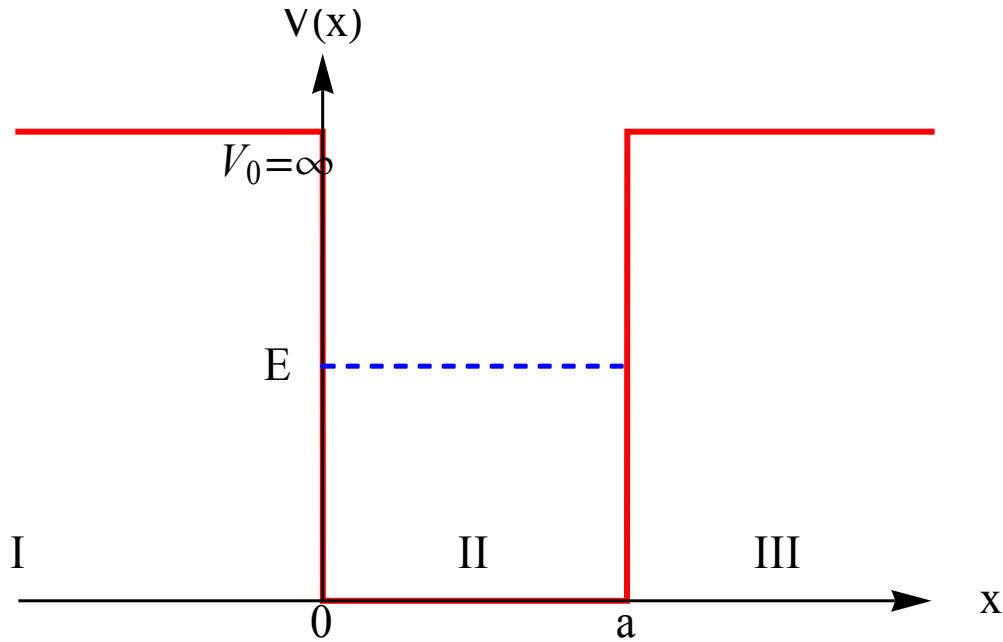


Quantum box with infinite well potential
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(Date: December 02, 2013)

1. 1D one-dimensional well potential



$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$$H\varphi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \varphi(x) = E\varphi(x) = \frac{\hbar^2 k^2}{2m} \varphi(x)$$

The solution of this equation is

$$\varphi(x) = A \sin(kx) + B \cos(kx)$$

where

$$E = \frac{\hbar^2 k^2}{2m}$$

Using the boundary condition:

$$\varphi(x=0) = \varphi(x=a) = 0$$

we have

$$B = 0 \text{ and } A \neq 0.$$

$$\sin(ka) = 0$$

$$ka = n\pi \quad (n = 1, 2, \dots)$$

Note that $n = 0$ is not included in our solution because the corresponding wave function becomes zero. The wave function is given by

$$\varphi_n(x) = \langle x | \varphi_n \rangle = A_n \sin\left(\frac{n\pi x}{a}\right) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

with

$$E_n = \frac{\hbar^2}{2m} \left(\frac{n\pi}{a}\right)^2$$

((Normalization))

$$1 = \int_0^a A_n^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2} A_n^2$$

2. Mathematica

$$|\varphi_n(x)|^2 = \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right]^2 = \frac{2}{a} \sin^2\left(\frac{n\pi x}{a}\right)$$

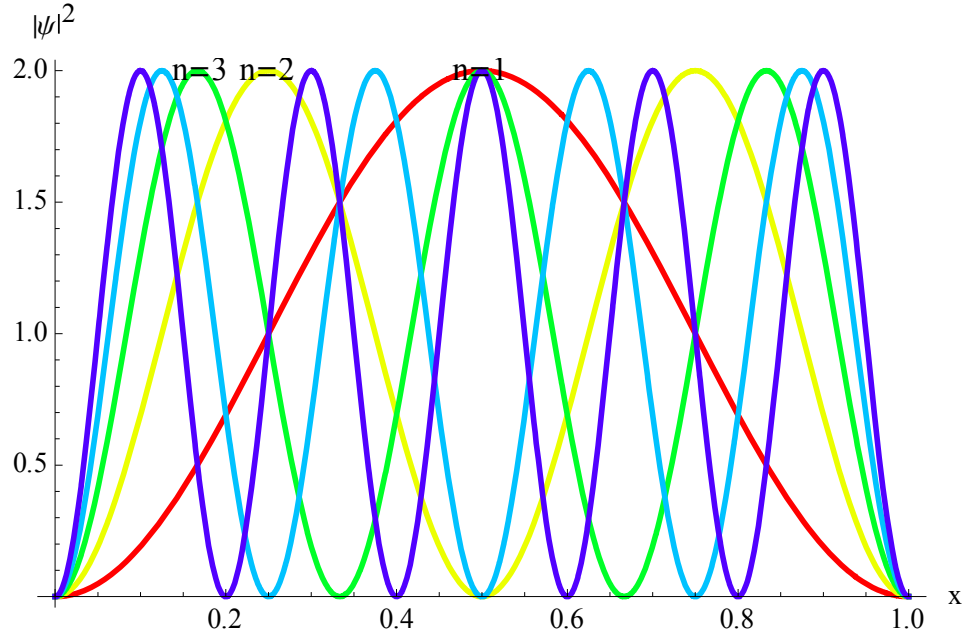


Fig. Plot of $|\varphi_n(x)|^2$ with $a = 1$, as a function of x . $n = 1$ (red), 2 (yellow), 3 (green), 4 (blue), and 5 (dark blue). There are n peaks for the state $|n\rangle$.

The expectation values and uncertainty

$$\langle x^m \rangle = \int_0^a \varphi_n^*(x) x^m \varphi_n(x) dx = \int_0^a \frac{2}{a} x^m \sin^n\left(\frac{n\pi x}{a}\right) dx$$

$$\langle p^m \rangle = \int_0^a \varphi_n^*(x) \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right)^m \varphi_n(x) dx$$

Since

$$\langle x \rangle = 0, \quad \langle x^2 \rangle = \frac{a^2}{6} \left(2 - \frac{3}{n^2 \pi^2} \right)$$

$$\langle p \rangle = 0, \quad \langle p^2 \rangle = \frac{n^2 \pi^2 \hbar^2}{a^2}$$

we have

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a \sqrt{\frac{1}{6} \left(2 - \frac{3}{n^2 \pi^2} \right)}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{n\pi\hbar}{a}$$

Then

$$\Delta p \Delta x = n\pi\hbar \sqrt{\frac{1}{6} \left(2 - \frac{3}{n^2\pi^2} \right)}$$

When $n = 1$,

$$\Delta p \Delta x = 1.67029 \hbar > 0.5 \hbar$$

((Mathematica))

$$\text{Clear["Global`*"]; } \phi[n_, x_] := \sqrt{\frac{2}{a}} \sin\left[\frac{n \pi x}{a}\right];$$

$$\text{xav1} = \int_0^a x \phi[n, x]^2 dx // \text{Simplify}[\#, n \in \text{Integers}] \&$$

$$\frac{a}{2}$$

$$\text{xav2} = \int_0^a x^2 \phi[n, x]^2 dx // \text{Simplify}[\#, n \in \text{Integers}] \&$$

$$\frac{1}{6} a^2 \left(2 - \frac{3}{n^2 \pi^2}\right)$$

$$\text{pav1} = \frac{\hbar}{i} \int_0^a \phi[n, x] D[\phi[n, x], x] dx //$$

$$\text{Simplify}[\#, n \in \text{Integers}] \&$$

$$0$$

$$\text{pav2} = \left(\frac{\hbar}{i}\right)^2 \int_0^a \phi[n, x] D[\phi[n, x], \{x, 2\}] dx //$$

$$\text{Simplify}[\#, n \in \text{Integers}] \&$$

$$\frac{n^2 \pi^2 \hbar^2}{a^2}$$

$$\Delta x \Delta p = \sqrt{\text{xav2}} \sqrt{\text{pav2}} // \text{Simplify}[\#, \{\hbar > 0, a > 0\}] \&$$

$$\frac{\sqrt{n^2} \sqrt{-\frac{3}{n^2} + 2 \pi^2} \hbar}{\sqrt{6}}$$

$$\frac{\Delta x \Delta p}{\hbar} /. n \rightarrow 1 // N$$

1.67029

3. Exercise: Townsend 6-16 problem

A particle of mass m is in lowest energy (ground) state of the infinite potential energy well

$$V(x) = 0 \text{ for } 0 < x < L \text{ and } \infty \text{ elsewhere.}$$

At time $t = 0$, the wall located at $x = L$ is suddenly pulled back to a position at $x = 2L$. This change occurs so rapidly that instantaneously the wave function does not change.

- (a) Calculate the probability that a measurement of the energy will yield the ground-state energy of the new well. What is the probability that a measurement of the energy will yield the first excited energy of the new well?
- (b) Describe the procedure you would use to determine the time development of the system. Is the system in a stationary state?

((Solution))

The old wave function of the ground state is given by

$$\varphi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \quad \text{only for } 0 < x < a \quad (0 \text{ otherwise}).$$

The new wave function is given by

$$\psi_{new}^{(n)}(x) = \sqrt{\frac{2}{2a}} \sin\left(\frac{n\pi x}{2a}\right) = \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right)$$

with the energy of

$$E_{new}^{(n)} = \frac{\hbar^2}{2m} \left(\frac{\pi n}{2a}\right)^2$$

(a)

$$\varphi_1(x) = \sum_n c_n \psi_{new}^{(n)}(x)$$

$$\begin{aligned} c_n &= \int_0^{2a} \psi_{new}^{(n)*}(x) \varphi_1(x) dx \\ &= \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) dx \\ &= \frac{4\sqrt{2}}{\pi(4-n^2)} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Note that

$$c_2 = \frac{1}{\sqrt{2}}$$

$$|c_1|^2 = \frac{32}{9\pi^2} = 0.360253, \quad |c_2|^2 = \frac{1}{2} = 0.5$$

(b)

The system is not stationary since $|\psi(t=0)\rangle$ is not an eigenstate of the new Hamiltonian \hat{H}_{new} , but is a superposition of the eigenstates $|\psi_{new}^{(n)}\rangle$ with various kinds of n .

$$|\psi(t=0)\rangle = |\varphi_1\rangle = \sum_n c_n |\psi_{new}^{(n)}\rangle$$

$$\begin{aligned} |\psi(t)\rangle &= \exp\left(-\frac{i}{\hbar}\hat{H}_{new}t\right)|\psi(t=0)\rangle \\ &= \sum_n c_n \exp\left(-\frac{i}{\hbar}\hat{H}_{new}t\right)|\psi_{new}^{(n)}\rangle \\ &= \sum_n c_n \exp\left(-\frac{i}{\hbar}E_{new}^{(n)}t\right)|\psi_{new}^{(n)}\rangle \end{aligned}$$

or

$$\psi(x,t) = \langle x|\psi(t)\rangle = \sum_n c_n \exp\left(-\frac{i}{\hbar}E_{new}^{(n)}t\right)\psi_{new}^{(n)}(x)$$

where c_n are determined as in (a)

$$c_n = \frac{4\sqrt{2}}{\pi(4-n^2)} \sin\left(\frac{n\pi}{2}\right)$$

and

$$E_{new}^{(n)} = \frac{\hbar^2}{2m} \left(\frac{\pi n}{2a}\right)^2,$$

$$\psi_{new}^{(n)}(x) = \sqrt{\frac{2}{2a}} \sin\left(\frac{n\pi x}{2a}\right) = \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right)$$

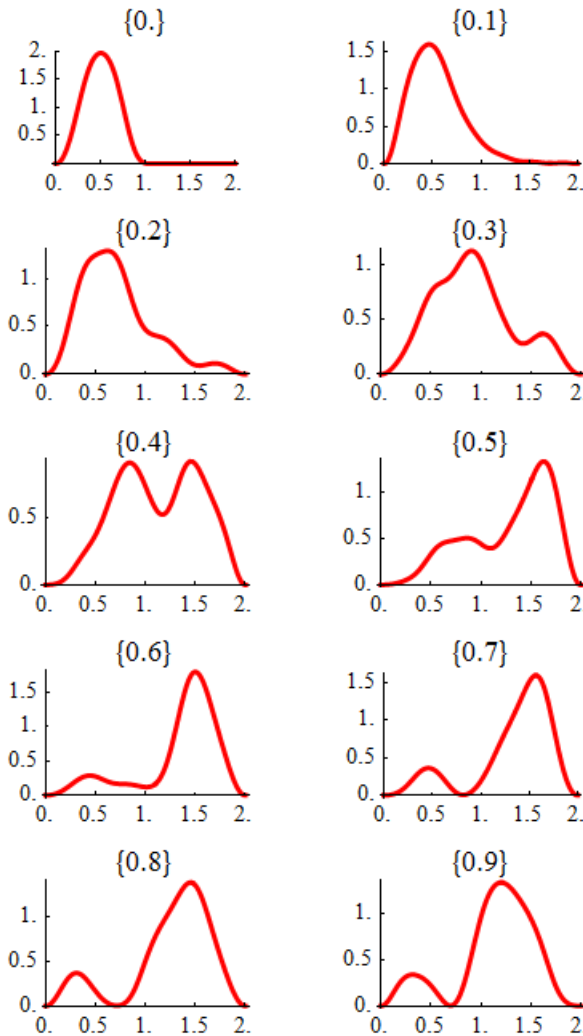
Then we get

$$\begin{aligned}
|\psi(x,t)|^2 &= \sum_m c_m^* \exp\left(\frac{i}{\hbar} E_{new}^{(m)} t\right) \psi_{new}^{(m)*}(x) \sum_n c_n \exp\left(-\frac{i}{\hbar} E_{new}^{(n)} t\right) \psi_{new}^{(n)}(x) \\
&= \sum_{n,m} c_m^* c_n \psi_{new}^{(m)*}(x) \psi_{new}^{(n)}(x) \exp\left[-\frac{i}{\hbar} (E_{new}^{(n)} - E_{new}^{(m)}) t\right]
\end{aligned}$$

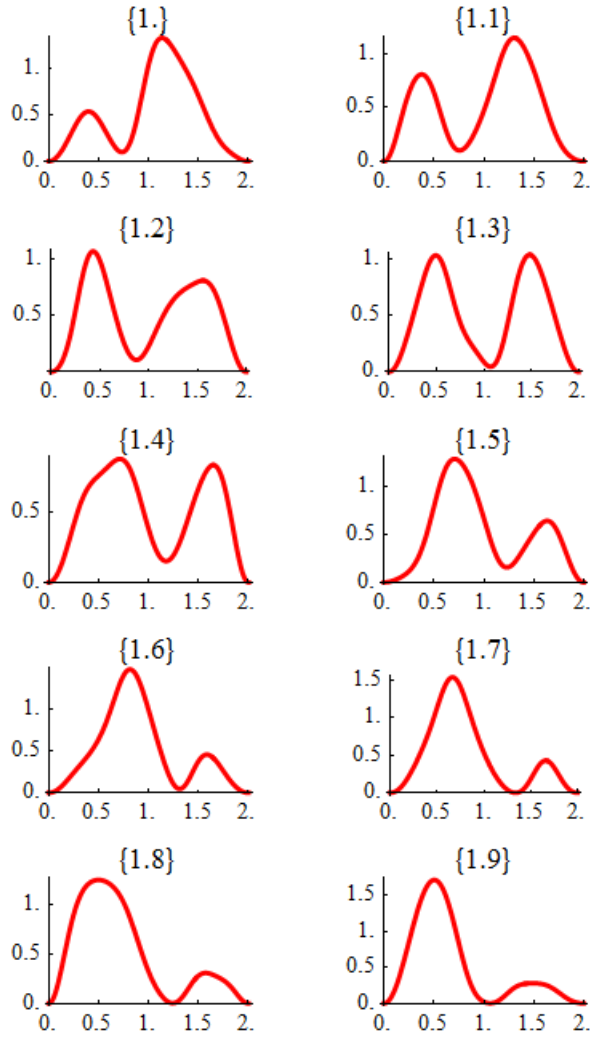
((Mathematica))

We use $m = \hbar = 1$. $a = 1$. Red (At $t = 0$). The Plot of $|\psi(x,t)|^2$ as a function of x ($0 < x < 2a$), where t is changed as parameter; $t = 0 - 3$ with $\Delta t = 0.1$. The summation over n ($n = 1 - 10$).

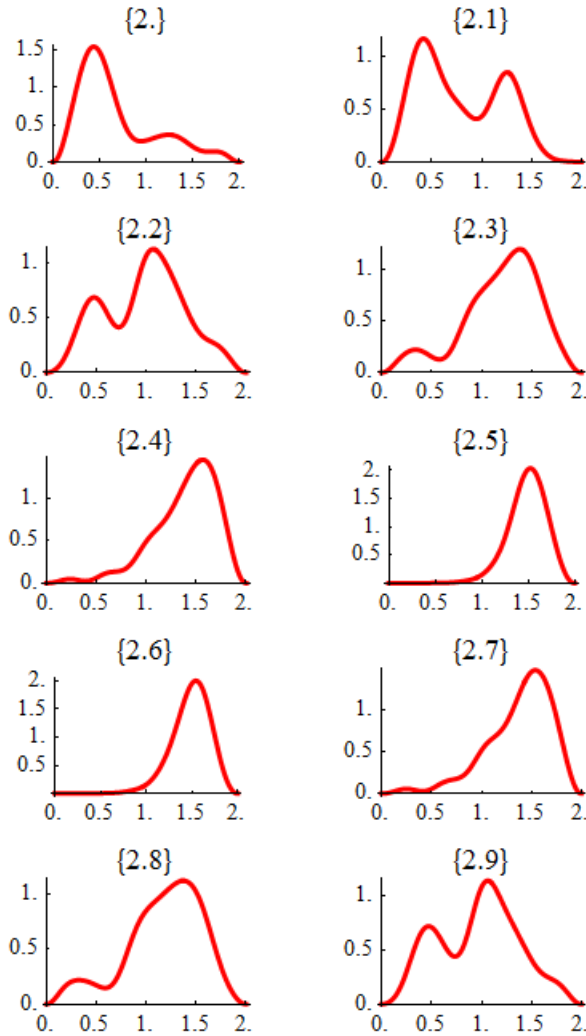
(a) $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$



(b) $t = 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9$



(c) $t = 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9$



4. 2D well potential

Next we consider a particle in a 2D well potential

The potential:

$$V(x,y) = 0 \text{ for } 0 \leq x \leq a \text{ and } 0 \leq y \leq a. \quad V(x,y) = \infty \text{ otherwise.}$$

$$H\varphi(x,y) = -\frac{\hbar^2}{2m} \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) \varphi(x,y) = E\varphi(x,y) = \frac{\hbar^2 k^2}{2m} \varphi(x,y)$$

$$E = \frac{\hbar^2}{2m} (k_x^2 + k_y^2)$$

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) \varphi(x,y) = -(k_x^2 + k_y^2) \varphi(x,y)$$

We use the method of the separation variables. Suppose that

$$\varphi(x, y) = X(x)Y(y)$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -(k_x^2 + k_y^2)$$

We assume that

$$X''(x) = -k_x^2 X(x)$$

$$Y''(y) = -k_y^2 Y(y)$$

Using the boundary condition

$$X(x=0) = X(x=a) = 0$$

and

$$Y(y=0) = Y(y=a) = 0$$

Then we have

$$\varphi_{n_x, n_y}(x, y) = \left(\sqrt{\frac{2}{a}} \right)^2 \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right)$$

5. Mathematica

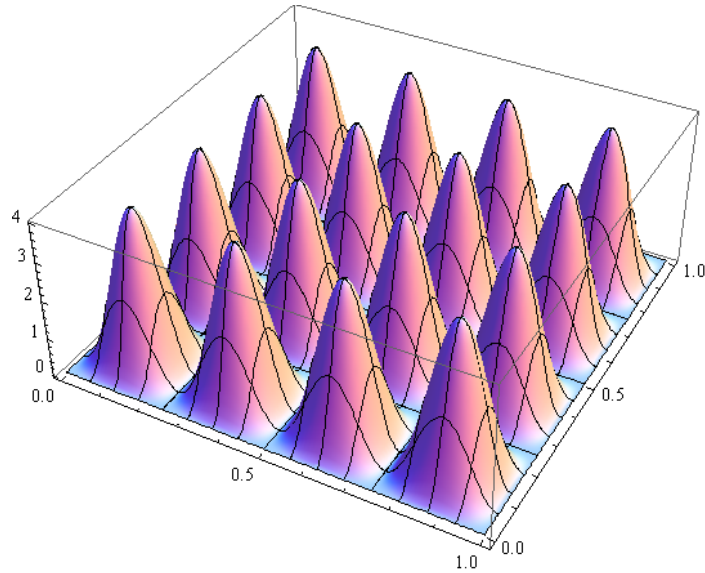
A particle in a two dimensional box

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Clear["Global`*"];
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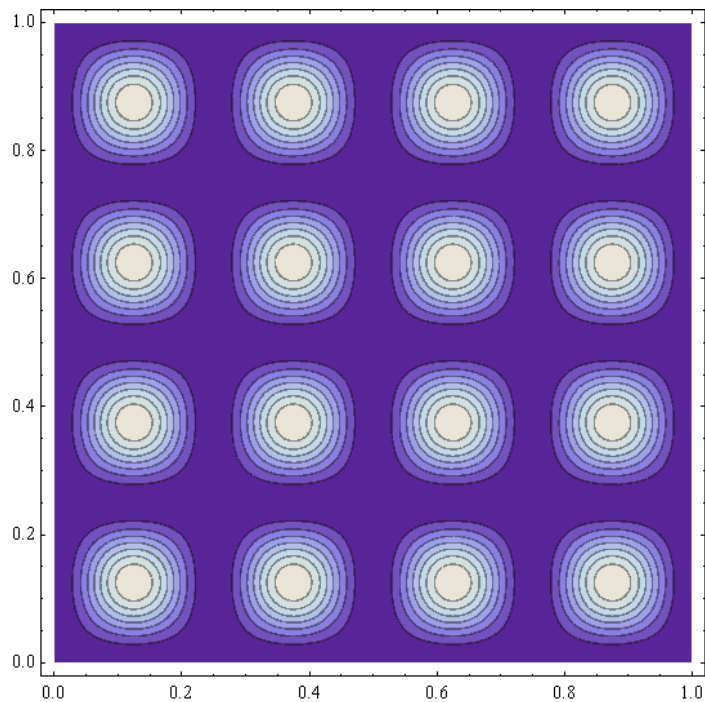
$$\psi = \sqrt{\frac{2}{a}} \sqrt{\frac{2}{b}} \sin\left[\frac{n\pi x}{a}\right] \sin\left[\frac{m\pi y}{b}\right];$$

```
prb =  $\psi^2$  /. {a -> 1, b -> 1};
```

```
pl3D1 = Plot3D[prb /. {n -> 4, m -> 4}, {x, 0, 1}, {y, 0, 1},  
PlotPoints -> 100]
```



```
cont1 = ContourPlot[prb /. {n -> 4, m -> 4}, {x, 0, 1},  
{y, 0, 1}, PlotPoints -> 100]
```



6. Standing wave solutions with a fixed boundary condition

We consider a free particle inside a box with length L_x , L_y , L_z along the x , y , and z axes, respectively. The Schrödinger equation of the system is given by

$$H\psi(x, y, z) = -\frac{\hbar^2}{2m}\nabla^2\psi(x, y, z) = E\psi(x, y, z)$$

under the boundary condition;

$$\begin{aligned}\psi(x = L_x, y, z) &= \psi(x = 0, y, z) = 0 \\ \psi(x, y = L_y, z) &= \psi(x, y = 0, z) = 0 \\ \psi(x, y, z = L_z) &= \psi(x, y, z = 0) = 0\end{aligned}$$

We use the method of separation variables. We assume that

$$\psi(x, y, z) = X(x)Y(y)Z(z)$$

with

$$X(0) = X(L_x) = 0, \quad Y(0) = Y(L_y) = 0, \quad Z(0) = Z(L_z) = 0$$

The substitution of the solution into the Schrödinger equation yields

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{2mE}{\hbar^2}$$

We assume that

$$\frac{X''(x)}{X(x)} = -k_x^2, \quad \frac{Y''(y)}{Y(y)} = -k_y^2, \quad \frac{Z''(z)}{Z(z)} = -k_z^2$$

The solution of these differential equations can be obtained as a standing wave solution,

$$X(x) = \sin(k_x x), \quad Y(y) = \sin(k_y y), \quad Z(z) = \sin(k_z z)$$

under the boundary conditions, where k_x , k_y , and k_z are constants. The resulting wave function is

$$\psi(x, y, z) = A \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

The condition that $\psi = 0$ at $x = L_x$ requires that

$$k_x = \frac{n_x \pi}{L_x}.$$

The values for the k_x , k_y , and k_z are

$$k_x = \frac{n_x \pi}{L_x}, \quad k_y = \frac{n_y \pi}{L_y}, \quad k_z = \frac{n_z \pi}{L_z}$$

where n_x , n_y , and n_z are positive integers.

((**Mathematica**)) ContourPlot3D

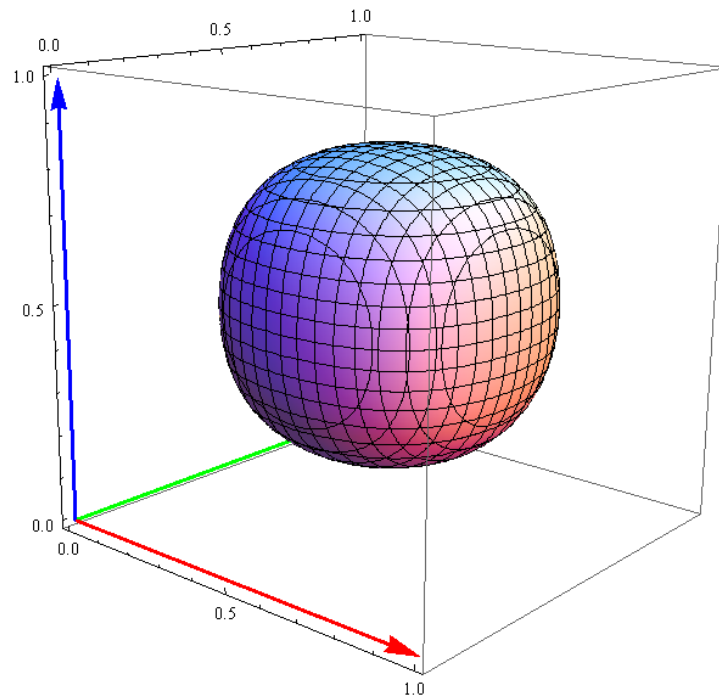


Fig. ContourPlot3D of $\sin^2(k_x x) \sin^2(k_y y) \sin^2(k_z z) = \text{const}$ in the 3D real space.

$$k_x x = \frac{n_x \pi}{L} x, \quad k_y y = \frac{n_y \pi}{L} y, \quad k_z z = \frac{n_z \pi}{L} z. \quad L = 1 \text{ for simplicity. } n_x = 1, n_y = 1, n_z = 1$$

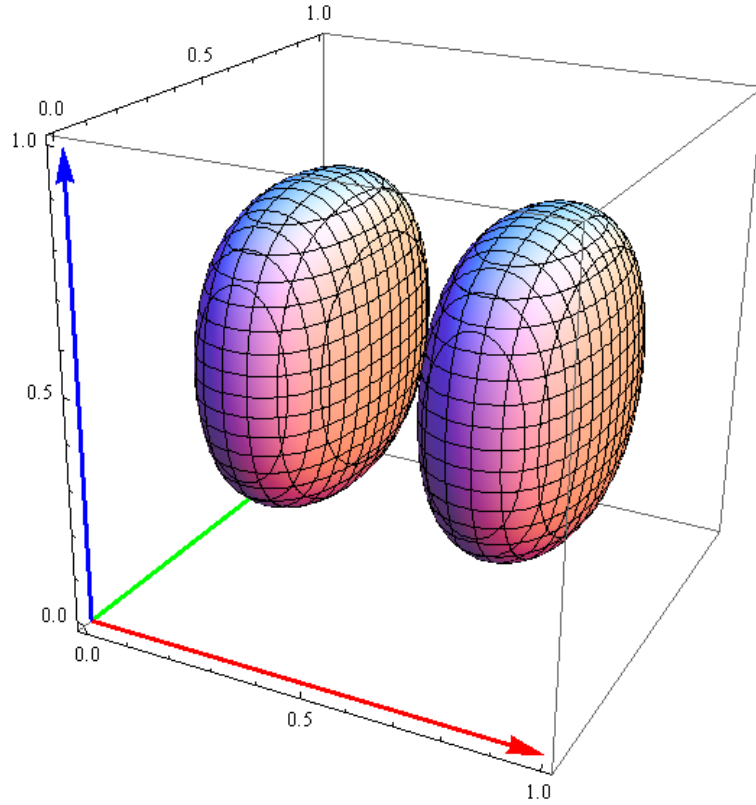


Fig. ContourPlot3D of $\sin^2(k_x x)\sin^2(k_y y)\sin^2(k_z z) = \text{const}$ in the 3D real space.

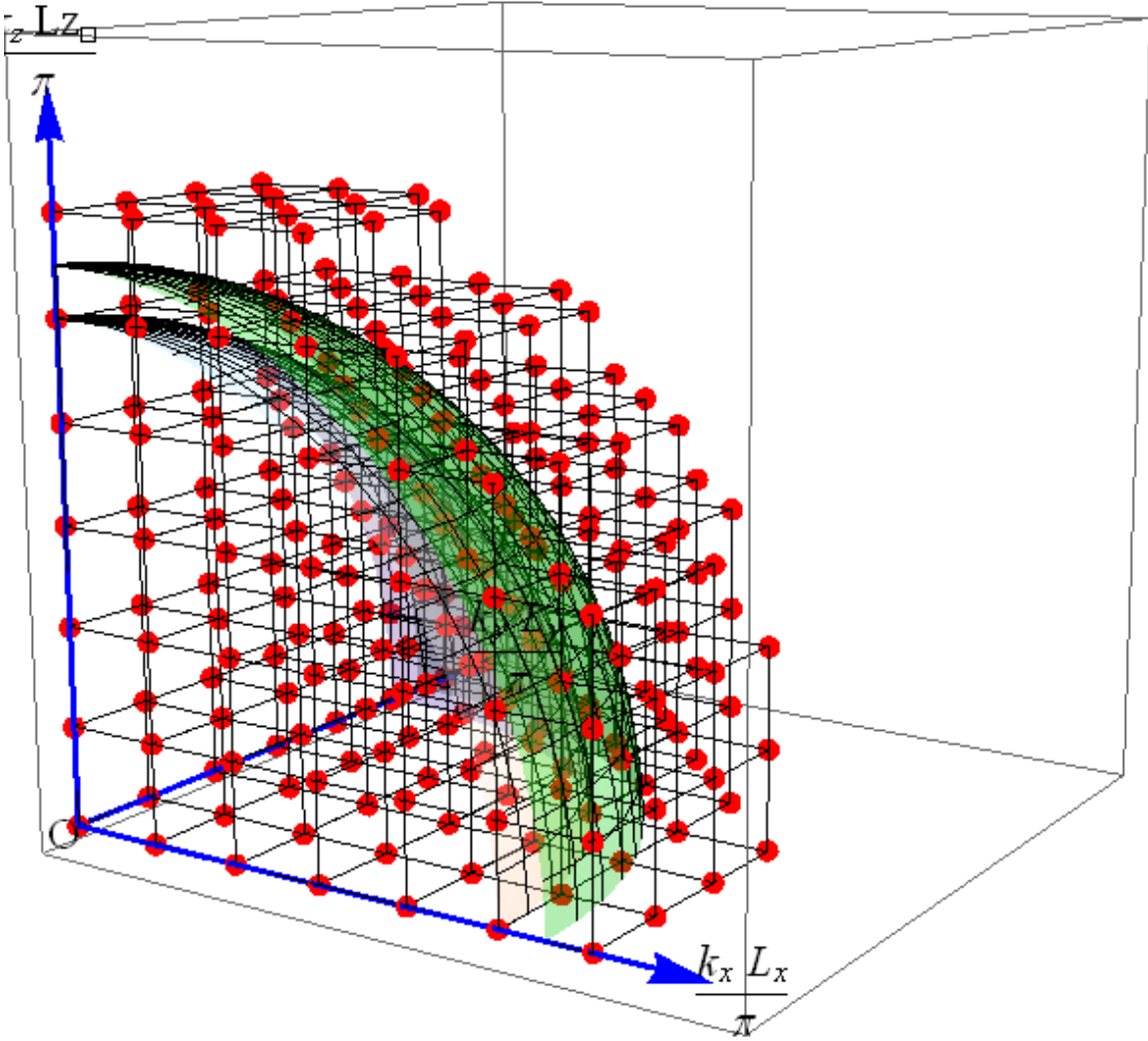
$$k_x x = \frac{n_x \pi}{L} x \cdot k_y y = \frac{n_y \pi}{L} y \cdot k_z z = \frac{n_z \pi}{L} z \cdot L = 1 \text{ for simplicity. } n_x = 2, n_y = 1, n_z = 1$$

((Density of states))

$$\begin{aligned} E(k_x, k_y, k_z) &= \varepsilon = \frac{\hbar^2}{2m} k^2 = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2) \\ &= \frac{\pi^2 \hbar^2}{2m} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right) \end{aligned}$$

There is one state per volume of the \mathbf{k} -space;

$$\frac{\pi}{L_x} \frac{\pi}{L_y} \frac{\pi}{L_z}$$



In the region of $k - k + dk$, the number of states is

$$\begin{aligned}
 D(\varepsilon)d\varepsilon &= 2 \frac{1}{8} \frac{4\pi k^2 dk}{\pi^3} \\
 &\quad \frac{L_x L_y L_z}{L_x L_y L_z} \\
 &= 2 \frac{V}{(2\pi)^3} 4\pi k^2 dk \\
 &= \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon} d\varepsilon
 \end{aligned}$$

where the factor 2 comes from the two allowed state $|+\rangle$ and $|-\rangle$ for the spin quantum number ($S = 1/2$); fermions such as electron. The density of state $D(\varepsilon)$ is obtained as

The total particle number N and total energy E can be described by

$$N = \int_0^{\varepsilon_F} D(\varepsilon) d\varepsilon = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\varepsilon_F} \sqrt{\varepsilon} d\varepsilon = \frac{2}{3} \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \varepsilon_F^{3/2}$$

and

$$E = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon) d\varepsilon = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon = \frac{2}{5} \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \varepsilon_F^{5/2}.$$

Then we have

$$\frac{E}{N} = \frac{\frac{2}{5} \varepsilon_F^{5/2}}{\frac{2}{3} \varepsilon_F^{3/2}} = \frac{3}{5} \varepsilon_F$$

((Note)) Fermi-Dirac distribution function

The Fermi-Dirac distribution gives the probability that an orbital at energy ε will be occupied in an ideal gas in thermal equilibrium

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}, \quad (12)$$

where μ is the chemical potential and $\beta = 1/(k_B T)$.

- (i) $\lim_{T \rightarrow 0} \mu = \varepsilon_F$.
- (ii) $f(\varepsilon) = 1/2$ at $\varepsilon = \mu$.
- (iii) For $\varepsilon - \mu \gg k_B T$, $f(\varepsilon)$ is approximated by $f(\varepsilon) = e^{-\beta(\varepsilon-\mu)}$. This limit is called the Boltzman or Maxwell distribution.
- (iv) For $k_B T \ll \varepsilon_F$, the derivative $-df(\varepsilon)/d\varepsilon$ corresponds to a Dirac delta function having a sharp positive peak at $\varepsilon = \mu$.

7. Plane wave solution with a periodic boundary condition

A. Energy level in 1D system

We consider a free electron gas in 1D system. The Schrödinger equation is given by

$$H\psi_k(x) = \frac{p^2}{2m} \psi_k(x) = -\frac{\hbar^2}{2m} \frac{d^2\psi_k(x)}{dx^2} = \varepsilon_k \psi_k(x), \quad (1)$$

where

$$p = \frac{\hbar}{i} \frac{d}{dx},$$

and ε_k is the energy of the electron in the orbital.

The orbital is defined as a solution of the wave equation for a system of only one electron: $\langle\langle$ one-electron problem $\rangle\rangle$.

Using a periodic boundary condition: $\psi_k(x+L) = \psi_k(x)$, we have the plane-wave solution

$$\psi_k(x) \sim e^{ikx}, \quad (2)$$

with

$$\varepsilon_k = \frac{\hbar^2}{2m} k^2 = \frac{\hbar^2}{2m} \left(\frac{2\pi}{L} n \right)^2,$$

$$e^{ikL} = 1 \text{ or } k = \frac{2\pi}{L} n,$$

where $n = 0, \pm 1, \pm 2, \dots$, and L is the size of the system.

B. Energy level in 3D system

We consider the Schrödinger equation of an electron confined to a cube of edge L .

$$H\psi_{\mathbf{k}} = \frac{\mathbf{p}^2}{2m} \psi_{\mathbf{k}} = -\frac{\hbar^2}{2m} \nabla^2 \psi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} \psi_{\mathbf{k}}. \quad (3)$$

It is convenient to introduce wavefunctions that satisfy periodic boundary conditions.

Boundary condition (Born-von Karman boundary conditions).

$$\psi_{\mathbf{k}}(x+L, y, z) = \psi_{\mathbf{k}}(x, y, z),$$

$$\psi_{\mathbf{k}}(x, y+L, z) = \psi_{\mathbf{k}}(x, y, z),$$

$$\psi_{\mathbf{k}}(x, y, z+L) = \psi_{\mathbf{k}}(x, y, z).$$

The wavefunctions are of the form of a traveling plane wave.

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (4)$$

with

$$k_x = (2\pi/L) n_x, (n_x = 0, \pm 1, \pm 2, \pm 3, \dots),$$

$$k_y = (2\pi/L) n_y, (n_y = 0, \pm 1, \pm 2, \pm 3, \dots),$$

$$k_z = (2\pi/L) n_z, (n_z = 0, \pm 1, \pm 2, \pm 3, \dots).$$

The components of the wavevector \mathbf{k} are the quantum numbers, along with the quantum number m_s of the spin direction. The energy eigenvalue is

$$\varepsilon(\mathbf{k}) = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2}{2m}\mathbf{k}^2. \quad (5)$$

Here

$$\mathbf{p}\psi_{\mathbf{k}}(\mathbf{r}) = \frac{\hbar}{i}\nabla_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{r}) = \hbar\mathbf{k}\psi_{\mathbf{k}}(\mathbf{r}). \quad (6)$$

So that the plane wave function $\psi_{\mathbf{k}}(\mathbf{r})$ is an eigenfunction of \mathbf{p} with the eigenvalue $\hbar\mathbf{k}$. The ground state of a system of N electrons, the occupied orbitals are represented as a point inside a sphere in \mathbf{k} -space.

Because we assume that the electrons are noninteracting, we can build up the N -electron ground state by placing electrons into the allowed one-electron levels we have just found.

((The Pauli's exclusion principle))

The one-electron levels are specified by the wavevectors \mathbf{k} and by the projection of the electron's spin along an arbitrary axis, which can take either of the two values $\pm\hbar/2$. Therefore associated with each allowed wave vector \mathbf{k} are two levels:

$$|\mathbf{k}, \uparrow\rangle, |\mathbf{k}, \downarrow\rangle.$$

In building up the N -electron ground state, we begin by placing two electrons in the one-electron level $k = 0$, which has the lowest possible one-electron energy $\varepsilon = 0$. We have

$$N = 2 \frac{L^3}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{V}{3\pi^2} k_F^3, \quad (7)$$

where the sphere of radius k_F containing the occupied one-electron levels is called the Fermi sphere, and the factor 2 is from spin degeneracy.

The electron density n is defined by

$$n = \frac{N}{V} = \frac{1}{3\pi^2} k_F^3. \quad (8)$$

The Fermi wavenumber k_F is given by

$$k_F = (3\pi^2 n)^{1/3}. \quad (9)$$

The Fermi energy is given by

$$\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}. \quad (10)$$

The Fermi velocity is

$$v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m} (3\pi^2 n)^{1/3}. \quad (11)$$

((Note))

The Fermi energy ε_F can be estimated using the number of electrons per unit volume as

$\varepsilon_F = 3.64645 \times 10^{-15} n^{2/3} \text{ [eV]} = 1.69253 n_0^{2/3} \text{ [eV]}$,
 where n and n_0 is in the units of (cm^{-3}) and $n = n_0 \times 10^{22}$. The Fermi wave number k_F is calculated as

$$k_F = 6.66511 \times 10^7 n_0^{1/3} \text{ [cm}^{-1}\text{]}.$$

The Fermi velocity v_F is calculated as

$$v_F = 7.71603 \times 10^7 n_0^{1/3} \text{ [cm/s]}.$$

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