# Probability current density

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In quantum mechanics, the **probability current** (sometimes called **probability flux**) is a mathematical quantity describing the flow of probability (i.e. probability per unit time per unit area). Intuitively, if one pictures the probability density as an inhomogeneous fluid, then the probability current is the rate of flow of this fluid. This is analogous to mass currents in hydrodynamics and electric currents in electromagnetism. It is a real vector, like electric current density. The notion of a probability current is useful in some of the formalism in quantum mechanics.
Wikipedia: http://en.wikipedia.org/wiki/Probability_current

## 1. Probability current density

The probability density is defined by

$$\rho(\boldsymbol{r},t) = \left|\langle \boldsymbol{r} | \psi(t)\rangle\right|^2 = \left|\psi(\boldsymbol{r},t)\right|^2 .$$

The integral

$$\int \rho(r,t) d\boldsymbol{r} = \int \left|\psi(\boldsymbol{r},t)\right|^2 d\boldsymbol{r} ,$$

taken over some finite volume *V*, is the probability of finding the particle in this volume. Let us calculate the derivative of the probability with respect to time *t*.

$$\begin{aligned}\frac{\partial}{\partial t}\int \left|\psi(\boldsymbol{r},t)\right|^2 d\boldsymbol{r} &= \int \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\right) d\boldsymbol{r} \\ &= -\frac{1}{i\hbar}\int [(H^*\psi^*)\psi - \psi^*(H\psi)] d\boldsymbol{r}\end{aligned}$$

Here

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi ,$$

where

$$\psi = \psi(\boldsymbol{r},t) = \langle \boldsymbol{r} | \psi(t)\rangle .$$

Complex conjugate of this equation

$$-i\hbar\frac{\partial\psi^*}{\partial t} = H^*\psi^* = H\psi^*,$$

where

$$H^* = H = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}),$$

$$(H^*\psi^*)\psi - \psi^*(H\psi) = (H\psi^*)\psi - \psi^*(H\psi)$$
$$= \{[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})]\psi^*\}\psi - \psi^*\{[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})]\psi\}$$

or

$$(H^*\psi^*)\psi - \psi^*(H\psi) = -\frac{\hbar^2}{2m}[(\nabla^2\psi^*)\psi - \psi^*(\nabla^2\psi)],$$

$$\frac{\partial}{\partial t}\int|\psi(\boldsymbol{r},t)|^2 d\boldsymbol{r} = \frac{\hbar}{2mi}\int[(\nabla^2\psi^*)\psi - \psi^*(\nabla^2\psi)]d\boldsymbol{r}.$$

Note that

$$(\nabla^2\psi^*)\psi - \psi^*(\nabla^2\psi) = \nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi).$$

((**Proof**))

We use the formula of vector analysis.

$$\nabla\cdot(\phi\boldsymbol{a}) = \nabla\phi\cdot\boldsymbol{a} + \phi\nabla\cdot\boldsymbol{a},$$

$$\nabla\cdot(\psi\nabla\psi^*) = \nabla\psi\cdot\nabla\psi^* + \psi\nabla^2\psi^*.$$

The complex conjugate of the above equation

$$\nabla\cdot(\psi^*\nabla\psi) = \nabla\psi^*\cdot\nabla\psi + \psi^*\nabla^2\psi.$$

From the above equations, we get

$$(\nabla^2\psi^*)\psi - \psi^*(\nabla^2\psi) = \nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi).$$

Then

$$\frac{\partial}{\partial t}\int|\psi(\boldsymbol{r},t)|^2 d\boldsymbol{r} = \frac{\hbar}{2mi}\int\nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi)d\boldsymbol{r}.$$

We note that

$$\int \frac{\partial}{\partial t}\rho d\boldsymbol{r} = -\int \nabla\cdot \boldsymbol{J} d\boldsymbol{r} = -\int \boldsymbol{J}\cdot d\boldsymbol{a}\,, \qquad \text{(Gauss's theorem)}$$

or

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\boldsymbol{J} = 0\,. \qquad \text{(Equation of continuity)}$$

Then the probability current density can be defined as

$$\boldsymbol{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = \frac{1}{m}\mathrm{Re}[\psi^*\frac{\hbar}{i}\nabla\psi],$$

since

$$\begin{aligned}\boldsymbol{J} &= \frac{1}{m}\mathrm{Re}[\psi^*\frac{\hbar}{i}\nabla\psi] \\ &= \frac{1}{2m}(\psi^*\frac{\hbar}{i}\nabla\psi - \psi\frac{\hbar}{i}\nabla\psi^*) \\ &= \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)\end{aligned}$$

((**Note-1**)) What is the units of ***J***?

The unit of ***J*** should be [cm$^{-2}$ s$^{-1}$]. Since $\int |\psi|^2 d\boldsymbol{r} = 1$, the unit of $\psi^*\nabla\psi$ is [cm$^{-4}$]. Then the unit of ***J*** is evaluated as

$$[\boldsymbol{J}] = \frac{erg\cdot s}{erg\cdot s^2/cm^2}\cdot\frac{1}{cm^4} = \frac{1}{s\cdot cm^2}$$

The unit of $\int J(\boldsymbol{r})d\boldsymbol{r}$ is [cm/s] which is the unit of the velocity.

((**Note-2**)) **Comment on the formula**
((**L.I. Schiff, Quantum Mechanics, McGraw-Hill, 1968**))

$$\boldsymbol{J}(\boldsymbol{r},t) = \frac{1}{m}\mathrm{Re}[\psi(\boldsymbol{r},t)^*\frac{\hbar}{i}\nabla\psi(\boldsymbol{r},t)]\,.$$

"Although this interpretation of ***J*** is suggestive, it must be realized that ***J*** is not susceptible to direct measurement in the sense in which the probability *P* is. It would be misleading, for example, to say that $J(\boldsymbol{r}, t)$ is the average measured particle flux at the point $\boldsymbol{r}$ and the time $t$, because a measurement of average local flux implies simultaneous high-precision measurements of position and velocity (which is equivalent to momentum) and is therefore inconsistent with the uncertainty relation. Nevertheless, it is sometimes helpful to think of $\boldsymbol{J}$ as a flux vector, especially when it depends only slightly or not at all on $r$, so that an accurate velocity determination can be made without impairing the usefulness of the concept of flux."

((**Note-3**)) The reason why $\boldsymbol{J}$ is called the probability current density.
**(J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics, 2nd edition, Addison-Wesley, 2011)**

What is the relation between the average of momentum and the probability current density?

We may intuitively expect that the average value of the momentum operator at time $t$ is related to the Probability current density $J(\boldsymbol{r},t)$ as

$$\frac{1}{m}\langle \boldsymbol{p}\rangle_t = \int d\boldsymbol{r}\boldsymbol{J}(\boldsymbol{r},t)$$

where $\langle \boldsymbol{p}\rangle_t$ is the average of the momentum operator and is defined by

$$\langle \boldsymbol{p}\rangle_t = \langle \psi|\hat{\boldsymbol{p}}|\psi\rangle .$$

This expectation value is in fact real since

$$\langle \psi|\boldsymbol{p}|\psi\rangle^* = \langle \psi|\hat{\boldsymbol{p}}^+|\psi\rangle = \langle \psi|\hat{\boldsymbol{p}}|\psi\rangle .$$

We now calculate

$$\mathrm{Re}[\frac{1}{m}\langle \psi|\hat{\boldsymbol{p}}|\psi\rangle],$$

which is in fact equal to $\frac{1}{m}\langle \psi|\hat{\boldsymbol{p}}|\psi\rangle$.

$$\frac{1}{m}\langle \boldsymbol{p}\rangle_t = \mathrm{Re}[\frac{1}{m}\langle \psi|\hat{\boldsymbol{p}}|\psi\rangle]$$

$$= \frac{1}{m}\mathrm{Re}[\int \langle \psi(t)|\boldsymbol{r}\rangle\langle \boldsymbol{r}|\hat{\boldsymbol{p}}|\psi(t)\rangle d\boldsymbol{r}]$$

$$= \mathrm{Re}[\int \psi^*(\boldsymbol{r},t)\frac{\hbar}{im}\nabla\psi(\boldsymbol{r},t)d\boldsymbol{r}]$$

$$= \int \frac{\hbar}{2im}[\psi^*(\boldsymbol{r},t)\nabla\psi(\boldsymbol{r},t) - \psi(\boldsymbol{r},t)\nabla\psi^*(\boldsymbol{r},t)]d\boldsymbol{r}$$

$$= \int \boldsymbol{J}(\boldsymbol{r},t)d\boldsymbol{r}$$

Thus we have

$$\boldsymbol{J}(\boldsymbol{r},t) = \frac{\hbar}{2im}[\psi^*(\boldsymbol{r},t)\nabla\psi(\boldsymbol{r},t) - \psi(\boldsymbol{r},t)\nabla\psi^*(\boldsymbol{r},t)].$$

((**Note-4**)) **E. Merzbacher, Quantum Mechanics (John Wiley & Sons, 1998)**

By using the continuity equation, the time derivative of the average x coordinate can be written as

$$\frac{d}{dt}\langle x\rangle = \int x\frac{\partial\rho}{\partial t}d\boldsymbol{r}$$

$$= -\int x(\nabla\cdot\boldsymbol{J})d\boldsymbol{r}$$

$$= \int \nabla\cdot(x\boldsymbol{J})d\boldsymbol{r} + \int j_x d\boldsymbol{r}$$

$$= \int (x\boldsymbol{j})\cdot d\boldsymbol{a} + \int j_x d\boldsymbol{r}$$

$$= \int j_x d\boldsymbol{r}$$

or

$$\frac{d}{dt}\langle \boldsymbol{r}\rangle = \int \boldsymbol{J}d\boldsymbol{r},$$

where the divergence term is removed under the assumption that $\psi$ vanishes sufficiently fast at infinity. Using the expression of $\boldsymbol{J}(\boldsymbol{r},t)$ and integration by parts, we obtain

$$m\frac{d}{dt}\langle \boldsymbol{r}\rangle = \langle \boldsymbol{p}\rangle = \int \psi^*\frac{\hbar}{i}\nabla\psi d\boldsymbol{r}.$$

## 2. Equation of continuity

The probability density is defined as

$$\rho = \psi^* \psi \ .$$

We take the derivative of $\rho$ with respect to $t$,

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\psi^* \psi) = \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t} .$$

Using the Schrödinger equation, we get

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = (\frac{\boldsymbol{p}^2}{2m} + V)\psi \ , \qquad -i\hbar \frac{\partial \psi^*}{\partial t} = H\psi^* = (\frac{\boldsymbol{p}^2}{2m} + V)\psi^* .$$

and

$$\boldsymbol{p} = \frac{\hbar}{i}\nabla \ ,$$

we have

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{1}{i\hbar}[(\frac{\boldsymbol{p}^2}{2m} + V)\psi^*]\psi + \frac{1}{i\hbar}\psi^*[(\frac{\boldsymbol{p}^2}{2m} + V)\psi] \\
&= -\frac{1}{i\hbar}(\frac{\boldsymbol{p}^2}{2m}\psi^*)\psi + \frac{1}{i\hbar}\psi^*(\frac{\boldsymbol{p}^2}{2m}\psi) \\
&= -\frac{1}{i\hbar}(-\frac{\hbar^2}{2m}\nabla^2\psi^*)\psi + \frac{1}{i\hbar}\psi^*(-\frac{\hbar^2}{2m}\nabla^2\psi) \\
&= -\frac{\hbar}{2mi}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)
\end{aligned}$$

We now calculate $\nabla \cdot \boldsymbol{J}$ ,

$$\begin{aligned}
\nabla \cdot \boldsymbol{J} &= \frac{\hbar}{2mi}\nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*] \\
&= \frac{\hbar}{2mi}(\nabla \psi^* \cdot \nabla \psi + \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* - \nabla \psi \cdot \nabla \psi^*) \\
&= \frac{\hbar}{2mi}(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*)
\end{aligned}$$

where we use the vector formula

$$\nabla \cdot (f\boldsymbol{A}) = \nabla f \cdot \boldsymbol{A} + f \nabla \cdot \boldsymbol{A} .$$

Then we have the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 ,$$

or

$$\int \frac{\partial \rho}{\partial t} d\boldsymbol{r} + \int (\nabla \cdot \boldsymbol{J}) d\boldsymbol{r} = 0 .$$

Using the Gauss's theorem, this can be rewritten as

$$\frac{\partial}{\partial t} \int \rho d\boldsymbol{r} + \int \boldsymbol{J} \cdot d\boldsymbol{a} = 0 ,$$

where ***da*** is the element of surface area vector whose direction is normal to the surface.

## 3. Examples

### (a) The plane wave

$$\psi(r,t) = C \exp[\frac{i}{\hbar}(\boldsymbol{p} \cdot \boldsymbol{r} - Et)] = C \exp[-\frac{i}{\hbar} Et] \varphi(\boldsymbol{r}) .$$

The probability current density is obtained as

$$\begin{aligned}
\boldsymbol{J} &= \frac{\hbar}{2mi} [\varphi^*(\boldsymbol{r}) \nabla \varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}) \nabla \varphi(\boldsymbol{r})] \\
&= \frac{\hbar}{2mi} [C^* \exp(-\frac{i}{\hbar} \boldsymbol{p} \cdot \boldsymbol{r}) C (\frac{i\boldsymbol{p}}{\hbar}) \exp(\frac{i}{\hbar} \boldsymbol{p} \cdot \boldsymbol{r}) - C^* (-\frac{i\boldsymbol{p}}{\hbar}) \exp(-\frac{i}{\hbar} \boldsymbol{p} \cdot \boldsymbol{r}) C \exp(\frac{i}{\hbar} \boldsymbol{p} \cdot \boldsymbol{r})] \\
&= \frac{\boldsymbol{p}}{m} |C|^2 \\
&= \boldsymbol{v} |\psi|^2
\end{aligned}$$

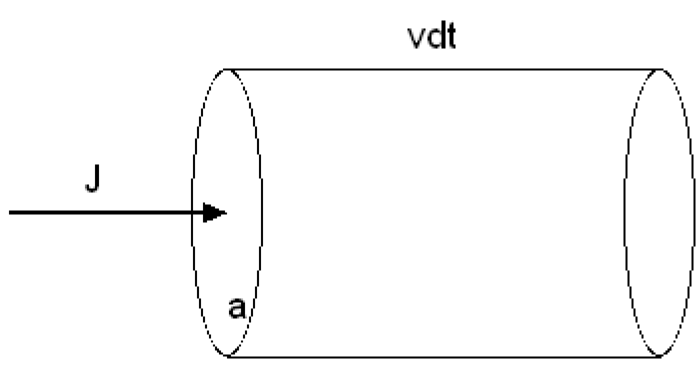



**Fig.** Flux density *J*: particles number flowing per unit area per unit time.

$$Jadt = |\psi|^2 avdt ,$$

or

$$J = |\psi|^2 v = \frac{p}{m}|\psi|^2 .$$

**(b) The superposition of the plane waves**

Suppose that

$$\psi = Ae^{ikx} + Be^{-ikx} .$$

Then we have

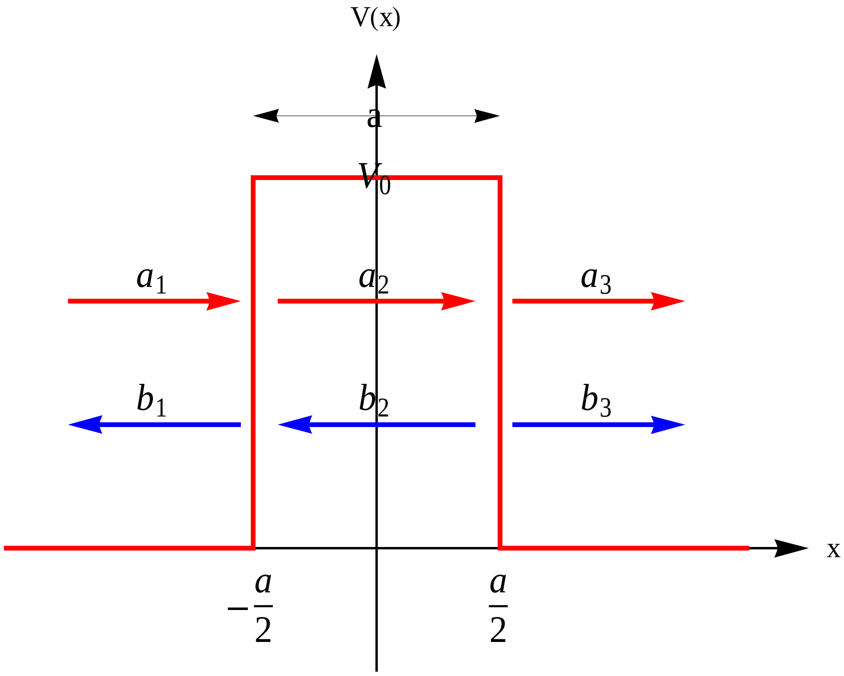
$$\begin{aligned} J_x &= \frac{1}{m}\mathrm{Re}[\psi^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi] \\ &= \frac{1}{m}\mathrm{Re}[(A^* e^{-ikx} + B^* e^{ikx})\frac{\hbar}{i}\frac{\partial}{\partial x}(Ae^{ikx} + Be^{-ikx})] \\ &= \frac{\hbar k}{m}\mathrm{Re}[A^* e^{-ikx} + B^* Be^{ikx})(Ae^{ikx} - Be^{-ikx})] \\ &= \frac{\hbar k}{m}(A^* A - B^* B) = \frac{\hbar k}{m}(|A|^2 - |B|^2) \end{aligned}$$

## 4. Conservation of the probability current density at the boundary for the one dimensional barrier problem

We consider the boundary condition one dimensional barrier problem as shown below.

The probability current density is defined by

$$J = \frac{\hbar}{2mi}(\psi^* \frac{d}{dx}\psi - \psi \frac{d}{dx}\psi^*) ,$$




Because of particle (flux) conservation, both $\psi$ and $\frac{d\psi}{dx}$ must be continuous at the barrier border at $x$ = -$a$/2 and $x$ = $a$/2. In other words, when $\psi$ and $\frac{d\psi}{dx}$ are continuous at the boundary, the flux conservation is valid.

(a) Region-I

$$\psi(x) = a_1 e^{ikx} + b_1 e^{-ikx}, \qquad \text{for } x < -\frac{a}{2}$$

$$\psi'(x) = ik(a_1 e^{ikx} - b_1 e^{-ikx}).$$

The probability current density is given by

$$J_I = \frac{\hbar k}{m}(|a_1|^2 - |b_1|^2).$$

(b) Region-II

$$\psi_{II}(x) = a_2 e^{-\rho x} + b_2 e^{\rho x}, \qquad \text{for } |x| < \frac{a}{2}$$

$$\psi_{II}'(x) = \rho(-a_2 e^{-\rho x} + b_2 e^{\rho x}).$$

The probability current density

$$J_{II} = -i\frac{\hbar\rho}{m}(a_2^* b_2 - a_2 b_2^*) .$$

(c) Region III

$$\psi_{III}(x) = a_3 e^{ikx} + b_3 e^{-ikx} , \qquad \text{for } x < -\frac{a}{2}$$

$$\psi_{III}'(x) = ik(a_3 e^{ikx} - b_3 e^{-ikx}) .$$

The probability current density

$$J_3 = \frac{\hbar k}{m}(|a_3|^2 - |b_3|^2) ,$$

where

$$\varepsilon = \frac{\hbar^2 k^2}{2m} , \qquad V_0 - \varepsilon = \frac{\hbar^2 \rho^2}{2m} .$$

From the continuity of $\psi$ and $\frac{d\psi}{dx}$, we get the relations between $a_1$, $b_1$, $a_2$, $b_2$, $a_3$, and $b_3$.

(i)

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = M(-\frac{a}{2})\begin{pmatrix} a_2 \\ b_2 \end{pmatrix},$$

where

$$M(-\frac{a}{2}) = \frac{1}{2k}\begin{pmatrix} (k+i\rho)\exp[\frac{a(ik+\rho)}{2}] & (k-i\rho)\exp[\frac{a(ik-\rho)}{2}] \\ (k-i\rho)\exp[\frac{a(-ik+\rho)}{2}] & (k+i\rho)\exp[-\frac{a(ik+\rho)}{2}] \end{pmatrix}.$$

(ii)

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = M(\frac{a}{2})\begin{pmatrix} a_2 \\ b_2 \end{pmatrix},$$

where

$$M(\frac{a}{2}) = \frac{1}{2k}\begin{pmatrix} (k+i\rho)\exp[-\frac{a(ik+\rho)}{2}] & (k-i\rho)\exp[\frac{a(-ik+\rho)}{2}] \\ (k-i\rho)\exp[\frac{a(ik-\rho)}{2}] & (k+i\rho)\exp[\frac{a(ik+\rho)}{2}] \end{pmatrix},$$

We find that

$$J_I = J_{II} = J_{III} ,$$

or

$$\frac{\hbar k}{m}(|a_1|^2 - |b_1|^2) = -i\frac{\hbar\rho}{m}(a_2{}^* b_2 - a_2 b_2{}^*) = \frac{\hbar k}{m}(|a_3|^2 - |b_3|^2) .$$

## 5. Lagrangian of particles with mass *m** and charge *q** in the presence of magnetic field

The Lagrangian $L$ for the motion of a particle in the presence of magnetic field and electric field is given by

$$L = \frac{1}{2}m^*\boldsymbol{v}^2 - q^*(\phi - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}),$$

where $m^*$ and $q^*$ are the mass and charge of the particle. $A$ is a vector potential and $\phi$ is a scalar potential.

The canonical momentum is defined as

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A} .$$

The mechanical momentum (the measurable quantity) is given by

$$\boldsymbol{\pi} = m^*\boldsymbol{v} = \boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A} .$$

The Hamiltonian $H$ is given by

$$H = \boldsymbol{p}\cdot\boldsymbol{v} - L = (m^*\boldsymbol{v} + \frac{q^*}{c}\boldsymbol{A})\cdot\boldsymbol{v} - L = \frac{1}{2}m^*\boldsymbol{v}^2 + q^*\phi = \frac{1}{2m^*}(\boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A})^2 + q^*\phi .$$

The Hamiltonian formalism uses $\boldsymbol{A}$ and $\phi$, and not $\boldsymbol{E}$ and $\boldsymbol{B}$, directly. The result is that the description of the particle depends on the gauge chosen.

## 6. Current density for the superconductors

We consider the current density for the superconductor. $\psi$ is the order parameter of the superconductor and $m^*$ and $q^*$ are the mass and charge of the Cooper pairs. The current density is invariant under the gauge transformation.

$$\boldsymbol{J}_s = \frac{q^*}{m^*}\mathrm{Re}[\psi^*(\frac{\hbar}{i}\nabla\psi - \frac{q^*}{c}\boldsymbol{A}\psi)].$$

This can be rewritten as

$$\begin{aligned}\boldsymbol{J}_s &= \frac{q^*}{2m^*}[(\psi^*\frac{\hbar}{i}\nabla\psi - \frac{q^*}{c}\boldsymbol{A}\psi^*\psi) + (-\psi\frac{\hbar}{i}\nabla\psi^* - \frac{q^*}{c}\boldsymbol{A}\psi^*\psi)] \\ &= \frac{q^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^{*2}|\psi|^2}{m^*c}\boldsymbol{A}\end{aligned}$$

The density is also gauge independent.

$$\rho_s = |\langle \boldsymbol{r}|\psi\rangle|^2.$$

## 7. Probability current density (general case)

Now we assume that

$$\psi(\boldsymbol{r}) = |\psi(\boldsymbol{r})|e^{i\theta(\boldsymbol{r})}.$$

We note that

$$\begin{aligned}\psi^*(\boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A})\psi(\boldsymbol{r}) - &= \psi^*\frac{\hbar}{i}\nabla[e^{i\theta(\boldsymbol{r})}|\psi(\boldsymbol{r})|] - \frac{q^*}{c}\boldsymbol{A}|\psi(\boldsymbol{r})|^2 \\ &= |\psi(\boldsymbol{r})|e^{-i\theta(\boldsymbol{r})}\frac{\hbar}{i}[i|\psi(\boldsymbol{r})|e^{i\theta(\boldsymbol{r})}\nabla\theta(\boldsymbol{r}) + e^{i\theta(\boldsymbol{r})}\nabla|\psi(\boldsymbol{r})|] - \frac{q^*}{c}\boldsymbol{A}|\psi(\boldsymbol{r})|^2. \\ &= \hbar|\psi(\boldsymbol{r})|^2[\nabla\theta(\boldsymbol{r}) - \frac{q^*}{c\hbar}\boldsymbol{A}] - i\hbar|\psi(\boldsymbol{r})|\nabla|\psi(\boldsymbol{r})|\end{aligned}$$

The last term is pure imaginary. Then the current density is obtained as

$$\boldsymbol{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\boldsymbol{A}) = q^*|\psi|^2\boldsymbol{v}_s,$$

or

$$\hbar\nabla\theta = \frac{q^*}{c}\mathbf{A} + m^*\mathbf{v}_s.$$

Since

$$\boldsymbol{\pi} = m^* \boldsymbol{v}_s = \boldsymbol{p} - \frac{q^*}{c}\boldsymbol{A},$$

we have

$$\boldsymbol{p} = \frac{q^*}{c}\boldsymbol{A} + m^* \boldsymbol{v}_s = \hbar\nabla\theta.$$

Note that $\boldsymbol{J}_s$ (or $\boldsymbol{v}_s$) is gauge-invariant. Under the gauge transformation, the wave function is transformed as

$$\psi'(\boldsymbol{r}) = \exp\left(\frac{iq^*\chi}{\hbar c}\right)\psi(\boldsymbol{r}).$$

This implies that

$$\theta \to \theta' = \theta + \frac{q^*\chi}{\hbar c},$$

Since $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$, we have

$$\begin{aligned}
\boldsymbol{J}_s' &= \hbar\left(\nabla\theta' - \frac{q^*}{c\hbar}\boldsymbol{A}'\right) \\
&= \hbar\left[\nabla\left(\theta + \frac{q^*\chi}{\hbar c}\right) - \frac{q^*}{c\hbar}(\boldsymbol{A} + \nabla\chi)\right]. \\
&= \hbar\left(\nabla\theta - \frac{q^*}{c\hbar}\boldsymbol{A}\right)
\end{aligned}$$

So the current density is invariant under the gauge transformation.

**((Note))** $\boldsymbol{v}_s = \dfrac{\hbar}{m}\nabla\theta$

Now we assume that

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\theta(\mathbf{r})}.$$

Noting that

$$\psi^* \boldsymbol{p}\psi(\boldsymbol{r}) = \psi^* \frac{\hbar}{i}\nabla[e^{i\theta(r)}|\psi(\boldsymbol{r})|]$$

$$= |\psi(\boldsymbol{r})|e^{-i\theta(r)}\frac{\hbar}{i}[i|\psi(\boldsymbol{r})|e^{i\theta(r)}\nabla\theta(\boldsymbol{r}) + e^{i\theta(r)}\nabla|\psi(\boldsymbol{r})|].$$

$$= \hbar|\psi(\boldsymbol{r})|^2[\nabla\theta(\boldsymbol{r})] - i\hbar|\psi(\boldsymbol{r})|\nabla|\psi(\boldsymbol{r})|$$

The last term is pure imaginary. Then the probability current density is obtained as

$$\boldsymbol{J} = \frac{\hbar}{m}|\psi|^2(\nabla\theta) = |\psi|^2\boldsymbol{v}_s,$$

or

$$\boldsymbol{v}_s = \frac{\hbar}{m}\nabla\theta.$$

## 8. Meissner effect: London equation

We start with

$$\mathbf{p} = \hbar\nabla\theta = \frac{q^*}{c}\mathbf{A} + m^*\mathbf{v}_s.$$

We assume that $|\psi|^2 = n_s$ is independent of $\boldsymbol{r}$. Then we get

$$\mathbf{J}_s = q^*|\psi|^2\mathbf{v}_s = q^* n_s \mathbf{v}_s.$$

Using the above two equations, we have

$$\boldsymbol{p} = \hbar\nabla\theta = \frac{q^*}{c}\boldsymbol{A} + \frac{m^*}{q^* n_s}\boldsymbol{J}_s.$$

Suppose that $\boldsymbol{p}$ = 0. Then we have a London's equation given by

$$\boldsymbol{J}_s = -\frac{q^{*2}n_s}{m^* c}\boldsymbol{A}.$$

From this equation, we have

$$\nabla\times\boldsymbol{J}_s = -\frac{q^{*2}n_s}{m^* c}\nabla\times\boldsymbol{A} = -\frac{q^{*2}n_s}{m^* c}\boldsymbol{B}.$$

Using the Maxwell's equation

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J}_s , \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B} = 0 ,$$

we get

$$\nabla \times (\nabla \times \boldsymbol{B}) = \frac{4\pi}{c} \nabla \times \boldsymbol{J}_s = -\frac{4\pi n_s q^{*2}}{m^* c^2} \boldsymbol{B} ,$$

where

$$n_s = |\psi|^2 = \text{constant} \qquad \text{(independent of } \boldsymbol{r}\text{)}$$

$$\lambda_L{}^2 = \frac{m^* c^2}{4\pi n_s q^{*2}} . \qquad \text{(penetration depth).}$$

Then

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = -\frac{1}{\lambda_L{}^2} \boldsymbol{B} ,$$

or

$$\nabla^2 \boldsymbol{B} = \frac{1}{\lambda_L{}^2} \boldsymbol{B} .$$

Inside the system, ***B*** become s zero, corresponding to the Meissner effect.

## 9. Flux quantization

We start with the current density

$$\boldsymbol{J}_s = \frac{q^* \hbar}{m^*} |\psi|^2 (\nabla \theta - \frac{q^*}{c\hbar} \boldsymbol{A}) = q^* |\psi|^2 \boldsymbol{v}_s .$$

Suppose that $n_s = |\psi|^2$ =constant. Then we have

$$\nabla \theta = \frac{m^*}{q^* \hbar n_s} \boldsymbol{J}_s + \frac{q^*}{c\hbar} \boldsymbol{A} ,$$

or

$$\oint \nabla\theta \cdot d\boldsymbol{l} = \frac{m^*}{q^* \hbar n_s^*} \oint \boldsymbol{J}_s \cdot d\boldsymbol{l} + \frac{q^*}{c\hbar} \oint \boldsymbol{A} \cdot d\boldsymbol{l} .$$

The path of integration can be taken inside the penetration depth where $\boldsymbol{J}_s = 0$.

$$\oint \nabla\theta \cdot d\boldsymbol{l} = \frac{q^*}{c\hbar} \oint \boldsymbol{A} \cdot d\boldsymbol{l} = \frac{q^*}{c\hbar} \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} = \frac{q^*}{c\hbar} \int \boldsymbol{B} \cdot d\boldsymbol{a} = \frac{q^*}{c\hbar} \Phi ,$$

where $\Phi$ is the magnetic flux. Then we find that

$$\Delta\theta = \theta_2 - \theta_1 = 2\pi n = \frac{q^*}{c\hbar} \Phi ,$$

where $n$ is an integer. The phase $\theta$ of the wave function must be unique, or differ by a multiple of $2\pi$ at each point,

$$\Phi = \frac{2\pi c\hbar}{|q^*|} n .$$

The flux is quantized. When $|q^*| = 2|e|$, we have a magnetic quantum fluxoid;

$$\Phi_0 = \frac{2\pi c\hbar}{2|e|} = \frac{ch}{2|e|} = 2.06783372 \times 10^{-7} \text{ Gauss cm}^2.$$

**REFERENCES**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