

Translation operator
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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Here we discuss the translation operator. The linear momentum is a generator of the translation. This is in contrast to the rotation operator where the angular momentum is a generator of the operation.

1 Definition of the translation operator

Here we discuss the transportation operator

$\hat{T}(a)$: translation operator (unitary operator)

$$|\psi'\rangle = \hat{T}(a)|\psi\rangle,$$

or

$$\langle\psi'| = \langle\psi|\hat{T}^\dagger(a).$$

(i) Analogy from classical mechanics for x

The average value of \hat{x} in the new state $|\psi'\rangle$ is equal to the average value of \hat{x} in the new state $|\psi\rangle$ plus the x -displacement a .

$$\langle\psi'|\hat{x}|\psi'\rangle = \langle\psi|\hat{x} + a|\psi\rangle,$$

or

$$\langle\psi|\hat{T}^\dagger(a)\hat{x}\hat{T}(a)|\psi\rangle = \langle\psi|\hat{x} + a|\psi\rangle,$$

or

$$\hat{T}^\dagger(a)\hat{x}\hat{T}(a) = \hat{x} + a\hat{1}. \quad (1)$$

Normalization condition:

$$\langle\psi'|\psi'\rangle = \langle\psi|\hat{T}^\dagger(a)\hat{T}(a)|\psi\rangle = \langle\psi|\psi\rangle,$$

or

$$\hat{T}^\dagger(a)\hat{T}(a) = \hat{1}, \quad (2)$$

((Unitary operator))

From Eqs.(1) and (2), we have

$$\hat{x}\hat{T}(a) = \hat{T}(a)(\hat{x} + a) = \hat{T}(a)\hat{x} + a\hat{T}(a).$$

((Commutation relation))

$$[\hat{x}, \hat{T}(a)] = a\hat{T}(a).$$

Here we note that

$$\hat{x}\hat{T}(a)|x\rangle = \hat{T}(a)\hat{x}|x\rangle + a\hat{T}(a)|x\rangle = (x+a)\hat{T}(a)|x\rangle.$$

Thus $\hat{T}(a)|x\rangle$ is the eigenket of \hat{x} with the eigenvalue $(x+a)$

$$\hat{T}(a)|x\rangle = |x+a\rangle.$$

We note that

$$\hat{T}^+(a)\hat{T}(a)|x\rangle = \hat{T}^+(a)|x+a\rangle = |x\rangle.$$

When x is replaced by $x-a$,

$$\hat{T}^+(a)|x\rangle = |x-a\rangle,$$

or

$$|x-a\rangle = \hat{T}^+(a)|x\rangle,$$

or

$$\langle x-a| = \langle x|\hat{T}(a).$$

Note that

$$\langle x|\psi'\rangle = \langle x|\hat{T}(a)|\psi\rangle = \langle x-a|\psi\rangle = \psi(x-a).$$

(ii) Analogy from the classical mechanics for p

The average value of \hat{p} in the new state $|\psi'\rangle$ is equal to the average value of \hat{p} in the new state $|\psi\rangle$.

$$\langle\psi'|\hat{p}|\psi'\rangle = \langle\psi|\hat{p}|\psi\rangle,$$

or

$$\langle\psi|\hat{T}^+(a)\hat{p}\hat{T}(a)|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle,$$

$$\hat{T}^+(a)\hat{p}\hat{T}(a) = \hat{p}.$$

So we have the commutation relation

$$[\hat{T}(a), \hat{p}] = 0.$$

From the above commutation relation, we have

$$\hat{p}\hat{T}(a)|p\rangle = \hat{T}(a)\hat{p}|p\rangle = p\hat{T}(a)|p\rangle.$$

Thus $\hat{T}(a)|p\rangle$ is the eigenket of \hat{p} associated with the eigenvalue p .

2 Infinitesimal translation operator

We now define the infinitesimal translation operator by

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar} \hat{G} dx,$$

where \hat{G} is called a generator of translation. The dimension of \hat{G} is that of the linear momentum.

The operator $\hat{T}(dx)$ satisfies the relations:

$$\hat{T}^+(dx)\hat{T}(dx) = \hat{1}, \tag{1}$$

$$\hat{T}^+(dx)\hat{x}\hat{T}(dx) = \hat{x} + dx,$$

or

$$\hat{x}\hat{T}(dx) - \hat{T}(dx)\hat{x} = dx\hat{T}(dx), \tag{2}$$

and

$$[\hat{T}(dx), \hat{p}] = 0, \quad (3)$$

Using the relation (1), we get

$$\left(\hat{1} - \frac{i}{\hbar} \hat{G} dx\right)^+ \left(\hat{1} - \frac{i}{\hbar} \hat{G} dx\right) = \hat{1},$$

or

$$\left(\hat{1} + \frac{i}{\hbar} \hat{G}^+ dx\right) \left(\hat{1} - \frac{i}{\hbar} \hat{G} dx\right) = \hat{1} + \frac{i}{\hbar} (\hat{G}^+ - \hat{G}) dx + O[(dx)^2] = \hat{1},$$

or

$$\hat{G}^+ = \hat{G}.$$

The operator \hat{G} is a Hermitian operator. Using the relation (2), we get

$$\hat{x} \left(\hat{1} - \frac{i}{\hbar} \hat{G} dx\right) - \left(\hat{1} - \frac{i}{\hbar} \hat{G} dx\right) \hat{x} = dx \left(\hat{1} - \frac{i}{\hbar} \hat{G} dx\right) = dx \hat{1} + O(dx)^2,$$

or

$$-\frac{i}{\hbar} [\hat{x}, \hat{G}] dx = dx \hat{1},$$

or

$$[\hat{x}, \hat{G}] = i\hbar \hat{1}.$$

Using the relation (3), we get

$$\left[\hat{1} - \frac{i}{\hbar} \hat{G} dx, \hat{p}\right] = 0.$$

Then we have

$$[\hat{G}, \hat{p}] = 0.$$

From these two commutation relations, we conclude that

$$\hat{G} = \hat{p},$$

and

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar} \hat{p} dx.$$

We see that the position operator and the momentum operator \hat{p} obeys the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \hat{1}.$$

which leads to the Heisenberg's principle of uncertainty.

3 Momentum operator \hat{p} in the position basis.

Using the relation

$$\hat{T}(\delta x)|x\rangle = |x + \delta x\rangle, \quad \hat{T}(\delta x) = \hat{1} - \frac{i}{\hbar} \hat{p} \delta x.$$

we get

$$\begin{aligned} \hat{T}(\delta x)|\psi\rangle &= \hat{T}(\delta x) \int dx' |x'\rangle \langle x'|\psi\rangle = \int dx' |x'+\delta x\rangle \langle x'|\psi\rangle \\ &= \int dx' |x'\rangle \langle x'-\delta x|\psi\rangle = \int dx' |x'\rangle \psi(x'-\delta x) \end{aligned}$$

We apply the Taylor expansion:

$$\psi(x'-\delta x) = \psi(x') - \delta x \frac{\partial}{\partial x'} \psi(x').$$

Substitution:

$$\begin{aligned} \hat{T}(\delta x)|\psi\rangle &= \int dx' |x'\rangle \psi(x'-\delta x) \\ &= \int dx' |x'\rangle [\psi(x') - \delta x \frac{\partial}{\partial x'} \psi(x')] \\ &= \int dx' |x'\rangle [\langle x'|\psi\rangle - \delta x \frac{\partial}{\partial x'} \langle x'|\psi\rangle] \\ &= |\psi\rangle - \delta x \int dx' |x'\rangle \frac{\partial}{\partial x'} \langle x'|\psi\rangle \end{aligned}$$

From the definition, we have

$$\hat{T}(\delta x)|\psi\rangle = \left(\hat{1} - \frac{i}{\hbar}\hat{p}\delta x\right)|\psi\rangle.$$

Comparing these two equations, we obtain the relation

$$\hat{p}|\psi\rangle = \frac{\hbar}{i} \int dx' |x'\rangle \frac{\partial}{\partial x'} \langle x'|\psi\rangle,$$

or

$$\begin{aligned} \langle x|\hat{p}|\psi\rangle &= \frac{\hbar}{i} \int dx' \langle x|x'\rangle \frac{\partial}{\partial x'} \langle x'|\psi\rangle \\ &= \frac{\hbar}{i} \int dx' \delta(x-x') \frac{\partial}{\partial x'} \langle x'|\psi\rangle \\ &= \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|\psi\rangle \end{aligned}$$

We obtain a very important formula

$$\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|\psi\rangle.$$

Note that

$$\begin{aligned} \langle \psi|\hat{p}|\psi\rangle &= \int dx \langle \psi|x\rangle \langle x|\hat{p}|\psi\rangle \\ &= \int dx \langle \psi|x\rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|\psi\rangle \\ &= \int dx \langle x|\psi\rangle^* \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|\psi\rangle \end{aligned}$$

These results suggest that in position space the momentum operator takes the form

$$\hat{p} \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial x}.$$

4. Position operator \hat{x} in the momentum basis.

$$\begin{aligned}
\langle p|\hat{x}|\psi\rangle &= \int dx \langle p|x\rangle \langle x|\hat{x}|\psi\rangle \\
&= \int dx x \langle p|x\rangle \langle x|\psi\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dx x e^{-\frac{ipx}{\hbar}} \langle x|\psi\rangle \\
&= i\hbar \frac{\partial}{\partial p} \frac{1}{\sqrt{2\pi\hbar}} \left(\int dx e^{-\frac{ipx}{\hbar}} \langle x|\psi\rangle \right) \\
&= i\hbar \frac{\partial}{\partial p} \int dx \langle p|x\rangle \langle x|\psi\rangle \\
&= i\hbar \frac{\partial}{\partial p} \langle p|\psi\rangle
\end{aligned}$$

Then we have

$$\langle p|\hat{x}|\psi\rangle = i\hbar \frac{\partial}{\partial p} \langle p|\psi\rangle.$$

Using this result, we get

$$\begin{aligned}
\langle \phi|\hat{x}|\psi\rangle &= \int dp \langle \phi|p\rangle \langle p|\hat{x}|\psi\rangle \\
&= \int dp \langle \phi|p\rangle i\hbar \frac{\partial}{\partial p} \langle p|\psi\rangle \\
&= \int dp \langle p|\phi\rangle^* i\hbar \frac{\partial}{\partial p} \langle p|\psi\rangle
\end{aligned}$$

These results suggest that in momentum space the position operator takes the form

$$\hat{x} \rightarrow i\hbar \frac{\partial}{\partial p}.$$

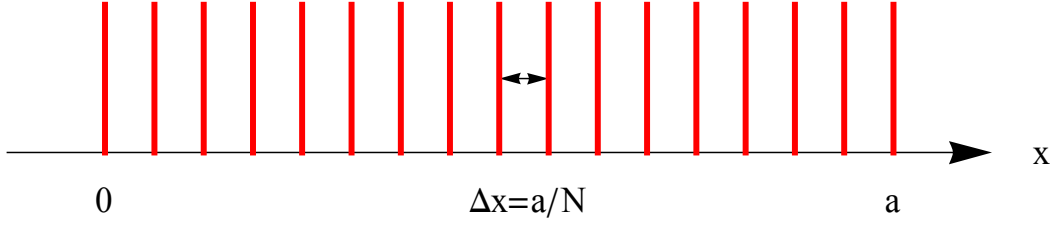
5. The finite translation operator

What is the operator $\hat{T}(a)$ corresponding to a finite translation a ? We find it by the following procedure. We divide the interval into N parts of size $dx = a/N$. As $N \rightarrow \infty$, a/N becomes infinitesimal.

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar} \hat{p} \left(\frac{a}{N} \right).$$

Since a translation by a equals N translations by a/N , we have

$$\hat{T}(a) = \lim_{N \rightarrow \infty} \left[\hat{1} - \frac{i}{\hbar} \hat{p} \left(\frac{a}{N} \right) \right]^N = \exp\left(-\frac{i}{\hbar} \hat{p} a\right).$$



Here we use the formula

$$\lim_{N \rightarrow \infty} \left(1 + \frac{1}{N}\right)^N = e, \quad \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N}\right)^N = e^{-1}.$$

$$\lim_{N \rightarrow \infty} \left[\left(1 - \frac{ax}{N}\right)^{\frac{N}{ax}} \right]^{ax} = \lim_{N \rightarrow \infty} \left(1 - \frac{ax}{N}\right)^N = (e^{-1})^{ax} = e^{-ax}.$$

In summary, we have

$$\hat{T}(a) = \exp\left(-\frac{i}{\hbar} \hat{p} a\right).$$

6. Discussion on the commutation relation

It is interesting to calculate

$$\hat{T}^+(a) \hat{x} \hat{T}(a) = e^{\frac{i}{\hbar} \hat{p} a} \hat{x} e^{-\frac{i}{\hbar} \hat{p} a},$$

by using the Baker-Hausdorff lemma:

$$\exp(\hat{A}x) \hat{B} \exp(-\hat{A}x) = \hat{B} + \frac{x}{1!} [\hat{A}, \hat{B}] + \frac{x^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{x^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots$$

When $x = 1$, we have

$$\exp(\hat{A}) \hat{B} \exp(-\hat{A}) = \hat{B} + \frac{1}{1!} [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots$$

Then we have

$$\hat{T}^+(a) \hat{x} \hat{T}(a) = e^{\frac{i}{\hbar} \hat{p} a} \hat{x} e^{-\frac{i}{\hbar} \hat{p} a} = \hat{x} + \left[\frac{i}{\hbar} \hat{p} a, \hat{x} \right] = \hat{x} + \frac{i}{\hbar} a [\hat{p}, \hat{x}] = \hat{x} + \frac{i}{\hbar} a \frac{\hbar}{i} = \hat{x} + a \hat{1}.$$

So we confirmed that the relation

$$\hat{T}^+(a)\hat{x}\hat{T}(a) = \hat{x} + a\hat{1},$$

holds for any finite translation operator.

7. Invariance of Hamiltonian under the translation

Now we consider the condition for the invariance of Hamiltonian \hat{H} under the translation.

The average value of \hat{H} in the new state $|\psi'\rangle$ is equal to the average value of \hat{H} in the new state $|\psi\rangle$.

$$\langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{H}|\psi\rangle,$$

or

$$\hat{T}^+(dx)\hat{H}\hat{T}(dx) = \hat{H}, \quad \text{or} \quad \hat{H}\hat{T}(dx) = \hat{T}(dx)\hat{H},$$

or

$$\hat{H}\left(\hat{1} - \frac{i}{\hbar}\hat{p}dx\right) = \left(\hat{1} - \frac{i}{\hbar}\hat{p}dx\right)\hat{H}.$$

Then we have

$$[\hat{H}, \hat{p}] = 0.$$

8. ((Sakurai 1-28))

- (a) Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket.

$$[x, F(p_x)]_{classical}.$$

- (b) Let \hat{x} and \hat{p}_x be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$[\hat{x}, \exp(\frac{i\hat{p}_x a}{\hbar})].$$

(c) Using the result obtained in (b), prove that

$$\exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle, \quad \hat{x}|x'\rangle = x'|x'\rangle$$

is an eigenstate of the coordinate operator x , What is the corresponding eigenvalue?

((Solution))

(a)

$$[x, F(p_x)]_{classical} = \frac{\partial x}{\partial x} \frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x} \frac{\partial F(p_x)}{\partial x} = \frac{\partial F(p_x)}{\partial p_x}.$$

(b)

We use the Gottfried's result

$$[\hat{x}, \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)] = i\hbar \frac{\partial}{\partial \hat{p}_x} \exp\left(\frac{i\hat{p}_x a}{\hbar}\right) = -a \exp\left(\frac{i\hat{p}_x a}{\hbar}\right).$$

(c)

$$\hat{x} \exp\left(\frac{i\hat{p}_x a}{\hbar}\right) = \exp\left(\frac{i\hat{p}_x a}{\hbar}\right) \hat{x} - a \exp\left(\frac{i\hat{p}_x a}{\hbar}\right).$$

Then we have

$$\begin{aligned} \hat{x} \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle &= \exp\left(\frac{i\hat{p}_x a}{\hbar}\right) \hat{x}|x'\rangle - a \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle \\ &= (x'-a) \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle \end{aligned}$$

The ket $\exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle$ is the eigenket of \hat{x} with an eigenvalue $(x'-a)$.

$$\exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle = |x'-a\rangle.$$

Therefore $\hat{T}_x(a) = \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)$ is a translation operator.

9. ((Sakurai 1-29))

(a) Gottfried (1966) states that

$$[\hat{x}_i, G(\hat{\mathbf{p}})] = i\hbar \frac{\partial}{\partial \hat{p}_i} G(\hat{\mathbf{p}}), \quad [\hat{p}_i, F(\hat{\mathbf{x}})] = -i\hbar \frac{\partial}{\partial \hat{x}_i} F(\hat{\mathbf{x}})$$

can be easily derived from the fundamental commutation relations for all functions of F and G can be expressed as power series in their arguments. Verify this statement.

(b) Evaluate $[\hat{x}^2, \hat{p}^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{\text{classic}}$.

((Solution))

(a)

(i)

$$\begin{aligned} \langle \mathbf{p} | [\hat{x}_i, G(\hat{\mathbf{p}})] | \alpha \rangle &= [i\hbar \frac{\partial}{\partial p_i} G(\mathbf{p}) - G(\mathbf{p}) i\hbar \frac{\partial}{\partial p_i}] \langle \mathbf{p} | \alpha \rangle \\ &= i\hbar \frac{\partial}{\partial p_i} [G(\mathbf{p}) \langle \mathbf{p} | \alpha \rangle] - i\hbar G(\mathbf{p}) \frac{\partial}{\partial p_i} \langle \mathbf{p} | \alpha \rangle \\ &= i\hbar \left(\frac{\partial}{\partial p_i} G(\mathbf{p}) \right) \langle \mathbf{p} | \alpha \rangle + i\hbar G(\mathbf{p}) \frac{\partial}{\partial p_i} \langle \mathbf{p} | \alpha \rangle - i\hbar G(\mathbf{p}) \frac{\partial}{\partial p_i} \langle \mathbf{p} | \alpha \rangle \\ &= i\hbar \left(\frac{\partial}{\partial p_i} G(\mathbf{p}) \right) \langle \mathbf{p} | \alpha \rangle \\ &= \langle \mathbf{p} | i\hbar \frac{\partial}{\partial \hat{p}_i} G(\hat{\mathbf{p}}) | \alpha \rangle \end{aligned}$$

Thus we have the final result

$$[\hat{x}_i, G(\hat{\mathbf{p}})] = i\hbar \frac{\partial}{\partial \hat{p}_i} G(\hat{\mathbf{p}}).$$

(ii)

$$\begin{aligned}
\langle \mathbf{r} | [\hat{p}_i, F(\hat{\mathbf{r}})] | \alpha \rangle &= \left[\frac{\hbar}{i} \frac{\partial}{\partial x_i} F(\mathbf{r}) - F(\mathbf{r}) \frac{\hbar}{i} \frac{\partial}{\partial x_i} \right] \langle \mathbf{r} | \alpha \rangle \\
&= \frac{\hbar}{i} \frac{\partial}{\partial x_i} [F(\mathbf{r}) \langle \mathbf{r} | \alpha \rangle] - \frac{\hbar}{i} F(\mathbf{r}) \frac{\partial}{\partial x_i} \langle \mathbf{r} | \alpha \rangle \\
&= \frac{\hbar}{i} \left(\frac{\partial}{\partial x_i} F(\mathbf{r}) \right) \langle \mathbf{r} | \alpha \rangle + \frac{\hbar}{i} F(\mathbf{r}) \frac{\partial}{\partial x_i} \langle \mathbf{r} | \alpha \rangle - \frac{\hbar}{i} F(\mathbf{r}) \frac{\partial}{\partial x_i} \langle \mathbf{r} | \alpha \rangle \\
&= \frac{\hbar}{i} \left(\frac{\partial}{\partial x_i} F(\mathbf{r}) \right) \langle \mathbf{r} | \alpha \rangle \\
&= \langle \mathbf{r} | \frac{\hbar}{i} \frac{\partial}{\partial \hat{x}_i} F(\hat{\mathbf{r}}) | \alpha \rangle
\end{aligned}$$

or

$$[\hat{p}_i, F(\hat{\mathbf{r}})] = \frac{\hbar}{i} \frac{\partial}{\partial \hat{x}_i} F(\hat{\mathbf{r}}).$$

(b)

$$\begin{aligned}
[\hat{x}^2, \hat{p}^2] &= \hat{x}[\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2]\hat{x} \\
&= \hat{x}i\hbar \frac{\partial}{\partial \hat{p}} \hat{p}^2 + i\hbar \left(\frac{\partial}{\partial \hat{p}} \hat{p}^2 \right) \hat{x} \\
&= 2i\hbar(\hat{x}\hat{p} + \hat{p}\hat{x})
\end{aligned}$$

The classical Poisson bracket is defined by

$$\begin{aligned}
[x^2, p^2]_{\text{classic}} &= \frac{\partial x^2}{\partial x} \frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p} \frac{\partial p^2}{\partial x} \\
&= 4xp \\
&= 2(xp + px)
\end{aligned}$$

10. ((Sakurai 1-30))

The translation operator for a finite (spatial) displacement is given by

$$\hat{T}(\mathbf{l}) = \exp\left(-\frac{i\hat{\mathbf{p}} \cdot \mathbf{l}}{\hbar}\right),$$

where $\hat{\mathbf{p}}$ is the momentum operator.

(a) Evaluate

$$[\hat{x}, \hat{T}(\mathbf{l})].$$

(b) Using (a) (or otherwise), demonstrate how the expectation value $\langle x \rangle$ changes under translation.

((Solution))

(a)

The translation operator is defined by

$$\hat{T}(\mathbf{l}) = \exp\left(-\frac{i\hat{\mathbf{p}} \cdot \mathbf{l}}{\hbar}\right),$$

$$[\hat{x}_i, \hat{T}(\mathbf{l})] = i\hbar \frac{\partial}{\partial \hat{p}_i} \hat{T}(\mathbf{l}) = l_i \exp\left(-\frac{i\hat{\mathbf{p}} \cdot \mathbf{l}}{\hbar}\right) = l_i \hat{T}(\mathbf{l})$$

or

$$[\hat{\mathbf{r}}, \hat{T}(\mathbf{l})] = \mathbf{l} \hat{T}(\mathbf{l}).$$

(b)

$$|\alpha'\rangle = \hat{T}(\mathbf{l})|\alpha\rangle,$$

$$\begin{aligned} \langle \alpha' | \hat{\mathbf{r}} | \alpha' \rangle &= \langle \alpha | \hat{T}^\dagger(\mathbf{l}) \hat{\mathbf{r}} \hat{T}(\mathbf{l}) | \alpha \rangle \\ &= \langle \alpha | \hat{T}^\dagger(\mathbf{l}) [\hat{\mathbf{r}} + \mathbf{l}] \hat{T}(\mathbf{l}) | \alpha \rangle \\ &= \langle \alpha | \hat{\mathbf{r}} + \mathbf{l} | \alpha \rangle \end{aligned}$$

or

$$\langle \alpha' | \hat{\mathbf{r}} | \alpha' \rangle = \langle \alpha | \hat{\mathbf{r}} | \alpha \rangle + \mathbf{l}.$$

11. ((Sakurai 1-31))

Prove

$$\langle \mathbf{r} \rangle \rightarrow \langle \mathbf{r} \rangle + d\mathbf{r}', \quad \langle \mathbf{p} \rangle \rightarrow \langle \mathbf{p} \rangle.$$

under infinitesimal translation.

((Solution))

We use the commutation relations

$$[\hat{\mathbf{r}}, \hat{T}(d\mathbf{r})] = d\mathbf{r}\hat{T}(d\mathbf{r})$$

and

$$[\hat{\mathbf{p}}, \hat{T}(d\mathbf{r})] = 0$$

We have

$$\begin{aligned} \langle \alpha | \hat{T}^+(d\mathbf{r}) \hat{\mathbf{r}} \hat{T}(d\mathbf{r}) | \alpha \rangle &= \langle \alpha | \hat{T}^+(d\mathbf{r}) [\hat{T}(d\mathbf{r}) \hat{\mathbf{r}} + d\mathbf{r} \hat{T}(d\mathbf{r})] | \alpha \rangle \\ &= \langle \alpha | \hat{\mathbf{r}} + d\mathbf{r} | \alpha \rangle \end{aligned}$$

or

$$\langle \alpha' | \hat{\mathbf{r}} | \alpha' \rangle = \langle \alpha | \hat{\mathbf{r}} | \alpha \rangle + d\mathbf{r}$$

Similarly

$$\langle \alpha | \hat{T}^+(d\mathbf{r}) \hat{\mathbf{p}} \hat{T}(d\mathbf{r}) | \alpha \rangle = \langle \alpha | \hat{T}^+(d\mathbf{r}) \hat{T}(d\mathbf{r}) \hat{\mathbf{p}} | \alpha \rangle = \langle \alpha | \hat{\mathbf{p}} | \alpha \rangle$$

12. ((Sakurai 1-33))

(a) Prove the following:

(i)

$$\langle p' | \hat{x} | \alpha \rangle = i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle.$$

(ii)

$$\langle \beta | \hat{x} | \alpha \rangle = \int dp \langle p' | \beta \rangle^* i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle.$$

(b) What is the physical significance of

$$\exp\left(\frac{i\hat{x}p_0}{\hbar}\right),$$

where \hat{x} is the position operator and p_0 is some number with the dimension of momentum? Justify your answer.

((Solution))

(a)

(i)

$$\begin{aligned} \langle p' | \hat{x} | \alpha \rangle &= \int dx' \langle p' | x' \rangle \langle x' | \hat{x} | \alpha \rangle \\ &= \int dx' x' \langle p' | x' \rangle \langle x' | \alpha \rangle \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int dx' x' e^{\frac{ip'x'}{\hbar}} \langle x' | \alpha \rangle \\ &= i\hbar \frac{\partial}{\partial p'} \frac{1}{\sqrt{2\pi\hbar}} \left(\int dx' e^{\frac{ip'x'}{\hbar}} \langle x' | \alpha \rangle \right) \\ &= i\hbar \frac{\partial}{\partial p'} \int dx' \langle p' | x' \rangle \langle x' | \alpha \rangle \\ &= i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle \end{aligned}$$

(ii)

$$\begin{aligned} \langle \beta | \hat{x} | \alpha \rangle &= \int dp' \langle \beta | p' \rangle \langle p' | \hat{x} | \alpha \rangle \\ &= \int dp' \langle \beta | p' \rangle \langle p' | \hat{x} | \alpha \rangle \\ &= \int dp' \langle \beta | p' \rangle i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle \end{aligned}$$

(b)

$$\begin{aligned}\hat{p} \exp\left(\frac{ip_0\hat{x}}{\hbar}\right)|p'\rangle &= \left\{ \left[\hat{p}, \exp\left(\frac{ip_0\hat{x}}{\hbar}\right) \right] + \exp\left(\frac{ip_0\hat{x}}{\hbar}\right) \hat{p} \right\} |p'\rangle \\ &= \{p_0 \exp\left(\frac{ip_0\hat{x}}{\hbar}\right) + \exp\left(\frac{ip_0\hat{x}}{\hbar}\right) p'\} |p'\rangle\end{aligned}$$

or

$$\hat{p} \exp\left(\frac{ip_0\hat{x}}{\hbar}\right)|p'\rangle = (p_0 + p') \exp\left(\frac{ip_0\hat{x}}{\hbar}\right)|p'\rangle.$$

Therefore $\exp\left(\frac{ip_0\hat{x}}{\hbar}\right)|p'\rangle$ is the eigenket of \hat{p} with an eigenvalue of $(p' + p_0)$.

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