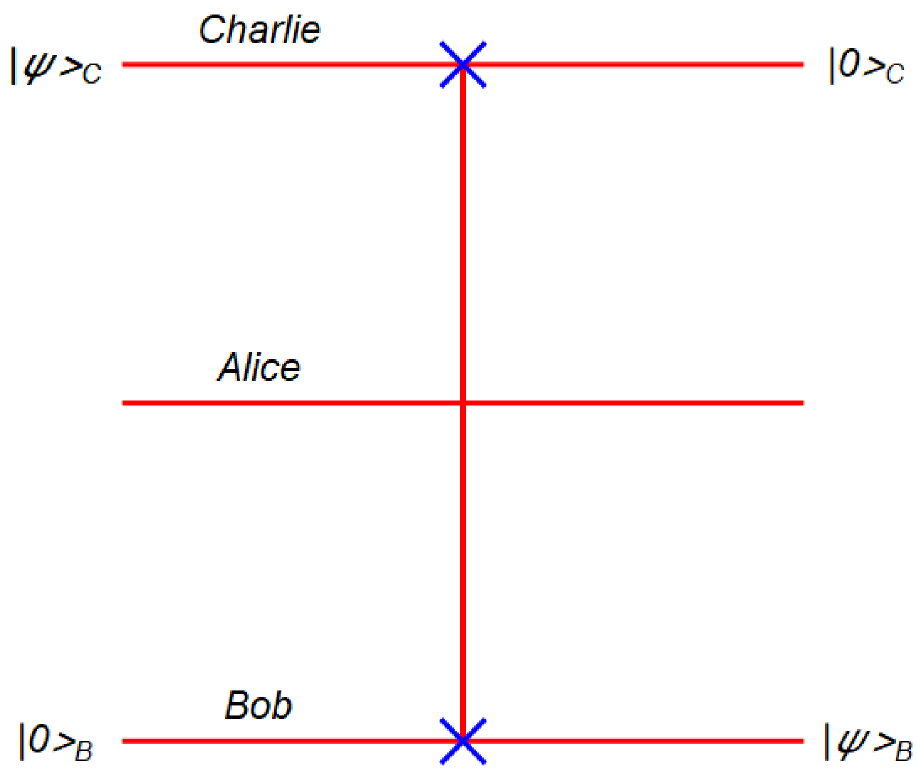

Fig.24 (a) SWAP circuit. Alice is not involved in this process. The result is independent of the choice of the state of Alice.

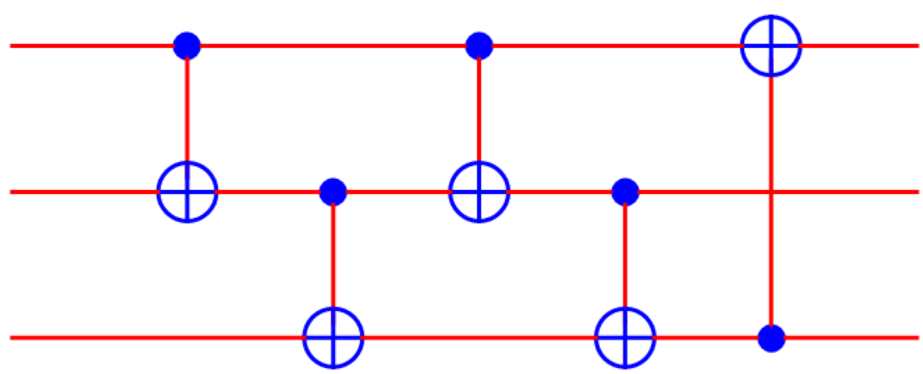

Fig.24(b) Equivalent circuit of SWAP, where the SWAP is replaced by CNOTs.

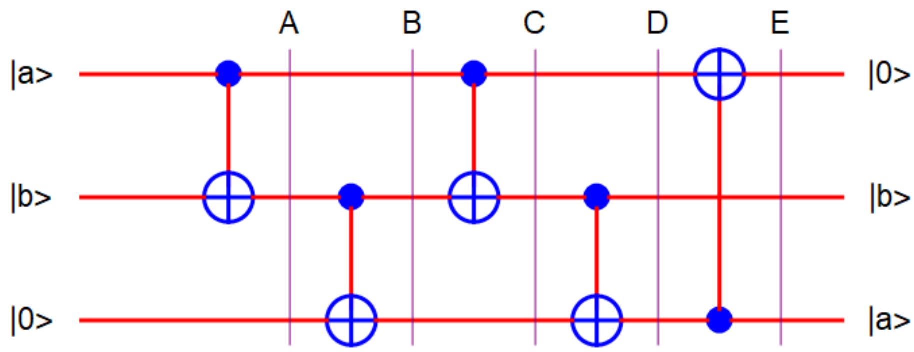

Fig.25 The same circuit as Fig.26(b), with input ($|a\rangle$ for Charlie, for Alice $|b\rangle$, and $|0\rangle$ for Bob).

Step-*A*: $C_{12}|a\rangle \otimes |b\rangle \otimes |0\rangle = |a\rangle \otimes |a \oplus b\rangle \otimes |0\rangle$

Step-*B*: $C_{23}|a\rangle \otimes |a \oplus b\rangle \otimes |0\rangle = |a\rangle \otimes |a \oplus b\rangle \otimes |a \oplus b\rangle$

Step-*C*:

$$C_{12}|a\rangle \otimes |a \oplus b\rangle \otimes |a \oplus b\rangle = |a\rangle \otimes |a \oplus a \oplus b\rangle \otimes |a \oplus b\rangle$$
$$= |a\rangle \otimes |b\rangle \otimes |a \oplus b\rangle$$

where $a \oplus a = 0$

Step-*D*:

$$C_{23}|a\rangle \otimes |b\rangle \otimes |a \oplus b\rangle = |a\rangle \otimes |b\rangle \otimes |b \oplus a \oplus b\rangle$$
$$= |a\rangle \otimes |b\rangle \otimes |a\rangle$$

Step-*E*:

$$C_{31}|a\rangle \otimes |b\rangle \otimes |a\rangle = |a \oplus a\rangle \otimes |b\rangle \otimes |a\rangle$$
$$= |0\rangle \otimes |b\rangle \otimes |a\rangle$$

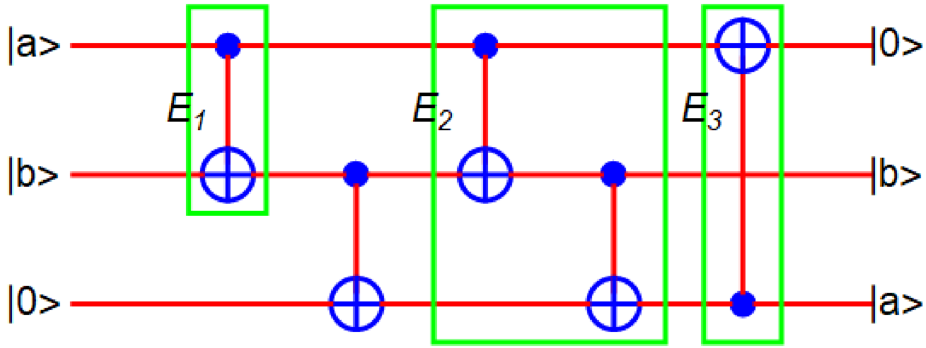

Fig.26 In SWAP, the circuit elements E_1, E_2, and E_3 can be replaced by the corresponding equivalent circuits.

12.2 Use of the equivalent circuits

The following two circuits are equivalent.

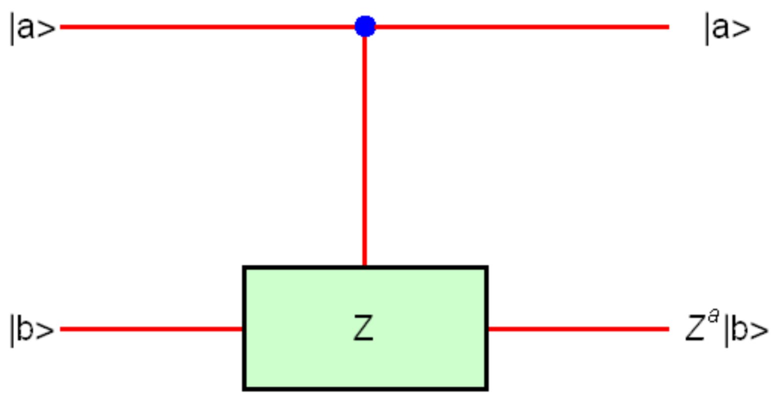

Fig.27 Controlled Z gate. $Z^a|b\rangle = (-1)^{ab}|b\rangle$.

$$Z^a|b\rangle = (-1)^{ab}|b\rangle, \quad \text{leading to } (-1)^{ab}|a\rangle \otimes |b\rangle$$

which is equivalent to a circuit (controlled Z gate)

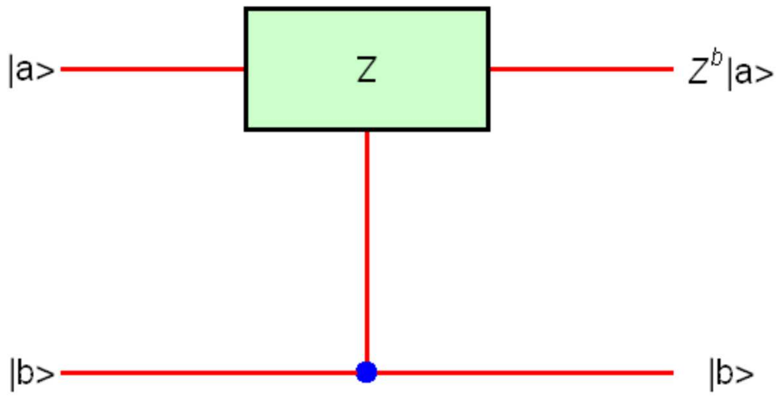

Fig.28 $Z^b \left|a\right\rangle = (-1)^{ab} \left|a\right\rangle$. CZ (controlled Z) is nonlocal, independent of the choice of the control and the target.

$Z^b \left|a\right\rangle = (-1)^{ab} \left|a\right\rangle$ leading to $(-1)^{ab} \left|a\right\rangle \otimes \left|b\right\rangle$

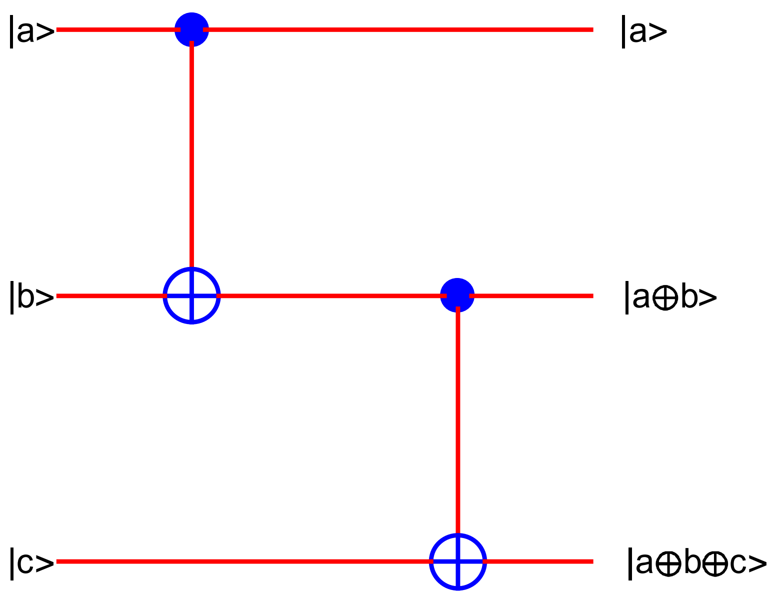

Fig.29 The circuit of Fig.31 is equivalent to that in Fig.32.

$$\left|a\right\rangle \otimes \left|a \oplus b\right\rangle \otimes \left|a \oplus b \oplus c\right\rangle$$

which is equivalent to

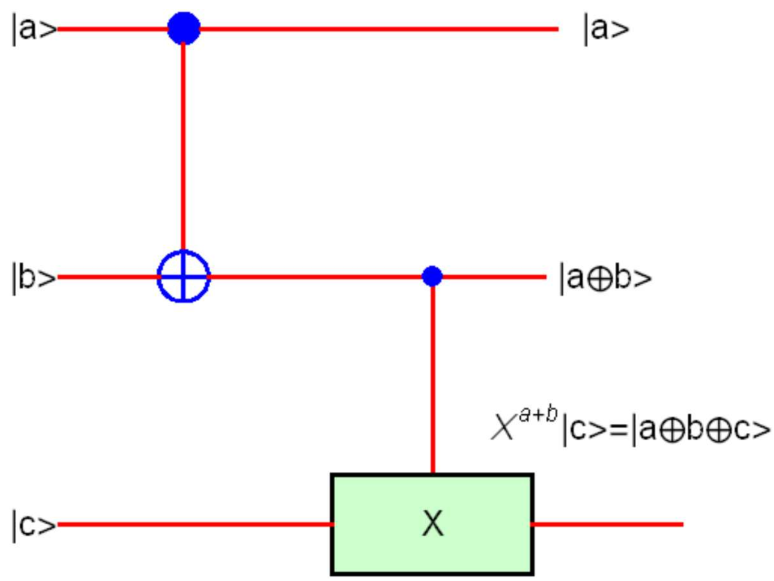

Fig.30 The circuit of Fig.32 is equivalent to that in Fig.31.

since

$$\left|a\right\rangle \otimes \left|a\oplus b\right\rangle \otimes X^{a+b}\left|c\right\rangle = \left|a\right\rangle \otimes \left|a\oplus b\right\rangle \otimes \left|a\oplus b\oplus c\right\rangle,$$

where

$$X^{1}\left|c\right\rangle = \left|\overline{c}\right\rangle = \left|c\oplus 1\right\rangle$$

$$X^{a+b}\left|c\right\rangle = \left|c\oplus a\oplus b\right\rangle$$

12.3 Modified swap circuit (which is equivalent to the original SWAP

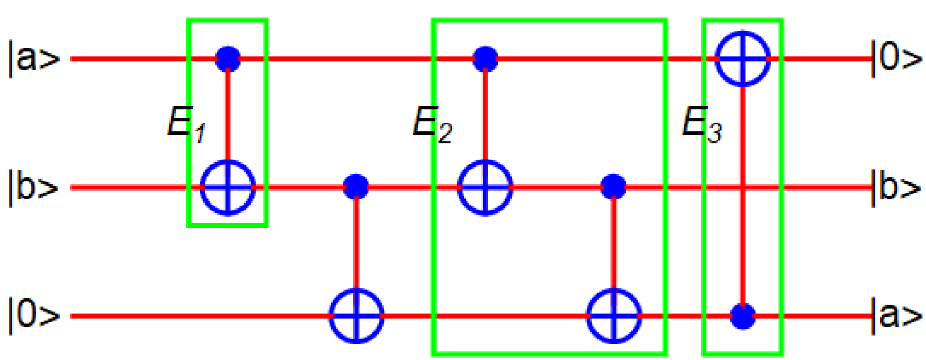

Fig.31(a) The staring quantum circuit (which is the same as Fig.28.

which is equivalent to the following circuit.The elements E_1 and E_3 are replaced by the corresponding equivalent circuits as follows.

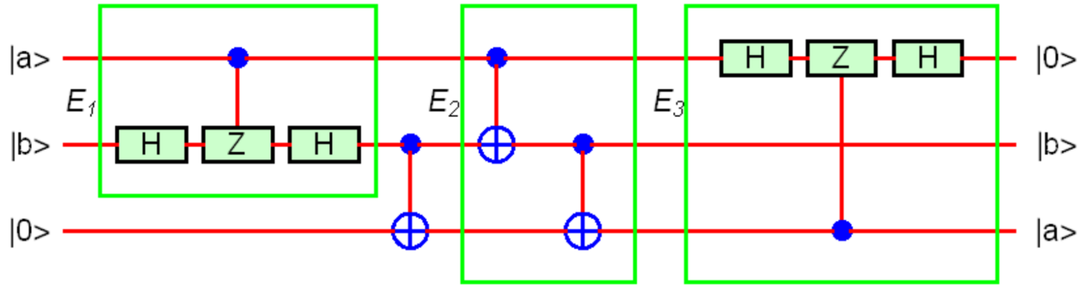

Fig.31 (b)

The element E_2 is replaced by the corresponding equivalent circuit as follows.

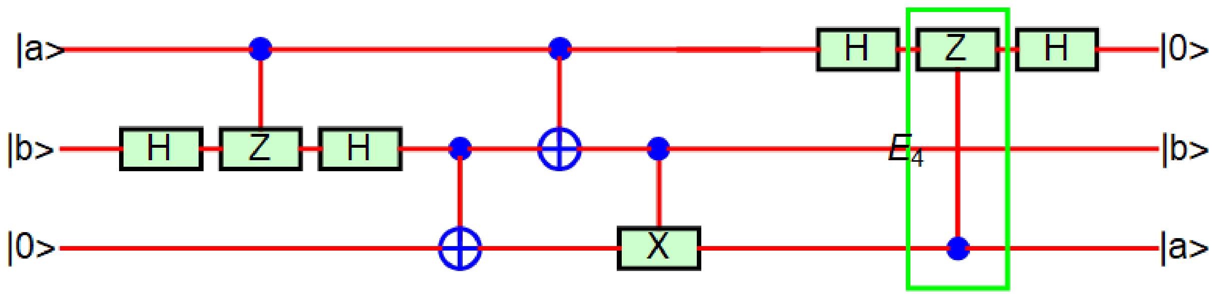

Fig.31 (c): Controlled Z is nonlocal in the elements E_4. Even if the location of the control and target is changed, the role of CZ remains unchanged.

The element E_4 is replaced by the corresponding equivalent circuit as follows.

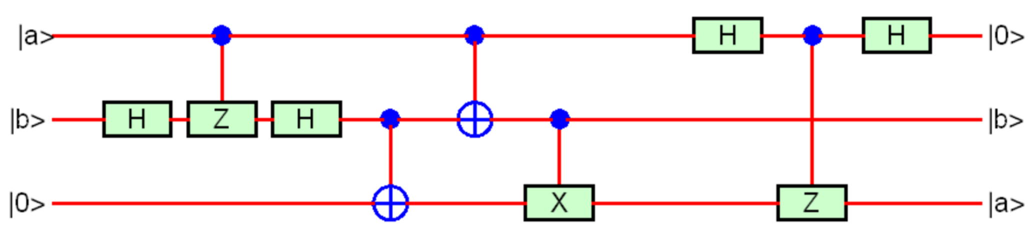

Fig.31(d)

It is obvious that

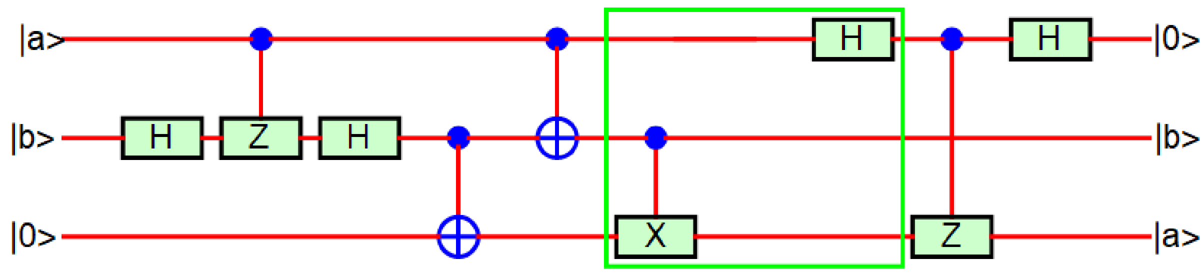

Fig.31(e): For the H and CX in the green box, with on change of the circuit, H can be shifted to the left, while CX can be shifted to the right within the box.

is equivalent to

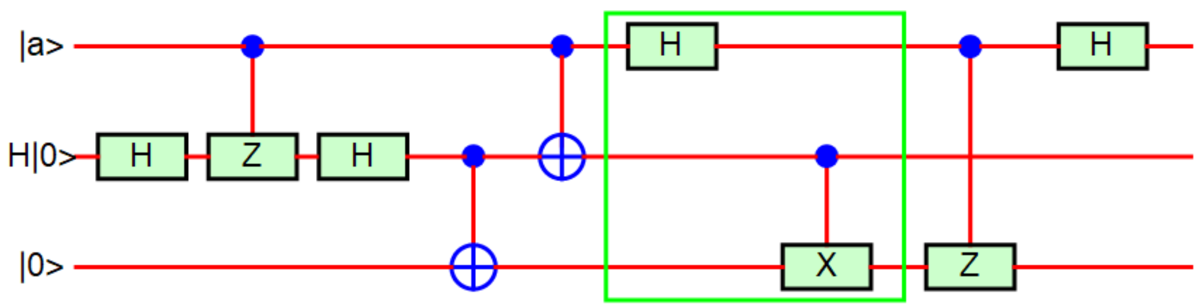

Fig.31(f): The circuit after shifting H and CX within the green box. The input: $|a\rangle$ for Charlie. The input: $|0\rangle$ for Bob. For simplicity, we choose the input $H|0\rangle$ for Alice. This input is arbitrary. $H|0\rangle_A = \frac{1}{\sqrt{2}}[|0\rangle_A + |1\rangle_A]$.

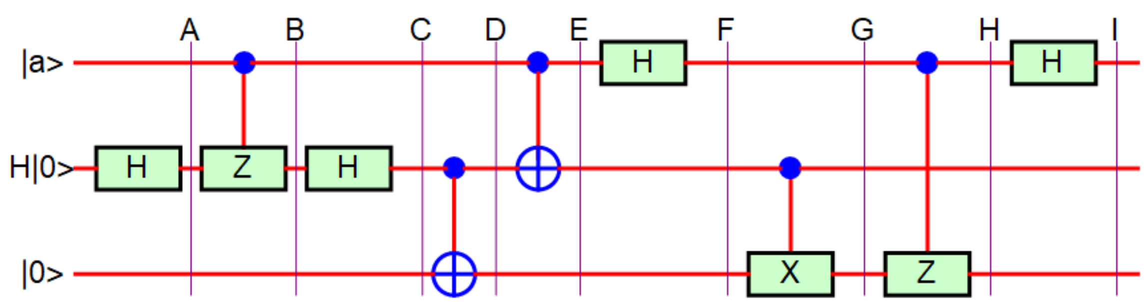

Fig.32 The equivalent quantum circuit for the SWAP.

We check the result after each step.

Step-*A*: $|a \oplus 0 \oplus 0\rangle$ (input)

Step-*B*: $|a\rangle Z^a |0\rangle \otimes |0\rangle = |a\rangle \otimes |0\rangle \otimes |0\rangle$ (the control Z gate is actually necessary)

Step-*C*:

$$|a\rangle \otimes H|0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}[|a\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle]$$
$$= \frac{1}{\sqrt{2}}[|a00\rangle + |a10\rangle]$$

Step-*D*:

$$\frac{1}{\sqrt{2}}|a\rangle \otimes C_{23}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}|a\rangle \otimes (|00\rangle + |11\rangle)$$

Step-*E*:

$$\frac{1}{\sqrt{2}}C_{12}(|a00\rangle + |a11\rangle) = \frac{1}{\sqrt{2}}(|a,a\rangle \otimes |0\rangle + |a, a \oplus 1\rangle \otimes |1\rangle)$$
$$= \frac{1}{\sqrt{2}}(|a,a\rangle \otimes |0\rangle + |a,\bar{a}\rangle \otimes |1\rangle)$$

Step-*F*:

$$\frac{1}{\sqrt{2}}(H|a\rangle \otimes |a\rangle \otimes |0\rangle + H|a\rangle \otimes |\bar{a}\rangle \otimes |1\rangle)$$
$$= \frac{1}{2}\sum_{a'}(-1)^{aa'}[|a'\rangle \otimes |a\rangle \otimes |0\rangle + |a'\rangle \otimes |\bar{a}\rangle \otimes |1\rangle]$$
$$= \frac{1}{2}|0\rangle \otimes |a\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle \otimes |\bar{a}\rangle \otimes |1\rangle$$
$$+ \frac{1}{2}(-1)^a|1\rangle \otimes |a\rangle \otimes |0\rangle + \frac{1}{2}(-1)^a|1\rangle \otimes |\bar{a}\rangle \otimes |1\rangle$$

Step-*G*:

$$\frac{1}{2}|0\rangle\otimes|a\rangle\otimes X^{a}|0\rangle+\frac{1}{2}|0\rangle\otimes|a\oplus1\rangle\otimes X^{a+1}|1\rangle$$

$$+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\rangle\otimes X^{a}|0\rangle+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\oplus1\rangle\otimes X^{a+1}|1\rangle$$

$$=\frac{1}{2}|0\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}|0\rangle\otimes|a\oplus1\rangle\otimes|a\oplus2\rangle$$

$$+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\oplus1\rangle\otimes|a\oplus2\rangle$$

$$=\frac{1}{2}|0\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}|0\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$

$$+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$

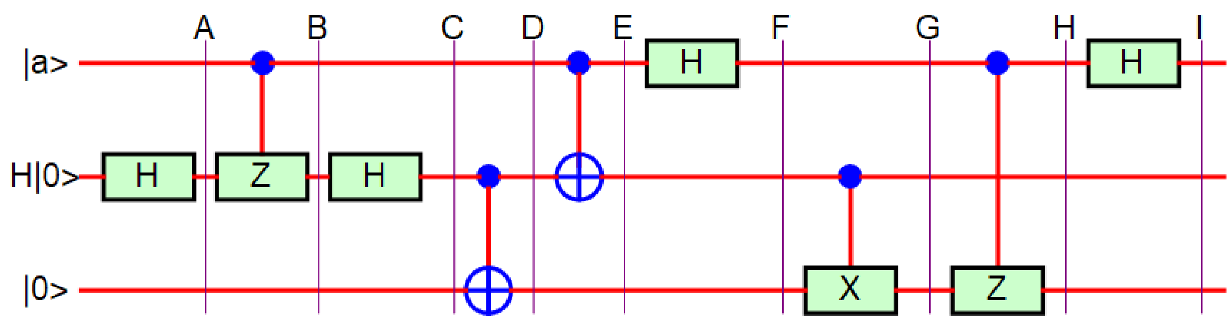

Fig.33

Step-*H*:

$$\frac{1}{2}|0\rangle\otimes|a\rangle\otimes Z^{0}|a\rangle+\frac{1}{2}|0\rangle\otimes|a\oplus1\rangle\otimes Z^{0}|a\rangle$$

$$+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\rangle\otimes Z|a\rangle+\frac{1}{2}(-1)^{a}|1\rangle\otimes|a\oplus1\rangle\otimes Z|a\rangle$$

$$=\frac{1}{2}|0\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}|0\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$

$$+\frac{1}{2}(-1)^{2a}|1\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}(-1)^{2a}|1\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$

$$=\frac{1}{2}|0\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}|0\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$

$$+\frac{1}{2}|1\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}|1\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$

Step-*I*:

$$\frac{1}{2}H|0\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}H|0\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$
$$+\frac{1}{2}H|1\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}H|1\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$
$$=\frac{1}{2}[H|0\rangle+H|1\rangle]\otimes|a\rangle\otimes|a\rangle+\frac{1}{2}[H|0\rangle+H|1\rangle]\otimes|a\oplus1\rangle\otimes|a\rangle$$
$$=\frac{1}{\sqrt{2}}|0\rangle\otimes|a\rangle\otimes|a\rangle+\frac{1}{\sqrt{2}}|0\rangle\otimes|a\oplus1\rangle\otimes|a\rangle$$
$$=|0\rangle\otimes[\frac{1}{\sqrt{2}}|a\rangle+\frac{1}{\sqrt{2}}|a\oplus1\rangle]\otimes|a\rangle$$

Here we note that

$$\begin{aligned}\frac{1}{\sqrt{2}}|a\rangle+\frac{1}{\sqrt{2}}|a\oplus1\rangle] &= X^a\frac{1}{\sqrt{2}}[|0\rangle+|1\rangle]\\ &= X^aH|0\rangle\\ &= H|0\rangle\end{aligned}$$

since

$$X^a|0\rangle=|a\rangle, \qquad X^{a+1}|0\rangle=|a\oplus1\rangle.$$

Using the relations

$$HXH=Z, \qquad H^2=1,$$

we get

$$\begin{aligned}(HXH)^a &= (HXH)(HXH)......(HXH)\\ &= HX^aH\end{aligned}$$

or

$$Z^a=(HXH)^a=HX^aH,$$

or

$$HZ^a = X^a H .$$

Thus, we have

$$X^a H|0\rangle = HZ^a|0\rangle = H|0\rangle .$$

So the output of the above quantum circuit is

$$|0\rangle \otimes H|0\rangle \otimes |a\rangle .$$

This is the equivalent circuit of swap. The final result is as follows.

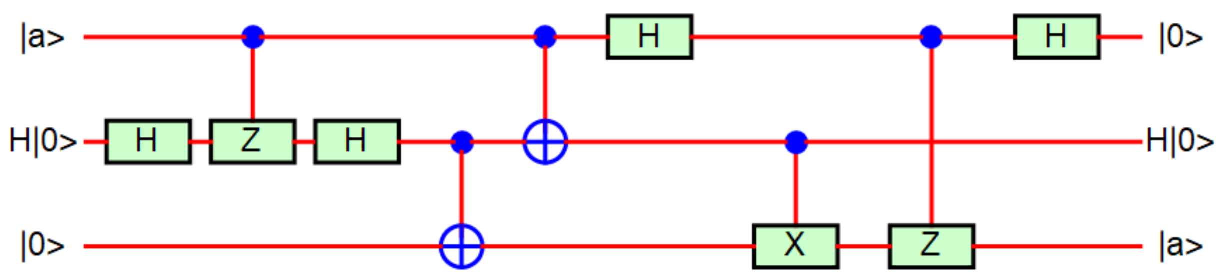

Fig.34 SWAP. The output: $|0\rangle_C$, $H|0\rangle_A$, and $|a\rangle_B$. The input: $|a\rangle_C$, $H|0\rangle_A$, and $|0\rangle_B$.

Even if the box with green can be removed from the circuit, both the input and output remain unchanged.

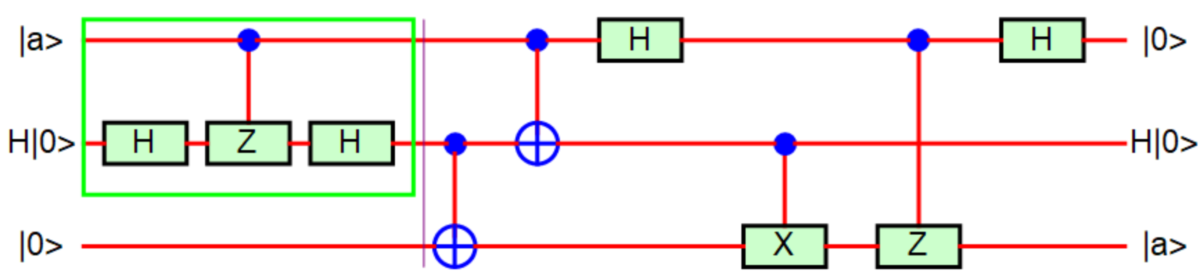

Fig.35 No change occurs to the circuit upon the removal of the circuit element enclosed by the green box.

Note that

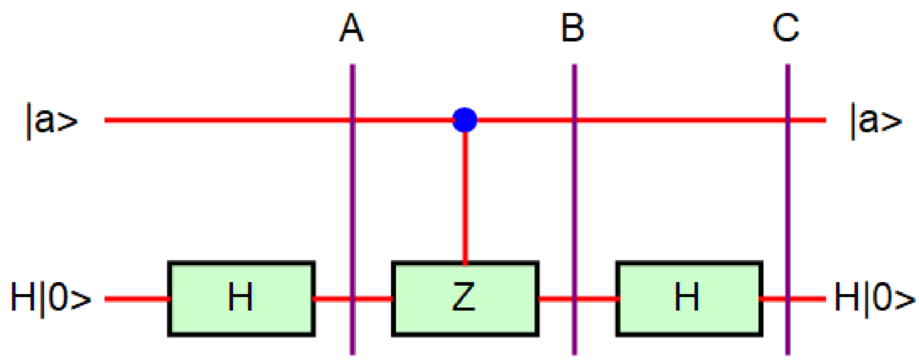

Fig.36 Checking the possibility of removing *H-Z-H* gate.

Input: $|a\rangle \otimes H|0\rangle$

Step-*A*: $|a\rangle \otimes H^2|0\rangle = |a\rangle \otimes |0\rangle$

Step-*B*: $|a\rangle \otimes Z^a|0\rangle = |a\rangle \otimes |0\rangle$

Step-*C*: $|a\rangle \otimes H|0\rangle$

So that the output is the same as the input. If we use the input $|a\rangle \otimes H|0\rangle$, the *H-Z-H* gate can be removed.

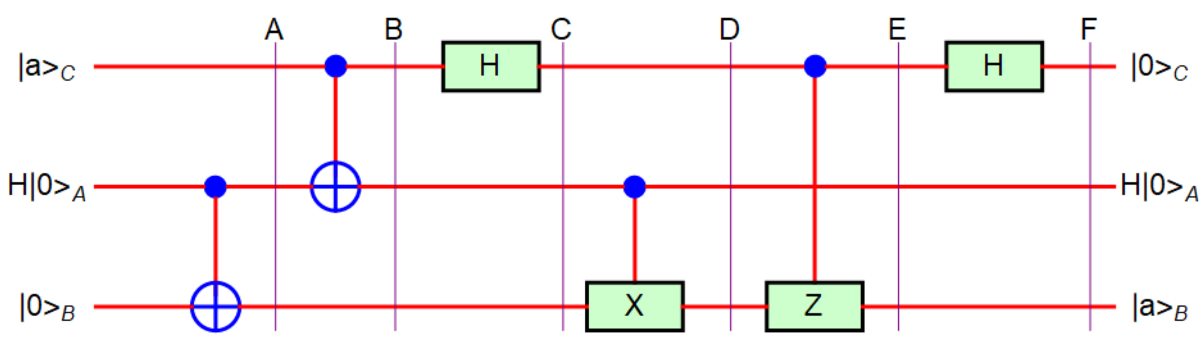

Fig.37 The output after the step E; $\frac{1}{\sqrt{2}}[|\beta_{00}\rangle_{CA} + |\beta_{01}\rangle_{CA}] \otimes |a\rangle$. Immediately after Alice measures one of the Bell states ($|\beta_{00}\rangle_{CA}$, $|\beta_{01}\rangle_{CA}$), the state of Bob collapse into the state $|a\rangle$. The output after the step F; $|0\rangle_C \otimes H|0\rangle_A \otimes |a\rangle_B$.

Input:

$$|a\rangle \otimes H|0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}|a\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}[|a\rangle \otimes |0\rangle \otimes |0\rangle + |a\rangle \otimes |1\rangle) \otimes |0\rangle]$$

Step-*A*:

$$\frac{1}{\sqrt{2}}C_{23}[|a\rangle \otimes |0\rangle \otimes |0\rangle + |a\rangle \otimes |1\rangle) \otimes |0\rangle] = \frac{1}{\sqrt{2}}[|a\rangle \otimes |0\rangle \otimes |0\rangle + |a\rangle \otimes |1\rangle) \otimes |1\rangle]$$
$$= |a\rangle \otimes \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}$$
$$= |a\rangle \otimes |\beta_{00}\rangle$$

where

$$|\beta_{00}\rangle_{AB} = \frac{|0\rangle \otimes |0\rangle + \otimes |1\rangle) \otimes |1\rangle}{\sqrt{2}} \qquad \text{(Bell state)}$$

Step-*B*:

$$\frac{1}{\sqrt{2}}C_{12}[|a\rangle \otimes |0\rangle \otimes |0\rangle + |a\rangle \otimes |1\rangle) \otimes |1\rangle]$$
$$= \frac{1}{\sqrt{2}}[|a\rangle \otimes |a\rangle \otimes |0\rangle + |a\rangle \otimes |a \oplus 1\rangle) \otimes |1\rangle]$$

Step-*C*:

$$\frac{1}{\sqrt{2}}H|a\rangle \otimes [|a\rangle \otimes |0\rangle + |a \oplus 1\rangle \otimes |1\rangle]$$
$$= \frac{1}{2}\sum_{a'}(-1)^{aa'}|a'\rangle \otimes [|a\rangle \otimes |0\rangle + |a \oplus 1\rangle) \otimes |1\rangle]$$
$$= \frac{1}{2}[|0\rangle + (-1)^{a}|1\rangle \otimes [|a\rangle \otimes |0\rangle + |a \oplus 1\rangle) \otimes |1\rangle]$$
$$= \frac{1}{2}|0\rangle \otimes |a\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle \otimes |a \oplus 1\rangle) \otimes |1\rangle$$
$$+ \frac{1}{2}(-1)^{a}|1\rangle \otimes |a\rangle \otimes |0\rangle + \frac{1}{2}(-1)^{a}|1\rangle \otimes |a \oplus 1\rangle) \otimes |1\rangle$$

Step-*D*:

$$\frac{1}{2}|0\rangle \otimes |a\rangle \otimes X^{a}|0\rangle + \frac{1}{2}|0\rangle \otimes |a\oplus 1\rangle) \otimes X^{a+1}|1\rangle$$
$$+\frac{1}{2}(-1)^{a}|1\rangle \otimes |a\rangle \otimes X^{a}|0\rangle + \frac{1}{2}(-1)^{a}|1\rangle \otimes |a\oplus 1\rangle) \otimes X^{a+1}|1\rangle$$
$$=\frac{1}{2}|0\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}|0\rangle \otimes |a\oplus 1\rangle) \otimes |a\oplus 1\oplus 1\rangle$$
$$+\frac{1}{2}(-1)^{a}|1\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}(-1)^{a}|1\rangle \otimes |a\oplus 1\rangle) \otimes |a\oplus 1\oplus 1\rangle$$
$$=[\frac{1}{2}|0\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}|0\rangle \otimes |a\oplus 1\rangle) \otimes |a\rangle$$
$$+\frac{1}{2}(-1)^{a}|1\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}(-1)^{a}|1\rangle \otimes |a\oplus 1\rangle) \otimes |a\rangle$$

Step-*E*:

$$\frac{1}{2}|0\rangle \otimes |a\rangle \otimes Z^{0}|a\rangle + \frac{1}{2}|0\rangle \otimes |a\oplus 1\rangle) Z^{0} \otimes |a\rangle$$
$$+\frac{1}{2}(-1)^{a}|1\rangle \otimes |a\rangle \otimes Z|a\rangle + \frac{1}{2}(-1)^{a}|1\rangle \otimes |a\oplus 1\rangle) \otimes Z|a\rangle$$
$$=\frac{1}{2}|0\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}|0\rangle \otimes |a\oplus 1\rangle) \otimes |a\rangle$$
$$+\frac{1}{2}(-1)^{2a}|1\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}(-1)^{2a}|1\rangle \otimes |a\oplus 1\rangle) \otimes |a\rangle$$
$$=\frac{1}{2}|0\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}|0\rangle \otimes |a\oplus 1\rangle) \otimes |a\rangle$$
$$+\frac{1}{2}|1\rangle \otimes |a\rangle \otimes |a\rangle + \frac{1}{2}|1\rangle \otimes |a\oplus 1\rangle) \otimes |a\rangle$$

where

$$\frac{1}{2}|0\rangle \otimes (|a\rangle + |a\oplus 1\rangle) \otimes |a\rangle + \frac{1}{2}|1\rangle \otimes (|a\rangle + |a\oplus 1\rangle) \otimes |a\rangle$$
$$=\frac{1}{\sqrt{2}}|0\rangle \otimes H|0\rangle \otimes |a\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes H|0\rangle \otimes |a\rangle$$
$$=H|0\rangle \otimes H|0\rangle \otimes |a\rangle$$
$$=\frac{1}{\sqrt{2}}[|\beta_{00}\rangle_{CA} + |\beta_{01}\rangle_{CA}] \otimes |a\rangle$$

Note that

$$H|0\rangle \otimes H|0\rangle = \frac{1}{2}[(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)]$$
$$= \frac{1}{2}[|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle]$$
$$= \frac{1}{\sqrt{2}}[\frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}} + \frac{|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle}{\sqrt{2}}]$$
$$= \frac{1}{\sqrt{2}}[|\beta_{00}\rangle_{CA} + |\beta_{01}\rangle_{CA}]$$

Step-*F*:

$$H^2|0\rangle \otimes H|0\rangle \otimes |a\rangle = |0\rangle \otimes H|0\rangle \otimes |a\rangle$$

or

$$|0\rangle \otimes H|0\rangle \otimes |a\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes [|0\rangle + |1\rangle] \otimes |a\rangle$$
$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle \otimes |a\rangle + \frac{1}{\sqrt{2}}|0\rangle \otimes |1\rangle] \otimes |a\rangle$$

Note that

$$\frac{1}{\sqrt{2}}(|a\rangle + |a \oplus 1\rangle) = H|0\rangle .$$

13. Summary

Using the equivalent circuits of the quantum computer, it is shown that the SWAP circuit is equivalent to the quantum circuit of the quantum teleportation. The equivalent circuit for the quantum teleportation and the SWAP is obtained as

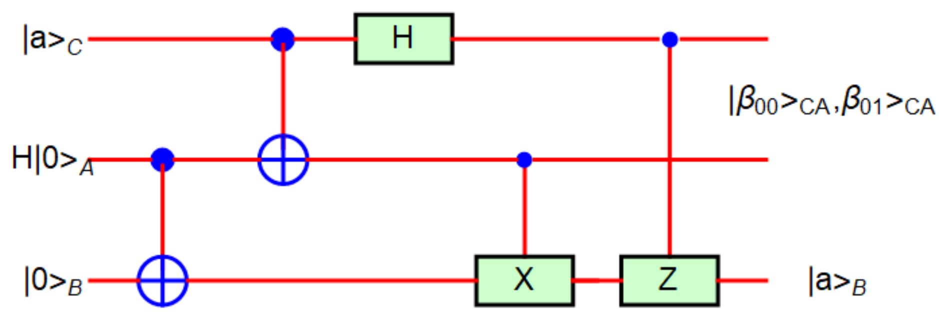

Fig.38 Equivalent circuit both for the quantum teleportation and the SWAP.

Immediately after one of the Bell states ($|\beta_{00}\rangle_{CA}$ or $|\beta_{01}\rangle_{CA}$) shared with Alice and Charlie, is measured by Alice, the system, collapses. Bob observes the state $|a\rangle_B$, which is the same state which Charlie has as an input ($|a\rangle_C$. Note that $|a\rangle$ is any linear combination of qubits $|0\rangle$ and $|1\rangle$.

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APPENDIX-A Fundamental properties

a=0 or $a = 1$

$a \oplus a = 0$, $, a \oplus \bar{a} = 1$ $(a \oplus (a \oplus 1) = (a \oplus a) \oplus 1 = 1$.

$a \oplus 0 = a$, $a \oplus 1 = \bar{a}$.

$\bar{a} \oplus 1 = (a \oplus 1) \oplus 1 = a \oplus 0 = a$.

$$H|a\rangle = \frac{1}{\sqrt{2}}\sum_{b=0}^{1}(-1)^{ab}|b\rangle, \qquad H = \frac{1}{\sqrt{2}}(X+Z).$$

$$H|a\rangle = \frac{1}{\sqrt{2}}\sum_{a'}(-1)^{aa'}|a'\rangle$$

$$\begin{aligned} Z^b H|a\rangle &= \frac{1}{\sqrt{2}}\sum_{a'}(-1)^{aa'}Z^b|a'\rangle \\ &= \frac{1}{\sqrt{2}}\sum_{a'}(-1)^{(a+b)a'}|a'\rangle \\ &= H|a\oplus b\rangle \end{aligned}$$

$$HZ^b H|a\rangle = H^2|a\oplus b\rangle = |a\oplus b\rangle.$$

$$X|a\rangle = |\bar{a}\rangle = |a\oplus 1\rangle.$$

$$Y|a\rangle = (-1)^a i|\bar{a}\rangle = (-1)^a i|a\oplus 1\rangle.$$

$$Z|a\rangle = (-1)^a|a\rangle.$$

$$X^a|b\rangle = |a\oplus b\rangle.$$

$$Z^a|b\rangle = (-1)^{ab}|b\rangle.$$

$$H^2 = 1.$$

$$HXH = Z, \quad HYH = -Y, \quad HZH = X.$$

$$HX^aH = Z^a, \quad HY^aH = (-)^aY^a, \quad HZ^aH = X^a.$$

$$\frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|a \oplus 1\rangle] = X^a \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$$
$$= X^a H|0\rangle$$
$$= HZ^a|0\rangle \qquad .$$
$$= H|0\rangle$$
$$= \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$$

$$C_{12}|a\rangle \otimes |b\rangle = |a\rangle \otimes |b \oplus a\rangle .$$

$$X^2 = Y^2 = Z^2 = 1 .$$

$$[X,Y] = 2iZ , \qquad [Y,Z] = 2iX , \qquad [Z,X] = 2iY .$$

Kronecker product:

$$H|0\rangle \otimes H|0\rangle = (H \otimes H)(|0\rangle \otimes |0\rangle) .$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$H \otimes H = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$H|0\rangle \otimes H|0\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Pauli matrix:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle,$$

$$Y|0\rangle = -i|1\rangle, \quad Y|1\rangle = i|0\rangle,$$

$$Z|0\rangle = |0\rangle, \qquad Z|1\rangle = -|1\rangle.$$

Qubits:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

APPENDIX B

B-1 Another example of quantum teleportation (II)

Here we consider the following simple quantum circuit (as a simple example). We find this example from a book [**C. Bernhardt, Quantum Computing for Everyone (MIT Press, 2019)].**

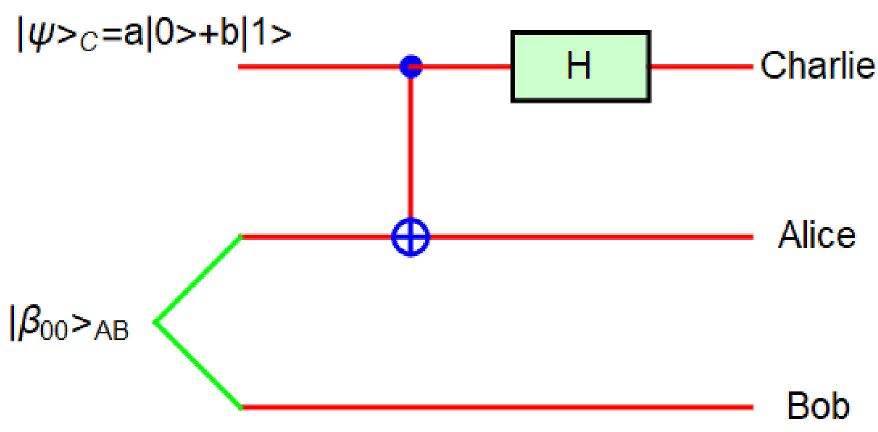

Fig.B1 Quantum teleportation. Alice, Bob, and Charlie

$$|\psi\rangle_C = a|0\rangle + b|1\rangle, \qquad \text{(the initial state of Charlie)}$$

$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B] \qquad \text{(Bell state between Alice and Bob)}$$

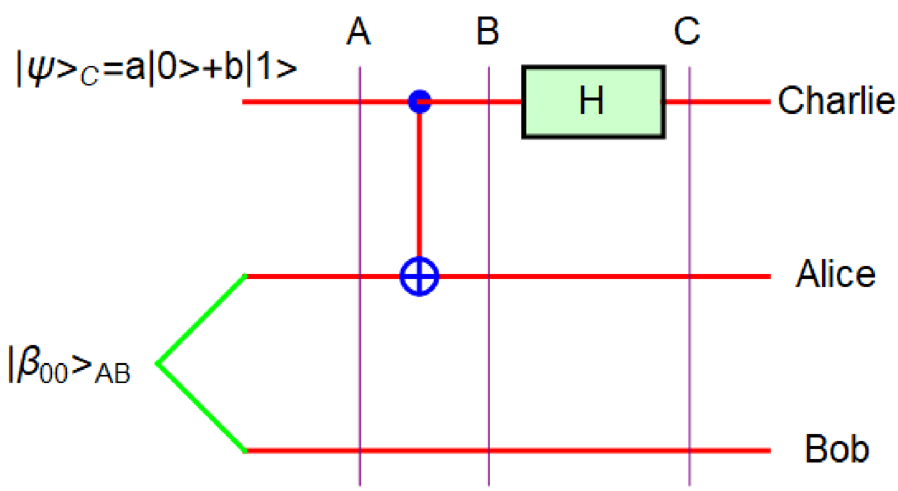

Fig.B2 Quantum circuit corresponding to Fig.B1.

Step-A:

$$\begin{aligned}
|\psi\rangle_C \otimes |\beta_{00}\rangle_{AB} &= [a|0\rangle + b|1\rangle]_C \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB} \\
&= \frac{1}{\sqrt{2}}[a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle] \\
&= \frac{1}{\sqrt{2}}[a|00\rangle \otimes |0\rangle + a|01\rangle \otimes |1\rangle + b|10\rangle \otimes |0\rangle + b|11\rangle \otimes |1\rangle]
\end{aligned}$$

Step-*B*: CNOT between channels 1 (Charlie) and 2 (Alice):

$$\begin{aligned}
&\frac{1}{\sqrt{2}}[aC_{12}|00\rangle \otimes |0\rangle + aC_{12}|01\rangle \otimes |1\rangle + bC_{12}|10\rangle \otimes |0\rangle + bC_{12}|11\rangle \otimes |1\rangle] \\
&= \frac{1}{\sqrt{2}}[a|00\rangle \otimes |0\rangle + a|01\rangle \otimes |1\rangle + b|11\rangle \otimes |0\rangle + b|10\rangle \otimes |1\rangle] \\
&= \frac{1}{\sqrt{2}}[a|0\rangle \otimes |00\rangle + a|0\rangle \otimes |11\rangle + b|1\rangle \otimes |10\rangle + b|1\rangle \otimes |01\rangle] \\
&= \frac{1}{\sqrt{2}}[a|0\rangle \otimes (|00\rangle + \otimes|11\rangle) + b|1\rangle \otimes (|10\rangle + |01\rangle)]
\end{aligned}$$

where we use the relation $C_{12}(|a\rangle_1 \otimes |b\rangle_2) = |a\rangle_1 \otimes |a \oplus b\rangle_2$

Step-*C*: Hadamard gate in the channel-1 (Charlie)

$$\frac{1}{\sqrt{2}} aH|0\rangle \otimes (|00\rangle + |11\rangle) + \frac{1}{\sqrt{2}} bH|1\rangle \otimes (|10\rangle + |01\rangle]$$

$$= \frac{1}{2} a(|0\rangle + |1\rangle) \otimes (|00\rangle + |11\rangle) + \frac{1}{2} b(|0\rangle - |1\rangle) \otimes (|10\rangle + |01\rangle]$$

$$= \frac{1}{2}|00\rangle \otimes a|0\rangle + \frac{1}{2}|01\rangle \otimes a|1\rangle + \frac{1}{2}|10\rangle \otimes a|0\rangle + \frac{1}{2}|11\rangle \otimes a|1\rangle$$

$$+ \frac{1}{2}|01\rangle \otimes b|0\rangle + \frac{1}{2}|00\rangle \otimes b|1\rangle - \frac{1}{2}|11\rangle \otimes b|0\rangle - \frac{1}{2}|10\rangle \otimes b|1\rangle$$

$$= \frac{1}{2}|00\rangle \otimes [a|0\rangle + b|1\rangle] + \frac{1}{2}|01\rangle \otimes [a|1\rangle + b|0\rangle]$$

$$+ \frac{1}{2}|10\rangle \otimes [a|0\rangle - b|1\rangle] + \frac{1}{2}|11\rangle \otimes [a|1\rangle - b|0\rangle]$$

where

$$H|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle], \qquad H|1\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$$

Alice now measures her two particles in the standard basis. She will get one of $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, each with probability 1/4.

If she gets $|00\rangle$, Bob's qubit will jump to state: $a|0\rangle + b|1\rangle$.
If she gets $|01\rangle$, Bob's qubit will jump to state: $a|1\rangle + b|0\rangle$.
If she gets $|10\rangle$, Bob's qubit will jump to state: $a|0\rangle - b|1\rangle$
If she gets $|11\rangle$, Bob's qubit will jump to state $a|1\rangle - b|0\rangle$.

Alice and Bob want Bob's qubit to be in the state $a|0\rangle + b|1\rangle$. It is almost there, but not quite. To sort things out, Alice has to let Bob know which of the four possible situations he is in. She sends Bob two classical bits of information, 00, 01, 10, or 11, corresponding to the results of her measurements, to let him know. These bits of information can be sent in any way, by text (by *classical communication*).

If Bob receives 00 from Alice, he knows that his qubit is in the correct form $a|0\rangle + b|1\rangle$ and so does nothing.

$$Z^0 X^0 [a|0\rangle + b|1\rangle] = a|0\rangle + b|1\rangle .$$

If Bob receives 01 from Alice, he knows that his qubit is $a|1\rangle + b|0\rangle$. He applies the gate *X* to it.

$$\begin{aligned} Z^0 X^1 [a|1\rangle + b|0\rangle] &= aX|1\rangle + bX|0\rangle \\ &= a|0\rangle + b|1\rangle \end{aligned} .$$

If Bob receives 10 from Alice, he knows that his qubit is $a|0\rangle - b|1\rangle$. He applies the gate *Z* to it.

$$\begin{aligned} Z^1 X^0 [a|0\rangle - b|1\rangle] &= aZ|0\rangle - bZ|1\rangle \\ &= a|0\rangle + b|1\rangle \end{aligned} .$$

If Bob receives 11 from Alice, he knows that his qubit is $a|1\rangle - b|0\rangle$. He applies the gate *ZX* to it.

$$\begin{aligned} Z^1 X^1 [a|1\rangle - b|0\rangle] &= Z[aX|1\rangle - bX|0\rangle] \\ &= Z[a|0\rangle - b|1\rangle] \\ &= a|0\rangle + b|1\rangle \end{aligned} .$$

In every case, Bob's qubit ends in state $a|0\rangle + b|1\rangle$, the original state of the qubit that Alice wanted to teleport. It is important to note that there is only one qubit in state $a|0\rangle + b|1\rangle$ at any point during the process. Initially, Alice has it. At the end Bob has it, but as the no cloning theorem tells us, we cannot copy, so only one of them can have it at a time. It is also interesting to observe that when Alice sends her qubits through her circuit Bob's qubit instantaneously jumps to one of the four states. He has to wait for Alice to send him the two classical bits before he can determine which of the four qubits correspond to Alice's original qubit.

B-2 Slight modification of the quantum circuit (II)

Suppose that we add the controlled *X*-gate and the controlled *Z*-gate after the output of the original circuit.

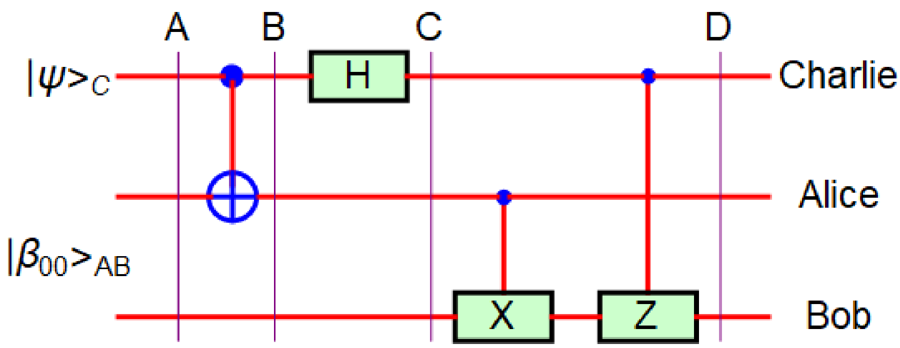

Fig.B3 The circuit with input-1 ($\left|\psi\right\rangle_C = [a\left|0\right\rangle + b\left|1\right\rangle]_C$ for Charlie, input-2 and input-3 (the Bell state $\left|\beta_{00}\right\rangle_{AB}$) shared with Alice and Bob. The output is $\frac{1}{\sqrt{2}}(\left|\beta_{00}\right\rangle_{CA} + \left|\beta_{01}\right\rangle_{CA}) \otimes [a\left|0\right\rangle + b\left|1\right\rangle]_B$. Immediately after Alice measures one of the Bell states ($\left|\beta_{00}\right\rangle_{CA}$, $\left|\beta_{0'}\right\rangle_{CA}$, the system collapses, and Bob observes the state $\left|\psi\right\rangle_B = [a\left|0\right\rangle + b\left|1\right\rangle]_B$.

We can change the order of measurement and control since the measurement commutes with controls.

$$\frac{1}{2}a\left|000\right\rangle + \frac{1}{2}a\left|011\right\rangle + \frac{1}{2}a\left|100\right\rangle + \frac{1}{2}a\left|111\right\rangle$$
$$+\frac{1}{2}b\left|010\right\rangle + \frac{1}{2}b\left|001\right\rangle - \frac{1}{2}b\left|110\right\rangle - \frac{1}{2}b\left|101\right\rangle$$

Step-*D*: **Controlled *Z* and *X***

$$\left|\alpha_1\alpha_2\right\rangle \otimes Z^{\alpha_1}X^{\alpha_2}\left|\alpha_3\right\rangle$$

where

$$\left|000\right\rangle \to \left|00\right\rangle \otimes Z^0X^0\left|0\right\rangle = \left|000\right\rangle,$$

$$\left|011\right\rangle \to \left|01\right\rangle \otimes Z^0X^1\left|1\right\rangle = \left|010\right\rangle,$$

$$|100\rangle \to |10\rangle \otimes Z^1 X^0 |0\rangle = |100\rangle,$$

$$|111\rangle \to |11\rangle \otimes Z^1 X^1 |1\rangle = |110\rangle,$$

$$|010\rangle \to |01\rangle \otimes Z^0 X^1 |0\rangle = |011\rangle,$$

$$|001\rangle \to |00\rangle \otimes Z^0 X^0 |1\rangle = |001\rangle,$$

$$|110\rangle \to |11\rangle \otimes Z^1 X^1 |0\rangle = -|111\rangle,$$

$$|101\rangle \to |10\rangle \otimes Z^1 X^0 |1\rangle = -|101\rangle.$$

Thus, we have

$$\begin{aligned}
&\frac{1}{2}a|000\rangle + \frac{1}{2}a|010\rangle + \frac{1}{2}a|101\rangle + \frac{1}{2}a|110\rangle \\
&+\frac{1}{2}b|011\rangle + \frac{1}{2}b|001\rangle + \frac{1}{2}b|111\rangle + \frac{1}{2}b|101\rangle \\
&= \frac{1}{2}[a|000\rangle + b|001\rangle] + \frac{1}{2}[a|010\rangle + b|011\rangle] \\
&+\frac{1}{2}[a|100\rangle + \frac{1}{2}b|101\rangle] + \frac{1}{2}[a|110\rangle + b|111\rangle] \\
&= \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle] \otimes [a|0\rangle + b|1\rangle]
\end{aligned}$$

Using the Bell states, we get the final result as

$$\begin{aligned}
&\frac{1}{\sqrt{2}}[(|00\rangle + |11\rangle) + (|01\rangle + |10\rangle)] \otimes \frac{a|0\rangle + b|1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(|\beta_{00}\rangle_{CA} + |\beta_{01}\rangle_{CA}) \otimes [a|0\rangle + b|1\rangle]_B
\end{aligned}$$

When Alice now measures her two electrons in the Bell states; $|\beta_{00}\rangle_{CA}$ and Then we find that Bob's qubit will always jump to the state $|\psi\rangle_B = [a|0\rangle + b|1\rangle]_B$, independent of the kinds of measurement by Alice.