Coherent state and squeezed state

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The coherent state is the quantum state of light most proximate to the classical radiation field. For the coherent state, the product of the fluctuations of two non-commutative physical quantities takes the minimum value. The electric field of the coherent state can be approximately expressed by a well-defined amplitude and phase as with classical waves. The Bogoliubov transformation will also be discussed for the squeezed state and squeezed coherent state.

((**Wikipedia**)) **Application of coherent squeezed state**

Squeezed states of the light field can be used to enhance precision measurements. For example phase-squeezed light can improve the phase read out of interferometric measurements (see for example gravitational waves). Amplitude-squeezed light can improve the readout of very weak spectroscopic signals. Various squeezed coherent states, generalized to the case of many degrees of freedom, are used in various calculations in quantum field theory, for example Unruh effect and Hawking radiation, and generally, particle production in curved backgrounds and Bogoliubov transformation.

Recently, the use of squeezed states for quantum information processing in the continuous variables (CV) regime has been increasing rapidly. Continuous variable quantum optics uses squeezing of light as an essential resource to realize CV protocols for quantum communication, unconditional quantum teleportation and one-way quantum computing. This is in contrast to quantum information processing with single photons or photon pairs as qubits. CV quantum information processing relies heavily on the fact that squeezing is intimately related to quantum entanglement, as the quadrature of a squeezed state exhibit sub-shot-noise quantum correlations.
http://en.wikipedia.org/wiki/Squeezed_coherent_state

1. Simple harmonics

We start with the quantum mechanics on the simple harmonics. The annihilation and creation operators are defined by

$$\hat{a}=\frac{\beta}{\sqrt{2}}(\hat{x}+\frac{i\hat{p}}{m\omega_0}),$$

$$\hat{a}^{+}=\frac{\beta}{\sqrt{2}}(\hat{x}-\frac{i\hat{p}}{m\omega_0}),$$

where

$$[\hat{a},\hat{a}^{+}]=\hat{1},$$

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}} .$$

The Hamiltonian of the simple harmonics is given by

$$\hat{H} = \hbar\omega_0(\hat{a}^+\hat{a} + \frac{1}{2}) = \hbar\omega_0(\hat{n} + \frac{1}{2}) = \hbar\omega_0(\hat{X}^2 + \hat{Y}^2),$$

where

$$\hat{X} = \frac{\beta}{\sqrt{2}}\hat{x} = \frac{\beta}{\sqrt{2}}\frac{1}{\sqrt{2}\beta}(\hat{a} + \hat{a}^+) = \frac{1}{2}(\hat{a} + \hat{a}^+),$$

$$\hat{Y} = \frac{\beta}{\sqrt{2}}\frac{\hat{p}}{m\omega_0} = \frac{1}{\sqrt{2}\hbar\beta}\hat{p} = \frac{1}{\sqrt{2}\hbar\beta}\frac{m\omega_0}{\sqrt{2}\beta i}(\hat{a} - \hat{a}^+) = \frac{1}{2i}(\hat{a} - \hat{a}^+),$$

or

$$\hat{a} = \hat{X} + i\hat{Y}, \qquad \hat{a}^+ = \hat{X} - i\hat{Y} .$$

Note that

$$[\hat{X}, \hat{Y}] = [\frac{\beta}{\sqrt{2}}\hat{x}, \frac{1}{\sqrt{2}\hbar\beta}\hat{p}] = \frac{\beta}{\sqrt{2}}\frac{1}{\sqrt{2}\hbar\beta}[\hat{x}, \hat{p}] = \frac{1}{2\hbar}[\hat{x}, \hat{p}] = \frac{1}{2}i\hat{1},$$

and

$$\hat{n} + \frac{1}{2}\hat{1} = \hat{X}^2 + \hat{Y}^2 .$$

The expectation value:

$$\langle n|\hat{X}|n\rangle = \langle n|\frac{1}{2}(\hat{a} + \hat{a}^+)|n\rangle = 0,$$

$$\langle n|\hat{Y}|n\rangle = \langle n|\frac{1}{2i}(\hat{a} - \hat{a}^+)|n\rangle = 0,$$

$$\langle n|\hat{X}^2|n\rangle = \frac{1}{4}\langle n|(\hat{a} + \hat{a}^+)^2|n\rangle = \frac{1}{2}(n + \frac{1}{2}),$$

$$\langle n|\hat{Y}^2|n\rangle = -\frac{1}{4}\langle n|(\hat{a}-\hat{a}^+)^2|n\rangle = \frac{1}{2}(n+\frac{1}{2}),$$

$$\langle n|\hat{X}\hat{Y}|n\rangle = \frac{i}{4}.$$

The uncertainty:

$$(\Delta X)^2 = \langle n|\hat{X}^2|n\rangle - \langle n|\hat{X}|n\rangle^2 = \frac{1}{2}(n+\frac{1}{2}),$$

$$(\Delta Y)^2 = \langle n|\hat{Y}^2|n\rangle - \langle n|\hat{Y}|n\rangle^2 = \frac{1}{2}(n+\frac{1}{2}).$$

Using the Schwarz inequality and $[\hat{X},\hat{Y}] = \frac{1}{2}i\hat{1}$, we have

$$\Delta X \Delta Y \geq \frac{1}{4},$$

where

$$\Delta X \Delta Y = \frac{1}{4},$$

for $\Delta X = \Delta Y = \dfrac{1}{2}$.

2. Single-mode field operator

From the discussion on the quantized electromagnetic field, the vector potential and electric field are given by

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} + \hat{a}^+{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\hat{\boldsymbol{A}}(\boldsymbol{r},t)$$
$$= \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}^+{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]$$

in the cgs units. The direction of propagation is taken to be the z axis and, with the mode subscripts $\boldsymbol{k}$ and λ removed. The scalar electric-field operator for the given direction of linear polarization is given by

$$\hat{E} = \sqrt{\frac{2\pi\hbar\omega}{V}}[i\hat{a}e^{i(kz-\omega t)} - i\hat{a}^{+}e^{-i(kz-\omega t)}]$$
$$= \sqrt{\frac{2\pi\hbar\omega}{V}}[\hat{a}e^{i(kz-\omega t+\frac{\pi}{2})} + \hat{a}^{+}e^{-i(kz-\omega t+\frac{\pi}{2})}]$$
$$= \sqrt{\frac{2\pi\hbar\omega}{V}}[\hat{a}e^{-i(k\omega t-kz-\frac{\pi}{2})} + \hat{a}^{+}e^{i(\omega t-kz-\frac{\pi}{2})}]$$

using $i = e^{\frac{i\pi}{2}}$ and $-i = e^{-\frac{i\pi}{2}}$. This can be rewritten as

$$\hat{E}(\chi) = \hat{E}^{+}(\chi) + \hat{E}^{-}(\chi) = \frac{E_0}{2}[\hat{a}e^{-i\chi} + \hat{a}^{+}e^{i\chi}],$$

$$\chi = \omega t - kz - \frac{\pi}{2}.$$

and $\omega = ck$ (dispersion relation of the photon), where

$$E_0 = \sqrt{\frac{2\pi\hbar\omega_0}{V}} \qquad \text{(in c.g.s. units)}$$

E_0 is the amplitude of the electric field, and E_A is the effective electric field,

$$E_A = \frac{E_0}{\sqrt{2}} = \sqrt{\frac{4\pi\hbar\omega_0}{V}}, \qquad \text{(in c.g.s. units)}$$

Note that the total energy is the sum of two contributions from electric field and magnetic energy, both are the same), and the factor 1/2 comes from the average, $\frac{1}{T}\int_0^T \cos^2(\omega t)dt = \frac{1}{2}$.

We consider the simple case (the single mode).

$$\hat{E}(\chi) = \frac{E_0}{2}[\hat{a}e^{-i\chi} + \hat{a}^{+}e^{i\chi}]$$
$$= \frac{E_0}{2}[\hat{a}e^{i(-\omega t+kz+\frac{\pi}{2})} + \hat{a}^{+}e^{-i(-\omega t+kz+\frac{\pi}{2})}],$$
$$= \frac{E_0}{2}i[\hat{a}e^{i(kz-\omega t)} - \hat{a}^{+}e^{-i(kz-\omega t)}]$$

((**Evaluation of the amplitude of electric field** E_0))

Suppose that in a volume *V*, there is one photon with the angular frequency ω_0 and energy $\hbar\omega_0$.

(a) SI units

Suppose that

$$V\langle u\rangle = V\frac{1}{2}\varepsilon_0 E_0{}^2 = \hbar\omega,$$

leading to

$$E_0 = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}},$$

where $\langle u\rangle$ is the energy density (see the topics of Maser Physics)

$$\langle u\rangle = \frac{1}{2}\varepsilon_0 E_0{}^2.$$

E_0 is the amplitude of the electric field; $E_A = E_0/\sqrt{2}$ is the effective electric field.

(b) c.g.s. units

Suppose that

$$V\langle u\rangle = V\frac{1}{8\pi}E_0{}^2 = \hbar\omega,$$

leading to

$$E_0 = \sqrt{\frac{8\pi\hbar\omega}{V}},$$

where $\langle u\rangle$ is the energy density (see the topics of Maser Physics)

$$\langle u\rangle = \frac{1}{8\pi}E_0{}^2.$$

Thus we get

$$\frac{E_0}{2} = \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}}, \qquad \text{(in SI units),}$$

$$\frac{E_0}{2} = \sqrt{\frac{2\pi\hbar\omega_0}{V}} \,. \qquad \text{(in cgs units),}$$

For convenience, the electric field is measured in units of $E_0 = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}$ (in SI units), or $E_0 = 2\sqrt{\frac{2\pi\hbar\omega}{V}}$ (in cgs units). By dividing the electric field by E_0, we get the expression

$$\begin{aligned}
\hat{E}(\chi) &= \hat{E}^{+}(\chi) + \hat{E}^{-}(\chi) \\
&= \frac{1}{2}\hat{a}e^{-i\chi} + \frac{1}{2}\hat{a}^{+}e^{i\chi} \\
&= \frac{1}{2}(\hat{X} + i\hat{Y})e^{-i\chi} + \frac{1}{2}(\hat{X} - i\hat{Y})e^{i\chi} \\
&= \hat{X}\cos\chi + \hat{Y}\sin\chi
\end{aligned}$$

where

$$\hat{a} = \hat{X} + i\hat{Y} \,,$$

$$\hat{a}^{+} = \hat{X} - i\hat{Y} \,,$$

with

$$[\hat{X},\hat{Y}] = \frac{i}{2}\hat{1} \,,$$

$$\begin{aligned}
[\hat{E}(\chi_1),\hat{E}(\chi_2)] &= [\hat{X}\cos\chi_1 + \hat{Y}\sin\chi_1, \hat{X}\cos\chi_2 + \hat{Y}\sin\chi_2] \\
&= [\hat{X},\hat{Y}]\cos\chi_1\sin\chi_2 - [\hat{X},\hat{Y}]\sin\chi_1\cos\chi_2 \\
&= -\frac{i}{2}\hat{1}\sin(\chi_1 - \chi_2)
\end{aligned}$$

$$\langle n|\hat{E}(\chi)|n\rangle = \langle n|\hat{X}\cos\chi + \hat{Y}\sin\chi|n\rangle = 0 \,,$$

$$\begin{aligned}
\langle n|\hat{E}^2(\chi)|n\rangle &= \langle n|(\hat{X}\cos\chi + \hat{Y}\sin\chi)^2|n\rangle \\
&= \langle n|\hat{X}^2|n\rangle\cos^2\chi + \langle n|\hat{Y}^2|n\rangle\sin^2\chi \\
&= \frac{1}{2}(n + \frac{1}{2})
\end{aligned}$$

$$[\Delta E(\chi)]^2 = \langle n|\hat{E}^2(\chi)|n\rangle - \langle n|\hat{E}(\chi)|n\rangle^2 \chi$$
$$= \frac{1}{2}(n+\frac{1}{2})$$

where

$$\langle n|\hat{X}\hat{Y}|n\rangle = \frac{i}{4}, \qquad \langle n|\hat{Y}\hat{X}|n\rangle = -\frac{i}{4},$$

$$\langle n|\hat{X}^2|n\rangle = \langle n|\hat{Y}^2|n\rangle = \frac{1}{2}(n+\frac{1}{2}).$$

((General formulation))　　More convenient method (complex plane)

Here we use more convenient expression which is familiar in the discusstion of AC circuit analysis. We consider the expectation value of $\hat{E}(\chi)$ for the state $|\psi\rangle$ (general state, including the vacuum state, the coherent state, the squeezed state, and so on).

$$\langle\psi|\hat{E}(\chi)|\psi\rangle = \frac{1}{2}\langle\psi|\hat{a}e^{-i\chi} + \hat{a}^+e^{i\chi}|\psi\rangle$$
$$= \frac{1}{2}[\langle\psi|\hat{a}|\psi\rangle e^{-i\chi} + \langle\psi|\hat{a}^+|\psi\rangle e^{i\chi}]$$

Noting that $\langle\psi|\hat{a}|\psi\rangle^* = \langle\psi|\hat{a}^+|\psi\rangle$, we have

$$\langle\psi|\hat{E}(\chi)|\psi\rangle = \frac{1}{2}[\langle\psi|\hat{a}|\psi\rangle e^{-i\chi} + \langle\psi|\hat{a}|\psi\rangle^* e^{i\chi}]$$
$$= \mathrm{Re}[\langle\psi|\hat{a}|\psi\rangle e^{-i\chi}]$$

where z is the complex number,

$$z = \langle\psi|\hat{a}|\psi\rangle$$
$$= \langle\psi|\hat{X}|\psi\rangle + i\langle\psi|\hat{Y}|\psi\rangle$$
$$= \langle X\rangle + i\langle Y\rangle$$
$$= X + iY$$

Since $\langle\psi|\hat{X}|\psi\rangle = \langle X\rangle = X$ and $\langle\psi|\hat{Y}|\psi\rangle = \langle Y\rangle = Y$ are real. This notation is very similar to that of the AC circuit analysis in the electricity and magnetism,

$$V(t) = \mathrm{Re}(\tilde{V}e^{i\omega t}) \qquad \text{(AC voltage)}$$

$$I(t) = \mathrm{Re}(\hat{I}e^{i\omega t}) \qquad \text{(AC current)}$$

where $\tilde{V}$ and $\hat{I}$ are complex numbers in the complex plane. So that we just consider how the complex number $z = \langle \psi | \hat{a} | \psi \rangle$ is located on the 2D complex plane.

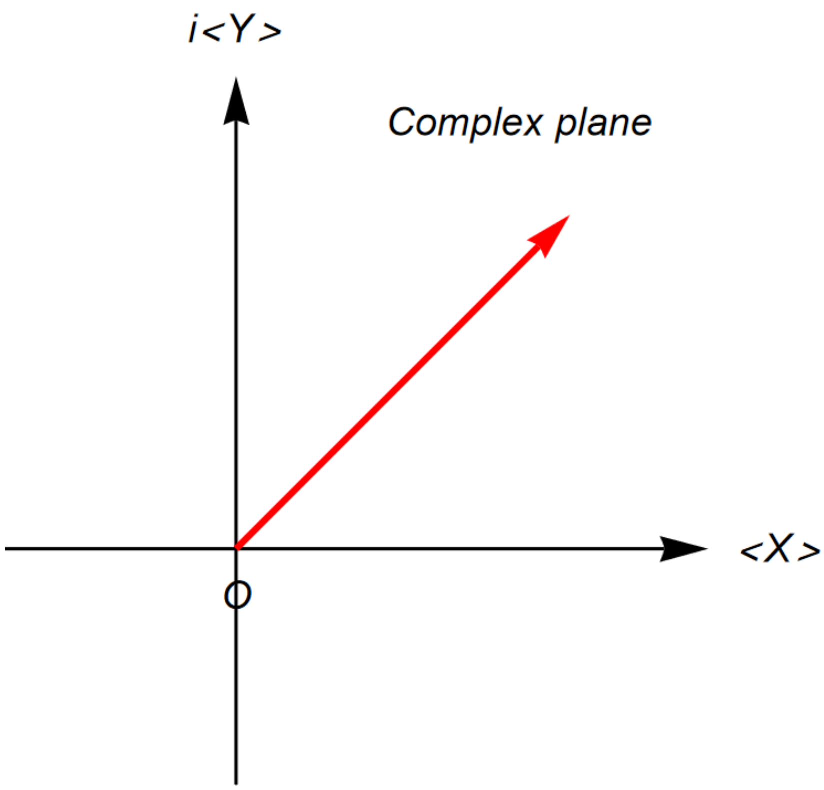


Fig.1 $\langle \psi | \hat{E}(\chi) | \psi \rangle = \mathrm{Re}[\langle \psi | \hat{a} | \psi \rangle e^{-i\chi}]$. $z = \langle \psi | \hat{a} | \psi \rangle = X + iY$ in the 2D complex plane. $|\psi\rangle$ is a given state.

(a)
As will be shown later, when $|\psi\rangle = |\alpha\rangle$ (coherent state),

$$z = \langle \alpha | \hat{a} | \alpha \rangle = \alpha = |\alpha| e^{i\theta}$$

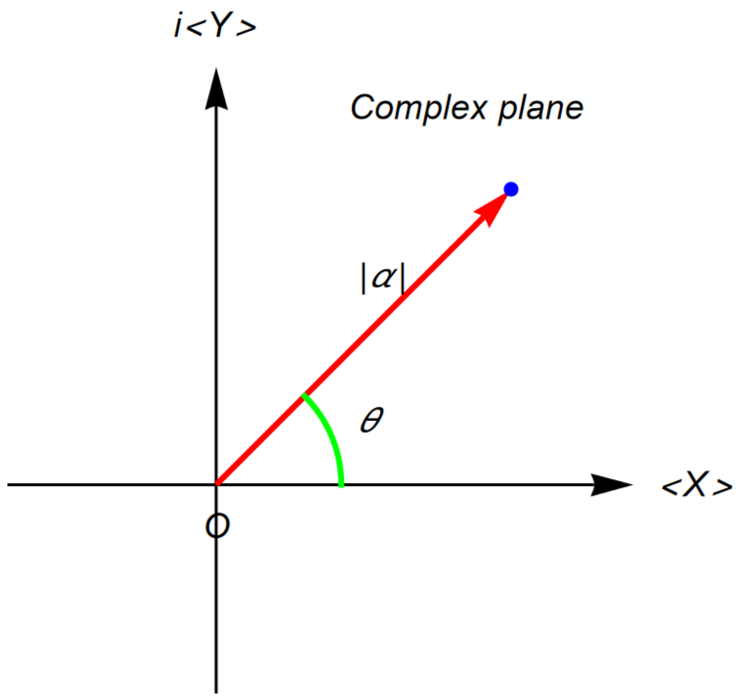


Fig.2 The 2D complex plane for the coherent state $|\psi\rangle = |\alpha\rangle$. The blue point indicates the location of $z = \langle\alpha|\hat{a}|\alpha\rangle = \alpha = |\alpha| e^{i\theta}$. The amplitude $|\alpha|$ and the phase θ for the coherent state with $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

(b) $|\psi\rangle = |0\rangle$, we have $z = \langle 0|\hat{a}|0\rangle = 0$

3. Displacement operator

Suppose that

$$\hat{A} = -\alpha\hat{a}^+ + \alpha^*\hat{a} \qquad \hat{B} = \hat{a},$$

where α is a complex number.

$$[\hat{A},\hat{B}] = [-\alpha\hat{a}^+ + \alpha^*\hat{a}, \hat{a}] = -\alpha[\hat{a}^+, \hat{a}] = \alpha\hat{1}$$

Using the Baker-Hausdorff lemma, we get

$$\exp(-\alpha\hat{a}^+ + \alpha^*\hat{a})\hat{a}\exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \hat{a} + \alpha\hat{1},$$

or

$$\hat{D}_\alpha^+\hat{a}\hat{D}_\alpha = \alpha\hat{1} + \hat{a},$$

where $\hat{D}_\alpha$ is called the displacement operator,

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}).$$

Note that

$$\hat{D}_\alpha^{\ +} = \hat{D}_{-\alpha} = \exp(-\alpha\hat{a}^+ + \alpha^*\hat{a}),$$

$$\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha = \alpha^*\hat{1} + \hat{a}^+,$$

$$\hat{D}_\alpha^{\ +}\hat{D}_\alpha = \hat{1}. \quad \text{(Unitary operator).}$$

4. Properties of $\hat{D}_\alpha$ (I)

Using the above theorem, we can derive the following formula.

$$\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}) = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a} + \frac{1}{2}|\alpha|^2)$$
$$= \exp(\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+ - \alpha^*\hat{a})$$

where

$$\hat{A} = \alpha\hat{a}^+, \qquad \hat{B} = -\alpha^*\hat{a},$$

$$[\hat{A},\hat{B}] = [\alpha\hat{a}^+, -\alpha^*\hat{a}] = |\alpha|^2[\hat{a},\hat{a}^+] = |\alpha|^2\hat{1},$$

with

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0.$$

Thus we get

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}),$$

where

$$[\hat{a}, \exp(\alpha\hat{a}^+)] = \alpha\exp(\alpha\hat{a}^+).$$

$$[\hat{a}^+, \exp(\alpha\hat{a})] = -\alpha\exp(\alpha\hat{a}).$$

5. Properties of $\hat{D}_\alpha$ (II)

$$\begin{aligned}\hat{D}_\alpha{}^+ \hat{n} \hat{D}_\alpha &= \hat{D}_\alpha{}^+ \hat{a}^+ \hat{a} \hat{D}_\alpha \\ &= \hat{D}_\alpha{}^+ \hat{a}^+ \hat{D}_\alpha \hat{D}_\alpha{}^+ \hat{a} \hat{D}_\alpha \\ &= (\hat{a}^+ + \alpha^* \hat{1})(\hat{a} + \alpha \hat{1}) \\ &= \hat{n} + |\alpha|^2 \hat{1} + \alpha^* \hat{a} + \alpha \hat{a}^+\end{aligned}$$

with

$$\hat{n} = \hat{a}^+ \hat{a} .$$

We also have

$$\hat{D}_\alpha{}^+ \hat{a} \hat{a} \hat{D}_\alpha = \hat{a}\hat{a} + \alpha^2 \hat{1} + 2\alpha \hat{a} = (\hat{a} + \alpha \hat{1})^2$$

$$\hat{D}_\alpha{}^+ \hat{a}^+ \hat{a}^+ \hat{D}_\alpha = \hat{a}^+ \hat{a}^+ + (\alpha^*)^2 \hat{1} + 2\alpha^* \hat{a}^+ = (\hat{a}^+ + \alpha^* \hat{1})^2$$

In general

$$\hat{D}_\alpha{}^+ f(\hat{a}, \hat{a}^+) \hat{D}_\alpha = f(\hat{a} + \alpha \hat{1}, \hat{a}^+ + \alpha^* \hat{1})$$

where *f* is any function of $\hat{a}$ and $\hat{a}^+$ with a power series expansion. We show that

$$\hat{D}_\alpha \hat{D}_\beta = \exp(\frac{\alpha\beta^* - \alpha^*\beta}{2}) \hat{D}_{\alpha+\beta}$$

((**Proof**))

$$\hat{D}_\alpha = \exp(\alpha \hat{a}^+ - \alpha^* \hat{a}) = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha \hat{a}^+) \exp(-\alpha^* \hat{a}) ,$$

$$\hat{D}_\beta = \exp(\beta \hat{a}^+ - \beta^* \hat{a}) = \exp(-\frac{1}{2}|\beta|^2) \exp(\beta \hat{a}^+) \exp(-\beta^* \hat{a}) .$$

Then we get

$$\hat{D}_{\alpha+\beta} = \exp[(\alpha+\beta)\hat{a}^+ - (\alpha+\beta)^* \hat{a})$$
$$= \exp(-\frac{1}{2}|\alpha+\beta|^2)\exp[(\alpha+\beta)\hat{a}^+)\exp[-(\alpha+\beta)^* \hat{a}]$$
$$= \exp(-\frac{1}{2}|\alpha+\beta|^2)\exp(\alpha\hat{a}^+)\exp(\beta\hat{a}^+)\exp(-\alpha^*\hat{a})\exp(-\beta^*\hat{a})$$

and

$$\hat{D}_\alpha \hat{D}_\beta = \exp(-\frac{1}{2}|\alpha|^2)\exp(-\frac{1}{2}|\beta|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})\exp(\beta\hat{a}^+)\exp(-\beta^*\hat{a}) .$$

Here we note that

$$\exp(-\alpha^*\hat{a})\exp(\beta\hat{a}^+) = \exp(-\alpha^*\hat{a}+\beta\hat{a}^+)\exp(-\frac{1}{2}\alpha^*\beta[\hat{a},\hat{a}^+])$$
$$= \exp(-\frac{1}{2}\alpha^*\beta])\exp(-\alpha^*\hat{a}+\beta\hat{a}^+)$$

$$\exp(\beta\hat{a}^+)\exp(-\alpha^*\hat{a}) = \exp(\beta\hat{a}^+ - \alpha^*\hat{a})\exp(\frac{1}{2}\alpha^*\beta[\hat{a},\hat{a}^+])$$
$$= \exp(\frac{1}{2}\alpha^*\beta])\exp(-\alpha^*\hat{a}+\beta\hat{a}^+)$$

From these two equations, we get

$$\exp(\frac{1}{2}\alpha^*\beta)\exp(-\alpha^*\hat{a})\exp(\beta\hat{a}^+) = \exp(-\frac{1}{2}\alpha^*\beta)\exp(\beta\hat{a}^+)\exp(-\alpha^*\hat{a}) ,$$

or

$$\exp(-\alpha^*\hat{a})\exp(\beta\hat{a}^+) = \exp(-\alpha^*\beta)\exp(\beta\hat{a}^+)\exp(-\alpha^*\hat{a}) .$$

Using this relation, we have

$$\hat{D}_\alpha \hat{D}_\beta = \exp(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta)\exp(\alpha\hat{a}^+)\exp(\beta\hat{a}^+)\exp(-\alpha^*\hat{a})\exp(-\beta^*\hat{a}) ,$$

$$\hat{D}_{\alpha+\beta} = \exp(-\frac{1}{2}|\alpha+\beta|^2)\exp(\alpha\hat{a}^+)\exp(\beta\hat{a}^+)\exp(-\alpha^*\hat{a})\exp(-\beta^*\hat{a}) ,$$

which leads to

$$\hat{D}_{\alpha+\beta} = \exp(-\frac{1}{2}|\alpha+\beta|^2 + \frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2 - \alpha^*\beta))\hat{D}_\alpha\hat{D}_\beta$$
$$= \exp(\frac{\alpha^*\beta - \alpha\beta^*}{2})\hat{D}_\alpha\hat{D}_\beta$$

or

$$\hat{D}_\alpha\hat{D}_\beta = \exp(\frac{\alpha\beta^* - \alpha^*\beta}{2})\hat{D}_{\alpha+\beta}.$$

We also note that

$$\hat{D}_\beta\hat{D}_\alpha = \exp(-\frac{\alpha\beta^* - \alpha^*\beta}{2})\hat{D}_{\alpha+\beta},$$

or

$$\hat{D}_\alpha\hat{D}_\beta = \exp(\alpha\beta^* - \alpha^*\beta)\hat{D}_\beta\hat{D}_\alpha.$$

From this we get

$$[\hat{D}_\alpha, \hat{D}_\beta] = 0,$$

only when

$$\alpha\beta^* - \alpha^*\beta = 2|\alpha||\beta|i\sin(\theta_\alpha - \theta_\beta) = 0,$$

or

$$\theta_\alpha = \theta_\beta.$$

6. Translation operator

Suppose that $\alpha^* = \alpha$. Then we have

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \exp[\alpha(\hat{a}^+ - \hat{a})],$$

where

$$\hat{a}^+ - \hat{a} = -i\sqrt{\frac{2}{m\hbar\omega_0}}\hat{p},$$

or

$$\hat{D}_\alpha = \exp[-\frac{i}{\hbar}\frac{\sqrt{2}\alpha}{\beta}\hat{p}],$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}, \qquad a = \frac{\sqrt{2}\alpha}{\beta}.$$

This operator coincides with the translation operator

$$\hat{T}(a) = \exp[-\frac{i}{\hbar}\hat{p}a],$$

where

$$\hat{T}(a)|x\rangle = |x+a\rangle,$$

or

$$\langle x|\hat{T}^{+}(a) = \langle x+a|.$$

Note that

$$\langle x|\hat{D}_\alpha|\psi\rangle = \langle x|\hat{D}_{-\alpha}{}^{+}|\psi\rangle = \left\langle x - \frac{\sqrt{2}\alpha}{\beta}\middle|\psi\right\rangle = \psi(x - \frac{\sqrt{2}\alpha}{\beta}).$$

When $|\psi\rangle = |0\rangle$,

$$\langle x|\alpha\rangle = \langle x|\hat{D}_\alpha|0\rangle = \left\langle x - \frac{\sqrt{2}\alpha}{\beta}\middle|0\right\rangle = \psi_0(x - x_0),$$

which is the Gaussian function where the center shifts from zero to $x = x_0$, where

$$x_0 = \frac{\sqrt{2}\alpha}{\beta}. \qquad \text{(real)}$$

7. Coherent state $|\alpha\rangle$

We define the coherent state $|\alpha\rangle$ as

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \text{and} \qquad \langle\alpha|\hat{a}^{+} = \langle\alpha|\alpha^{*}$$

Suppose that

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle .$$

Then

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle .$$

This is equal to

$$\alpha|\alpha\rangle = \sum_{n=0}^{\infty} c_n \alpha |n\rangle .$$

Then we get the recursion relation

$$c_{n+1} = c_n \frac{\alpha}{\sqrt{n+1}} ,$$

which leads to

$$c_n = c_0 \frac{\alpha^n}{\sqrt{n!}} .$$

Consequently, we have

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

The bra vector $\langle\alpha|$ is given by

$$\langle\alpha| = c_0^{*} \sum_{n=0}^{\infty} \frac{(\alpha^{*})^n}{\sqrt{n!}} \langle n| .$$

From the normalization requirement, we determine c_0;

$$\begin{aligned}\langle\alpha|\alpha\rangle &= |c_0|^2\sum_{m=0}^{\infty}\frac{\left(\alpha^*\right)^m}{\sqrt{m!}}\langle m|\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\ &= |c_0|^2\sum_{m=0}^{\infty}\frac{\left(\alpha^*\right)^m}{\sqrt{m!}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle m|n\rangle \\ &= |c_0|^2\sum_{m=0}^{\infty}\frac{\left(\alpha^*\right)^m}{\sqrt{m!}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\delta_{n,m} \\ &= |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \\ &= 1\end{aligned}$$

yields

$$|c_0| = e^{-\frac{|\alpha|^2}{2}}.$$

Finally we have the coherent state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^+)^n}{n!}|0\rangle.$$

where

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle.$$

8. Property of $|\alpha\rangle$

Here we show that $|\alpha\rangle$ is expressed by

$$|\alpha\rangle = \hat{D}_\alpha|0\rangle.$$

where

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}), \qquad \hat{D}_\alpha^{\ +} = \exp(\alpha^*\hat{a} - \alpha\hat{a}^+)$$

We know that

$$\hat{D}_\alpha^{\ +}\hat{a}\hat{D}_\alpha = \hat{a} + \alpha\hat{1},$$

or

$$\hat{a}\hat{D}_\alpha = \hat{D}_\alpha \hat{a} + \alpha \hat{D}_\alpha .$$

From this we get

$$\hat{a}\hat{D}_\alpha |0\rangle = \hat{D}_\alpha \hat{a}|0\rangle + \alpha \hat{D}_\alpha |0\rangle ,$$

or

$$\hat{a}\hat{D}_\alpha |0\rangle = \alpha \hat{D}_\alpha |0\rangle .$$

Then $\hat{D}_\alpha |0\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue α,

$$|\alpha\rangle = \hat{D}_\alpha |0\rangle .$$

The form of $|\alpha\rangle$ can be expressed as follows.

$$|\alpha\rangle = \hat{D}_\alpha |0\rangle = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha \hat{a}^+) \exp(-\alpha^* \hat{a})|0\rangle .$$

Using

$$\exp(-\alpha^* \hat{a})|0\rangle = |0\rangle ,$$

$$[\exp(\alpha \hat{a}^+), \hat{a}] = -\exp(\alpha \hat{a}^+)\alpha ,$$

or

$$\exp(\alpha \hat{a}^+)(\alpha \hat{1} + \hat{a}) = \hat{a} \exp(\alpha \hat{a}^+)$$

we have

$$
\begin{aligned}
\hat{a}\left|\alpha\right\rangle &= \hat{a}\hat{D}_\alpha\left|0\right\rangle \\
&= \exp(-\frac{1}{2}\left|\alpha\right|^2)\hat{a}\exp(\alpha\hat{a}^+)\left|0\right\rangle \\
&= \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)(\alpha\hat{1}+\hat{a})\left|0\right\rangle \\
&= \alpha\exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)\left|0\right\rangle \\
&= \alpha\left|\alpha\right\rangle
\end{aligned}
$$

or

$$
\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle,
$$

which means that $\left|\alpha\right\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue α.

((**Note**))

$$
\begin{aligned}
\hat{D}_\alpha &= \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) \\
&= \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}) \\
&= \exp(\frac{1}{2}\left|\alpha\right|^2)\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^+)
\end{aligned}
$$

$$
\begin{aligned}
\left|\alpha\right\rangle &= \hat{D}_\alpha\left|0\right\rangle \\
&= \exp(\alpha\hat{a}^+ - \alpha^*\hat{a})\left|0\right\rangle \\
&= \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})\left|0\right\rangle \\
&= \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)\left|0\right\rangle \\
&= \exp(-\frac{1}{2}\left|\alpha\right|^2)\sum_{n=0}\frac{(\alpha\hat{a}^+)^n}{n!}\left|0\right\rangle
\end{aligned}
$$

9. The explicit expression of $\left|\alpha\right\rangle$ in terms of $\left|n\right\rangle$

$$|\alpha\rangle = \hat{D}_\alpha |0\rangle$$
$$= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)|0\rangle$$
$$= \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^+)^n}{n!}|0\rangle$$

Since

$$|n>=\frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0>,$$

we get

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n(\hat{a}^+)^n}{n!}|0\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

or

$$\langle n|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\frac{\alpha^n}{\sqrt{n!}},$$

$$P_n = |\langle n|\alpha\rangle|^2 = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}.$$

Note that the coherent states do not have a definite value of the quantum number *n*, simply because they are not eigenstates of the number operator. In fact, the probability of observing the values of *n* in a quantum measurement of state $|\alpha\rangle$ is actually a Poisson distribution.

Now we now consider the single mode coherent state

$$<n>=\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2,$$

$$<n^2>=\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}+\hat{a}^+\hat{a}^+\hat{a}\hat{a}|\alpha\rangle = |\alpha|^2+|\alpha|^4 = \langle n\rangle[1+\langle n\rangle],$$

$$\Delta n = \sqrt{\langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2} = |\alpha|,$$

or

$$\frac{\Delta n}{\langle n\rangle} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{\langle n\rangle}},$$

or

$$(\Delta n)^2 = |\alpha|^2 = \langle n \rangle .$$ (Maxwell-Boltzmann distribution)

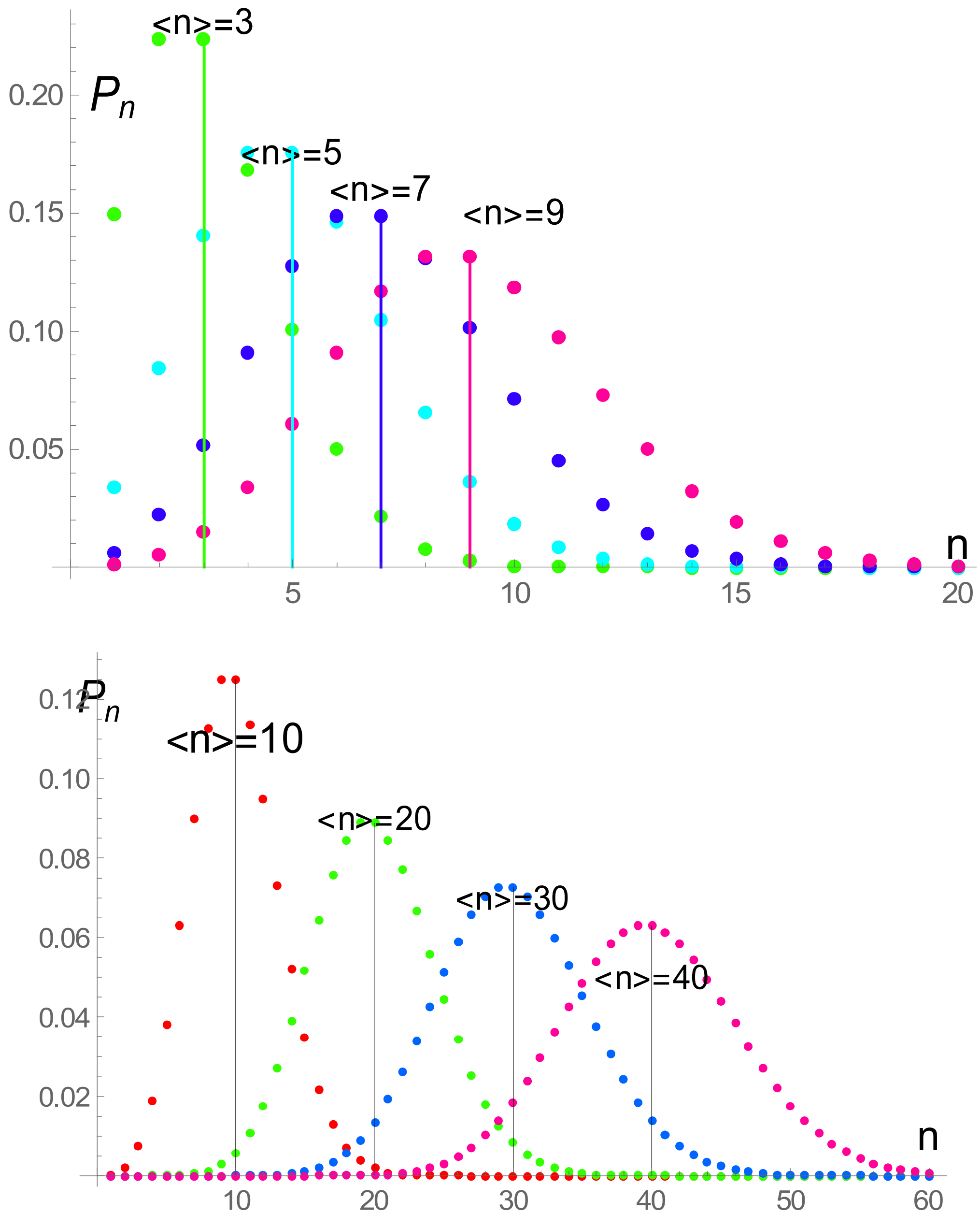


Fig.3 Poisson distribution function. The average value $\langle n \rangle$ is changed as a parameter. $|\alpha| = \sqrt{\langle n \rangle}$. In the limit of $|\alpha| \to \infty$, the Poisson distribution may be approximated as a Gaussian distribution.

((**Note**))

(i) Bose-Einstein (BE) distribution function

$$(\Delta n_{BE})^2 = n_{BE}(1+n_{BE}) .$$

(ii) Fermi-Dirac (FD) distribution function

$$(\Delta n_{FD})^2 = n_{FD}(1-n_{FD}) .$$

(iii) Maxwell-Boltzman (MB) distribution function

$$(\Delta n_{MB})^2 = n_{MB} .$$

10. Time evolution of the coherent state

The coherent state is defined by

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

The time evolution of the coherent state is given by

$$\begin{aligned}
|\alpha(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t} |\alpha\rangle \\
&= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}\hat{H}t} |n\rangle \\
&= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{[\alpha^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}\hbar\omega_0(n+\frac{1}{2})t} |n\rangle \\
&= e^{-\frac{i\omega_0 t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{[(\alpha e^{-i\omega_0 t})^n}{\sqrt{n!}} |n\rangle \\
&= e^{-\frac{i\omega_0 t}{2}} |\alpha e^{-i\omega_0 t}\rangle
\end{aligned}$$

or

$$|\alpha(t)\rangle = e^{-\frac{i\omega_0 t}{2}} |\alpha e^{-i\omega_0 t}\rangle .$$

((**Note**))

$$\hat{a}|\alpha(t)\rangle = e^{-\frac{i\omega_0 t}{2}} \hat{a}|\alpha e^{-i\omega_0 t}\rangle = (\alpha e^{-i\omega_0 t}) e^{-\frac{i\omega_0 t}{2}} |\alpha e^{-i\omega_0 t}\rangle = (\alpha e^{-i\omega_0 t})|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle .$$

Then we have

$$\alpha(t) = \alpha e^{-i\omega t} .$$

((**Example**)) **Ehrenfest theorem for simple harmonics**

Using

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^{+})$$

we get

$$\begin{aligned}
\langle x \rangle &= \langle \alpha(t) | \hat{x} | \alpha)t) \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha(t) | (\hat{a} + \hat{a}^{+}) | \alpha(t) \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} (\alpha e^{-i\omega t} + \alpha^{*} e^{i\omega t}) \\
&= \sqrt{\frac{\hbar}{2m\omega}} 2\,\mathrm{Re}[\alpha e^{-i\omega t}] \\
&= \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}(\alpha e^{-i\omega t})
\end{aligned}$$

We note that

$$\alpha = |\alpha| e^{-i\delta}$$

$$\begin{aligned}
\langle p \rangle &= \langle \alpha(t) | \hat{p} | \alpha(t) \rangle \\
&= -i\sqrt{\frac{m\hbar\omega}{2}} \langle \alpha(t) | \hat{a} - \hat{a}^{+} | \alpha() \rangle \\
&= \sqrt{\frac{m\hbar\omega}{2}} (-i\alpha e^{-i\omega t} + i\alpha^{*} e^{i\omega t}) \\
&= \sqrt{\frac{m\hbar\omega}{2}} 2\,\mathrm{Re}(-i\alpha e^{-i\omega t}) \\
&= \sqrt{2m\hbar\omega}\, \mathrm{Re}[(-i\alpha) e^{-i\omega t}]
\end{aligned}$$

The time dependence of $\langle x \rangle$ and $\langle p \rangle$;

$$\frac{d\langle x\rangle}{dt} = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}[(-i\omega\alpha)e^{-i\omega t}]$$
$$= \sqrt{\frac{2\hbar\omega}{m}}\,\mathrm{Re}[(-i\alpha)e^{-i\omega t}]$$

$$\frac{d\langle p\rangle}{dt} = \sqrt{2m\hbar\omega}\,\mathrm{Re}[(-i\alpha)(-i\omega)e^{-i\omega t}]$$
$$= \sqrt{2m\hbar\omega}\,\mathrm{Re}[(-\alpha\omega)e^{-i\omega t}]$$
$$= -\omega\sqrt{2m\hbar\omega}\,\mathrm{Re}[-\alpha e^{-i\omega t}]$$

Note that

$$\frac{\langle p\rangle}{m} = \frac{\sqrt{2m\hbar\omega}}{m}\,\mathrm{Re}[(-i\alpha)e^{-i\omega t}]$$
$$= \sqrt{\frac{2\hbar\omega}{m}}\,\mathrm{Re}[(-i\alpha)e^{-i\omega t}]$$

$$\left\langle \frac{dV}{dx}\right\rangle = m\omega^2\langle x\rangle$$
$$= \omega\sqrt{2\hbar m\omega}\,\mathrm{Re}(\alpha e^{-i\omega t})$$

Thus, we have the relations

$$\frac{d\langle x\rangle}{dt} = \frac{1}{m}\langle p\rangle, \qquad \frac{d\langle p\rangle}{dt} = -\left\langle \frac{dV}{dx}\right\rangle$$

(Ehrenfest theorem)

---

11. Unitary operator $\hat{R}_\lambda$

We consider the unitary operator:

$$\hat{R}_\lambda = \exp(i\lambda \hat{a}^+ \hat{a})$$

where λ is real. We note that

$$\hat{R}_\lambda^+ \hat{a} \hat{R}_\lambda = \exp(i\lambda)\hat{a}.$$

((**Proof**))

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + \frac{[\hat{A},\hat{B}]}{1!} + \frac{[\hat{A},[\hat{A},\hat{B}]]}{2!} + \frac{[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]}{3!} + ...$$

$$\hat{A} = -i\lambda \hat{a}^+ \hat{a}, \qquad \hat{B} = \hat{a},$$

$$[\hat{n},\hat{a}] = -\hat{a},$$

$$[\hat{A},\hat{B}] = -i\lambda[\hat{a}^+\hat{a},\hat{a}] = -i\lambda[\hat{n},\hat{a}] = i\lambda\hat{a},$$

$$[\hat{A},[\hat{A},\hat{B}]] = -i\lambda[\hat{a}^+\hat{a}, i\lambda\hat{a}] = -(i\lambda)^2[\hat{n},\hat{a}] = (i\lambda)^2\hat{a}$$

$$[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] = -i\lambda[\hat{a}^+\hat{a},(i\lambda)^2\hat{a}] = -(i\lambda)^3[\hat{n},\hat{a}] = (i\lambda)^3\hat{a}$$

and so on.

$$\hat{R}_\lambda^+ \hat{a}\hat{R}_\lambda = \hat{a}[1 + \frac{i\lambda}{1!} + \frac{(i\lambda)^2}{2!} + ...] = e^{i\lambda}\hat{a}.$$

Now let us show that

$$\hat{R}_\lambda|\alpha\rangle = c|e^{i\lambda}\alpha\rangle.$$

We note that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

$$\hat{R}_\lambda^+\hat{a}\hat{R}_\lambda|\alpha\rangle = e^{i\lambda}\hat{a}|\alpha\rangle = \alpha e^{i\lambda}|\alpha\rangle,$$

or

$$\hat{a}(\hat{R}_\lambda|\alpha\rangle) = \alpha e^{i\lambda}(\hat{R}_\lambda|\alpha\rangle),$$

which means that $\hat{R}_\lambda|\alpha\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue $\alpha e^{i\lambda}$. Thus we have

$$\hat{R}_\lambda|\alpha\rangle = c|e^{i\lambda}\alpha\rangle.$$

$|e^{i\lambda}\alpha\rangle$ is the coherent state. c is the phase factor.

12. Non-orthogonality

The coherent state ($|\alpha\rangle$) do not form an orthogonal state.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

$$\hat{a}|\beta\rangle = \beta|\beta\rangle,$$

$$\begin{aligned}|\alpha\rangle &= \hat{D}_\alpha|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)|0\rangle\end{aligned}$$

$$\langle\beta| = \langle 0|\hat{D}_\beta{}^+ = \langle 0|\hat{D}_{-\beta} = \langle 0|\exp(-\frac{1}{2}|\beta|^2)\exp(\beta^*\hat{a}),$$

$$\langle\beta|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\exp(-\frac{1}{2}|\beta|^2)\langle 0|\exp(\beta^*\hat{a})\exp(\alpha\hat{a}^+)|0\rangle.$$

Note that

$$\exp(\beta^*\hat{a})\exp(\alpha\hat{a}^+)\exp(-\beta^*\hat{a}) = \exp(\alpha\hat{a}^+)\exp(\alpha\beta^*),$$

or

$$\exp(\beta^*\hat{a})\exp(\alpha\hat{a}^+) = \exp(\alpha\beta^*)\exp(\alpha\hat{a}^+)\exp(\beta^*\hat{a}),$$

$$\langle\beta|\alpha\rangle = \exp(-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha\beta^*)\langle 0|\exp(\alpha\hat{a}^+)\exp(\beta^*\hat{a})|0\rangle,$$

or

$$\langle\beta|\alpha\rangle = \exp(-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha\beta^*).$$

We also note that

$$|\alpha-\beta|^2 = (\alpha-\beta)(\alpha-\beta)^* = |\alpha|^2+|\beta|^2-(\alpha\beta^*+\alpha^*\beta),$$

$$|\langle\beta|\alpha\rangle|^2 = \exp(-|\alpha|^2-|\beta|^2+\alpha\beta^*+\alpha^*\beta) = \exp(-|\alpha-\beta|^2).$$

The distance $|\alpha-\beta|^2$ measures the degree to which the two eigenstates are approximately orthogonal.

Note that

$$\hat{D}_\alpha = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}) = \exp(\alpha\hat{a}^+-\alpha^*\hat{a}),$$

using the theorem

$$\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}) = \exp(\frac{1}{2}|\alpha|^2])\exp(-\alpha^*\hat{a}+\alpha\hat{a}^+).$$

((**Note**))

$$\begin{aligned}|\alpha\rangle &= \hat{D}_\alpha|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(\alpha^*\hat{a})|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)|0\rangle\end{aligned}$$

$$\begin{aligned}\hat{a}^+|\alpha\rangle &= \exp(-\frac{1}{2}|\alpha|^2)\hat{a}^+\exp(\alpha\hat{a}^+)|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\frac{\partial}{\partial\alpha}\exp(\alpha\hat{a}^+)|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\frac{\partial}{\partial\alpha}\exp(\frac{1}{2}|\alpha|^2)\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)|0\rangle \\ &= \exp(-\frac{1}{2}|\alpha|^2)\frac{\partial}{\partial\alpha}\exp(\frac{1}{2}|\alpha|^2)|\alpha\rangle\end{aligned}$$

13. Over-closure relation

$$\hat{K} = \frac{1}{\pi}\int |\alpha\rangle d^2\alpha \langle\alpha| = \hat{1}.$$

The integration is extended over the entire α plane with a real element of area. Let us give a proof for this.

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

$$\langle\alpha| = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\langle n|.$$

We calculate $\hat{K}$ defined by

$$\hat{K} = \frac{1}{\pi}\int |\alpha\rangle d^2\alpha \langle\alpha|$$
$$= \frac{1}{\pi}\int \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \exp(-|\alpha|^2) d^2\alpha \sum_{m=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m|$$

We assume that

$$\alpha = |\alpha|e^{i\theta} = re^{i\theta}, \qquad (r = |\alpha|)$$

then we have

$$\alpha^n(\alpha^*)^m = |\alpha|^{n+m}e^{i(n-m)\theta},$$

$$J = \int d^2\alpha \exp(-r^2) r^{n+m} \exp[i(n-m)\theta],$$

with

$$d^2\alpha = r d\theta dr .$$

Then

$$J = \iint r d\theta dr \exp(-r^2) r^{n+m} \exp[i(n-m)\theta],$$

where $0 \le r < \infty$ and $0 \le \theta \le 2\pi$. Note that

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi\delta_{n,m},$$

$$J = 2\pi\delta_{n,m}\int_0^\infty dr\,\exp(-r^2)r^{2n} = \pi n!\delta_{n,m}\,.$$

Thus

$$\hat{K} = \frac{1}{\pi}\sum_{n.m}|n\rangle\langle m|\frac{\pi n!}{n!}\delta_{n,m} = \sum_n |n\rangle\langle n| = \hat{1}\,.$$

Using the closure relation, we have

$$|\alpha\rangle = \frac{1}{\pi}\int d^2\beta|\beta\rangle\langle\beta|\alpha\rangle = \frac{1}{\pi}\int d^2\beta|\beta\rangle\exp(-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha\beta^*)\,,$$

where

$$\langle\beta|\alpha\rangle^2 = \exp(-|\alpha-\beta|^2)\,.$$

This is an expression of over-closure. This means that a coherent state forms an complete set and that the simultaneous measurement of $\hat{a}_1$ and $\hat{a}_2$, represented by the projection operator $|\alpha\rangle\langle\alpha|$ is not an exact measurement but instead an approximate measurement with a finite error measurement.

14. $|x\rangle$-representation of the coherent state

If we multiply the left-hand side of

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,,$$

by $\langle x|$ and use

$$\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i\hat{p}}{m\omega_0}\right), \qquad \hat{a}^+ = \frac{\beta}{\sqrt{2}}\left(\hat{x}-\frac{i\hat{p}}{m\omega_0}\right)$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}\,.$$

Then we have

$$\langle x|\hat{a}|\alpha\rangle = \alpha\langle x|\alpha\rangle\,,$$

or

$$\langle x|\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i\hat{p}}{m\omega_0}\right)|\alpha\rangle=\alpha\langle x|\alpha\rangle,$$

or

$$\frac{\hbar}{m\omega_0}\frac{\partial}{\partial x}\langle x|\alpha\rangle+(x-\sqrt{2}\frac{\alpha}{\beta})\langle x|\alpha\rangle=0.$$

The solution for $\langle x|\alpha\rangle$ can be obtained as

$$\begin{aligned}\langle x|\alpha\rangle &= C\exp[-\frac{m\omega_0}{2\hbar}(x-\sqrt{2}\frac{\alpha}{\beta})^2] \\ &= C'\exp[-\frac{m\omega_0}{2\hbar}(x-\sqrt{2}\frac{\operatorname{Re}\alpha}{\beta})^2+i\frac{m\omega_0}{\hbar}x\frac{\sqrt{2}\operatorname{Im}\alpha}{\beta})] \\ &= C'\exp[-\frac{m\omega_0}{2\hbar}(x-x_0)^2+i\frac{p_0}{\hbar}x)]\end{aligned}$$

where

$$x_0=\sqrt{2}\frac{\operatorname{Re}\alpha}{\beta},\qquad p_0=m\omega_0\frac{\sqrt{2}\operatorname{Im}\alpha}{\beta}=\hbar\beta\sqrt{2}\operatorname{Im}\alpha,$$

$$\int_{-\infty}^{\infty}|\langle x|\alpha\rangle|^2dx=1,$$

$$C'=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}.$$

Note that

$$\langle x\rangle=\int_{-\infty}^{\infty}x|\langle x|\alpha\rangle|^2dx=x_0,$$

$$(\Delta x)^2=\int_{-\infty}^{\infty}(x-x_0)^2|\langle x|\alpha\rangle|^2dx=\frac{\hbar}{2m\omega_0},$$

$$\langle p\rangle = \frac{\hbar}{i}\int_{-\infty}^{\infty}\langle x|\alpha\rangle \frac{\partial}{\partial x}\langle x|\alpha\rangle dx = p_0,$$

$$\langle p^2\rangle = \left(\frac{\hbar}{i}\right)^2\int_{-\infty}^{\infty}\langle x|\alpha\rangle \frac{\partial^2}{\partial x^2}\langle x|\alpha\rangle dx = p_0{}^2 + \frac{m\hbar\omega_0}{2},$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega_0}{2}.$$

Then we have

$$\Delta x\Delta p = \sqrt{\frac{\hbar}{2m\omega_0}}\sqrt{\frac{m\hbar\omega_0}{2}} = \frac{\hbar}{2}.$$

The wavefunction $\langle x|\alpha\rangle$ is indeed the minimum uncertainty wave-packet with stationary quantum uncertainty.

The wavefunction for the coherent state is

$$\langle x|\alpha\rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\exp[-\frac{(x-\langle x\rangle)^2}{4(\Delta x)^2} + i\frac{\langle p\rangle}{\hbar}x)].$$

The time dependence of the coherent state:

$$|\alpha(t)\rangle = \exp(-\frac{i\omega_0 t}{2})\left|\alpha\exp(-\frac{i\omega_0 t}{2})\right\rangle,$$

$$\langle x|\alpha(t)\rangle = C\exp[-\frac{m\omega_0}{2\hbar}(x-\sqrt{2}\frac{1}{\beta}\alpha\exp(-\frac{i\omega_0 t}{2}))^2].$$

((**Note**)) **$|\xi\rangle$-representation of the coherent state**

Here we use

$$\xi = \beta x, \qquad |\xi\rangle = \frac{1}{\sqrt{\beta}}|x\rangle,$$

$$\kappa = \frac{k}{\beta} = \frac{p}{\hbar\beta}, \qquad |\kappa\rangle = \sqrt{\beta\hbar}|p\rangle,$$

where

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}.$$

We start with the differential equation

$$\frac{\hbar}{m\omega_0}\frac{\partial}{\partial x}\langle x|\alpha\rangle + (x - \sqrt{2}\frac{\alpha}{\beta})\langle x|\alpha\rangle = 0,$$

or

$$\frac{1}{\beta^2}\sqrt{\beta}\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\langle\xi|\alpha\rangle + \sqrt{\beta}(\frac{\xi}{\beta} - \sqrt{2}\frac{\alpha}{\beta})\langle\xi|\alpha\rangle = 0,$$

or

$$\frac{\partial}{\partial\xi}\langle\xi|\alpha\rangle + (\xi - \sqrt{2}\alpha)\langle\xi|\alpha\rangle = 0.$$

The solution

$$\begin{aligned}\langle\xi|\alpha\rangle &= C\exp[-\frac{1}{2}(\xi - \sqrt{2}\alpha)^2] \\ &= C'\exp[-\frac{1}{2}(\xi - \sqrt{2}\,\mathrm{Re}\,\alpha)^2 + i\xi\sqrt{2}\,\mathrm{Im}\,\alpha)] \\ &= \frac{1}{\pi^{1/4}}\exp[-\frac{1}{2}(\xi - \xi_0)^2 + i\kappa_0\xi]\end{aligned}$$

where

$$\xi_0 = \sqrt{2}\,\mathrm{Re}\alpha, \qquad \kappa_0 = \sqrt{2}\,\mathrm{Im}\alpha,$$

$$(\Delta\xi)^2 = \frac{1}{2}, \qquad (\Delta\kappa)^2 = \frac{1}{2}.$$

$$\int_{-\infty}^{\infty} d\xi \exp[-(\xi - \xi_0)^2] = \pi^{1/2}$$

$$\langle\xi|\kappa\rangle = \frac{1}{\sqrt{2\pi}}e^{i\kappa\xi}.$$

((**Note**))

$$\left|\alpha(t)\right\rangle = e^{-\frac{i\omega_0 t}{2}}\left|\alpha e^{-i\omega_0 t}\right\rangle,$$

with

$$\left\langle \xi \middle| \alpha(t)\right\rangle = \frac{1}{\pi^{1/4}}\exp[-\frac{1}{2}(\xi-\xi_0)^2 + i\kappa_0\xi],$$ (Gaussian)

where

$$\xi_0 = \sqrt{2}\,\mathrm{Re}[\alpha e^{-i\omega_0 t}] = \sqrt{2}\left|\alpha\right|\cos(\omega_0 t-\theta),$$

$$\kappa_0 = \sqrt{2}\,\mathrm{Im}[[\alpha e^{-i\omega_0 t}] = -\sqrt{2}\left|\alpha\right|\sin(\omega_0 t-\theta),$$

where

$$\alpha = \left|\alpha\right| e^{i\theta}.$$

Suppose that $\left|\alpha\right| = \sqrt{n}$, then we have

$$\xi_0 = \sqrt{2n}\cos(\omega_0 t-\theta).$$

The classical turning point is

$$\xi_{class} = \sqrt{2n+1} \approx \sqrt{2n}.$$

So the coherent state remains a coherent state under the free field evolution. A coherent state-wave function moves through the harmonic oscillator potential, between the classical turning points, without dispersion.

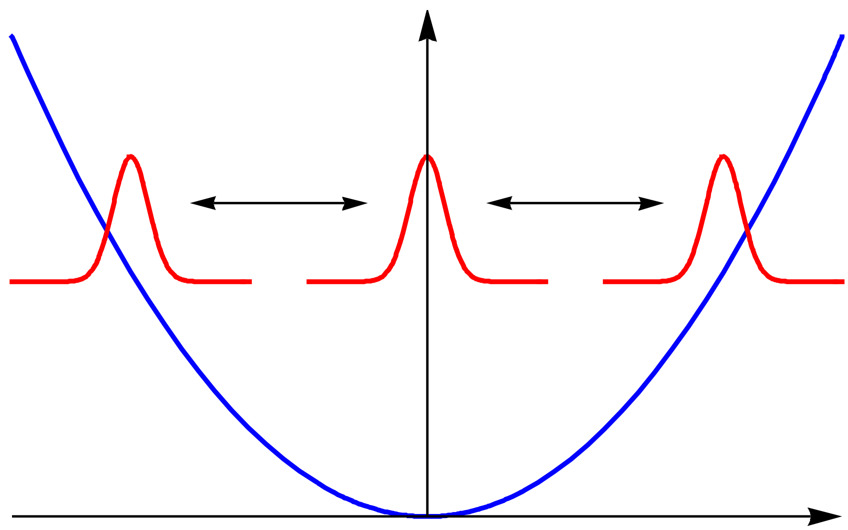


Fig.4 The movement of a coherent state-wave function through the harmonic oscillator potential, between the classical turning points, without dispersion (C.C. Gerry and P.L. Knight, Introductory Quantum Optics).

15. Canonical coordinate and momentum

From

$$\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right), \qquad \hat{a}^+ = \frac{\beta}{\sqrt{2}}\left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right),$$

we get

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a} + \hat{a}^+\right),$$

$$\hat{p} = \frac{\beta\hbar}{\sqrt{2}i}(\hat{a} - \hat{a}^+).$$

The expectation values of the position and momentum in the coherent state are given by

$$\langle\alpha|\hat{x}|\alpha\rangle = \frac{1}{\sqrt{2}\beta}\left(\alpha + \alpha^*\right) = \frac{\sqrt{2}|\alpha|}{\beta}\cos\theta,$$

$$\langle\alpha|\hat{p}|\alpha\rangle = \frac{\beta\hbar}{\sqrt{2}i}(\alpha - \alpha^*) = \sqrt{2}\beta\hbar|\alpha|\sin\theta,$$

where

$$\alpha = |\alpha| e^{i\theta}.$$

The expectation values of $\hat{x}^2$ and $\hat{p}^2$ can be calculated as

$$\langle \alpha | \hat{x}^2 | \alpha \rangle = \frac{1}{2\beta^2} [(\alpha + \alpha^*)^2 + 1],$$

$$\langle \alpha | \hat{p}^2 | \alpha \rangle = \frac{\beta^2 \hbar^2}{2} [1 - (\alpha^* - \alpha)^2],$$

where we use

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \langle \alpha | \hat{a}^+ = \langle \alpha | \alpha^*, \qquad [\hat{a}, \hat{a}^+] = \hat{1}.$$

The variances Δp and Δx are obtained by

$$\begin{aligned} (\Delta x)^2 &= \langle \alpha | \hat{x}^2 | \alpha \rangle - \langle \alpha | \hat{x} | \alpha \rangle^2 \\ &= \frac{1}{2\beta^2} [(\alpha + \alpha^*)^2 + 1] - \frac{1}{2\beta^2} (\alpha + \alpha^*)^2 \\ &= \frac{1}{2\beta^2} \end{aligned}$$

and

$$\begin{aligned} (\Delta p)^2 &= \langle \alpha | \hat{p}^2 | \alpha \rangle - \langle \alpha | \hat{p} | \alpha \rangle^2 \\ &= \frac{\beta^2 \hbar^2}{2} [1 - (\alpha^* - \alpha)^2] + \frac{\beta^2 \hbar^2}{2} (\alpha - \alpha^*)^2 \\ &= \frac{\beta^2 \hbar^2}{2} \end{aligned}$$

The minimum uncertainty relation

$$\Delta x \Delta p = \sqrt{\frac{1}{2\beta^2}} \sqrt{\frac{\beta^2 \hbar^2}{2}} = \frac{\hbar}{2},$$

is satisfied in the coherent state.

16. Phase operator

We define the phase operator as

$$\hat{a} = (\hat{n} + \hat{1})^{1/2} e^{i\hat{\phi}},$$

$$\hat{a}^{+} = e^{-i\hat{\phi}} (\hat{n} + \hat{1})^{1/2},$$

or

$$e^{i\hat{\phi}} = (\hat{n} + \hat{1})^{-1/2} \hat{a},$$

$$e^{-i\hat{\phi}} = \hat{a}^{+} (\hat{n} + \hat{1})^{-1/2},$$

Then we get

$$\begin{aligned} e^{i\hat{\phi}} e^{-i\hat{\phi}} &= (\hat{n} + \hat{1})^{-1/2} \hat{a}\hat{a}^{+} (\hat{n} + \hat{1})^{-1/2} \\ &= (\hat{n} + \hat{1})^{-1/2} (\hat{a}^{+}\hat{a} + \hat{1})(\hat{n} + \hat{1})^{-1/2} \\ &= \hat{1} \end{aligned}$$

However, the reverse-order product is not equal to unity. The phase operator has the number expansion

$$\begin{aligned} e^{i\hat{\phi}} &= \sum_{n=0}^{\infty} (\hat{n} + 1)^{-1/2} \hat{a} |n\rangle\langle n| \\ &= \sum_{n=0}^{\infty} (n)^{-1/2} \sqrt{n} |n - 1\rangle\langle n| \\ &= \sum_{n=1}^{\infty} |n\rangle\langle n + 1| \end{aligned}$$

The Hermite conjugate is

$$(e^{i\hat{\phi}})^{+} = \sum_{n=1}^{\infty} |n + 1\rangle\langle n| = e^{-i\hat{\phi}},$$

since

$$
\begin{aligned}
e^{-i\hat{\phi}} &= \hat{a}^{+}(\hat{n}+\hat{1})^{-1/2} \\
&= \sum_{n=0}^{\infty} \hat{a}^{+}(\hat{n}+\hat{1})^{-1/2}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} \hat{a}^{+}(n+1)^{-1/2}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} (n+1)^{-1/2}\hat{a}^{+}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} (n+1)^{-1/2}(n+1)^{1/2}|n+1\rangle\langle n| \\
&= \sum_{n=0}^{\infty} |n+1\rangle\langle n| = (e^{i\hat{\phi}})^{+}
\end{aligned}
$$

It is easy to see that

$$e^{i\hat{\phi}}(e^{i\hat{\phi}})^{+} = (e^{i\hat{\phi}})^{+}e^{i\hat{\phi}} = \hat{1}.$$

We also note that

$$e^{i\hat{\phi}}|n\rangle = (\hat{n}+\hat{1})^{-1/2}\hat{a}|n\rangle = (\hat{n}+\hat{1})^{-1/2}\sqrt{n}|n-1\rangle = |n-1\rangle, \qquad \text{for } n\neq 0,$$

$$e^{-i\hat{\phi}}|n\rangle = \hat{a}^{+}(\hat{n}+\hat{1})^{-1/2}|n\rangle = (n+1)^{-1/2}\hat{a}^{+}|n\rangle = |n+1\rangle.$$

The phase operator $e^{i\hat{\phi}}$ is not Hermitian and it cannot represent observable properties of the electromagnetic field. Here we define the Hermitian operator as

$$\cos\hat{\phi} = \frac{1}{2}(e^{i\hat{\phi}} + e^{-i\hat{\phi}}),$$

and

$$\sin\hat{\phi} = \frac{1}{2i}(e^{i\hat{\phi}} - e^{-i\hat{\phi}}).$$

((**Note**)) **Phase operator**
Luca Salasnich, Quantum Physics of Light and Matter; A Modern Introduction to Photons, Atoms and Many-Body Systems (Springer, 2014).

Here we discuss the origin of the definition for the phase operator.

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad \hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{N} = \hat{a}^{+}\hat{a}$$

$$\begin{aligned}\hat{a} &= \sum_{m=0}^{\infty} \hat{a}|m\rangle\langle m| \\ &= \sum_{m=0}^{\infty} \sqrt{m}\,|m-1\rangle\langle m| \\ &= \sum_{m=0}^{\infty} \sqrt{m+1}\,|m\rangle\langle m+1| \\ &= |0\rangle\langle 1| + \sqrt{2}\,|1\rangle\langle 2| + \sqrt{3}\,|2\rangle\langle 3| + \ldots\end{aligned}$$

$$\begin{aligned}\hat{a}^{+} &= \sum_{m=0}^{\infty} \hat{a}^{+}|m\rangle\langle m| \\ &= \sum_{m=0}^{\infty} \sqrt{m+1}\,|m+1\rangle\langle m| \\ &= |1\rangle\langle 0| + \sqrt{2}\,|2\rangle\langle 1| + \sqrt{3}\,|3\rangle\langle 2| + \ldots\end{aligned}$$

Phase operator

$$\hat{f} = \hat{a}\hat{N}^{-1/2}, \qquad \hat{f}^{+} = \hat{N}^{-1/2}\hat{a}^{+},$$

$$\begin{aligned}\hat{f} &= \sum_{m=0}^{\infty} \sqrt{m+1}\,|m\rangle\langle m+1|\,\hat{N}^{-1/2} \\ &= \sum_{m=0}^{\infty} \sqrt{m+1}\,\frac{1}{\sqrt{m+1}}\,|m\rangle\langle m+1| \\ &= \sum_{m=0}^{\infty} |m\rangle\langle m+1| \\ &= |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3| + \ldots\end{aligned}$$

$$\begin{aligned}\hat{f}^{+} &= \sum_{m=0}^{\infty} \hat{N}^{-1/2}\sqrt{m+1}\,|m+1\rangle\langle m| \\ &= \sum_{m=0}^{\infty} \sqrt{m+1}\,\frac{1}{\sqrt{m+1}}\,|m+1\rangle\langle m+1| \\ &= \sum_{m=0}^{\infty} |m+1\rangle\langle m| \\ &= |1\rangle\langle 0| + |2\rangle\langle 1| + |3\rangle\langle 2| + \ldots\end{aligned}$$

$$\hat{f}\left|n\right\rangle = \hat{a}\hat{N}^{-1/2}\left|n\right\rangle = n^{-1/2}\hat{a}\left|n\right\rangle = n^{-1/2}\sqrt{n}\left|n-1\right\rangle = \left|n-1\right\rangle$$

$$\hat{f}^{+}\left|n\right\rangle = \hat{N}^{-1/2}\hat{a}^{+}\left|n\right\rangle = \hat{N}^{-1/2}\sqrt{n+1}\left|n+1\right\rangle = \left|n+1\right\rangle$$

These operators are similar to the translation operators such that

$$\hat{T}_a\left|x\right\rangle = \left|x+a\right\rangle, \qquad \hat{T}_a^{+}\left|x+a\right\rangle = \left|x\right\rangle, \qquad \hat{T}_a^{+}\left|x\right\rangle = \left|x-a\right\rangle$$

where

$$\hat{T}_a = \exp(-\frac{i}{\hbar}\hat{p}a)$$

We assume that

$$\hat{f} = e^{i\hat{\theta}}$$

with

$$\hat{f}\left|n\right\rangle = \left|n-1\right\rangle, \qquad \hat{f}^{+}\left|n\right\rangle = \left|n+1\right\rangle$$

$$\hat{f}\,\hat{f}^{+}\left|n\right\rangle = \hat{f}\left|n+1\right\rangle = \left|n\right\rangle$$

$$\hat{f} = \sum_{n=0}^{\infty}\hat{f}\left|n\right\rangle\left\langle n\right| = \sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n+1\right|$$

$$\hat{f}^{+} = \sum_{n=0}^{\infty}\hat{f}^{+}\left|n\right\rangle\left\langle n\right| = \sum_{n=0}^{\infty}\left|n+1\right\rangle\left\langle n\right|$$

$$\begin{aligned}\hat{f}\,\hat{f}^{+} &= \sum_{n,m=0}^{\infty}\left|n\right\rangle\left\langle n+1\middle|m+1\right\rangle\left\langle m\right| \\ &= \sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n\right| \\ &= \hat{1}\end{aligned}$$

$$\hat{f}^{+}\hat{f} = \sum_{n,m=0}^{\infty} \left|m+1\right\rangle\left\langle m\middle|n\right\rangle\left\langle n+1\right|$$
$$= \sum_{n=0}^{\infty}\left|n+1\right\rangle\left\langle n+1\right|$$
$$= \hat{1} - \left|0\right\rangle\left\langle 0\right|$$

$$\left\langle\alpha\right|\hat{f}\left|\alpha\right\rangle = \left\langle\alpha\right|\hat{a}\hat{N}^{-1/2}\left|\alpha\right\rangle$$

Note that

$$\left|\alpha\right\rangle = \exp(-\frac{\left|\alpha\right|^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle$$

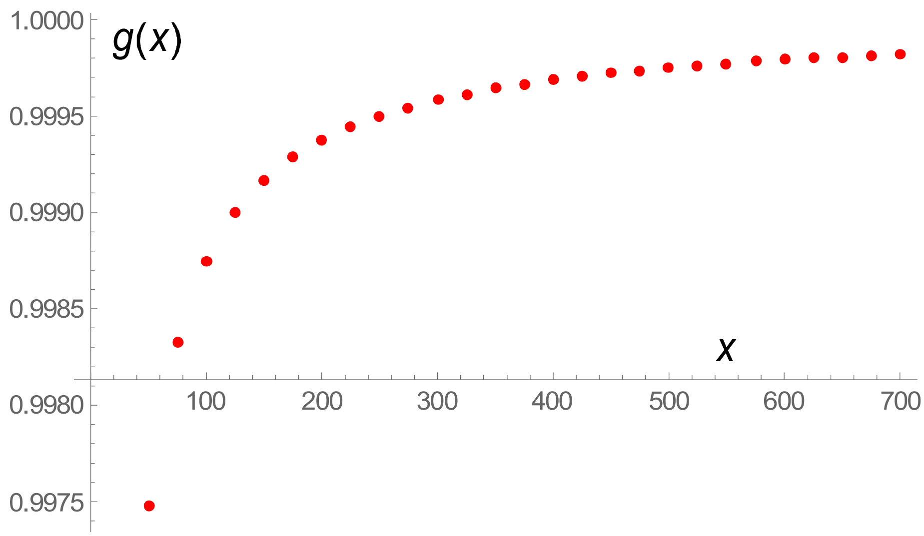
$$\hat{a}\hat{N}^{-1/2}\left|\alpha\right\rangle = \hat{a}\hat{N}^{-1/2}\exp(-\frac{\left|\alpha\right|^2}{2})\sum_{m=0}^{\infty}\frac{\left(\alpha^*\right)^m}{\sqrt{m!}}\hat{N}^{-1/2}\left|m\right\rangle$$
$$= \exp(-\frac{\left|\alpha\right|^2}{2})\sum_{m=0}^{\infty}\frac{\left(\alpha^*\right)^m}{\sqrt{m!}}m^{-1/2}\hat{a}\left|m\right\rangle$$
$$= \exp(-\frac{\left|\alpha\right|^2}{2})\sum_{n=0}^{\infty}\frac{\left(\alpha^*\right)^m}{\sqrt{m!}}\left|m-1\right\rangle$$
$$= \exp(-\frac{\left|\alpha\right|^2}{2})\sum_{n=0}^{\infty}\frac{\left(\alpha^*\right)^{m+1}}{\sqrt{(m+1)!}}\left|m\right\rangle$$

$$\left\langle\alpha\right|\hat{f}\left|\alpha\right\rangle = \exp(-\left|\alpha\right|^2)\sum_{m,n=0}^{\infty}\frac{(\alpha)^n}{\sqrt{n!}}\langle n|\frac{\left(\alpha^*\right)^{m+1}}{\sqrt{(m+1)!}}\left|m\right\rangle$$
$$= \exp(-\left|\alpha\right|^2)\sum_{m,n=0}^{\infty}\frac{(\alpha)^n\left(\alpha^*\right)^{m+1}}{\sqrt{n!}\sqrt{(m+1)!}}\left\langle n\middle|m\right\rangle$$
$$= \exp(-\left|\alpha\right|^2)\sum_{m,n=0}^{\infty}\frac{(\alpha)^n\left(\alpha^*\right)^{m+1}}{\sqrt{n!}\sqrt{(m+1)!}}\delta_{n,m}$$
$$= \alpha\exp(-\left|\alpha\right|^2)\sum_{n=0}^{\infty}\frac{\left|\alpha\right|^{2n}}{\sqrt{n!}\sqrt{(n+1)!}}$$

Here we define a function

$$g(x) = \sqrt{x}\exp(-x)\sum_{n=0}^{N}\frac{x^n}{\sqrt{n!(n+1)!}}$$

When we choose $N = 1000$. We make a plot of $g(x)$ as a function of x. For x>>1, $g(x)$ tends to approach unity.



Fig,5 Plot of $g(x)$ as a function of x.

When $x = |\alpha|^2 >> 1$, we get

$$g(x = |\alpha|^2) = |\alpha| \exp(-|\alpha|^2) \sum_{n=0}^{N} \frac{|\alpha|^{2n}}{\sqrt{n!(n+1)!}} \approx 1$$

or

$$\exp(-|\alpha|^2) \sum_{n=0}^{N} \frac{|\alpha|^{2n}}{\sqrt{n!(n+1)!}} \approx \frac{1}{|\alpha|}.$$

Using this approximation,

$$\langle \alpha | \hat{f} | \alpha \rangle = \frac{\alpha}{|\alpha|} = e^{i\theta}.$$

for $\langle \alpha | \hat{N} | \alpha \rangle = |\alpha|^2 >> 1$, with $\alpha = |\alpha| e^{i\theta} = \sqrt{\langle N \rangle} e^{i\theta}$. Thus, for a large average number $\langle N \rangle$ of photons the expectation value of the phase operator $\hat{f}$ on the coherent state $|\alpha\rangle$ gives the phase factor $e^{i\theta}$ of the complex eigenvalue α of the coherent state.

17. Commutation relation

$$[\hat{n},\hat{a}] = -\hat{a}\,, \qquad [\hat{n},\hat{a}^{+}] = \hat{a}^{+}\,,$$

Using the relations

$$\hat{a} = (\hat{n}+\hat{1})^{1/2} e^{i\hat{\phi}}\,,$$

$$\hat{a}^{+} = e^{-i\hat{\phi}} (\hat{n}+\hat{1})^{1/2}\,,$$

we get

$$[\hat{n}, e^{i\hat{\phi}}] = -e^{i\hat{\phi}}\,, \qquad [\hat{n}, e^{-i\hat{\phi}}] = e^{-i\hat{\phi}}\,,$$

since

$$\hat{n} = \hat{a}^{+}\hat{a} = e^{-i\hat{\phi}}(\hat{n}+\hat{1})^{1/2}(\hat{n}+\hat{1})^{1/2}e^{i\hat{\phi}} = e^{-i\hat{\phi}}(\hat{n}+\hat{1})e^{i\hat{\phi}} = e^{-i\hat{\phi}}\hat{n}e^{i\hat{\phi}} + +\hat{1}.$$

(i)

$$e^{i\hat{\varphi}}\hat{n} = \hat{n}e^{i\hat{\varphi}} + e^{i\hat{\varphi}}, \qquad \text{leading to} \qquad [\hat{n}, e^{i\hat{\phi}}] = -e^{i\hat{\phi}}\,.$$

(ii)

$$\hat{n}e^{-i\hat{\varphi}} = e^{-i\hat{\varphi}}\hat{n} + e^{-i\hat{\varphi}}, \qquad \text{leading to} \qquad [\hat{n}, e^{-i\hat{\phi}}] = e^{-i\hat{\phi}}\,.$$

Then we have

$$[\hat{n}, \cos\hat{\phi}] = -i\sin\hat{\phi}, \qquad [\hat{n}, \sin\hat{\phi}] = i\cos\hat{\phi}.$$

These equations lead to the inequalities,

$$(\Delta n)(\Delta \cos\phi) \geq \frac{1}{2}\left|\langle \sin\phi \rangle\right|,$$

$$(\Delta n)(\Delta \sin\phi) \geq \frac{1}{2}\left|\langle \cos\phi \rangle\right|,$$

using the Schwarz inequality. Then we get

$$(\Delta n)^2 \frac{[(\Delta\cos\phi)^2 + (\Delta\sin\phi)^2]}{\langle\sin\phi\rangle^2 + \langle\cos\phi\rangle^2} \geq \frac{1}{4}.$$

18. Coherent state

Coherent state:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

$$\langle n\rangle = \langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2,$$

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}\hat{a}^+\hat{a}|\alpha\rangle = \langle\alpha|\hat{a}^+(\hat{a}^+\hat{a}+\hat{1})\hat{a}|\alpha\rangle = |\alpha|^4 + |\alpha|^2.$$

Then we have

$$(\Delta n)^2 = \langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2 = |\alpha|^2.$$

or

$$\Delta n = |\alpha|.$$

The fraction uncertainty in the number is

$$\frac{\Delta n}{\langle n\rangle} = \frac{|\alpha|}{|\alpha|^2} = \frac{1}{|\alpha|}.$$

For $|\alpha| >> 1$, (see the note below)

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = \cos\theta(1 - \frac{1}{8|\alpha|^2}) + ...$$

$$\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle = \cos^2\theta - \frac{(\cos^2\theta - \frac{1}{2})}{2|\alpha|^2} - ...$$

where

$$\alpha = |\alpha|e^{i\theta}.$$

((**Note**))

From the above, we have the relations,

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

$$e^{i\hat{\phi}} = (\hat{n}+\hat{1})^{-1/2}\hat{a},$$

$$e^{-i\hat{\phi}} = \hat{a}^+(\hat{n}+\hat{1})^{-1/2}.$$

Then we get

$$\begin{aligned}
\cos\hat{\phi}|\alpha\rangle &= (\frac{e^{i\hat{\phi}}+e^{-i\hat{\phi}}}{2})|\alpha\rangle \\
&= \frac{1}{2}[(\hat{n}+\hat{1})^{-1/2}\hat{a}+\hat{a}^+(\hat{n}+\hat{1})^{-1/2}]|\alpha\rangle \\
&= \frac{1}{2}\exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}[(\hat{n}+\hat{1})^{-1/2}\hat{a}+\hat{a}^+(\hat{n}+\hat{1})^{-1/2}]\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\
&= \frac{1}{2}\exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}[|n-1\rangle+|n+1\rangle]
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\cos\hat{\phi}|\alpha\rangle &= \frac{1}{2}\exp(-|\alpha|^2)\sum_{n,m=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}[\langle m|n-1\rangle+\langle m|n+1\rangle] \\
&= \frac{1}{2}\exp(-|\alpha|^2)\sum_{n,m=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}[\delta_{m,n-1}+\delta_{m,n+1}] \\
&= \frac{1}{2}\exp(-|\alpha|^2)\sum_{n}^{\infty}[\frac{\alpha^{*n-1}\alpha^n}{\sqrt{n!(n-1)!}}+\frac{\alpha^{*n+1}\alpha^n}{\sqrt{n!(n+1)!}}] \\
&= \frac{1}{2}\exp(-|\alpha|^2)\sum_{n}^{\infty}\frac{\alpha^{*n}\alpha^{n+1}+\alpha^{*n+1}\alpha^n}{\sqrt{n!(n+1)!}} \\
&= \frac{1}{2}(\alpha+\alpha^*)\exp(-|\alpha|^2)\sum_{n}^{\infty}\frac{|\alpha|^{2n}}{\sqrt{n!(n+1)!}} \\
&= |\alpha|\cos\theta\exp(-|\alpha|^2)\sum_{n}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}
\end{aligned}$$

or

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = |\alpha|\cos\theta\exp(-|\alpha|^2)\sum_{n}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}. \qquad (1)$$

We also get

$$\langle n|\cos\hat{\phi}|\alpha\rangle = \frac{1}{2}\exp(-\frac{1}{2}|\alpha|^2)\langle n|\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}[|m-1\rangle + |m+1\rangle]$$
$$= \frac{1}{2}\exp(-\frac{1}{2}|\alpha|^2)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}[\langle n|m-1\rangle + \langle n|m+1\rangle]$$
$$= \frac{1}{2}\exp(-\frac{1}{2}|\alpha|^2)[\frac{\alpha^{n+1}}{\sqrt{(n+1)!}} + \frac{\alpha^{n-1}}{\sqrt{(n-1)!}}]$$

$$\langle n|\cos\hat{\phi}|\alpha\rangle^* = \langle\alpha|\cos\hat{\phi}|n\rangle$$
$$= \frac{1}{2}\exp(-\frac{1}{2}|\alpha|^2)[\frac{\alpha^{*n+1}}{\sqrt{(n+1)!}} + \frac{\alpha^{*n-1}}{\sqrt{(n-1)!}}]$$

$$\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle = \sum_{n=0}^{\infty}\langle\alpha|\cos\hat{\phi}|n\rangle\langle n|\cos\hat{\phi}|\alpha\rangle$$
$$= \frac{1}{4}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}(\frac{\alpha^{n+1}}{\sqrt{(n+1)!}} + \frac{\alpha^{n-1}}{\sqrt{(n-1)!}})(\frac{\alpha^{*n+1}}{\sqrt{(n+1)!}} + \frac{\alpha^{*n-1}}{\sqrt{(n-1)!}})$$
$$= \frac{1}{4}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}[\frac{|\alpha|^{2(n+1)}}{(n+1)!} + \frac{|\alpha|^{2(n-1)}}{(n-1)!} + \frac{\alpha^{n+1}\alpha^{*n-1} + \alpha^{n-1}\alpha^{*n+1}}{\sqrt{(n-1)!(n+1)!}}]$$
$$= \frac{1}{4}\exp(-|\alpha|^2)[2\exp(-|\alpha|^2) - 1] + \frac{1}{4}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}[\frac{\alpha^{n+1}\alpha^{*n-1} + \alpha^{n-1}\alpha^{*n+1}}{\sqrt{(n-1)!(n+1)!}}]$$
$$= \frac{1}{2} - \frac{1}{4}\exp(-|\alpha|^2) + \frac{1}{4}\exp(-|\alpha|^2)(\alpha^2 + \alpha^{*2})\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}$$

Noting that

$$\alpha^2 + \alpha^{*2} = |\alpha|^2\exp(2i\theta) + |\alpha|^2\exp(-2i\theta)$$
$$= 2|\alpha|^2\cos(2\theta)$$
$$= 4|\alpha|^2(\cos^2\theta - \frac{1}{2})$$

we have

$$\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle = \frac{1}{2} - \frac{1}{4}\exp(-|\alpha|^2) + |\alpha|^2(\cos^2\theta - \frac{1}{2})\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}} \qquad (2)$$

Here we use the asymptotic expansions for $|\alpha|^2 >> 1$ (R. Loudon, 2nd edition)

$$\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} = \frac{\exp(|\alpha|^2)}{|\alpha|}(1 - \frac{1}{8|\alpha|^2} + ...) \ ,$$

$$\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}} = \frac{\exp(|\alpha|^2)}{|\alpha|^2}(1 - \frac{1}{2|\alpha|^2} + ...).$$

((**Note**)) Mathematics formula; C.C. Gerry and P.L. Knight, Introductory Quantum Optics

When $|\alpha| \to \infty$,

$$\lim_{|\alpha|\to\infty}\exp(-|\alpha|^2) = 0,$$

$$\lim_{|\alpha|\to\infty}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}} = 1,$$

$$\lim_{|\alpha|\to\infty}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{n!\sqrt{n+1}\sqrt{n+2}} = 1.$$

Thus for large mean numbers of photons, the phase expectation values become

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = |\alpha|\cos\theta\exp(-|\alpha|^2)\frac{\exp(|\alpha|^2)}{|\alpha|}(1 - \frac{1}{8|\alpha|^2} + ...)$$

$$= \cos\theta(1 - \frac{1}{8|\alpha|^2} + ...)$$

$$\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle = \frac{1}{2} - \frac{1}{4}\exp(-|\alpha|^2) + (\cos^2\theta - \frac{1}{2})(1 - \frac{1}{2|\alpha|^2} + ...)$$
$$= \frac{1}{2} + (\cos^2\theta - \frac{1}{2})(1 - \frac{1}{2|\alpha|^2} + ...)$$
$$= \cos^2\theta - \frac{1}{2|\alpha|^2}(\cos^2\theta - \frac{1}{2}) - ...$$

Note that

$$(\Delta\cos\phi)^2 = \langle\alpha|\cos^2\hat{\phi}|\alpha\rangle - \langle\alpha|\cos\hat{\phi}|\alpha\rangle^2$$
$$= \cos^2\theta - \frac{(\cos^2\theta - \frac{1}{2})}{2|\alpha|^2} - \cos^2\theta(1 - \frac{1}{8|\alpha|^2})^2$$
$$\approx \cos^2\theta - \frac{(\cos^2\theta - \frac{1}{2})}{2|\alpha|^2} - \cos^2\theta(1 - \frac{1}{4|\alpha|^2})$$
$$= \frac{\sin^2\theta}{4|\alpha|^2}$$

The phase uncertainty is

$$\Delta\cos\phi = \frac{\sin\theta}{2|\alpha|},$$

where

$$\alpha = |\alpha|e^{i\theta}.$$

The product of uncertainties is

$$\Delta n(\Delta\cos\phi) = |\alpha|\frac{\sin\theta}{2|\alpha|} = \frac{\sin\theta}{2}.$$

We also note that

$$\langle\alpha|\sin\hat{\phi}|\alpha\rangle = \sin\theta,$$

for $|\alpha| >> 1$. Thus we have

$$\Delta n(\Delta\cos\phi) = \frac{1}{2}\left|\langle\alpha|\sin\hat{\phi}|\alpha\rangle\right|,$$

which means that the coherent state has the minimum uncertainty product.

If we use the relation $\langle\alpha|\sin\hat{\phi}|\alpha\rangle = \sin\theta$ (whose proof will be givem below), we get

$$\Delta n(\Delta\cos\phi) = \frac{1}{2}\sin\theta.$$

((**Proof**)) $\langle\alpha|\sin\hat{\phi}|\alpha\rangle = \sin\theta$.

$$\begin{aligned}
\sin\hat{\phi}|\alpha\rangle &= \left(\frac{e^{i\hat{\phi}} - e^{-i\hat{\phi}}}{2i}\right)|\alpha\rangle \\
&= \frac{1}{2i}[(\hat{n}+\hat{1})^{-1/2}\hat{a} - \hat{a}^{+}(\hat{n}+\hat{1})^{-1/2}]|\alpha\rangle \\
&= \frac{1}{2i}\exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}[(\hat{n}+\hat{1})^{-1/2}\hat{a} - \hat{a}^{+}(\hat{n}+\hat{1})^{-1/2}]\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\
&= \frac{1}{2i}\exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}[|n-1\rangle - |n+1\rangle]
\end{aligned}$$

and

$$\begin{aligned}
\langle\alpha|\sin\hat{\phi}|\alpha\rangle &= \frac{1}{2i}\exp(-|\alpha|^2)\sum_{n,m=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}[\langle m|n-1\rangle - \langle m|n+1\rangle] \\
&= \frac{1}{2i}\exp(-|\alpha|^2)\sum_{n,m=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}[\delta_{m,n-1} - \delta_{m,n+1}] \\
&= \frac{1}{2i}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\left[\frac{\alpha^{*n-1}\alpha^n}{\sqrt{n!(n-1)!}} - \frac{\alpha^{*n+1}\alpha^n}{\sqrt{n!(n+1)!}}\right] \\
&= \frac{1}{2i}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^{*n}\alpha^{n+1} - \alpha^{*n+1}\alpha^n}{\sqrt{n!(n+1)!}} \\
&= \frac{1}{2i}(\alpha - \alpha^*)\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{\sqrt{n!(n+1)!}} \\
&= |\alpha|\sin\theta\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \sin\theta(1 - \frac{1}{8|\alpha|^2} + ...) \approx \sin\theta
\end{aligned}$$

19. The relation of $\Delta X = \Delta Y = \frac{1}{2}$ for the coherent state

$$\hat{X} = \frac{\beta}{\sqrt{2}}\hat{x} = \frac{1}{2}(\hat{a}+\hat{a}^{+}), \qquad \hat{Y} = \frac{1}{2i}(\hat{a}-\hat{a}^{+}).$$

Since

$$\hat{X}\left|\alpha\right\rangle = \frac{1}{2}(\hat{a}+\hat{a}^{+})\left|\alpha\right\rangle = \frac{1}{2}(\alpha+\alpha^{*})\left|\alpha\right\rangle = \left|\alpha\right|\cos\theta\left|\alpha\right\rangle,$$

$$\hat{Y}\left|\alpha\right\rangle = \frac{1}{2i}(\hat{a}-\hat{a}^{+})\left|\alpha\right\rangle = \frac{1}{2i}(\alpha-\alpha^{*})\left|\alpha\right\rangle = \left|\alpha\right|\sin\theta\left|\alpha\right\rangle.$$

$\left|\alpha\right\rangle$ is the eigenket of $\hat{X}$ with the eigenvalue $\left|\alpha\right|\cos\theta$, and is also the eigenket of $\hat{Y}$ with the eigenvalue $\left|\alpha\right|\sin\theta$, where

$$\alpha = \left|\alpha\right|e^{i\theta}.$$

We note that

$$\left\langle\alpha\right|\hat{X}\left|\alpha\right\rangle = \left\langle\alpha\right|\frac{1}{2}(\hat{a}+\hat{a}^{+})\left|\alpha\right\rangle = \frac{1}{2}(\alpha+\alpha^{*}) = \frac{1}{2}\left|\alpha\right|(2\cos\theta) = \left|\alpha\right|\cos\theta,$$

$$\left\langle\alpha\right|\hat{Y}\left|\alpha\right\rangle = \left\langle\alpha\right|\frac{1}{2i}(\hat{a}-\hat{a}^{+})\left|\alpha\right\rangle = \frac{1}{2i}(\alpha-\alpha^{*}) = \frac{1}{2i}\left|\alpha\right|(2i\sin\theta) = \left|\alpha\right|\sin\theta,$$

$$\begin{aligned}\left\langle\alpha\right|\hat{X}^{2}\left|\alpha\right\rangle &= \left\langle\alpha\right|\frac{1}{4}(\hat{a}+\hat{a}^{+})(\hat{a}+\hat{a}^{+})\left|\alpha\right\rangle \\ &= \frac{1}{4}\left\langle\alpha\right|(\hat{a})^{2}+(\hat{a}^{+})^{2}+2\hat{a}^{+}\hat{a}+\hat{1})\left|\alpha\right\rangle \\ &= \frac{1}{4}(\alpha^{2}+\alpha^{*2}+2\left|\alpha\right|^{2}+1)\end{aligned}$$

$$\begin{aligned}\left\langle\alpha\right|\hat{Y}^{2}\left|\alpha\right\rangle &= -\left\langle\alpha\right|\frac{1}{4}(\hat{a}-\hat{a}^{+})(\hat{a}-\hat{a}^{+})\left|\alpha\right\rangle \\ &= -\frac{1}{4}\left\langle\alpha\right|(\hat{a})^{2}+(\hat{a}^{+})^{2}-2\hat{a}^{+}\hat{a}-\hat{1})\left|\alpha\right\rangle \\ &= -\frac{1}{4}(\alpha^{2}+\alpha^{*2}-2\left|\alpha\right|^{2}-1)\end{aligned}$$

Then we have

$$(\Delta X)^2 = \langle\alpha|\hat{X}^2|\alpha\rangle - \langle\alpha|\hat{X}|\alpha\rangle^2 = \frac{1}{4},$$

$$(\Delta Y)^2 = \langle\alpha|\hat{Y}^2|\alpha\rangle - \langle\alpha|\hat{Y}|\alpha\rangle^2 = \frac{1}{4}.$$

So we get

$$\Delta X = \Delta Y = \frac{1}{2}.$$

((**Note**)) Vacuum state $|0\rangle$

$$(\Delta X)^2 = \langle 0|\hat{X}^2|0\rangle - \langle 0|\hat{X}|0\rangle^2 = \frac{1}{4},$$

$$(\Delta Y)^2 = \langle 0|\hat{Y}^2|0\rangle - \langle 0|\hat{Y}|0\rangle^2 = \frac{1}{4}.$$

$$\Delta X = \frac{1}{2}, \qquad \Delta Y = \frac{1}{2}$$

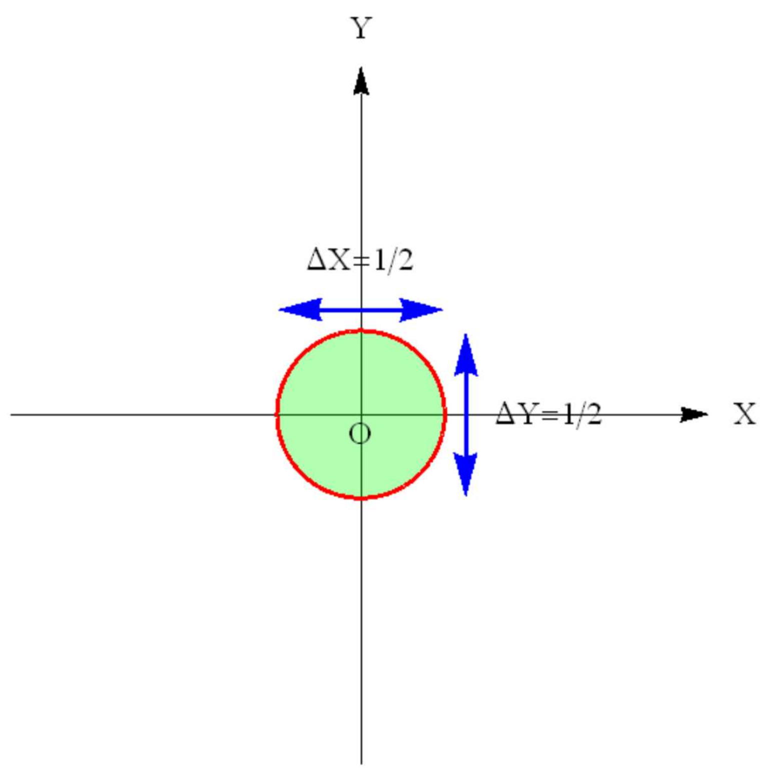


Fig.6 The 2D complex plane of $z = \langle \psi | \hat{a} | \psi \rangle = \langle X \rangle + i \langle Y \rangle = X + iY$. with $|\psi\rangle = |0\rangle$. $\Delta X = \frac{1}{2}$ and $\Delta Y = \frac{1}{2}$. $X = Y = 0$.

We also have several relations

$$\langle \alpha | \sin\hat{\phi} | \alpha \rangle = \sin\theta , \qquad \langle \alpha | \cos\hat{\phi} | \alpha \rangle = \cos\theta ,$$

$$\Delta n = |\alpha| , \qquad \Delta \cos\phi = \frac{\sin\theta}{2|\alpha|} ,$$

$$\langle n \rangle = |\alpha|^2 .$$

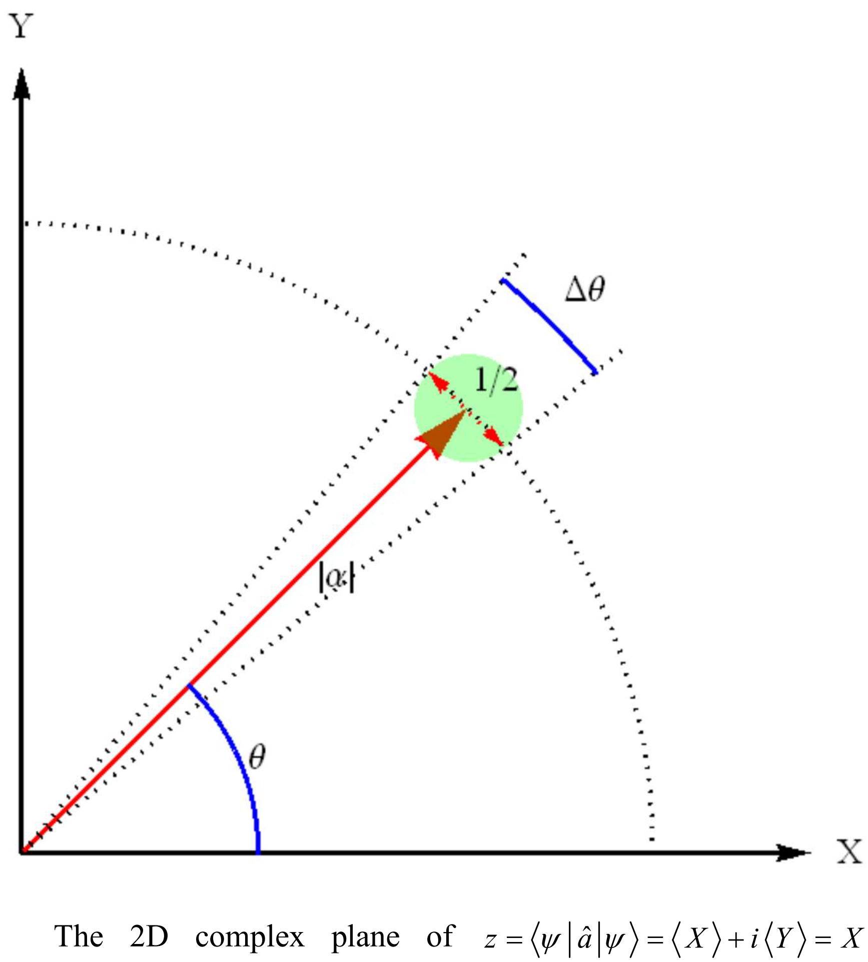


Fig.7 The 2D complex plane of $z = \langle \psi | \hat{a} | \psi \rangle = \langle X \rangle + i \langle Y \rangle = X + iY$ with $|\psi\rangle = |\alpha\rangle$. The uncertainty circle of a coherent state. $|\alpha| = \sqrt{\langle n \rangle}$. $\Delta\theta$ is the phase uncertainty. The state (denoted by green circle) is expressed by $|\alpha\rangle = \hat{D}_\alpha |0\rangle$. $\langle \alpha | \hat{X} | \alpha \rangle = |\alpha| \cos\theta$, $\langle \alpha | \hat{Y} | \alpha \rangle = |\alpha| \sin\theta$

At α>>1, simple geometry gives

$$\Delta\theta|\alpha| = \frac{1}{2}.$$

From this we can see that there is a tradeoff between number uncertainty and phase uncertainty

$$\Delta\theta\Delta n = \Delta\theta|\alpha| = \frac{1}{2},$$

which sometimes can be interpreted as the number-phase uncertainty.

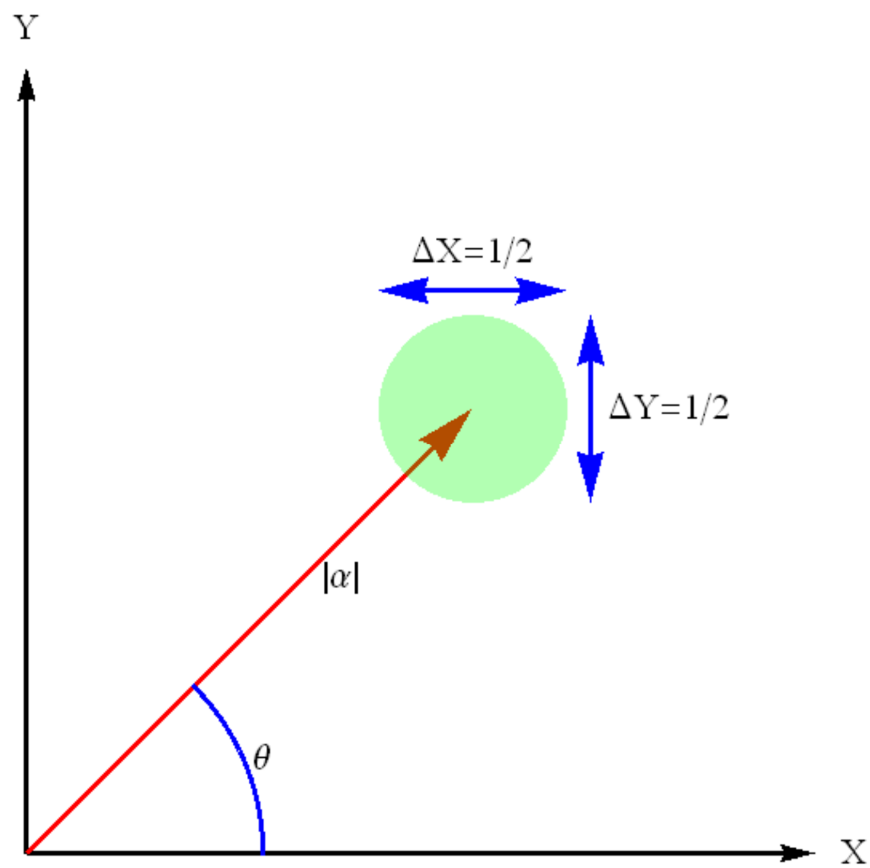


Fig.8 The phasor diagram for the coherent state $|\alpha\rangle$. The length of the phasor is equal to $|\alpha|$, and the angle from the X-axis is the phase θ. $\alpha = |\alpha|e^{i\theta}$ with $|\alpha| = \sqrt{\langle n\rangle}$. The quantum uncertainty is shown by a circle of diameter 1/2 at the end of the phasor. The state is defined by $\hat{D}_\alpha|0\rangle$.

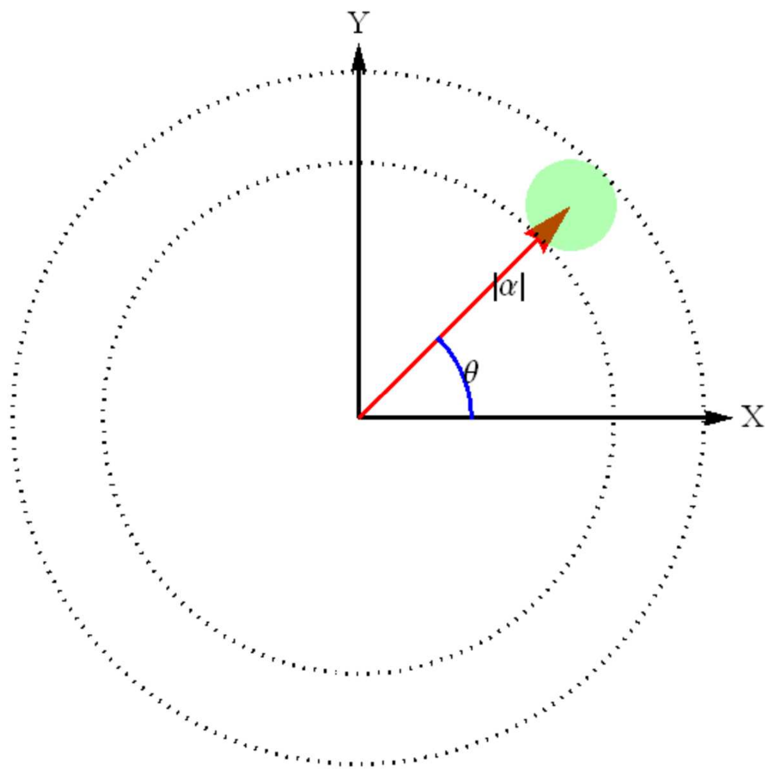


Fig.9 Coherent stated defined by $\hat{D}_\alpha|0\rangle$, where $\alpha = |\alpha| e^{i\theta}$.

20. Electric field variation

$$\hat{E}(\chi) = \hat{E}^+(\chi) + \hat{E}^-(\chi) = \frac{1}{2}[\hat{a}e^{-i\chi} + \hat{a}^+ e^{i\chi}] = \hat{X}\cos\chi + \hat{Y}\sin\chi\,,$$

where

$$\chi = \omega t - kz - \frac{\pi}{2}\,.$$

The average of the electric field is

$$\begin{aligned}\langle\alpha|\hat{E}(\chi)|\alpha\rangle &= \frac{1}{2}(\alpha e^{-i\chi} + \alpha^* e^{i\chi}) \\ &= |\alpha|\cos(\chi-\theta) \\ &= \mathrm{Re}[|\alpha| e^{i(\theta-\chi)}] \\ &= \mathrm{Re}[|\alpha| e^{i\theta} e^{-i\chi}]\end{aligned}\quad .$$

or more generally, we define

$$E = \mathrm{Re}[\tilde{E}e^{-i\chi}].$$

where $\tilde{E}$ is the phasor of electric field and is the complex number. For the coherent state, we have

$$\tilde{E} = \alpha = |\alpha| e^{i\theta}$$

We use the phasor $|\alpha| e^{i\theta}$ in the complex plane.

$$\mathrm{Re}[|\alpha| e^{i\theta}] = |\alpha|\cos\theta = \langle X\rangle, \qquad \mathrm{Im}[|\alpha| e^{i\theta}] = |\alpha|\sin\theta = \langle Y\rangle$$

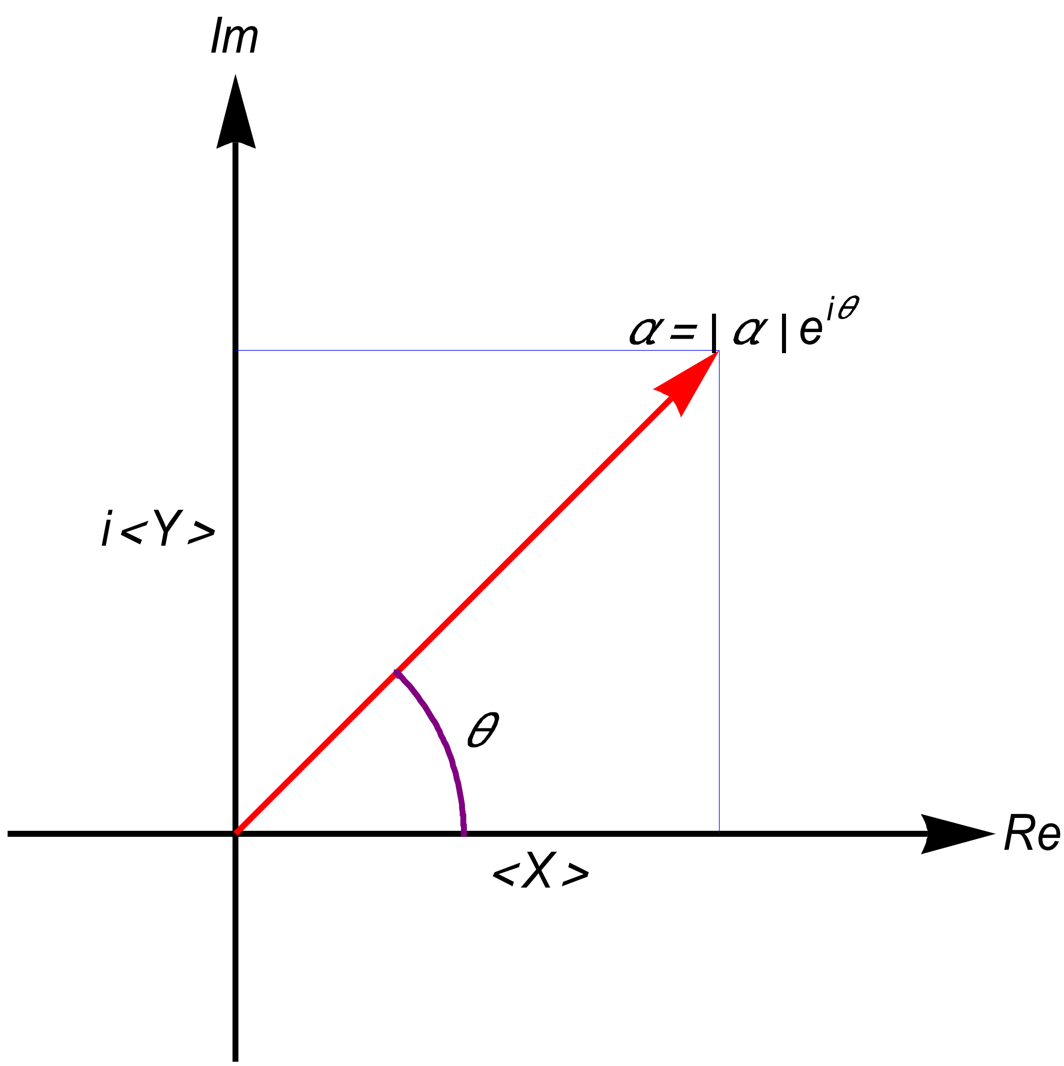


Fig.10 Phasor diagram of electric field in the complex plane. $\langle X\rangle = |\alpha|\cos\theta$. $\langle Y\rangle = |\alpha|\sin\theta$. Coherent state.

The variation:

$$\langle\alpha|\hat{E}^2(\chi)|\alpha\rangle = \frac{1}{4}\langle\alpha|(\hat{a}e^{-i\chi} + \hat{a}^+e^{i\chi})(\hat{a}^+e^{i\chi} + \hat{a}e^{-i\chi})|\alpha\rangle$$
$$= \frac{1}{4}\langle\alpha|(\hat{a}^+\hat{a} + \hat{1}) + \hat{a}^+\hat{a}) + \hat{a}\hat{a} + \hat{a}^+\hat{a}^+|\alpha\rangle$$
$$= \frac{1}{4}[2|\alpha|^2 + 1 + (\alpha^2 e^{-2i\chi} + \alpha^{*2}e^{2i\chi})]$$

Then we get

$$(\Delta E)^2 = \langle\alpha|\hat{E}^2(\chi)|\alpha\rangle - \langle\alpha|\hat{E}(\chi)|\alpha\rangle^2 = \frac{1}{4}.$$

The root-mean-square deviation in electric field is

$$\Delta E = \sqrt{\langle\alpha|\hat{E}^2(\chi)|\alpha\rangle - \langle\alpha|\hat{E}(\chi)|\alpha\rangle^2} = \frac{1}{2}.$$

21. Squeezed state (over view)

A squeezed state of light is a special form of light that is studied in the field of quantum optics. The light's quantum noise is a direct consequence of the existence of photons. When light is detected with an ideal photo diode, every photon is translated into a photo-electron. For squeezed light the resulting photo-current exhibits surprisingly low noise. The noise is lower than the minimum noise one expects from the existence of independent photons and their statistical arrival times.

The quantum noise of light with independent (uncorrelated) photons is often called shot-noise. The light itself is then in a so-called coherent state, or Glauber state. Shot-noise could be expected as the minimum noise possible. And highly stabilized laser sources almost achieve this low noise level. However, squeezed light can even show less noise than Glauber states. Squeezed states of light belong to the class of non-classical states of light.

Having less noise, squeezed light has applications in optical communication and optical measurements. Using squeezed light, weaker signals can be transmitted with the same signal to noise ratio and the same light power. Squeezed light can be used to distribute secret keys to two distant parties to perform quantum cryptography. Squeezed light composes an attractive field of research in fundamental quantum physics. In particular it allows the production of *entangled states of light* and the experimental demonstration of the so-called Einstein-Podolosky-Rosen (EPR) paradox. Squeezed light has been used to demonstrate quantum teleportation.

22. Origin of the expression of $\hat{S}_\zeta$

What is the origin of the squeezed operator $\hat{S}_\varsigma$. The wave function for the vacuum state is given by

$$\langle x|0\rangle = \left(\frac{1}{2\pi(\Delta x)^2}\right)^{1/4} \exp[-\frac{x^2}{4(\Delta x)^2}] \qquad (|x\rangle \text{ representation})$$

$$\langle p|0\rangle = \left(\frac{1}{2\pi(\Delta p)^2}\right)^{1/4} \exp[-\frac{p^2}{4(\Delta p)^2}] \qquad (|p\rangle \text{ representation})$$

where

$$\Delta x = \sqrt{\langle 0|\hat{x}^2|0\rangle} = \frac{1}{\sqrt{2}\beta}, \qquad \Delta p = \sqrt{\langle 0|\hat{p}^2|0\rangle} = \frac{m\omega}{\sqrt{2}\beta} = \frac{\hbar\beta^2}{\sqrt{2}\beta} = \frac{\hbar\beta}{\sqrt{2}}$$

and

$$\Delta x\Delta p = \frac{1}{\sqrt{2}\beta}\frac{\hbar\beta}{\sqrt{2}} = \frac{\hbar}{2} \qquad \text{(Heisenberg's principle of uncertainty)}$$

We parameterize the deviation of the variances from their vacuum values by a real number ζ called the squeezing parameter

$$\Delta x = \frac{1}{\sqrt{2}\beta}e^{-\varsigma}, \qquad \Delta p = \frac{\hbar\beta}{\sqrt{2}}e^{\varsigma}$$

Obviously, the product of Δx and Δp equals the minimal value $\frac{\hbar}{2}$;

$$\Delta x\Delta p = \frac{1}{\sqrt{2}\beta}e^{-\varsigma}\frac{\beta\hbar}{\sqrt{2}}e^{\varsigma} = \frac{\hbar}{2}$$

How can we squeeze the vacuum? Mathematically, we could just scale the position wave function $\varphi_0(x)$ for the squeezed state

$$\varphi_0(x) = e^{\varsigma/2}\varphi_0(e^{\varsigma}x) = e^{\varsigma/2}\varphi_0(\xi). \qquad (1)$$

or

$$\langle x|\varphi_0\rangle = e^{\varsigma/2}\langle e^{\varsigma}x|\varphi_0\rangle = e^{\varsigma/2}\langle \xi|\varphi_0\rangle$$

where

$$|e^{\varsigma}x\rangle = |\xi\rangle = e^{-\varsigma/2}|x\rangle .$$

The pre-factor $e^{\varsigma/2}$ in this scaling serves for maintaining the normalization of $\varphi_0(x)$. The momentum wave function $\varphi_0(p)$ is the Fourier transformed position wave function $\varphi_0(x)$. Consequently,

$$\varphi_0(p) = e^{-\varsigma/2}\varphi_0(e^{-\varsigma}p)$$

or

$$\langle p|\varphi_0\rangle = e^{-\varsigma/2}\langle e^{-\varsigma}p|\varphi_0\rangle$$

where

$$|e^{-\varsigma}p\rangle = e^{\varsigma/2}|p\rangle .$$

This formula implies that the momentum wave function is stretched when the position wave function is squeezed and vice versa. We differentiate $\varphi_0(x)$ in Eq.(1) with respect to the squeezing parameter ζ and obtain

$$\begin{aligned}
\frac{\partial\varphi_0(x)}{\partial\varsigma} &= \frac{1}{2}e^{\varsigma/2}\varphi_0(\xi) + e^{\varsigma/2}\frac{\partial\varphi_0(\xi)}{\partial\xi}\frac{\partial\xi}{\partial\varsigma} \\
&= \frac{1}{2}e^{\varsigma/2}\varphi_0(\xi) + e^{\varsigma/2}\frac{\partial\varphi_0(\xi)}{\partial\xi}e^{\varsigma}x \\
&= e^{\varsigma/2}[\frac{1}{2}\varphi_0(\xi) + \xi\frac{\partial\varphi_0(\xi)}{\partial\xi}] \\
&= \frac{1}{2}e^{\varsigma/2}(\frac{\partial}{\partial\xi}\xi + \xi\frac{\partial}{\partial\xi})\varphi_0(\xi) \\
&= \frac{i}{2\hbar}e^{\varsigma/2}(\frac{\hbar}{i}\frac{\partial}{\partial\xi}\xi + \xi\frac{\hbar}{i}\frac{\partial}{\partial\xi})\varphi_0(\xi)
\end{aligned}$$

where

$$\varphi_0(\xi) = \frac{\partial}{\partial \xi}[\xi \varphi_0(\xi)] - \xi \frac{\partial \varphi_0(\xi)}{\partial \xi}$$

Thus, we have

$$\begin{aligned}\frac{\hbar}{i}\frac{\partial}{\partial \varsigma} e^{\varsigma/2} \langle \xi | \varphi_0 \rangle &= \frac{i}{2\hbar} e^{\varsigma/2} (\frac{\hbar}{i}\frac{\partial}{\partial \xi}\xi + \xi \frac{\hbar}{i}\frac{\partial}{\partial \xi}) \langle \xi | \varphi_0 \rangle \\ &= \frac{1}{2} e^{\varsigma/2} \langle \xi | (\hat{p}\hat{x} + \hat{x}\hat{p}) | \varphi_0 \rangle\end{aligned}$$

or

$$\begin{aligned}\frac{\hbar}{i}\frac{\partial}{\partial \varsigma} e^{\varsigma/2} \langle \xi | \varphi_0 \rangle &= \frac{1}{2} (\frac{\hbar}{i}\frac{\partial}{\partial \xi}\xi + \xi \frac{\hbar}{i}\frac{\partial}{\partial \xi}) \langle \xi | \varphi_0 \rangle \\ &= \frac{1}{2} e^{\varsigma/2} \langle \xi | (\hat{p}\hat{x} + \hat{x}\hat{p}) | \varphi_0 \rangle\end{aligned} \qquad (2)$$

where

$$\xi = e^{\varsigma} x, \qquad |\xi\rangle = e^{-\varsigma/2} |x\rangle$$

Thus we have

$$\frac{\hbar}{i}\frac{\partial}{\partial \varsigma} \langle x | \varphi_0 \rangle = \frac{1}{2} \langle x | (\hat{p}\hat{x} + \hat{x}\hat{p}) | \varphi_0 \rangle$$

We note that

$$\frac{\hat{p}\hat{x} + \hat{x}\hat{p}}{2\hbar} = \frac{m\omega_0}{2\hbar\beta^2 i}[\hat{a}^2 - (\hat{a}^+)^2] = \frac{1}{2i}[\hat{a}^2 - (\hat{a}^+)^2]$$

Here we note that

$$|\xi\rangle = e^{-\varsigma/2} |x\rangle$$

and

$$\frac{1}{i}\frac{\partial}{\partial \varsigma} \langle x | \varphi_0 \rangle = \langle x | \frac{1}{2\hbar} (\hat{p}\hat{x} + \hat{x}\hat{p}) | \varphi_0 \rangle = \frac{1}{2i} \langle x | \hat{a}^2 - (\hat{a}^+)^2 | \varphi_0 \rangle$$

or

$$\frac{\partial}{\partial \varsigma}\langle x|\varphi_0\rangle = \frac{1}{2}\langle x|\hat{a}^2 - (\hat{a}^+)^2|\varphi_0\rangle$$

In the limit of $\varsigma \to 0$,

$$\langle x|\varphi_0\rangle \approx \langle x|0\rangle + \frac{1}{2}\varsigma\langle x|\hat{a}^2 - (\hat{a}^+)^2|0\rangle$$

$$|\varphi_0\rangle \approx \{1 + \frac{1}{2}\varsigma[\hat{a}^2 - (\hat{a}^+)^2]\}|0\rangle$$
$$= \exp[\frac{1}{2}\varsigma[\hat{a}^2 - (\hat{a}^+)^2]|0\rangle$$

By solving the differential equation for Eq.(2), we get the squeezed state

$$|\varphi_0\rangle = \hat{S}_\varsigma|0\rangle$$

with

$$\hat{S}_\varsigma = \exp[\frac{\varsigma}{2}\hat{a}^2 - \frac{\varsigma}{2}(\hat{a}^+)^2]$$

In general, using a complex number ς, the squeezed operator can be given by

$$\hat{S}_\varsigma = \exp(\frac{1}{2}\varsigma^*\hat{a}^2 - \frac{1}{2}\varsigma\hat{a}^{+2}).$$

---

23. Pure squeezed state $|\varsigma\rangle = \hat{S}_\zeta|0\rangle$

A pure squeezed state is defined as produced by the sole action of the unitary operator $\hat{S}_\varsigma$ on the vacuum state $|0\rangle$;

$$|\varsigma\rangle = \hat{S}_\varsigma|0\rangle,$$

where $|\varsigma\rangle$ is the squeezed-vacuum state. The squeezed operator is defined by

$$\hat{S}_\varsigma = \exp[\frac{1}{2}\varsigma^*\hat{a}^2 - \frac{1}{2}\varsigma(\hat{a}^+)^2],$$

and ς is the complex squeeze parameter with amplitude and phase defined by

$$\varsigma = se^{i\vartheta},$$

where s>0 is the squeezed parameter and ϑ is real. It is seen by inspection that $\hat{S}_\varsigma$ is a unitary operator.

$$\hat{S}_\varsigma{}^+\hat{S}_\varsigma = \hat{S}_\varsigma\hat{S}_\varsigma{}^+ = \hat{1},$$

where

$$\hat{S}_\varsigma{}^+ = \hat{S}_{-\varsigma}.$$

The squeezed vacuum state can be expressed by the superposition of number states,

$$\left|\varsigma\right\rangle = [\sec hs]^{-1/2}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{n!}[-\frac{1}{2}e^{i\vartheta}\tanh s]^n\left|2n\right\rangle.$$

Here we use the formula given by

$$\hat{S}_\varsigma{}^+\hat{a}\hat{S}_\varsigma = \hat{a}\cosh(s) - \hat{a}^+e^{i\vartheta}\sinh(s),$$

$$\hat{S}_\varsigma{}^+\hat{a}^+\hat{S}_\varsigma = \hat{a}^+\cosh(s) - \hat{a}e^{-i\vartheta}\sinh(s).$$

The expectation number:

$$\begin{aligned}\left\langle\varsigma\right|\hat{n}\left|\varsigma\right\rangle &= \left\langle\varsigma\right|\hat{a}^+\hat{a}\left|\varsigma\right\rangle \\ &= \left\langle 0\right|\hat{S}_\varsigma{}^+\hat{a}^+\hat{a}\hat{S}_\varsigma\left|0\right\rangle \\ &= \left\langle 0\right|\hat{S}_\varsigma{}^+\hat{a}^+\hat{S}_\varsigma\hat{S}_\varsigma{}^+\hat{a}\hat{S}_\varsigma\left|0\right\rangle \\ &= \left\langle 0\right|[\hat{a}^+\cosh(s) - \hat{a}e^{-i\vartheta}\sinh(\varsigma)][\hat{a}\cosh(s) - \hat{a}^+e^{i\vartheta}\sinh(s)]\left|0\right\rangle \\ &= \sinh^2(s)\end{aligned}$$

This means that $\langle\hat{n}\rangle = 0$, when s = 0 (in the absence of any squeezing). Then the squeezed vacuum state reduces to the ordinary vacuum state. The number $\langle\hat{n}\rangle$ increases sharply as the magnitude of the squeezed parameter is increased.

$$\left\langle n^2\right\rangle = 3\sinh^4(s) + 2\sinh^2(s) = 3\left\langle n\right\rangle^2 + 2\left\langle n\right\rangle.$$

Then the variance of number is

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = 2\langle n \rangle(\langle n \rangle + 1)\,,$$

$$\langle \varsigma | \hat{a} | \varsigma \rangle = \langle 0 | \hat{S}_\varsigma^{\ +} \hat{a} \hat{S}_\varsigma | 0 \rangle = \langle 0 | \hat{a}\cosh(s) - \hat{a}^+ e^{i\vartheta} \sinh(s) | 0 \rangle = 0\,,$$

$$\langle \varsigma | \hat{a}^+ | \varsigma \rangle = \langle 0 | \hat{S}_\varsigma^{\ +} \hat{a}^+ \hat{S}_\varsigma | 0 \rangle = \langle 0 | \hat{a}^+ \cosh(s) - \hat{a} e^{-i\vartheta} \sinh(s) | 0 \rangle = 0\,,$$

$$\begin{aligned}\langle \varsigma | \hat{a}\hat{a} |, \varsigma \rangle &= \langle 0 | \hat{S}_\varsigma^{\ +} \hat{a}\hat{a} \hat{S}_\varsigma | 0 \rangle \\ &= \langle 0 | [\hat{a}\cosh(s) - \hat{a}^+ e^{i\vartheta} \sinh(s)][\hat{a}\cosh(s) - \hat{a}^+ e^{i\vartheta} \sinh(s)] | 0 \rangle \\ &= \langle 0 | [-\hat{a}\hat{a}^+ e^{-i\vartheta} \cosh(s)\sinh(s) | 0 \rangle \\ &= -e^{i\vartheta} \sinh(\varsigma)\cosh(\varsigma)\end{aligned}$$

$$\begin{aligned}\langle \varsigma | \hat{a}^+ \hat{a}^+ | \varsigma \rangle &= \langle 0 | \hat{S}_\varsigma^{\ +} \hat{a}^+ \hat{a}^+ \hat{S}_\varsigma | 0 \rangle \\ &= -e^{--i\vartheta} \sinh(\varsigma)\cosh(\varsigma)\end{aligned}$$

We note that

$$\hat{X} = \frac{\beta}{\sqrt{2}}\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^+)\,,$$

$$\hat{Y} = \frac{1}{\sqrt{2}\hbar\beta}\hat{p} = \frac{1}{2i}(\hat{a} - \hat{a}^+)\,,$$

$$\begin{aligned}\hat{S}_\zeta^{\ +} \hat{X} \hat{S}_\zeta &= \frac{1}{2}\hat{S}_\zeta^{\ +}(\hat{a} + \hat{a}^+)\hat{S}_\zeta \\ &= \frac{1}{2}[\hat{a}\cosh(s) - \hat{a}^+ e^{i\vartheta}\sinh(s) + \hat{a}^+\cosh(s) - \hat{a}e^{-i\vartheta}\sinh(s)]\end{aligned}$$

$$\begin{aligned}\hat{S}_\zeta^{\ +} \hat{Y} \hat{S}_\zeta &= \frac{1}{2i}\hat{S}_\zeta^{\ +}(\hat{a} - \hat{a}^+)\hat{S}_\zeta \\ &= \frac{1}{2i}[\hat{a}\cosh(s) - \hat{a}^+ e^{i\vartheta}\sinh(s) - \hat{a}^+\cosh(s) + \hat{a}e^{-i\vartheta}\sinh(s)]\end{aligned}$$

Then we have

$$\langle \varsigma | \hat{X} | \varsigma \rangle = \langle 0 | \hat{S}_\zeta^{\ +} \hat{X} \hat{S}_\zeta | 0 \rangle = 0\,,$$

$$\langle \varsigma | \hat{Y} | \varsigma \rangle = \langle 0 | \hat{S}_\zeta^{\ +} \hat{Y} \hat{S}_\zeta | 0 \rangle = 0 ,$$

$$(\Delta X)^2 = \langle \varsigma | \hat{X}^2 | \varsigma \rangle = \frac{1}{4} [e^{2s} \sin^2 \frac{\vartheta}{2} + e^{-2s} \cos^2 \frac{\vartheta}{2}] ,$$

$$(\Delta Y)^2 = \langle \varsigma | \hat{Y}^2 | \varsigma \rangle = \frac{1}{4} [e^{2s} \cos^2 \frac{\vartheta}{2} + e^{-2s} \sin^2 \frac{\vartheta}{2}] ,$$

$$\hat{E}(\chi) = \hat{E}^+(\chi) + \hat{E}^-(\chi) = \frac{1}{2} [\hat{a} e^{-i\chi} + \hat{a}^+ e^{i\chi}] ,$$

$$\langle \varsigma | \hat{E}(\chi) | \varsigma \rangle = 0 ,$$

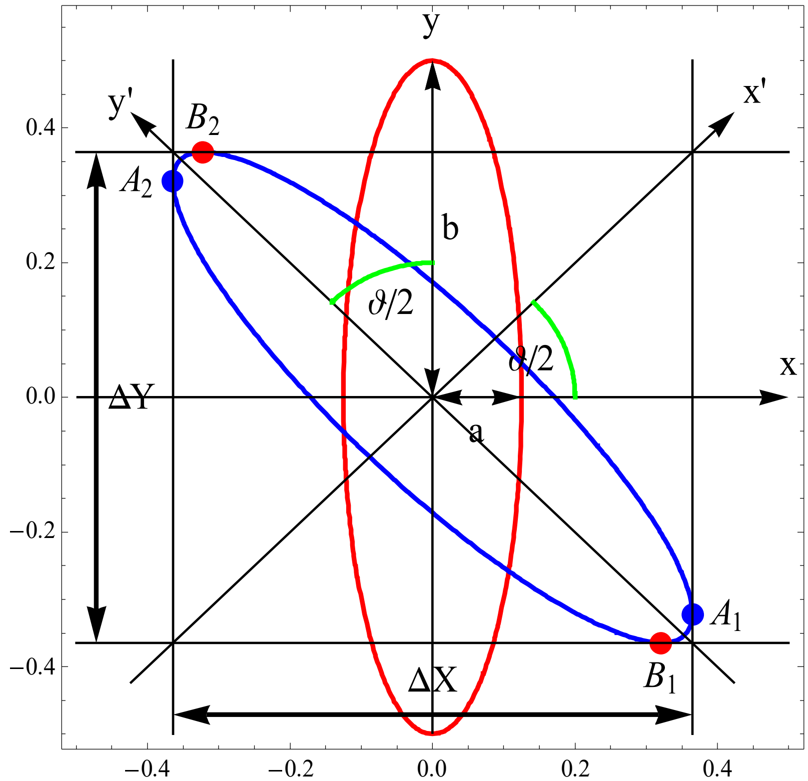
$$[\Delta E(\chi)]^2 = \frac{1}{4} [e^{2s} \sin^2 (\chi - \frac{1}{2} \vartheta) + e^{-2s} \cos^2 (\chi - \frac{1}{2} \vartheta)] .$$

When $\vartheta = 0$,

$$\Delta X = \frac{1}{2} e^{-s} , \quad \Delta Y = \frac{1}{2} e^{s} ,$$

$$\Delta X \Delta Y = \frac{1}{4} ,$$

satisfying the Heisenberg's principle with minimum uncertainty.



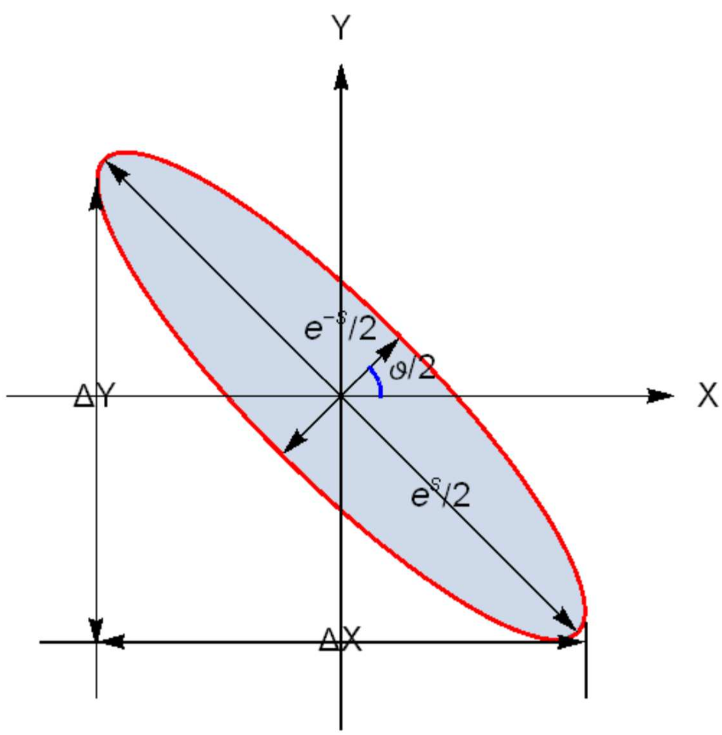


Fig.11 The 2D complex plane of $z = \langle\psi|\hat{a}|\psi\rangle = \langle X\rangle + i\langle Y\rangle = X + iY$ with $|\psi\rangle = |\varsigma\rangle = \hat{S}_\varsigma|0\rangle$. Representation of the quadrature operator means and

uncertainties for the squeezed vacuum state with s parameter given by exp(s) = 2. The points A_1 and A_2 ($dx/dy = 0$ on the ellipse). The point B_1 and B_2 ($dy/dx = 0$ on the ellipse).

$a = \frac{1}{4}e^{-s}$, $\quad b = \frac{1}{4}e^{s}$ where $e^{s} = 2$.

$\Delta X = 2a = \frac{1}{2}e^{-s}$. $\quad \Delta Y = 2b = \frac{1}{2}e^{s}$ (the red line)

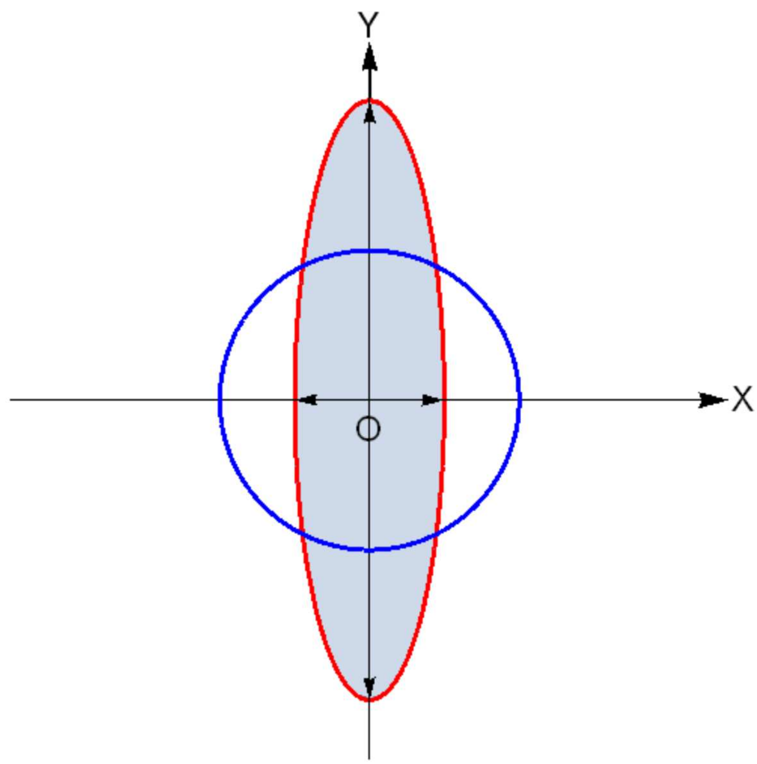


Fig.12 Error ellipse for a squeezed vacuum state where the squeezing is in the X direction. The state is defined by $|\varsigma\rangle = \hat{S}_{\varsigma}|0\rangle$.

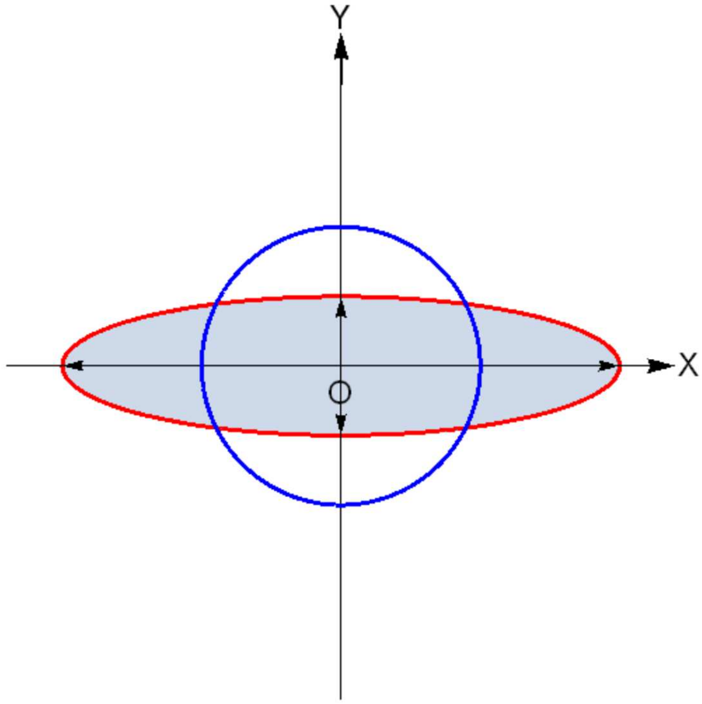


Fig.13 Error ellipse for a squeezed vacuum state where the squeezing is in the Y direction. The state is defined by $|\varsigma\rangle = \hat{S}_{\varsigma}|0\rangle$.

The new co-ordinates of the point on the ellipse (x', y') is related to the old co-ordinate of the same point on the ellipse (x, y) by the relation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\vartheta}{2} & -\sin\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

((**Note**)) The calculation of ΔX and ΔY for the ellipse

(a) Calculation of ΔY

Starting with the equation of ellipse given by

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

in the {x', y'} co-ordinates. The relation between (x', y') and (x, y):

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y = x'\boldsymbol{e}_x' + y'\boldsymbol{e}_y',$$

where

$$x' = \boldsymbol{e}_x' \cdot (x\boldsymbol{e}_x + y\boldsymbol{e}_y) = x(\boldsymbol{e}_x' \cdot \boldsymbol{e}_x) + y(\boldsymbol{e}_x' \cdot \boldsymbol{e}_y) = x\cos\theta + y\sin\theta,$$

$$y' = \boldsymbol{e}_y' \cdot (x\boldsymbol{e}_x + y\boldsymbol{e}_y) = x(\boldsymbol{e}_y' \cdot \boldsymbol{e}_x) + y(\boldsymbol{e}_y' \cdot \boldsymbol{e}_y) = -x\sin\theta + y\cos\theta.$$

Where

$$\boldsymbol{e}_x' \cdot \boldsymbol{e}_x = \cos\theta, \qquad \boldsymbol{e}_x' \cdot \boldsymbol{e}_y = \sin\theta,$$

$$\boldsymbol{e}_y' \cdot \boldsymbol{e}_x = -\sin\theta, \qquad \boldsymbol{e}_y' \cdot \boldsymbol{e}_y = \cos\theta,$$

Then the equation of the ellipse can be rewritten as

$$\frac{(x\cos\theta + y\sin\theta)^2}{a^2} + \frac{(-x\sin\theta + y\cos\theta)^2}{b^2} = 1. \tag{1}$$

We now calculate the value of Δy. We take the derivative of Eq.(1) with respect to x.

$$\frac{(x\cos\theta + y\sin\theta)(\cos\theta + y'\sin\theta)}{a^2} + \frac{(-x\sin\theta + y\cos\theta)(-\sin\theta + y'\cos\theta)}{b^2} = 0.$$

When $y' = 0$ we get the value of x as a function of y,

$$x = \frac{(a-b)(a+b)y\sin\theta\cos\theta}{b^2\cos^2\theta + a^2\sin^2\theta}.$$

Substituting this value into Eq.(1), we get

$$y = \sqrt{b^2\cos^2\theta + a^2\sin^2\theta}.$$

Then we have

$$\Delta Y = 2\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}, \qquad \text{or} \qquad (\Delta Y)^2 = 4(b^2\cos^2\theta + a^2\sin^2\theta)$$

(b) Derivation of ΔX

We now calculate the value of ΔX. We take the derivative of Eq.(1) with respect to y.

$$\frac{(x\cos\theta + y\sin\theta)(x'\cos\theta + \sin\theta)}{a^2} + \frac{(-x\sin\theta + y\cos\theta)(-x'\sin\theta + \cos\theta)}{b^2} = 0.$$

When $y' = 0$ we get the value of y as a function of x,

$$y = \frac{(a-b)(a+b)x\sin\theta\cos\theta}{a^2\cos^2\theta + b^2\sin^2\theta}.$$

Substituting this value into Eq.(1), we get

$$x = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}.$$

Then we have

$$\Delta X = 2\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}, \qquad \text{or} \qquad (\Delta X)^2 = 4(a^2\cos^2\theta + b^2\sin^2\theta).$$

(c)

$$(\Delta X)^2 = \langle\varsigma|\hat{X}^2|\varsigma\rangle = \frac{1}{4}(e^{-2s}\cos^2\frac{\vartheta}{2} + e^{2s}\sin^2\frac{\vartheta}{2}),$$

$$(\Delta Y)^2 = \langle\varsigma|\hat{Y}^2|\varsigma\rangle = \frac{1}{4}(e^{-2s}\sin^2\frac{\vartheta}{2} + e^{2s}\cos^2\frac{\vartheta}{2}).$$

Then we have

$$a = \frac{1}{4}e^{-s}, \qquad b = \frac{1}{4}e^{s}, \qquad \theta = \frac{\vartheta}{2}.$$

24. Probability $|\langle\xi|\varsigma\rangle|^2$ for the pure squeezed state $|\varsigma\rangle = \hat{S}_\varsigma|0\rangle$

A pure squeezed state is defined as produced by the sole action of the unitary operator $\hat{S}_\varsigma$ on the vacuum state $|0\rangle$;

$$|\varsigma\rangle = \hat{S}_\varsigma|0\rangle,$$

and

$$|\varsigma\rangle = [\sec hs]^{-1/2}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{n!}[-\frac{1}{2}e^{i\vartheta}\tanh s]^n|2n\rangle$$
$$= [\sec hs]^{-1/2}\sum_{n=0}^{\infty}(-1)^n\frac{[(2n)!]^{1/2}}{2^n n!}e^{in\vartheta}[\tanh s]^n|2n\rangle$$

The probability of finding the state $|2n\rangle$ state (in other words, $2m$ photons) in the field is

$$P_{2n} = \left|\langle 2n|\varsigma\rangle\right|^2 = \frac{(2n)!}{2^{2n}(n!)^2}\frac{(\tanh s)^{2n}}{\cosh s},$$

while for finding the state $|2n+1\rangle$ state, it is

$$P_{2n+1} = 0.$$

So the photon probability distribution for a squeezed vacuum state is oscillatory, vanishing for all odd photon numbers.

We now make a plot of the function $\langle\xi|\varsigma\rangle$ as s is chosen as a parameter.

$$\begin{aligned}\langle\xi|\varsigma\rangle &= [\sec hs]^{-1/2}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{n!}[-\frac{1}{2}e^{i\vartheta}\tanh s]^n\langle\xi|2n\rangle \\ &= [\sec hs]^{-1/2}\sum_{n=0}^{\infty}(-1)^m\frac{[(2n)!]^{1/2}}{2^n n!}e^{im\vartheta}[\tanh s]^n\langle\xi|2n\rangle\end{aligned}$$

The probability is

$$\left|\langle\xi|\varsigma\rangle\right|^2 = \langle\xi|\varsigma\rangle\langle\xi|\varsigma\rangle^*,$$

where

$$\xi = \beta x, \qquad |\xi\rangle = \frac{1}{\sqrt{\beta}}|x\rangle.$$

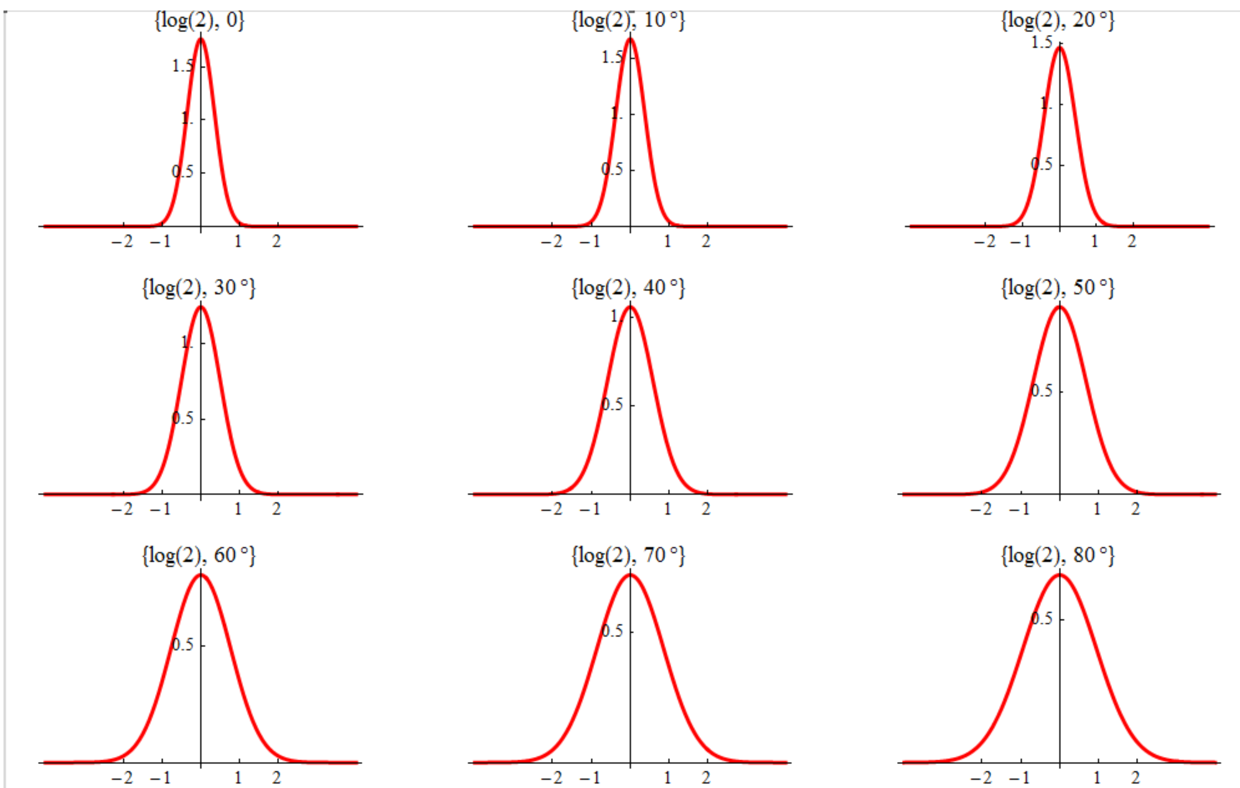
((**Mathematica**)) We make a plot

$$\left|\langle\xi|\varsigma\rangle\right|^2,$$

as a function ξ, where s and ϑ are changes as parameters.

Squeezed vacuum state

```
Clear["Global`*"]; Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]}; p1 = 80;
ψHOz[n_, z_] := 2^(-n/2) π^(-1/4) (n!)^(-1/2) Exp[-z^2/2] HermiteH[n, z];
f1 = 1/Sqrt[Sech[s]] Sum[((2 n)!)^(1/2)/n! (-1/2 Exp[-ⅈ ϑ] Tanh[s])^n ψHOz[2 n, ξ],
    {n, 0, p1}];
g1 = Abs[f1]^2;
pt1[s_, ϑ_] := Plot[g1, {ξ, -4, 4}, PlotLabel -> {s, ϑ},
    PlotStyle -> {Red, Thick},
    Ticks -> {Range[-2, 2, 1], Range[0, 2, 0.5]}, PlotPoints -> 100,
    PlotRange -> All, DisplayFunction -> Identity];
s = Log[2]; pt2 = Table[pt1[s, ϑ], {ϑ, 0 °, 180 °, 10 °}];
Show[GraphicsGrid[Partition[pt2, 3]]]
```



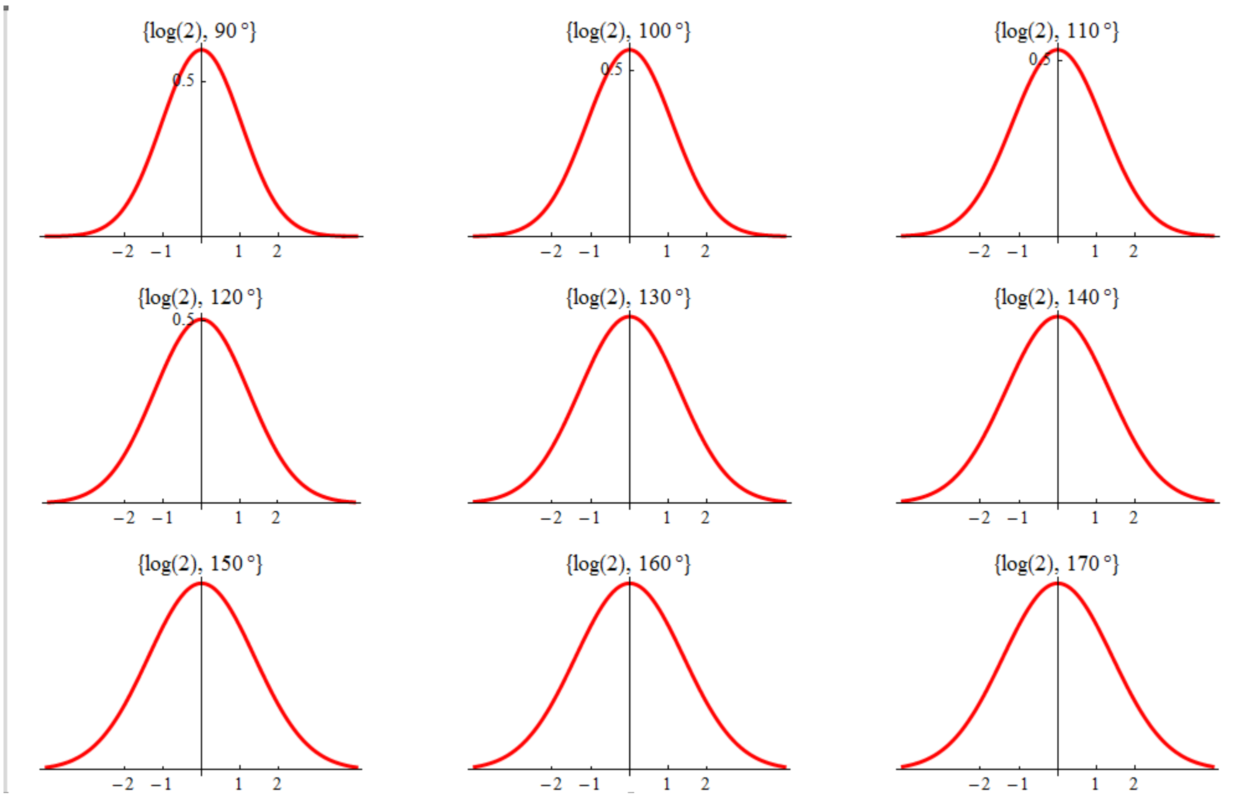


Fig.14 Plot of $|\langle\xi|\varsigma\rangle|^2$ for the case of {s, ϑ } where s=ln2, and ϑ =0° - 180°.

25. Bogoliubov transformation: squeezed state ((Merzbaxher, Quantum Mechanics))

In the coherent state

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega_0}}, \qquad \Delta p = \sqrt{\frac{m\hbar\omega_0}{2}},$$

where

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

We now construct the coherent state with narrower width Δx, so-called squeezed state, where

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega_0'}}, \qquad \Delta p = \sqrt{\frac{m\hbar\omega_0'}{2}}.$$

When $\omega_0' >> \omega_0$, we can reduce Δx arbitrarily at the expense of letting Δp grow correspondingly. We define the raising and lowering operator

$$\hat{b} = \sqrt{\frac{m\omega_0'}{2\hbar}}(\hat{x} + \frac{i\hat{p}}{m\omega_0'}),$$

$$\hat{b}^+ = \sqrt{\frac{m\omega_0'}{2\hbar}}(\hat{x} - \frac{i\hat{p}}{m\omega_0'}),$$

using an arbitrarily chosen positive parameter ω_0'. Note that

$$[\hat{b}, \hat{b}^+] = \hat{1}.$$

Then we get

$$\begin{aligned}\hat{b} &= \frac{1}{2}\sqrt{\frac{m\omega_0'}{2\hbar}}[\frac{\hat{a}+\hat{a}^+}{\sqrt{\frac{m\omega_0}{2\hbar}}} + \frac{1}{m\omega_0'}\frac{m\omega_0(\hat{a}-\hat{a}^+)}{\sqrt{\frac{m\omega_0}{2\hbar}}}] \\ &= \frac{1}{2}\sqrt{\frac{\omega_0'}{\omega_0}}(\hat{a}+\hat{a}^+) + \frac{1}{2}\sqrt{\frac{\omega_0}{\omega_0'}}(\hat{a}-\hat{a}^+)] \\ &= \frac{1}{2}(\sqrt{\frac{\omega_0'}{\omega_0}} + \sqrt{\frac{\omega_0}{\omega_0'}})\hat{a} + \frac{1}{2}(\sqrt{\frac{\omega_0'}{\omega_0}} - \sqrt{\frac{\omega_0}{\omega_0'}})\hat{a}^+\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{b}^+ &= \frac{1}{2}\sqrt{\frac{m\omega_0'}{2\hbar}}[\frac{\hat{a}+\hat{a}^+}{\sqrt{\frac{m\omega_0}{2\hbar}}} - \frac{1}{m\omega_0'}\frac{m\omega_0(\hat{a}-\hat{a}^+)}{\sqrt{\frac{m\omega_0}{2\hbar}}}] \\ &= \frac{1}{2}(\sqrt{\frac{\omega_0'}{\omega_0}} + \sqrt{\frac{\omega_0}{\omega_0'}})\hat{a}^+ + \frac{1}{2}(\sqrt{\frac{\omega_0'}{\omega_0}} - \sqrt{\frac{\omega_0}{\omega_0'}})\hat{a}\end{aligned}$$

or

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+, \qquad \hat{b}^+ = \mu\hat{a}^+ + \nu\hat{a}.$$

(**Bogoliubov transformation**), where μ and ν are real numbers,

$$\mu = \frac{1}{2}\left(\sqrt{\frac{\omega_0'}{\omega_0}} + \sqrt{\frac{\omega_0}{\omega_0'}}\right), \qquad \nu = \frac{1}{2}\left(\sqrt{\frac{\omega_0'}{\omega_0}} - \sqrt{\frac{\omega_0}{\omega_0'}}\right),$$

and

$$\mu^2 - \nu^2 = 1,$$

(with μ>1).

$$\hat{a} = \mu\hat{b} - \nu\hat{b}^+, \qquad \hat{a}^+ = \mu\hat{b}^+ - \nu\hat{b}.$$

Inversely, we have

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+, \qquad \hat{b}^+ = \mu\hat{a}^+ + \nu\hat{a}.$$

((**Note**))

$$\begin{pmatrix} \mu & -\nu \\ -\nu & \mu \end{pmatrix}^{-1} = \begin{pmatrix} \mu & \nu \\ \nu & \mu \end{pmatrix}.$$

Then we have

$$(\Delta x)^2 = \frac{\hbar^2}{2m\omega_0'} = \frac{\hbar^2}{2m\omega_0}\frac{\omega_0}{\omega_0'} = \frac{\hbar^2}{2m\omega_0}(\mu - \nu)^2,$$

$$(\Delta p)^2 = \frac{m\hbar\omega_0}{2}\frac{\omega_0'}{\omega_0} = \frac{m\hbar\omega_0}{2}(\mu + \nu)^2.$$

So that

$$\Delta x \Delta p = \frac{\hbar}{2},$$

as it should be for a minimum state.

26. Application of Bogoliubov transformation to the squeezed state

We start with the vacuum state $|0\rangle$, satisfying

$$\hat{a}|0\rangle = 0.$$

Multiplying by $\hat{S}_\varsigma$ from the left and using the fact that this operator is unitary, we may write

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma^{\ +} \hat{S}_\varsigma |0\rangle = 0 ,$$

or

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma^{\ +} |\varsigma\rangle = 0 .$$

Here we use

$$\hat{S}_{-\varsigma}^{\ +} \hat{a} \hat{S}_{-\varsigma} = \hat{a} \cosh s + \hat{a}^+ e^{i\vartheta} \sinh s ,$$

or

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma^{\ +} = \hat{a} \cosh s + \hat{a}^+ e^{i\vartheta} \sinh s ,$$

since $\hat{S}_\varsigma^{\ +} = \hat{S}_{-\varsigma}$ and $-\varsigma = -se^{i\vartheta}$. Then we get

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma^{\ +} |\varsigma\rangle = (\hat{a} \cosh s + \hat{a}^+ e^{i\vartheta} \sinh s) |\varsigma\rangle$$
$$= (\mu \hat{a} + \nu \hat{a}^+) |\varsigma\rangle = 0$$

or

$$\hat{b} |\varsigma\rangle = 0 ,$$

where

$$\hat{b} = \mu \hat{a} + \nu \hat{a}^+ ,$$

and

$$\mu = \cosh s , \qquad \nu = e^{i\vartheta} \sinh s ,$$

with

$$|\mu|^2 - |\nu|^2 = 1 .$$

Thus the squeezed vacuum state is the eigenstate of $\hat{b}$ with the eigenvalue zero.
From

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma{}^+ \hat{S}_\varsigma |0\rangle = 0 .$$

((**Note**)) **Bogolyubov transformation**

It is said that the Bogolyubov transformation is a canonical transformation, as it preserves the commutation relations. Let $\hat{b}$ and $\hat{b}^+$ be the operators

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+ , \qquad \hat{b}^+ = \mu^*\hat{a}^+ + \nu^*\hat{a} ,$$

where the complex numbers μ and ν satisfy

$$|\mu|^2 - |\nu|^2 = 1 .$$

The operators $\hat{b}$ and $\hat{b}^+$ satisfy the commutation relation

$$[\hat{b}, \hat{b}^+] = \hat{1} ,$$

since

$$\begin{aligned}
[\hat{b}, \hat{b}^+] &= [\mu\hat{a} + \nu\hat{a}^+ , \mu^*\hat{a}^+ + \nu^*\hat{a}] \\
&= \mu\mu^*[\hat{a}, \hat{a}^+] - \nu\nu^*[\hat{a}, \hat{a}^+] \\
&= (|\mu|^2 - |\nu|^2)\hat{1} = \hat{1}
\end{aligned}$$

We also note that

$$\hat{a} = \mu^*\hat{b} - \nu\hat{b}^+ , \qquad \hat{a}^+ = -\nu^*\hat{b} + \mu\hat{b}^+ .$$

27. Calculations of the fluctuation $(\Delta X)_s^2$ and $(\Delta Y)_s^2$

The operators $\hat{X}$ and $\hat{Y}$ are given by

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^+) = \frac{(\mu^* - \nu^*)\hat{b} + (\mu - \nu)\hat{b}^+}{2} ,$$

$$\hat{Y} = \frac{1}{2i}(\hat{a} - \hat{a}^+) = \frac{(\mu^* + \nu^*)\hat{b} - (\mu + \nu)\hat{b}^+}{2i} .$$

(i)

$$\langle X\rangle_s = \langle\varsigma|\hat{X}|\varsigma\rangle = \langle\varsigma|\frac{(\mu^* - \nu^*)\hat{b} + (\mu - \nu)\hat{b}^+}{2}|\varsigma\rangle = 0,$$

$$\langle Y\rangle_s = \langle\varsigma|\hat{Y}|\varsigma\rangle = \langle\varsigma|\frac{(\mu^* + \nu^*)\hat{b} - (\mu + \nu)\hat{b}^+}{2i}|\varsigma\rangle = 0,$$

since

$$\hat{b}|\varsigma\rangle = 0, \qquad \langle\varsigma|\hat{b}^+ = 0.$$

(ii)

$$\begin{aligned}\langle X^2\rangle_s &= \langle\varsigma|\hat{X}^2|\varsigma\rangle \\ &= \langle\varsigma|\hat{X}\hat{X}|\varsigma\rangle \\ &= \langle\psi|\psi\rangle\end{aligned}$$

or

$$\begin{aligned}\langle X^2\rangle_s &= \frac{1}{4}|\mu - \nu|^2 \langle\varsigma|\hat{b}^+\hat{b}|\varsigma\rangle \\ &= \frac{1}{4}|\mu - \nu|^2 \langle\varsigma|\hat{b}\hat{b}^+ + \hat{1}|\varsigma\rangle \\ &= \frac{1}{4}|\mu - \nu|^2\end{aligned}$$

$$\begin{aligned}\langle Y^2\rangle_s &= \langle\varsigma|\hat{Y}^2|\varsigma\rangle \\ &= \langle\varsigma|\hat{Y}\hat{Y}|\varsigma\rangle \\ &= \langle\psi|\psi\rangle\end{aligned}$$

$$\begin{aligned}\langle Y^2\rangle_s &= \frac{1}{4}|\mu + \nu|^2 \langle\varsigma|\hat{b}^+\hat{b}|\varsigma\rangle \\ &= \frac{1}{4}|\mu + \nu|^2 \langle\varsigma|\hat{b}\hat{b}^+ + \hat{1}|\varsigma\rangle \\ &= \frac{1}{4}|\mu + \nu|^2\end{aligned}$$

Then we have

$$(\Delta X)_s^2 = \langle X^2\rangle_s - \langle X\rangle_s^2 = \frac{1}{4}|\mu - \nu|^2 = \frac{1}{4}(e^{2s}\sin^2\frac{\vartheta}{2} + e^{-2s}\cos^2\frac{\vartheta}{2}),$$

$$(\Delta Y)_s^{\ 2} = \langle Y^2 \rangle_s - \langle Y \rangle_s^{\ 2} = \frac{1}{4}|\mu + \nu|^2 = \frac{1}{4}(e^{2s}\cos^2\frac{\vartheta}{2} + e^{-2s}\sin^2\frac{\vartheta}{2}).$$

28. Squeezed coherent state

We assume the squeezed coherent state as

$$|\alpha,\varsigma\rangle = \hat{D}_\alpha \hat{S}_\varsigma |0\rangle, \qquad \text{or} \qquad \langle\alpha,\varsigma| = \langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}.$$

Using the formula shown in the **APPENDIX**, we have

$$\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}\hat{D}_\alpha\hat{S}_\varsigma = \hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1},$$

$$\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma = \hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1},$$

with

$$\alpha = |\alpha|e^{i\theta}, \qquad \varsigma = se^{i\vartheta}.$$

(i)

$$\langle n\rangle = |\alpha|^2 + \sinh^2 s.$$

((**Proof**))

$$\begin{aligned}\langle n\rangle &= \langle\alpha,\varsigma|\hat{n}|\alpha,\varsigma\rangle \\ &= \langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\ &= \langle\psi|\psi\rangle\end{aligned}$$

$$\begin{aligned}\langle\psi|\psi\rangle &= \langle 0|-\hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(-\hat{a}^+e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle \\ &= \langle 0|(\hat{a}^+\hat{a} + \hat{1})\sinh^2 s + |\alpha|^2|0\rangle \\ &= |\alpha|^2 + \sinh^2 s\end{aligned}$$

where

$$
\begin{aligned}
|\psi\rangle &= \hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{a}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\
&= (\hat{a}\cosh s - \hat{a}^+e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle \\
&= (-\hat{a}^+e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle
\end{aligned}
$$

and

$$
\langle\psi| = \langle 0|\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma = \langle 0|(-\hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1}).
$$

(ii)

$$
(\Delta n)^2 = |\alpha|^2[e^{2s}\sin^2(\theta - \frac{1}{2}\vartheta) + e^{-2s}\cos^2(\theta - \frac{1}{2}\vartheta)] + 2\sinh^2 s(\sinh^2 s + 1).
$$

((**Proof**))

$$
\begin{aligned}
\langle n^2\rangle &= \langle\alpha,\varsigma|\hat{n}\hat{n}|\alpha,\varsigma\rangle \\
&= \langle 0|[\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{n}\hat{D}_\alpha\hat{S}_\varsigma\hat{S}_\varsigma{}^+\hat{D}_\alpha\hat{n}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\
&= \langle\psi|\psi\rangle
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
|\psi\rangle &= \hat{S}_\varsigma{}^+\hat{D}_\alpha\hat{n}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\
&= \hat{S}_\varsigma{}^+\hat{D}_\alpha\hat{a}^+\hat{a}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\
&= \hat{S}_\varsigma{}^+\hat{D}_\alpha\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma\hat{S}_\varsigma{}^+\hat{D}_\alpha\hat{a}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\
&= (\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(\hat{a}\cosh s - \hat{a}^+e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle \\
&= (\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(-\hat{a}^+e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle \\
&= (-\hat{a}^+\hat{a}^+e^{i\vartheta}\cosh s\sinh s + \alpha\hat{a}^+\cosh s + \hat{a}\hat{a}^+\sinh^2 s - \alpha^*\hat{a}^+e^{i\vartheta}\sinh s + |\alpha|^2)|0\rangle \\
&= (-\hat{a}^+\hat{a}^+e^{i\vartheta}\cosh s\sinh s + \alpha\hat{a}^+\cosh s - \alpha^*\hat{a}^+e^{i\vartheta}\sinh s + \sinh^2 s + |\alpha|^2)|0\rangle
\end{aligned}
$$

and

$$
\langle\psi| = \langle 0|(-\hat{a}\hat{a}e^{-i\vartheta}\cosh s\sinh s + \alpha^*\hat{a}\cosh s - \alpha\hat{a}e^{-i\vartheta}\sinh s + \sinh^2 s + |\alpha|^2).
$$

Then we have

$$\langle\psi|\psi\rangle = \langle 0|\hat{a}^2\hat{a}^{+2}\sinh^2 s\cosh^2 s + |\alpha|^2\hat{a}\hat{a}^+\cosh^2 s - (\alpha^{*2}e^{i\vartheta} + \alpha^2 e^{-i\vartheta})\hat{a}\hat{a}^+\sinh s\cosh s$$
$$+|\alpha|^2\hat{a}\hat{a}^+\sinh^2 s + (\sinh^2 s + |\alpha|^2)^2|0\rangle$$
$$= 2\sinh^2 s\cosh^2 s + (\sinh^2 s + |\alpha|^2)^2 + |\alpha|^{2+}(\cosh^2 s + \sinh^2 s)$$
$$-(\alpha^{*2}e^{i\vartheta} + \alpha^2 e^{-i\vartheta})\sinh s\cosh s$$

where

$$[(\hat{a})^2, (\hat{a}^+)^2] = 2(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) = 2(2\hat{a}^+\hat{a} + \hat{1}).$$

Thus, we get

$$(\Delta n)^2 = <n^2> - <n>^2$$
$$= 2\sinh^2 s\cosh^2 s + |\alpha|^2(\cosh^2 s + \sinh^2 s) - (\alpha^{*2}e^{i\vartheta} + \alpha^2 e^{-i\vartheta})\sinh s\cosh s$$
$$= 2\sinh^2 s(\sinh^2 s + 1) + |\alpha|^2(\cosh^2 s + \sinh^2 s) - 2|\alpha|^2\cos(2\theta - \vartheta)\sinh s\cosh s$$
$$= 2\sinh^2 s(\sinh^2 s + 1) + |\alpha|^2(\cosh^2 s + \sinh^2 s)$$
$$-2|\alpha|^2[\cos^2(\theta - \frac{\vartheta}{2}) - \sin^2(\theta - \frac{\vartheta}{2})]\sinh s\cosh s$$
$$= 2\sinh^2 s(\sinh^2 s + 1) + |\alpha|^2(\cosh^2 s + \sinh^2 s)$$
$$-4|\alpha|^2[\cos^2(\theta - \frac{\vartheta}{2}) - \sin^2(\theta - \frac{\vartheta}{2})]\sinh s\cosh s$$
$$= |\alpha|^2\frac{1}{2}(e^{2s} + e^{-2s}) - |\alpha|^2\frac{1}{2}[\cos^2(\theta - \frac{\vartheta}{2}) - \sin^2(\theta - \frac{\vartheta}{2})](e^{2s} - e^{-2s})$$
$$= |\alpha|^2[e^{2s}\sin^2(\theta - \frac{1}{2}\vartheta) + e^{-2s}\cos^2(\theta - \frac{1}{2}\vartheta)] + 2\sinh^2 s(\sinh^2 s + 1)$$

(iii)

$$\langle\alpha,\varsigma|\hat{X}|\alpha,\varsigma\rangle = |\alpha|\cos\theta = \mathrm{Re}[\alpha].$$

((**Proof**))
Using the expression of $\hat{X}$

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^+),$$

we get

$$\langle \alpha,\varsigma|\hat{X}|\alpha,\varsigma\rangle = \frac{1}{2}\langle 0|\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+(\hat{a}+\hat{a})\hat{D}_\alpha\hat{S}_\varsigma|0\rangle$$

$$= \frac{1}{2}\langle 0|\hat{a}\cosh s - e^{i\vartheta}\hat{a}^+\sinh s + \alpha\hat{1}$$

$$+\hat{a}^+\cosh s - e^{-i\vartheta}\hat{a}\sinh s + \alpha^*\hat{1}|0\rangle$$

$$= \frac{1}{2}(\alpha + \alpha^*)$$

$$= \mathrm{Re}[\alpha]$$

where

$$\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{a}\hat{D}_\alpha\hat{S}_\varsigma = \hat{a}\cosh s - e^{i\vartheta}\hat{a}^+\sinh(s) + \alpha\hat{1},$$

$$\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma = \hat{a}^+\cosh s - e^{-i\vartheta}\hat{a}\sinh s + \alpha^*\hat{1}.$$

(iv)

$$\langle \alpha,\varsigma|\hat{Y}|\alpha,\varsigma\rangle = |\alpha|\sin\theta = \mathrm{Im}[\alpha].$$

((**Proof**))

$$\hat{Y} = \frac{1}{2i}(\hat{a} - \hat{a}^+),$$

we get

$$\langle \alpha,\varsigma|\hat{Y}|\alpha,\varsigma\rangle = \frac{1}{2i}\langle 0|\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+(\hat{a}-\hat{a})\hat{D}_\alpha\hat{S}_\varsigma|0\rangle$$

$$= \frac{1}{2i}\langle 0|(\hat{a}\cosh s - e^{i\vartheta}\hat{a}^+\sinh s + \alpha\hat{1})$$

$$-(\hat{a}^+\cosh s - e^{-i\vartheta}\hat{a}\sinh s + \alpha^*\hat{1})|0\rangle$$

$$= \frac{1}{2i}(\alpha - \alpha^*)$$

$$= \mathrm{Im}[\alpha]$$

(v)

$$(\Delta X)^2 = \frac{1}{4}(e^{-2s}\cos^2\frac{\vartheta}{2} + e^{2s}\sin^2\frac{\vartheta}{2}).$$

((**Proof**))

$$< X^2 > = \langle \alpha,\varsigma | \hat{X}^2 | \alpha,\varsigma \rangle = \langle 0 | \hat{S}_\varsigma{}^+ \hat{D}_\alpha{}^+ \hat{X}\hat{D}_\alpha \hat{S}_\varsigma \hat{S}_\varsigma{}^+ \hat{D}_\alpha{}^+ \hat{X}\hat{D}_\alpha \hat{S}_\varsigma | 0 \rangle = \langle \chi | \gamma \rangle,$$

where

$$\begin{aligned}
|\psi\rangle &= \hat{S}_\varsigma{}^+ \hat{D}_\alpha \hat{X}\hat{D}_\alpha \hat{S}_\varsigma |0\rangle \\
&= \frac{1}{2}\hat{S}_\varsigma{}^+ \hat{D}_\alpha (\hat{a} + \hat{a}^+)\hat{D}_\alpha \hat{S}_\varsigma |0\rangle \\
&= \frac{1}{2}(\hat{a}^+ \cosh s - \hat{a}e^{-i\vartheta} \sinh s + \alpha^* \hat{1} + \hat{a}\cosh s - \hat{a}^+ e^{i\vartheta} \sinh s + \alpha\hat{1})|0\rangle \\
&= \frac{1}{2}[(\cosh s - e^{i\vartheta} \sinh s)\hat{a}^+ + (\alpha + \alpha^*)\hat{1}]|0\rangle
\end{aligned}$$

$$\langle \psi | = \frac{1}{2}\langle 0 | [(\cosh s - e^{-i\vartheta} \sinh s)\hat{a} + (\alpha + \alpha^*)\hat{1}] .$$

So we get

$$\begin{aligned}
\langle \psi | \psi \rangle &= \frac{1}{4}\langle 0 | [(\cosh s - e^{-i\vartheta} \sinh s)\hat{a} + (\alpha + \alpha^*)\hat{1}][(\cosh s - e^{i\vartheta} \sinh s)\hat{a}^+ + (\alpha + \alpha^*)\hat{1}]|0\rangle \\
&= \frac{1}{4}[(\cosh s - e^{-i\vartheta} \sinh s)(\cosh s - e^{i\vartheta} \sinh s)\langle 0 | \hat{a}\hat{a}^+ | 0 \rangle + (\alpha + \alpha^*)^2 \langle 0 | 0 \rangle] \\
&= \frac{1}{4}[(\cosh s - e^{-i\vartheta} \sinh s)(\cosh s - e^{-i\vartheta} \sinh s) + (\alpha + \alpha^*)^2] \\
&= \frac{1}{4}(\cosh^2 s + \sinh^2 s - 2\cos\vartheta \sinh s \cosh s + (\alpha + \alpha^*)^2] \\
&= \frac{1}{4}[\left(\frac{e^{2s} + e^{-2s}}{2}\right) - (\cos^2 \frac{\vartheta}{2} - \sin^2 \frac{\vartheta}{2})\left(\frac{e^{2s} - e^{-2s}}{2}\right)] + \frac{1}{4}(\alpha + \alpha^*)^2] \\
&= \frac{1}{4}(e^{-2s} \cos^2 \frac{\vartheta}{2} + e^{2s} \sin^2 \frac{\vartheta}{2}) + \frac{1}{4}(\alpha + \alpha^*)^2
\end{aligned}$$

Then

$$\begin{aligned}
(\Delta X)^2 &= < X^2 > - < X >^2 \\
&= \frac{1}{4}(e^{2s} \sin^2 \frac{\vartheta}{2} + e^{2-s} \sin^2 \frac{\vartheta}{2}) + \frac{1}{4}(\alpha + \alpha^*)^2 - \frac{1}{4}(\alpha + \alpha^*)^2 \\
&= \frac{1}{4}(e^{-2s} \cos^2 \frac{\vartheta}{2} + e^{2s} \sin^2 \frac{\vartheta}{2})
\end{aligned}$$

(vi)

$$\left(\Delta Y\right)^2 = \frac{1}{4}(e^{-2s}\sin^2\frac{\vartheta}{2} + e^{2s}\cos^2\frac{\vartheta}{2}).$$

((**Proof**))

$$< Y^2 >= \langle \alpha,\varsigma | \hat{Y}^2 | \alpha,\varsigma \rangle = \langle 0 | \hat{S}_\varsigma{}^+ \hat{D}_\alpha{}^+ \hat{Y}\hat{D}_\alpha \hat{S}_\varsigma \hat{S}_\varsigma{}^+ \hat{D}_\alpha{}^+ \hat{Y}\hat{D}_\alpha \hat{S}_\varsigma | 0 \rangle = \langle \chi | \gamma \rangle ,$$

where

$$\begin{aligned}
|\psi\rangle &= \hat{S}_\varsigma{}^+ \hat{D}_\alpha \hat{Y} \hat{D}_\alpha \hat{S}_\varsigma |0\rangle \\
&= \frac{1}{2i}\hat{S}_\varsigma{}^+ \hat{D}_\alpha (\hat{a} - \hat{a}^+)\hat{D}_\alpha \hat{S}_\varsigma |0\rangle \\
&= \frac{1}{2i}(\hat{a}^+ \cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1} - \hat{a}\cosh s + \hat{a}^+ e^{i\vartheta}\sinh s - \alpha\hat{1})|0\rangle \\
&= \frac{1}{2i}[(\cosh s + e^{i\vartheta}\sinh s)\hat{a}^+ + (\alpha^* - \alpha)\hat{1}]|0\rangle
\end{aligned}$$

$$\langle \psi | = \frac{1}{2}\langle 0 | [(\cosh\ s + e^{-i\vartheta}\sinh\ s)\hat{a} + (\alpha - \alpha^*)\hat{1}].$$

So we get

$$\begin{aligned}
\langle \psi | \psi \rangle &= -\frac{1}{4}\langle 0 | [(\cosh s + e^{-i\vartheta}\sinh s)\hat{a} + (\alpha - \alpha^*)\hat{1}](\cosh s + e^{i\vartheta}\sinh s)\hat{a}^+ + (\alpha^* - \alpha)\hat{1}] | 0 \rangle \\
&= -\frac{1}{4}[(\cosh s + e^{-i\vartheta}\sinh s)(\cosh s + e^{i\vartheta}\sinh s)\langle 0 | \hat{a}\hat{a}^+ | 0 \rangle - (\alpha - \alpha^*)^2 \langle 0 | 0 \rangle] \\
&= -\frac{1}{4}[(\cosh s + e^{-i\vartheta}\sinh s)(\cosh s + e^{-i\vartheta}\sinh s) - (\alpha - \alpha^*)^2] \\
&= -\frac{1}{4}(\cosh^2 s + \sinh^2 s + 2\cos\vartheta \sinh s \cosh s - (\alpha - \alpha^*)^2] \\
&= -\frac{1}{4}[\left(\frac{e^{2s} + e^{-2s}}{2}\right) + (\cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2})\left(\frac{e^{2s} - e^{-2s}}{2}\right)] - \frac{1}{4}(\alpha - \alpha^*)^2] \\
&= \frac{1}{4}(e^{-2s}\sin^2\frac{\vartheta}{2} + e^{2s}\cos^2\frac{\vartheta}{2}) - \frac{1}{4}(\alpha - \alpha^*)^2
\end{aligned}$$

Then, we have

$$\begin{aligned}(\Delta Y)^2 &= < Y^2 > - < Y >^2 \\ &= \frac{1}{4}(e^{2s}\sin^2\frac{\vartheta}{2} + e^{2-s}\sin^2\frac{\vartheta}{2}) - \frac{1}{4}(\alpha-\alpha^*)^2 + \frac{1}{4}(\alpha-\alpha^*)^2 \\ &= \frac{1}{4}(e^{-2s}\cos^2\frac{\vartheta}{2} + e^{2s}\sin^2\frac{\vartheta}{2})\end{aligned}$$

(vi)

$$S = \langle E(\chi)\rangle = \langle \alpha,\varsigma | E(\chi) | \alpha,\varsigma\rangle = |\alpha|\cos(\chi-\theta),$$

where

$$\hat{E}(\chi) = \frac{1}{2}[\hat{a}e^{-i\chi} + \hat{a}^+ e^{i\chi}].$$

((**Proof**))

$$\begin{aligned}S &= \frac{1}{2}\langle \alpha,\varsigma|\hat{a}|\alpha,\varsigma\rangle e^{-i\chi} + \frac{1}{2}\langle \alpha,\varsigma|\hat{a}^+|\alpha,\varsigma\rangle e^{i\chi} \\ &= \frac{1}{2}(\alpha e^{-i\chi} + \alpha^* e^{i\chi}) \\ &= |\alpha|\cos(\chi-\theta)\end{aligned}$$

where

$$\begin{aligned}\langle \alpha,\varsigma|\hat{a}|\alpha,\varsigma\rangle &= \langle 0|\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{a}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\ &= \langle 0|\hat{a}\cosh s - \hat{a}^+e^{i\vartheta}\sinh s + \alpha\hat{1}|0\rangle \\ &= \alpha\end{aligned}$$

$$\begin{aligned}\langle \alpha,\varsigma|\hat{a}^+|\alpha,\varsigma\rangle &= \langle 0|\hat{S}_\varsigma{}^+\hat{D}_\alpha{}^+\hat{a}\hat{D}_\alpha\hat{S}_\varsigma|0\rangle \\ &= \langle 0|\hat{a}\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1}|0\rangle \\ &= \alpha^*\end{aligned}$$

(vii)

$$N = [\Delta E(\chi)]^2 = \frac{1}{4}[e^{-2s}\cos^2(\chi - \frac{1}{2}\vartheta) + e^{2s}\sin^2(\chi - \frac{1}{2}\vartheta).$$

((**Proof**))

$$\hat{E}(\chi)^2 = \frac{1}{4}(\hat{a}e^{-i\chi} + \hat{a}^+ e^{i\chi})(\hat{a}e^{-i\chi} + \hat{a}^+ e^{i\chi})$$
$$= \frac{1}{4}(\hat{a}^2 e^{-2i\chi} + \hat{a}^{+2} e^{2i\chi} + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$$

$$\left\langle \hat{E}(\chi)^2 \right\rangle = \frac{1}{4}[\langle \varsigma,\alpha | \hat{a}^2 | \varsigma,\alpha \rangle e^{-2i\chi} + \langle \varsigma,\alpha | \hat{a}^{+2} | \varsigma,\alpha \rangle e^{2i\chi} + \langle \varsigma,\alpha | \hat{a}^+\hat{a} | \varsigma,\alpha \rangle + \langle \varsigma,\alpha | \hat{a}\hat{a}^+ | \varsigma,\alpha \rangle]$$
$$= \frac{1}{4}[(-e^{i\vartheta}\sinh s\cosh s + \alpha^2)e^{-2i\chi} + (-e^{-i\vartheta}\sinh s\cosh s + \alpha^{*2})e^{2i\chi}$$
$$+ (\sinh^2 s + |\alpha|^2) + (\cosh^2 s + |\alpha|^2)]$$
$$= \frac{1}{4}[-2\sinh s\cosh s\cos(2\chi - \vartheta) + 2|\alpha|^2\cos(2\chi - \vartheta) + \sinh^2 s + \cosh^2 s + 2|\alpha|^2]$$

Then we have

$$\left\langle \hat{E}(\chi)^2 \right\rangle - \left\langle \hat{E}(\chi) \right\rangle^2 = \frac{1}{4}[(-2\sinh s\cosh s\cos(2\chi - \vartheta) + 4|\alpha|^2\cos(\chi - \frac{\vartheta}{2}) - 2|\alpha|^2$$
$$+ (\sinh^2 s + \cosh^2 s + 2|\alpha|^2)] - |\alpha|^2\cos^2(\chi - \frac{\vartheta}{2})$$
$$= \frac{1}{4}[(-2\sinh s\cosh s\cos(2\chi - \vartheta) + (\sinh^2 s + \cosh^2 s+)]$$
$$= \frac{1}{4}\{-\frac{(e^{2s} - e^{-2s})}{2}[\cos^2(\chi - \frac{\vartheta}{2}) - \sin^2(\chi - \frac{\vartheta}{2})] + \frac{e^{2s} + e^{-2s}}{2}\}$$
$$= \frac{1}{4}[e^{-2s}\cos^2(\chi - \frac{\vartheta}{2}) + e^{2s}\sin^2(\chi - \frac{\vartheta}{2})]$$

where

$$\langle 0 | \hat{S}_\varsigma^{\,+}\hat{D}_\alpha\hat{a}^+\hat{a}\hat{D}_\alpha\hat{S}_\varsigma | 0 \rangle = \langle 0 | \hat{S}_\varsigma^{\,+}\hat{D}_\alpha\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma\hat{S}_\varsigma^{\,+}\hat{D}_\alpha\hat{a}\hat{D}_\alpha\hat{S}_\varsigma | 0 \rangle$$
$$= \langle 0 | (\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(\hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1}) | 0 \rangle$$
$$= \langle 0 | (\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(-\hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1}) | 0 \rangle$$
$$= \langle 0 | (\hat{a}\hat{a}^+\sinh^2 s + |\alpha|^2) | 0 \rangle$$
$$= \sinh^2 s + |\alpha|^2$$

$$\langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}\hat{a}^+\hat{D}_\alpha \hat{S}_\varsigma|0\rangle = \langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}\hat{D}_\alpha \hat{S}_\varsigma \hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}^+\hat{D}_\alpha \hat{S}_\varsigma|0\rangle$$
$$= \langle 0|(\hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1})(\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})|0\rangle$$
$$= \langle 0|(\hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1})(\hat{a}^+\cosh s + \alpha^*\hat{1})|0\rangle$$
$$= \langle 0|(\hat{a}\hat{a}^+\cosh^2 s + |\alpha|^2)|0\rangle$$
$$= \cosh^2 s + |\alpha|^2$$

$$\langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}^+\hat{a}^+\hat{D}_\alpha \hat{S}_\varsigma|0\rangle = \langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}^+\hat{D}_\alpha \hat{S}_\varsigma \hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}^+\hat{D}_\alpha \hat{S}_\varsigma|0\rangle$$
$$= \langle 0|(\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})|0\rangle$$
$$= \langle 0|(\hat{a}^+\cosh s - \hat{a}e^{-i\vartheta}\sinh s + \alpha^*\hat{1})(\hat{a}^+\cosh s + \alpha^*\hat{1})|0\rangle$$
$$= \langle 0| - e^{-i\vartheta}\hat{a}\hat{a}^+\sinh s\cosh s + \alpha^{*2})|0\rangle$$
$$= -e^{-i\vartheta}\sinh s\cosh s + \alpha^{*2}$$

$$\langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}\hat{a}\hat{D}_\alpha \hat{S}_\varsigma|0\rangle = \langle 0|\hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}\hat{D}_\alpha \hat{S}_\varsigma \hat{S}_\varsigma^{\ +}\hat{D}_\alpha \hat{a}\hat{D}_\alpha \hat{S}_\varsigma|0\rangle$$
$$= \langle 0|(\hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1})(\hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle$$
$$= \langle 0|(\hat{a}\cosh s - \hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1})(-\hat{a}^+ e^{i\vartheta}\sinh s + \alpha\hat{1})|0\rangle$$
$$= \langle 0| - \hat{a}\hat{a}^+ e^{i\vartheta}\sinh s\cosh s + \alpha^2)|0\rangle$$
$$= -e^{i\vartheta}\sinh s\cosh s + \alpha^2$$

29. Application of the Bogoliubov transformation to squeezed coherent state

We start with the vacuum state $|0\rangle$, satisfying

$$\hat{a}|0\rangle = 0 .$$

Multiplying by $\hat{D}_\alpha \hat{S}_\varsigma$ from the left and using the fact that this operator is unitary, we may write

$$\hat{D}_\alpha \hat{S}_\varsigma \hat{a}\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{D}_\alpha \hat{S}_\varsigma|0\rangle = 0 ,$$

or

$$\hat{D}_\alpha \hat{S}_\varsigma \hat{a}\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}|\alpha,\varsigma\rangle = 0 .$$

Here we use the relation

$$\hat{D}_\alpha \hat{S}_\varsigma \hat{a} \hat{S}_\varsigma^{\ +} \hat{D}_\alpha^{\ +} = \hat{D}_\alpha (\lambda \hat{a} + \mu \hat{a}^+) \hat{D}_\alpha^{\ +}$$
$$= \lambda(\hat{a} - \alpha \hat{1}) + \mu(\hat{a}^+ - \alpha^* \hat{1})$$
$$= (\lambda \hat{a} + \mu \hat{a}^+) - (\lambda \alpha + \mu \alpha^*) \hat{1}$$

since

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma^{\ +} = \lambda \hat{a} + \mu \hat{a}^+ ,$$

and

$$\hat{D}_\alpha \hat{a} \hat{D}_\alpha^{\ +} = \hat{a} - \alpha \hat{1}, \qquad \hat{D}_\alpha \hat{a}^+ \hat{D}_\alpha^{\ +} = \hat{a}^+ - \alpha^* \hat{1}.$$

Then we have

$$(\lambda \hat{a} + \mu \hat{a}^+) |\alpha, \varsigma\rangle = (\lambda \alpha + \mu \alpha^*) |\alpha, \varsigma\rangle = \gamma |\alpha, \varsigma\rangle ,$$

or

$$\hat{b} |\alpha, \varsigma\rangle = \gamma |\alpha, \varsigma\rangle, \qquad \langle \alpha, \varsigma | \hat{b} = \langle \alpha, \varsigma | \gamma^*,$$

where

$$\gamma = \lambda \alpha + \mu \alpha^* = \alpha \cosh s + \alpha^* e^{i\vartheta} \sinh s .$$

This means that $|\alpha, \varsigma\rangle$ is the eigenket of $\hat{b} = \lambda \hat{a} + \mu \hat{a}^+$ with the eigenvalue γ.

30. Calculations of the fluctuation $(\Delta X)_{\alpha,\varsigma}^{\ \ 2}$ and $(\Delta Y)_{\alpha,\varsigma}^{\ \ 2}$

$$\langle \alpha, \varsigma | \hat{X} | \alpha, \varsigma\rangle = \langle \alpha, \varsigma | \frac{(\lambda^* - \mu^*)\hat{b} + (\lambda - \mu)\hat{b}^+}{2} | \alpha, \varsigma\rangle$$
$$= \frac{(\lambda^* - \mu^*)}{2} \langle \alpha, \varsigma | \hat{b} | \alpha, \varsigma\rangle + \frac{(\lambda - \mu)}{2} \langle \alpha, \varsigma | \hat{b}^+ | \alpha, \varsigma\rangle$$
$$= \frac{\gamma(\lambda^* - \mu^*)}{2} + \frac{\gamma^*(\lambda - \mu)}{2}$$

$$\langle \alpha, \varsigma | \hat{X}\hat{X} | \alpha, \varsigma\rangle = \langle \psi | \psi\rangle ,$$

where

$$
\begin{aligned}
|\psi\rangle &= \hat{X}|\alpha,\varsigma\rangle \\
&= \frac{(\lambda^* - \mu^*)\hat{b} + (\lambda - \mu)\hat{b}^+}{2}|\alpha,\varsigma\rangle \\
&= \frac{\gamma(\lambda^* - \mu^*)}{2}|\alpha,\varsigma\rangle + \frac{(\lambda - \mu)}{2}\hat{b}^+|\alpha,\varsigma\rangle
\end{aligned}
$$

since $\hat{X}^+ = \hat{X}$,

$$
\begin{aligned}
\langle\psi| &= \langle\alpha,\varsigma|\hat{X}^+ \\
&= \langle\alpha,\varsigma|\hat{X} \\
&= \langle\alpha,\varsigma|\frac{(\lambda - \mu)}{2}\hat{b}^+ + \langle\alpha,\varsigma|\frac{(\lambda^* - \mu^*)}{2}\hat{b} \\
&= \langle\alpha,\varsigma|[\frac{(\lambda - \mu)}{2}\gamma^* + \frac{(\lambda^* - \mu^*)}{2}\hat{b}]
\end{aligned}
$$

Then

$$
\begin{aligned}
\langle\psi|\psi\rangle &= \frac{1}{4}|\gamma|^2|\lambda-\mu|^2 + \frac{\gamma(\lambda^* - \mu^*)^2}{4}\langle\alpha,\varsigma|\hat{b}|\alpha,\varsigma\rangle + \frac{\gamma^*(\lambda-\mu)^2}{4}\langle\alpha,\varsigma|\hat{b}^+|\alpha,\varsigma\rangle \\
&\quad + \frac{1}{4}|\lambda-\mu|^2\langle\alpha,\varsigma|\hat{b}\hat{b}^+|\alpha,\varsigma\rangle \\
&= \frac{1}{4}|\gamma|^2|\lambda-\mu|^2 + \frac{\gamma^2(\lambda^* - \mu^*)^2}{4} + \frac{\gamma^{*2}(\lambda-\mu)^2}{4} \\
&\quad + \frac{1}{4}|\lambda-\mu|^2\langle\alpha,\varsigma|\hat{b}^+\hat{b}+\hat{1}|\alpha,\varsigma\rangle \\
&= \frac{1}{4}|\gamma|^2|\lambda-\mu|^2 + \frac{\gamma^2(\lambda^* - \mu^*)^2}{4} + \frac{\gamma^{*2}(\lambda-\mu)^2}{4} \\
&\quad + \frac{1}{4}|\lambda-\mu|^2(1+|\gamma|^2)
\end{aligned}
$$

The fluctuations $(\Delta X)_{\alpha,\varsigma}{}^2$ and $(\Delta Y)_{\alpha,\varsigma}{}^2$:

$$
\begin{aligned}
(\Delta X)_{\alpha,\varsigma}{}^2 &= \langle \alpha,\varsigma|\hat{X}\hat{X}|\alpha,\varsigma\rangle - \langle \alpha,\varsigma|\hat{X}|\alpha,\varsigma\rangle^2 \\
&= \frac{1}{4}|\gamma|^2|\lambda-\mu|^2 + \frac{\gamma^2(\lambda^*-\mu^*)^2}{4} + \frac{\gamma^{*2}(\lambda-\mu)^2}{4} \\
&\quad + \frac{1}{4}|\lambda-\mu|^2(1+|\gamma|^2) - [2\frac{|\gamma||\lambda-\mu|^2}{4} + \frac{\gamma^{*2}(\lambda-\mu)^2}{4} + \frac{\gamma^2(\lambda^*-\mu^*)^2}{4}] \\
&= \frac{1}{4}|\lambda-\mu|^2 = \frac{1}{4}(e^{2s}\sin^2\frac{\vartheta}{2} + e^{-2s}\cos^2\frac{\vartheta}{2})
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(\Delta Y)_{\alpha,\varsigma}{}^2 &= \langle \alpha,\varsigma|\hat{Y}\hat{Y}|\alpha,\varsigma\rangle - \langle \alpha,\varsigma|\hat{Y}|\alpha,\varsigma\rangle^2 \\
&= \frac{1}{4}|\lambda+\mu|^2 = \frac{1}{4}(e^{2s}\cos^2\frac{\vartheta}{2} + e^{-2s}\sin^2\frac{\vartheta}{2})
\end{aligned}
$$

which are identical to the results for the squeezed vacuum state.

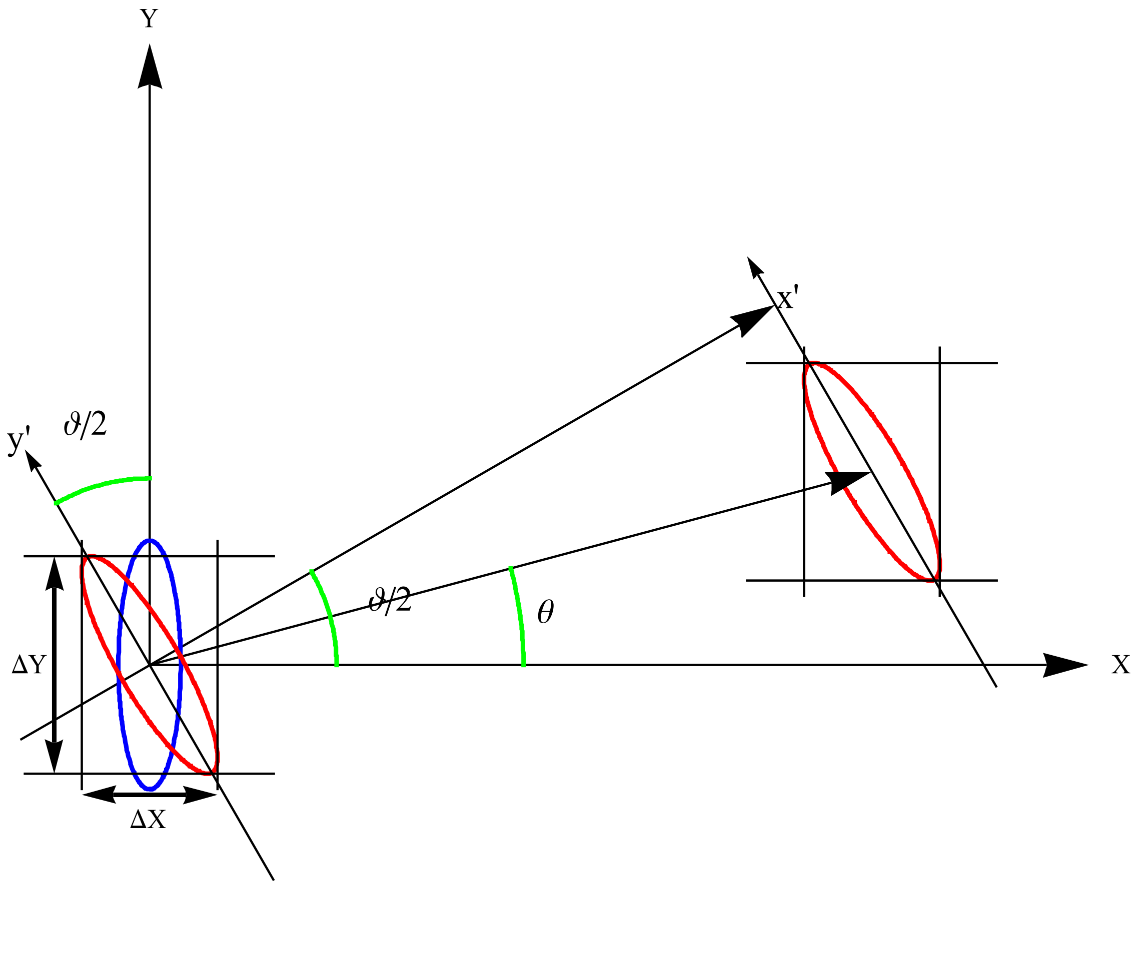


Fig.15(a) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. Representations of the quadrature operator means and variance for the squeezed vacuum state and the squeezed vacuum state. $\theta < \vartheta/2$ The ellipse (blue) at the origin is one for $\vartheta/2 = 0$. (a) The squeezed vacuum state. (b) The squeezed coherent state. $|\alpha,\varsigma\rangle = \hat{D}_{\alpha}\hat{S}_{\varsigma}|0\rangle$.

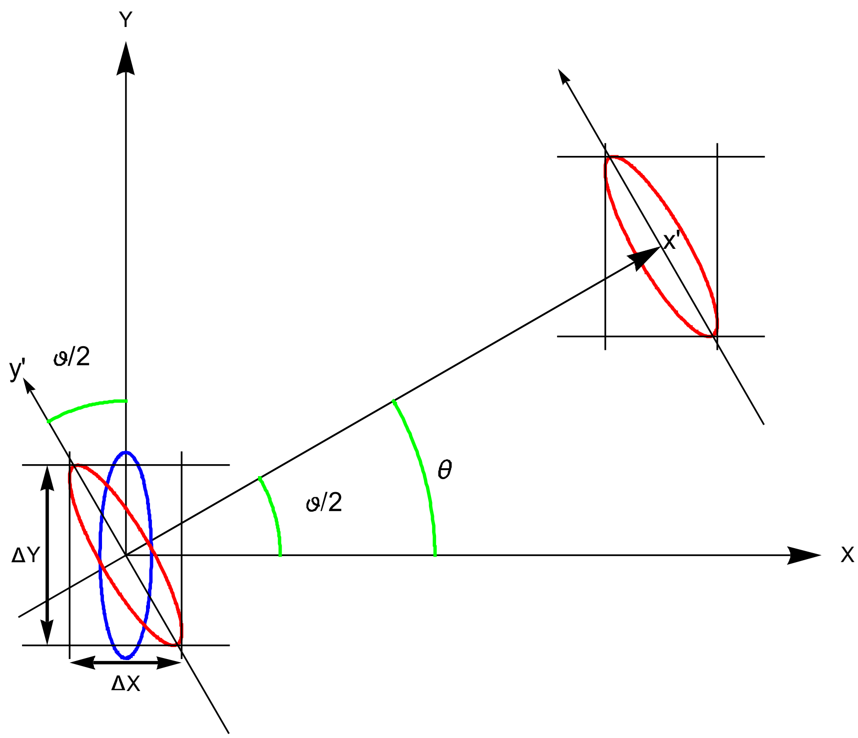


Fig.15(b) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. Rotated error ellipse of a displaced squeezed vacuum state. $\theta = \vartheta/2$. The state is defined by $|\alpha,\varsigma\rangle = \hat{D}_{\alpha}\hat{S}_{\varsigma}|0\rangle$. The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. $|\psi\rangle$ is the corresponding squeezed coherent state. .$\langle\psi|\hat{E}(\chi)|\psi\rangle = \mathrm{Re}[\langle\psi|\hat{a}|\psi\rangle e^{-i\chi}]$

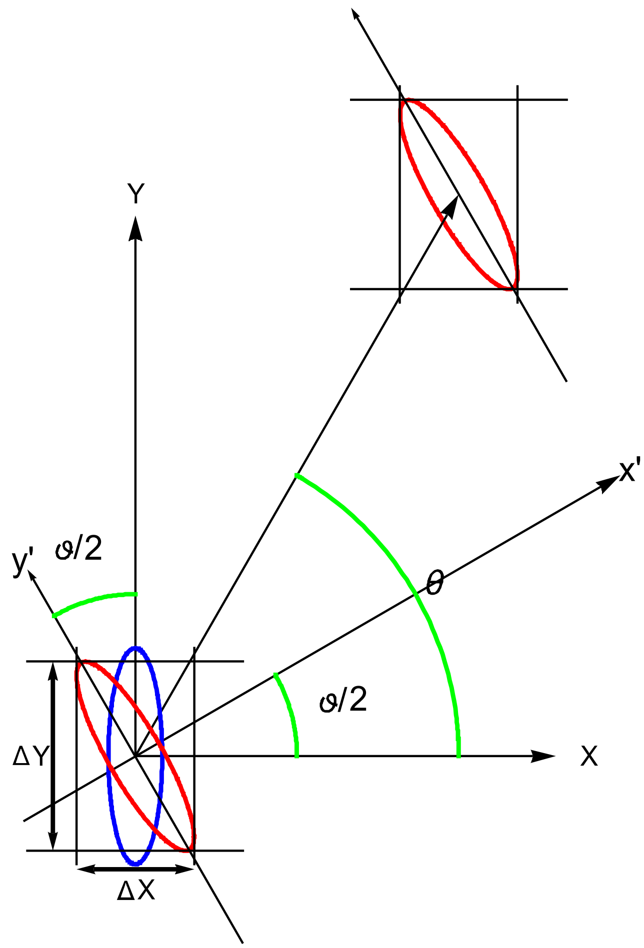
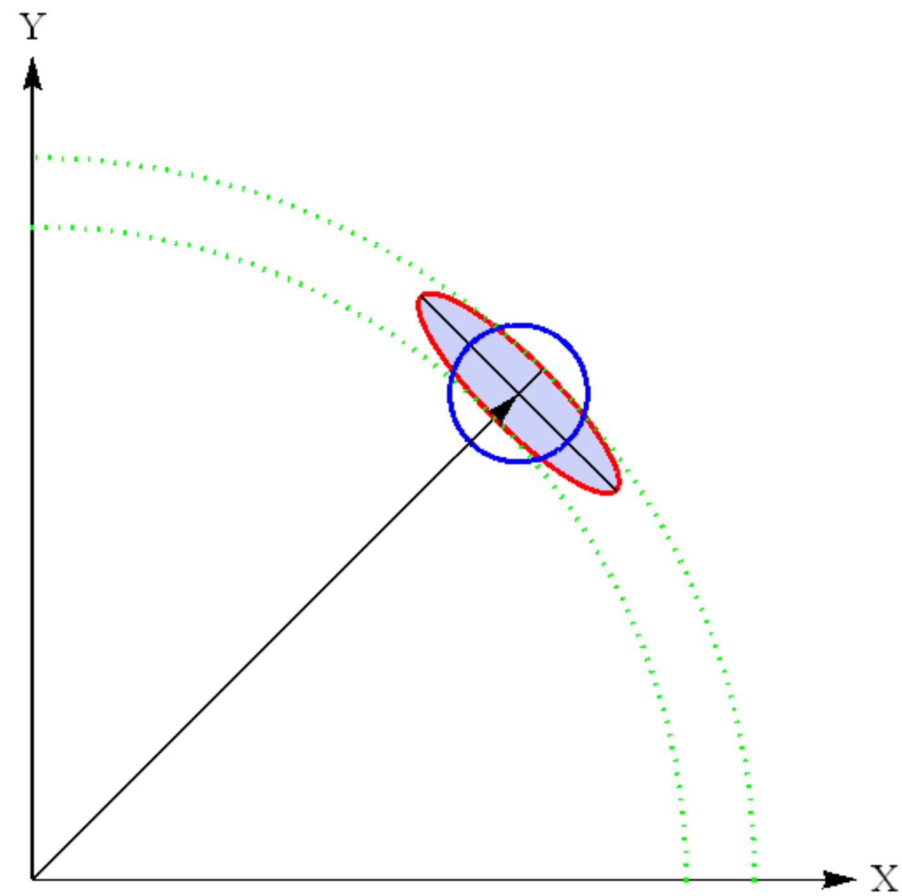


Fig.15(c) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. Rotated error ellipse of a displaced squeezed vacuum state. $\theta > \vartheta/2$. The state is defined by $|\alpha,\varsigma\rangle = \hat{D}_\alpha \hat{S}_\varsigma |0\rangle$.


Y
X

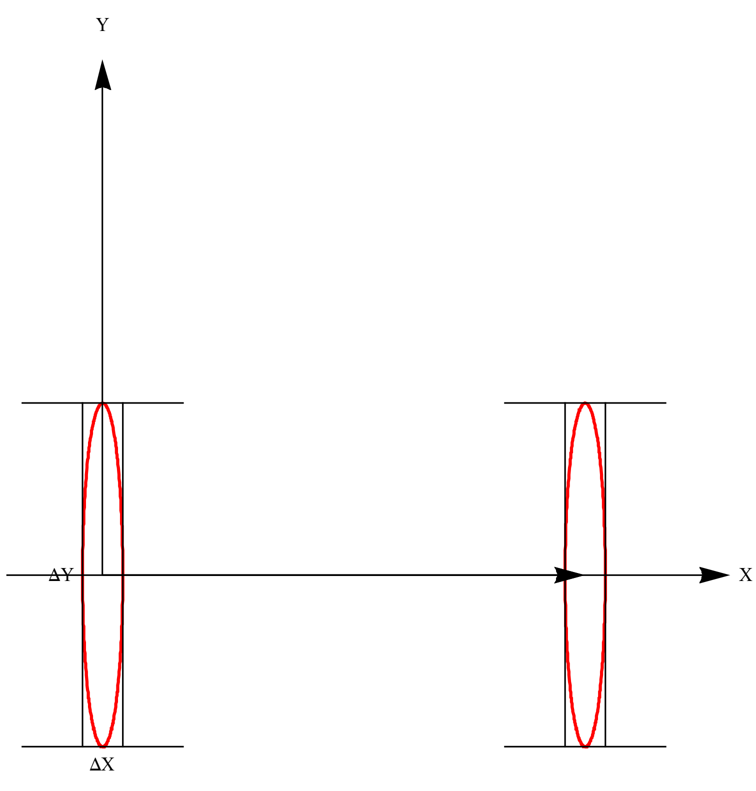

Y
ΔY
X
ΔX

Fig.16 Quadrature squeezed state. The phase-squeezed light.

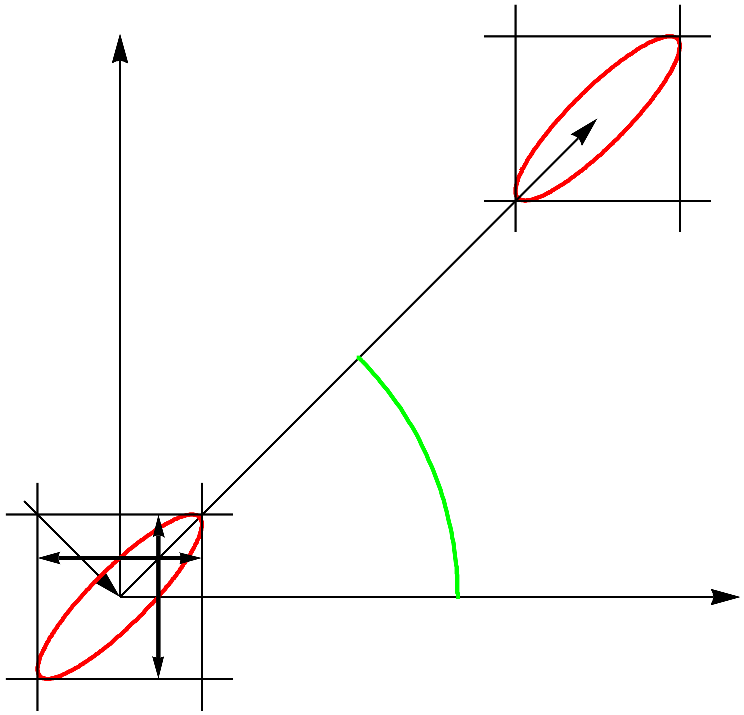


Fig.17(a) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. Quadrature squeezed state and squeezed coherent state. Amplitude-squeezed light

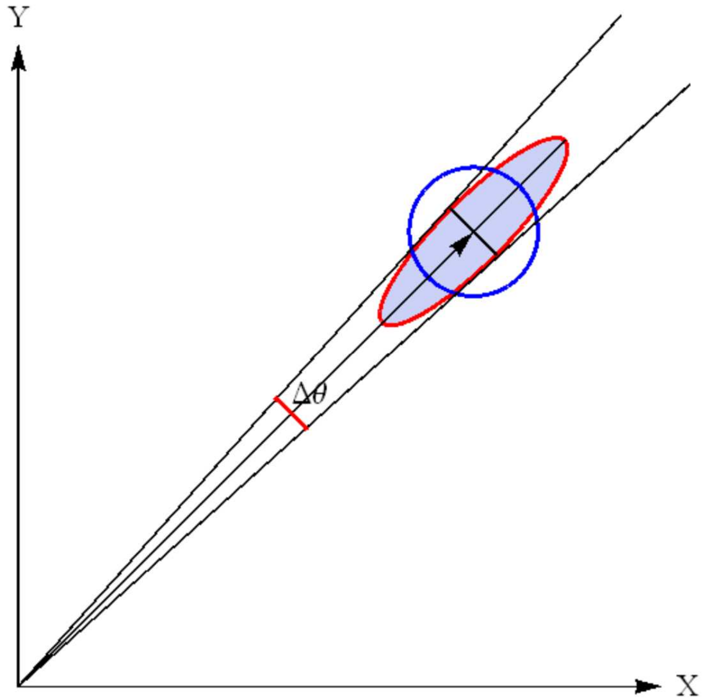


Fig.17(b) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. $|\psi\rangle$ is the corresponding squeezed coherent state. Amplitude-squeezed light.

31. The electric field

(a) The case of fixed phase

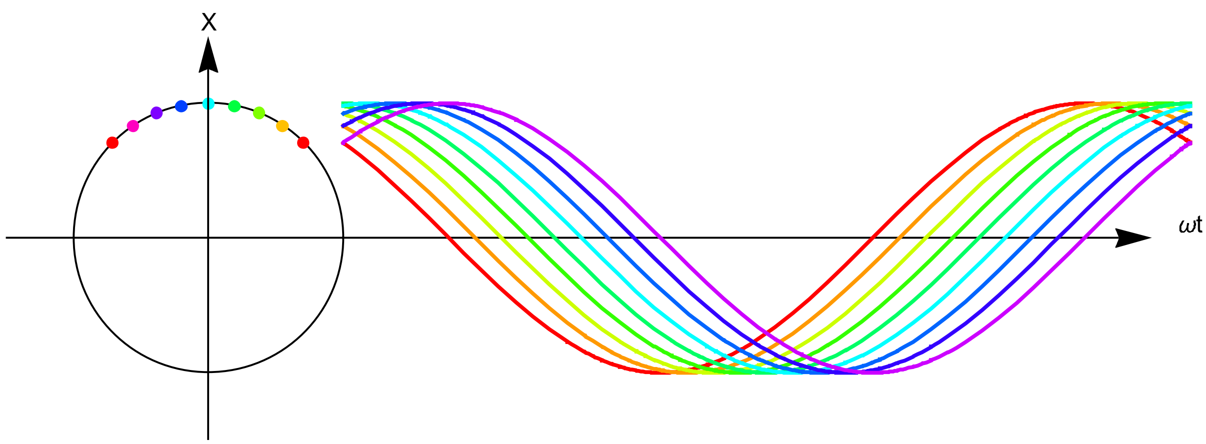


Fig.18 The error circle of a coherent state revolve about the origin of the phase space at the oscillator angular frequency ω and the expectation value of the electric field is the projection onto an axis parallel with x.

We assume that the ellipse is approximated by a line along the X axis. In this case, the phase of the electric field is fixed. Only the amplitude of the electric field changes over the limited region of X.

$$E(t) = a\sin(2\pi ft + \phi).$$

While a is the amplitude and ϕ is the phase. While ϕ is fixed, the amplitude a is changed.

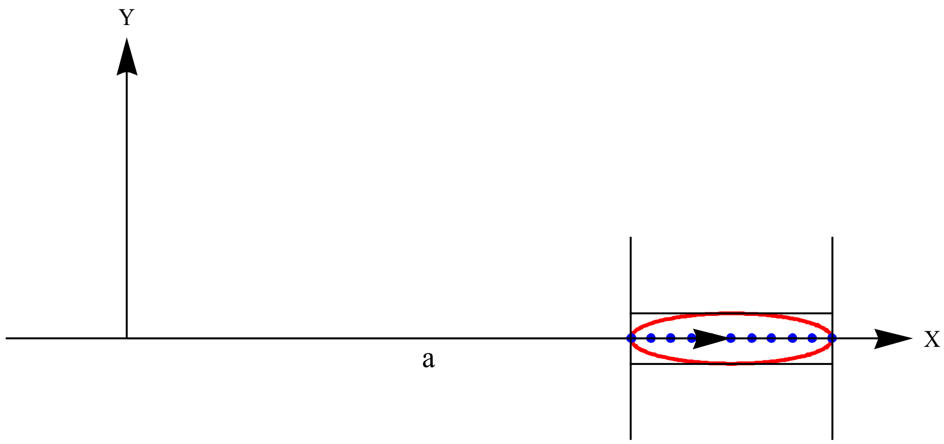


Fig.19(a) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$. $a = 0.8$ - 1.2 $\Delta a = 0.05$, f= 1.

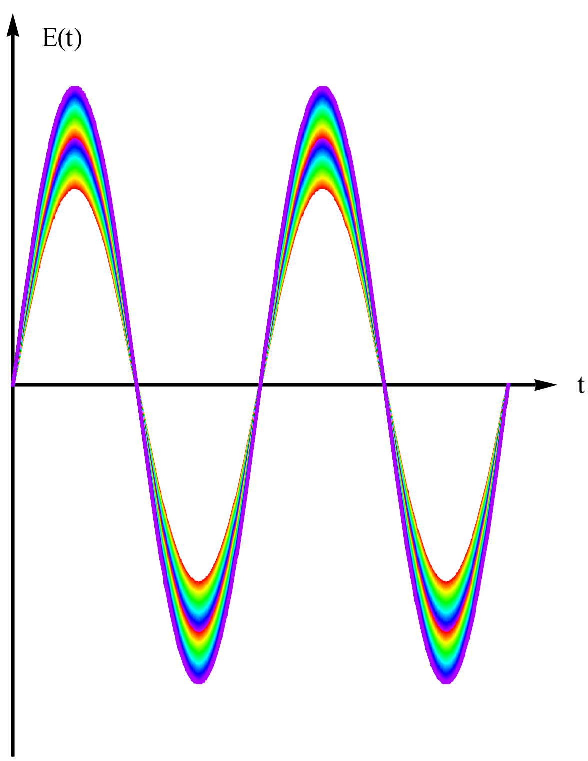


Fig.19(b) Plot of $E(t)$ vs t.

(b) The case of fixed amplitude

We assume that the ellipse is approximated by a line along the Y axis. In this case, the phase of the electric field changes, while the amplitude of the electric field almost remains unchanged over the limited region of Y.

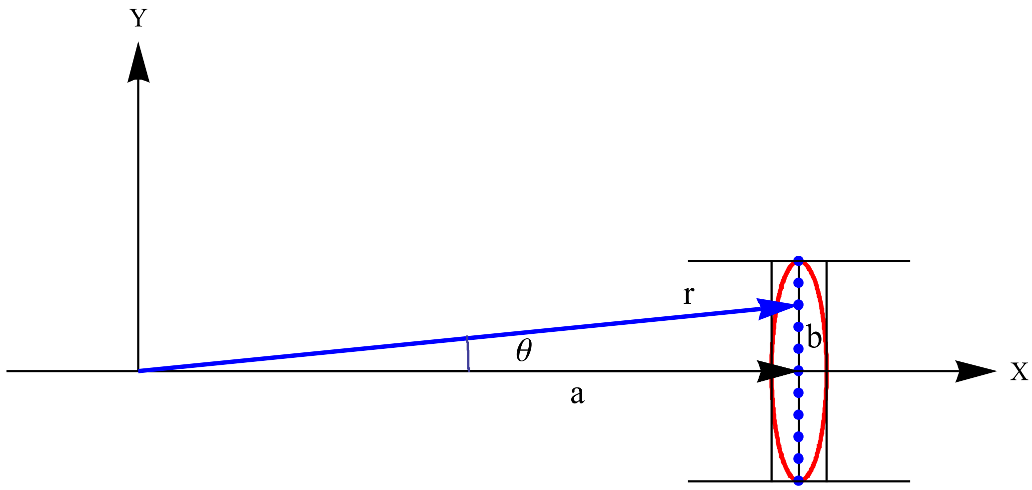


Fig.19(a) The 2D complex plane of $\langle\psi|\hat{a}|\psi\rangle = X + iY$ with $|\psi\rangle = |\alpha,\varsigma\rangle$ (the squeezed coherent state).

$$E(t) = r\sin(2\pi ft + \theta),$$

with

$$r = \sqrt{a^2 + b^2}, \quad \tan\theta = \frac{b}{a}.$$

While a is fixed, the parameter b is changed.

$$a = 1, \qquad b = -\frac{\pi}{5} - \frac{\pi}{5}, \ \Delta b = \frac{\pi}{50}. \qquad f = 1.$$

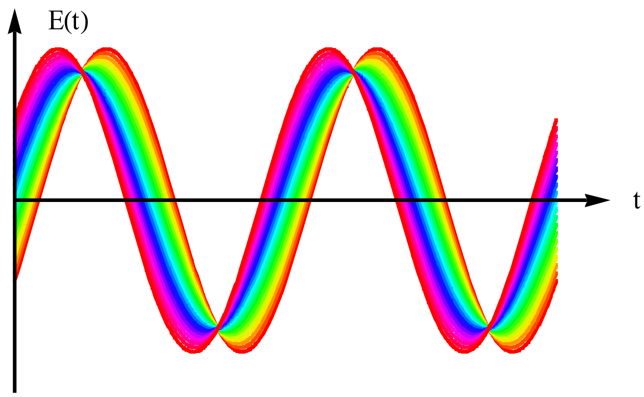


Fig.19(b) Electric field vs *t*.

((**Mathematica**))

```
Clear["Global`*"]; a1 = 1;
h1 = Graphics[{Black, Thick, Arrow[{{0, 0}, {2.2, 0}}],
   Arrow[{{0, -1.5}, {0, 1.5}}],
   Text[Style["E(t)", Black, 15], {0.2, 1.4}],
   Text[Style["t", Black, 15], {2.3, 0}]}];
g1 =
 Plot[Evaluate[Table[Sqrt[a1^2 + b1^2] Sin[2 π t + ArcTan[b1/a1]],
    {b1, -π/5, π/5, π/50}]], {t, 0, 2},
  PlotStyle → Table[{Hue[0.05 i], Thick}, {i, 0, 40}],
  Axes → False]; Show[g1, h1, PlotRange → All]
```

(c) Coherent state

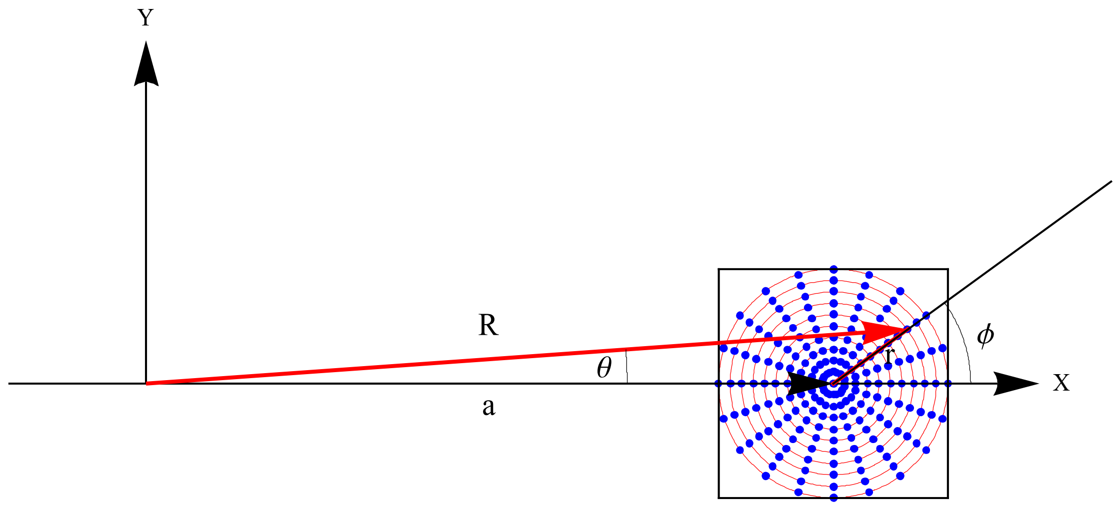


Fig.20(a) The 2D complex plane for the squeezed coherent state $|\psi\rangle = |\varsigma, \alpha\rangle$. Each point inside a circle contributes to the superposition of electromagnetic waves.

$$E(t) = R(r,\phi)\sin(2\pi ft + \theta),$$

with

$$R(r,\phi) = \sqrt{a^2 + r^2 + 2ar\cos\phi},$$

$$\sin\theta = \frac{r\sin\phi}{R(r,\phi)}.$$

While a is fixed, r and ϕ are changed as parameters.

$$a = 1, \qquad r = 0 - 0.2,\ \Delta r = 0.02,\ \phi = .0 - 2\pi. \qquad \Delta\phi = \frac{2\pi}{24} f = 1.$$

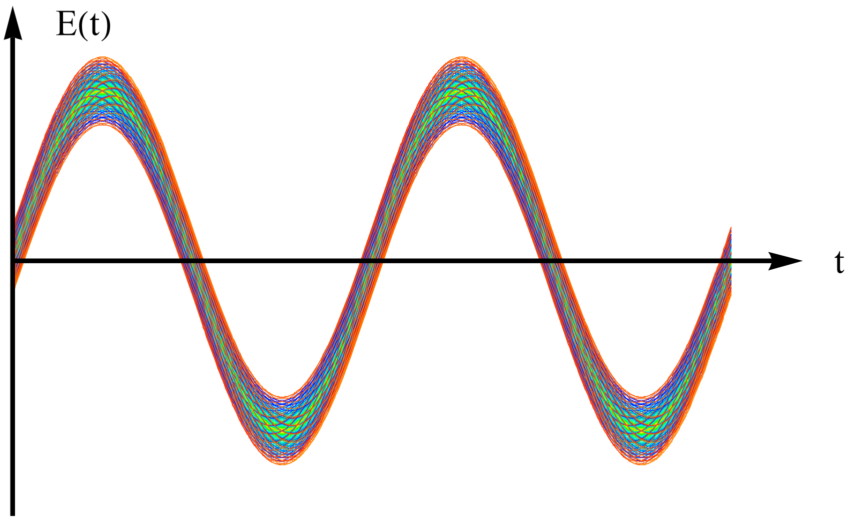


Fig.20(b) Electric field vs t.

```
Clear["Global`*"]; a = 1;
h1 = Graphics[{Black, Thick, Arrow[{{0, 0}, {2.2, 0}}],
   Arrow[{{0, -1.5}, {0, 1.5}}],
   Text[Style["E(t)", Black, 15], {0.2, 1.4}],
   Text[Style["t", Black, 15], {2.3, 0}]}];
R1[r_, φ_] := Sqrt[a^2 + r^2 + 2 a r Cos[φ]];
θ1[r_, φ_] := ArcSin[(r Sin[φ])/R1[r, φ]];
g1 =
 Plot[Evaluate[Table[ R1[r, φ] Sin[2 π t + θ1[r, φ] ],
    {r, 0, 0.2, 0.02}, {φ, 0, 2 π, (2 π)/24}]], {t, 0, 2},
  PlotStyle → Table[{Hue[i/360], Thin}, {i, 0, 240}],
  Axes → False];
Show[g1, h1, PlotRange → All]
```

32. Summary

Important results are summarized as follows. The electric field is expressed by

$$E = \mathrm{Re}[\tilde{E} e^{-i\chi}]$$

with $\tilde{E} = |\tilde{E}| e^{i\theta}$, where Re is the real part and $\tilde{E}$ is the complex amplitude of the electric field, θ is the phase angle, and $\chi = kz - \omega t - \frac{\pi}{2}$.

(a) Vacuum state $|0\rangle$

$$(\Delta X)_0 = \frac{1}{2}, \quad (\Delta Y)_0 = \frac{1}{2} \qquad\qquad (\Delta X)_0 (\Delta Y)_0 = \frac{1}{4}$$

$$\langle X\rangle_0 = \langle 0|\hat{X}|0\rangle = 0, \qquad\qquad \langle Y\rangle_0 = \langle 0|\hat{Y}|0\rangle = 0$$

(b) Coherent state $|\alpha\rangle = \hat{D}_\alpha|0\rangle$

Displacement operator:

$$\begin{aligned}\hat{D}_\alpha &= \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}^+) \\ &= \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}^+)\end{aligned}$$

$$E_\alpha = \mathrm{Re}[\tilde{E}_\alpha e^{-i\chi}]$$

with

$$\tilde{E}_\alpha = |\alpha|e^{i\theta} = \langle X\rangle_\alpha + i\langle Y\rangle_\alpha$$

$$\langle X\rangle_\alpha = \langle\alpha|\hat{X}|\alpha\rangle, \qquad \langle Y\rangle_\alpha = \langle\alpha|\hat{Y}|\alpha\rangle$$

$$(\Delta X)_\alpha = \frac{1}{2}, \quad (\Delta Y)_\alpha = \frac{1}{2}$$

(c) Squeezed state

The squeezed operator

$$\hat{S}_\varsigma = \exp[\frac{\varsigma^*}{2}\hat{a}^2 - \frac{\varsigma}{2}(\hat{a}^+)^2]$$

$$|\varsigma\rangle = \hat{S}_\varsigma|0\rangle$$

$$(\Delta X)_\varsigma (\Delta Y)_\varsigma = \frac{1}{4}$$

where $(\Delta X)_\varsigma \neq (\Delta Y)_\varsigma$.

$$\langle X\rangle_\varsigma = \langle \varsigma | \hat{X} | \varsigma\rangle = 0, \qquad \langle Y\rangle_\varsigma = \langle \varsigma | \hat{Y} | \varsigma\rangle = 0$$

(d) Coherent squeezed state

$$|\alpha,\varsigma\rangle = \hat{D}_\alpha \hat{S}_\varsigma |0\rangle$$

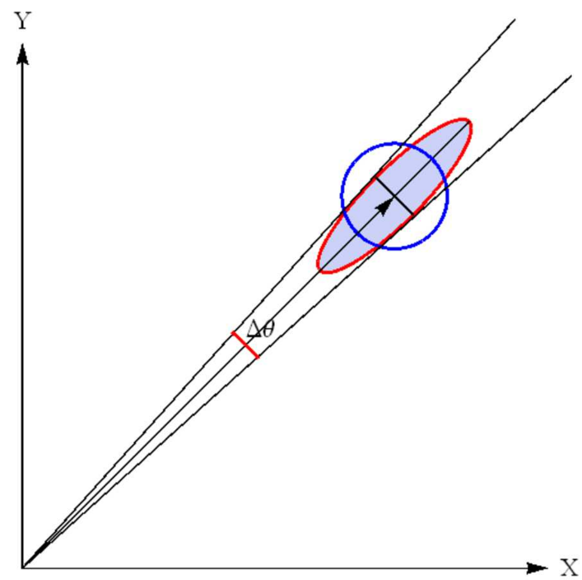


Fig.21 The 2D complex plane for the coherent state $|\psi\rangle = |\varsigma,\alpha\rangle$.

REFERENCES

S.M. Barnett and P.M. Radmore, *Methods in Theoretical Quantum Optics* (Oxford, 1997).

H.-A. Bachor and T.C. Ralph, *A Guide to Experiments in Quantum Optics*, Second, Revised and Enlarged Edition (Wiley-VCH, 2004).

P.R. Berman and V.S. Malinovsky, *Principles of Laser Spectroscopy and Quantum Optics* (Princeton, 2011).

M. Combescure and D. Robert, *Cohere*
nt states and Applications in Mathematical Physics (Springer 2012).

C. Gerry and P. Knight, *Introductory quantum optics* (Cambridge University Press, 2004).

M. Fox, *Quantum Optics An Introduction* (Oxford University Press, 2006).

J.-P. Gazeau, *Coherent States in Quantum Physics* (Wiley_VCH, 2009).

G. Grynberg, A. Aspect, and C. Fabre, *Introduction to Quantum Optics From the Semi-classical Approach to Quantized Light* (Cambridge University Press, 2010).

E. Hanamura, Y. Kawabe, and A. Yamanaka, *Quantum Nonlinear optics* (Springer, 2007).

H.A. Haus, *Electromagnetic Noise and Quantum Optical Measurements* (Springer 2000).
J.R. Klauder and E.C.G. Sudarshan, *Fundamentals of Quantum Optics* (W.A. Benjamin, 1968).
P. Lambropoulos and D. Petrosyan, *Fundamentals of Quantum Optics and Quantum Information* (Springer, 2007).
R. Loudon, *The Quantum Theory of Light*, 3rd Edition (Oxford, 2000).
L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge, 1995).
E. Merzbacher, *Quantum Mechanics*, 3rd edition (John Wiley & Sons, New York, 1998).
P. Meystre and M. Sargent III, *Elements of Quantum Optics*, 4th edition (Springer, 2007).
H. Paul, *Introduction to Quantum Optics* (Cambridge, 2004).
A. Perelomov, *Generalized Coherent States and Applications* (Springer, 1986).
J. Peřina, Z. Hradil, and B. Jurčo, *Quantum Optics and Fundamentals of Physics* (Kluwer Academic, 1994).
S.C. Rand, *Lectures on Light, Nonlinear and Quantum Optics using the Density Matrix* (Oxford, 2010).
M.O. Scully and M.S. Zubairy, *Quantum Optics* (Cambridge, 2001).
V. Vedral, *Modern Foundations of Quantum Optics* (Imperial College Press, 2005).
W. Vogel and D.-G. Welsch, *Quantum Optics*, Third, revised and extended edition (Wiley-VCH, 2006)
D.F. Walls and G.J. Milburn, *Quantum Optics*, 2nd edition (Springer, 2008).
H.M. Wiseman and G.J. Millburn, *Quantum Measurement and Control* (Cambridge, 2010).

APPENDIX

A: Schwarz inequality

$\hat{A}$ and $\hat{B}$ are two Hermitian operators with the condition

$$[\hat{A},\hat{B}]=i\hat{C}.$$

Then we have a Heisenberg's principle of uncertainty:

$$\Delta A\Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|.$$

B Commutation relations for coherent state and squeezed state

The derivation of all the formula were discussed previously. Here we only list up the formula used in the section.

B-1 Operators for the simple harmonics

$$\hat{a}\,|\,n>=\sqrt{n}\,|\,n-1>,$$

$$\hat{a}^{+}\,|\,n>=\sqrt{n+1}\,|\,n+1>,$$

$$\hat{n}\,|\,n>=n\,|\,n>,$$

$$|\,n>=\frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}\,|\,0>,$$

$$[\hat{n},\hat{a}]=-\hat{a}\,,$$

$$[\hat{n},\hat{a}^{+}]=\hat{a}^{+}\,.$$

B-2 Commutation relations related to the creation and annihilation operators

$$[\hat{a},\hat{a}^{+}]=\hat{1},$$

$$[\hat{a},(\hat{a}^{+})^{2}]=2\hat{a}^{+}\,,$$

$$[\hat{a},(\hat{a}^{+})^{3}]=3(\hat{a}^{+})^{2}\,,$$

$$[\hat{a},(\hat{a}^{+})^{n}]=n(\hat{a}^{+})^{n-1}\,,$$

$$[\hat{a},f(\hat{a}^{+})]=f\,'\left(\hat{a}^{+}\right),$$

$$[\hat{a}^{+},(\hat{a})^{2}]=-2\hat{a}\,,$$

$$[\hat{a}^{+},(\hat{a})^{3}]=-3\left(\hat{a}\right)^{2},$$

$$[\hat{a}^{+},(\hat{a})^{n}]=-n\left(\hat{a}\right)^{n-1},$$

or

$$[\hat{a}^{+},f(\hat{a})]=-f\,'\left(\hat{a}\right),$$

$$[(\hat{a})^{2},(\hat{a}^{+})^{2}]=2(\hat{a}^{+}\hat{a}+\hat{a}\hat{a}^{+})=2(2\hat{a}^{+}\hat{a}+\hat{1}),$$

$$[(\hat{a}^{+})^{2},(\hat{a})^{2}]=-2(\hat{a}^{+}\hat{a}+\hat{a}\hat{a}^{+})=-2(2\hat{a}^{+}\hat{a}+\hat{1}),$$

$$[\hat{a}^{2},(\hat{a}^{+})^{3}]=2\left(\hat{a}^{+}\right)^{2}\hat{a}+2\hat{a}\left(\hat{a}^{+}\right)^{2}+2\hat{a}^{+}\hat{a}\hat{a}^{+}.$$

$$[\hat{a}^2, \hat{n}] = \hat{a}^2\hat{a}^+\hat{a} - \hat{a}^+\hat{a}\hat{a}^2$$
$$= (\hat{a}^2\hat{a}^+ - \hat{a}^+\hat{a}^2)\hat{a}$$
$$= -[\hat{a}^+, \hat{a}^2]\hat{a}$$
$$= 2\hat{a}^2$$

$$[\hat{a}^{+2}, \hat{n}] = \hat{a}^{+2}\hat{a}^+\hat{a} - \hat{a}^+\hat{a}\hat{a}^{+2}$$
$$= -\hat{a}^+[\hat{a}, \hat{a}^{+2}]$$
$$= -2\hat{a}^{+2}$$

In general.

$[\hat{a}^p, \hat{n}] = p\hat{a}^p$, $[\hat{a}^{+p}, \hat{n}] = -p\hat{a}^{+p}$ (Messiah, p.460).

$[\hat{n}, \hat{a}^p] = -p\hat{a}^p$, $[\hat{n}, \hat{a}^{+p}] = p\hat{a}^{+p}$

$$[\hat{a}^+, f(\hat{a}, \hat{a}^+)] = -\frac{\partial}{\partial \hat{a}} f(\hat{a}, \hat{a}^+) .$$

$$[\hat{a}, f(\hat{a}, \hat{a}^+)] = \frac{\partial}{\partial \hat{a}} f(a, \hat{a}^+) \cdot$$

B-3 Baker-Hausdorff theorem

(i) In general, we get

$$\exp(\hat{A}x)\hat{B}\exp(-\hat{A}x) = \hat{B} + \frac{x}{1!}[\hat{A}, \hat{B}] + \frac{x^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{x^3}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + ...$$

(ii) Suppose that

$$[\hat{A}, [\hat{A}, \hat{B}]] = 0,$$

then we have

$$\exp(\hat{A}x)\hat{B}\exp(-\hat{A}x) = \hat{B} + x[\hat{A}, \hat{B}]$$

(ii) If the operators satisfy the relation (β: constant),

$$[\hat{A}, [\hat{A}, \hat{B}]] = \beta \hat{B}$$

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B}\cosh(x\sqrt{\beta}) + \frac{1}{\sqrt{\beta}}[\hat{A},\hat{B}]\sinh(x\sqrt{\beta}).$$

(iii) Suppose that $\beta = 0$, $[\hat{A},[\hat{A},\hat{B}]] = 0$. Then we get

$$e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \hat{B} + x[\hat{A},\hat{B}]$$

since

$$\lim_{\beta\to 0}\frac{\sinh(x\sqrt{\beta})}{\sqrt{\beta}} = x.$$

B-4 Baker-Campbell-Hausdorff formula

If the commutator of two operators $\hat{A}$ and $\hat{B}$ commutes with each of them ($\hat{A}$ and $\hat{B}$)

$$[\hat{A},[\hat{A},\hat{B}]] = \hat{0}$$

$$[\hat{B},[\hat{A},\hat{B}]] = \hat{0}$$

One has an identity

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-\frac{1}{2}[\hat{A},\hat{B}]). \qquad (2)$$

This is known as the Baker-Campbell-Hausdorff theorem.

((**Proof by Glauber**))

Roy J. Glauber (**A. Messiah, Quantum Mechanics p.422**)

We start with

.

Taking a derivative of $f(x)$ with respect to x, we get

$$\begin{aligned}\frac{df(x)}{dx} &= \hat{A}\exp(x\hat{A})\exp(x\hat{B}) + \exp(x\hat{A})\hat{B}\exp(x\hat{B})\\ &= (\hat{A} + \exp(x\hat{A})\hat{B}\exp(-x\hat{A}))\exp(x\hat{A})\exp(x\hat{B})\\ &= [\hat{A} + \exp(x\hat{A})\hat{B}\exp(-x\hat{A})]f(x)\end{aligned}$$

or

$$\begin{aligned}\frac{df(x)}{dx} &= \exp(x\hat{A})\exp(x\hat{B})\hat{B} + \hat{A}\exp(x\hat{A})\exp(x\hat{B}) \\ &= \exp(x\hat{A})\exp(x\hat{B})[\hat{B} + \exp(-x\hat{B})\hat{A}\exp(x\hat{B})] \\ &= f(x)[\hat{B} + \exp(-x\hat{B})\hat{A}\exp(x\hat{B})]\end{aligned}$$

We note the commutation relations which is derived above. If

$$[\hat{A},[\hat{A},\hat{B}]] = \hat{0} \quad \text{and} \quad [\hat{B},[\hat{A},\hat{B}]] = \hat{0}$$

then we have

$$\hat{A} + \exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{A} + \hat{B} + [\hat{A},\hat{B}]x$$

$$\hat{B} + \exp(-x\hat{B})\hat{A}\exp(x\hat{B}) = \hat{A} + \hat{B} + [\hat{A},\hat{B}]x$$

Using this relation, we get

$$\frac{df(x)}{dx} = (\hat{A} + \hat{B} + [\hat{A},\hat{B}]x)f(x) = f(x)(\hat{A} + \hat{B} + [\hat{A},\hat{B}]x),$$

with $f(x=0) = \hat{1}$. The operators $\hat{A} + \hat{B}$ and $[\hat{A},\hat{B}]$ commute,

$$[\hat{A} + \hat{B},[\hat{A},\hat{B}]] = [\hat{A},[\hat{A},\hat{B}]] + [\hat{B},[\hat{A},\hat{B}]] = 0$$

The function $f(x)$ and $\hat{A} + \hat{B} + [\hat{A},\hat{B}]x$ commutes. Then they can be considered as quantities of ordinary algebra,

$$\int \frac{1}{f} df = \int (\hat{A} + \hat{B} + [\hat{A},\hat{B}]x)dx$$

or

$$\ln(f) = (\hat{A} + \hat{B})x + \frac{x^2}{2}[\hat{A},\hat{B}],$$

or

$$f(x) = \exp(\hat{A}x)\exp(\hat{B}x) = \exp[(\hat{A} + \hat{B})x + \frac{x^2}{2}[\hat{A},\hat{B}]).$$

When $x = 1$,

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]),$$

or

$$\exp[(\hat{A} + \hat{B}) - \frac{1}{2}[\hat{A}, \hat{B}]) = \exp(\hat{A})\exp(\hat{B})$$

where

$$[\hat{A},[\hat{A},\hat{B}]] = \hat{0}, \qquad [\hat{B},[\hat{A},\hat{B}]] = \hat{0}.$$

In general case,

$$\exp(\hat{A})\exp(\hat{B}) = \exp[(\hat{A} + \hat{B}) + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{12}[\hat{B},[\hat{B},\hat{A}]] + ...]$$

B-6 Displacement operator $\hat{D}_\alpha$

(i) For the displacement operator

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}),$$

we have

$$\hat{D}_\alpha^+\hat{a}\hat{D}_\alpha = \hat{a} + \alpha\hat{1},$$

$$\hat{D}_\alpha^+\hat{a}^+\hat{D}_\alpha = \hat{a}^+ + \alpha^*\hat{1},$$

(ii)

$$\hat{D}_\alpha^+ = \hat{D}_{-\alpha} = \exp(-\alpha\hat{a}^+ + \alpha^*\hat{a}),$$

$$\hat{D}_\alpha^+\hat{D}_\alpha = \hat{1}. \quad \text{(Unitary operator).}$$

(iii)

$$\hat{D}_\alpha^+\hat{a}^+\hat{a}\hat{D}_\alpha = (\alpha^*\hat{1} + \hat{a}^+)(\alpha\hat{1} + \hat{a}) = |\alpha|^2 + \alpha\hat{a}^+ + \alpha^*\hat{a} + \hat{a}^+\hat{a}.$$

(iv) $\hat{D}_\alpha^{\ +} = \hat{D}_{-\alpha}$

$$\hat{D}_\alpha \hat{a}^+ \hat{D}_\alpha^{\ +} = \hat{a}^+ - \alpha^* \hat{1}$$

$$\hat{D}_\alpha \hat{a} \hat{D}_\alpha^{\ +} = \hat{a} - \alpha \hat{1}$$

B-7 Formula for $\hat{D}_\alpha$

$$\hat{D}_\alpha = \exp(\alpha \hat{a}^+ - \alpha^* \hat{a}) = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha \hat{a}^+)\exp(-\alpha^* \hat{a}),$$

where

$$[\hat{a}, \exp(\alpha \hat{a}^+)] = \alpha \exp(\alpha \hat{a}^+),$$

$$[\hat{a}^+, \exp(\alpha \hat{a})] = -\alpha \exp(\alpha \hat{a}).$$

$$\left[\alpha \hat{a}^+, -\alpha^* \hat{a}\right] = |\alpha|^2 \hat{1}.$$

B-8 Formula for the squeezed state

We define the operator

$$\hat{S}_\varsigma = \exp(\frac{1}{2}\varsigma^* \hat{a}^2 - \frac{1}{2}\varsigma \hat{a}^{+2}),$$

where

$\varsigma = s e^{i\vartheta}$, (ζ, complex number, s and ϑ are real)

$$\hat{S}_\varsigma^{\ +} \hat{a} \hat{S}_\varsigma = \hat{a}\cosh(s) - e^{i\vartheta} \hat{a}^+ \sinh(s),$$

$$\hat{S}_\varsigma^{\ +} \hat{a}^+ \hat{S}_\varsigma = \hat{a}^+ \cosh(s) - e^{-i\vartheta} \hat{a} \sinh(s).$$

$$\hat{S}_{-\varsigma} = \hat{S}_\varsigma^{\ +}$$

((**Proof**))

$$\hat{S}_\varsigma^{\ +} \hat{a} \hat{S}_\varsigma = \exp(\hat{A})\hat{B}\exp(-\hat{A})$$

with $A = -\frac{1}{2}\varsigma^* \hat{a}^2 + \frac{1}{2}\varsigma \hat{a}^{+2}$, $\hat{B} = \hat{a}$

$$
\begin{aligned}
[\hat{A}, \hat{B}] &= [-\frac{1}{2}\varsigma^* \hat{a}^2 + \frac{1}{2}\varsigma \hat{a}^{+2}, \hat{a}] \\
&= \frac{1}{2}\varsigma[\hat{a}^{+2}, \hat{a}] \\
&= -\frac{1}{2}\varsigma[\hat{a}, \hat{a}^{+2}] \\
&= -\varsigma \hat{a}^{+}
\end{aligned}
$$

$$
\begin{aligned}
[\hat{A},[\hat{A}, \hat{B}]] &= [-\frac{1}{2}\varsigma^* \hat{a}^2 + \frac{1}{2}\varsigma \hat{a}^{+2}, -\varsigma \hat{a}^{+}] \\
&= [-\frac{1}{2}\varsigma^* \hat{a}^2, -\varsigma \hat{a}^{+}] \\
&= -\frac{1}{2}|\varsigma|^2 [\hat{a}^{+}, \hat{a}^2] \\
&= |\varsigma|^2 \hat{a}
\end{aligned}
$$

$$
\begin{aligned}
[\hat{A},[\hat{A},[\hat{A}, \hat{B}]]] &= [-\frac{1}{2}\varsigma^* \hat{a}^2 + \frac{1}{2}\varsigma \hat{a}^{+2}, |\varsigma|^2 \hat{a}] \\
&= -\frac{1}{2}\varsigma |\varsigma|^2 [\hat{a}, \hat{a}^{+2}] \\
&= -\frac{1}{2}\varsigma |\varsigma|^2 2\hat{a}^{+} \\
&= -\varsigma |\varsigma|^2 \hat{a}^{+}
\end{aligned}
$$

$$
\begin{aligned}
[\hat{A},[\hat{A},[\hat{A},[\hat{A}, \hat{B}]]]] &= [-\frac{1}{2}\varsigma^* \hat{a}^2 + \frac{1}{2}\varsigma \hat{a}^{+2}, -\varsigma |\varsigma|^2 \hat{a}^{+}] \\
&= -\frac{1}{2}|\varsigma|^4 [\hat{a}^{+}, \hat{a}^2] \\
&= |\varsigma|^4 \hat{a}
\end{aligned}
$$

Finally, we get

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]$$

$$+\frac{1}{4!}[\hat{A},[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]] + ...$$

$$= \hat{a} - \varsigma\hat{a}^{+} + \frac{1}{2!}|\varsigma|^{2}\hat{a} - \frac{1}{3!}\varsigma|\varsigma|^{2}\hat{a}^{+} + \frac{1}{4!}|\varsigma|^{4}\hat{a} + ...$$

$$= (1 + \frac{1}{2!}|\varsigma|^{2} + \frac{1}{4!}|\varsigma|^{4} + ...)\hat{a} - e^{i\vartheta}[|\varsigma| + \frac{1}{3!}|\varsigma|^{3} + \frac{1}{5!}|\varsigma|^{5} + ...)\hat{a}^{+}$$

$$= \cosh(|\varsigma|)\hat{a} - e^{i\vartheta}\sinh(|\varsigma|)\hat{a}^{+}$$

or

$$\hat{b}_{1} = \hat{S}_{\varsigma}^{+}\hat{a}\hat{S}_{\varsigma}$$
$$= \mu\hat{a} - \nu\hat{a}^{+}$$
$$= \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^{+}\sinh(s)$$

with

$$\varsigma = se^{i\vartheta}, \qquad \varsigma = |\varsigma|e^{i\vartheta} = se^{i\vartheta} .$$

Similarly, we get

$$\hat{b}_{1}^{+} = \hat{S}_{\varsigma}^{+}\hat{a}^{+}\hat{S}_{\varsigma}$$
$$= \mu\hat{a}^{+} - \nu^{*}\hat{a}$$
$$= \hat{a}^{+}\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s)$$

since

$$[\hat{S}_{\varsigma}^{+}\hat{a}\hat{S}_{\varsigma}]^{+} = [\hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^{+}\sinh(s)]^{+}$$

or

$$\hat{S}_{\varsigma}^{+}\hat{a}^{+}\hat{S}_{\varsigma} = \mu\hat{a}^{+} - \nu^{*}\hat{a}$$
$$= \hat{a}^{+}\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s)$$

We note that using the relation $\hat{S}_{-\varsigma} = \hat{S}_{\varsigma}^{+}$

$$\hat{S}_{-\varsigma}\hat{a}^{+}\hat{S}_{-\varsigma}{}^{+} = \mu\hat{a}^{+} - \nu^{*}\hat{a}$$
$$= \hat{a}^{+}\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s)$$

When $\varsigma \to -\varsigma = s(-e^{i\vartheta})$, we have

$$\hat{S}_{\varsigma}\hat{a}^{+}\hat{S}_{\varsigma}{}^{+} = \mu\hat{a}^{+} + \nu^{*}\hat{a}$$
$$= \hat{a}^{+}\cosh(s) + e^{-i\vartheta}\hat{a}\sinh(s)$$

where the phase of $-\varsigma$ is $(-e^{i\vartheta}) = e^{i(\pi+\vartheta)}$. Taking the Hermitian conjugate for both the right hand side and the left hand side, we also get

$$\hat{S}_{\varsigma}\hat{a}\hat{S}_{\varsigma}{}^{+} = \mu\hat{a} + \nu\hat{a}^{+}$$
$$= \hat{a}\cosh(s) + e^{i\vartheta}\hat{a}^{+}\sinh(s)$$

where

$$\hat{b} = \hat{S}_{\varsigma}\hat{a}\hat{S}_{\varsigma}{}^{+} = \mu\hat{a} + \nu\hat{a}^{+}$$

and

$$\hat{b}^{+} = \hat{S}_{\varsigma}\hat{a}^{+}\hat{S}_{\varsigma}{}^{+} = \mu\hat{a}^{+} + \nu^{*}\hat{a}$$

B-9 Formula for the squeezed coherent state

$$\alpha = |\alpha|e^{i\theta}, \qquad \varsigma = se^{i\vartheta},$$

$$\hat{S}_{\varsigma}{}^{+}\hat{D}_{\alpha}{}^{+}\hat{a}\hat{D}_{\alpha}\hat{S}_{\varsigma} = \mu\hat{a} - \nu\hat{a}^{+} + \alpha\hat{1}$$
$$= \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^{+}\sinh(s) + \alpha\hat{1},$$

$$\hat{S}_{\varsigma}{}^{+}\hat{D}_{\alpha}{}^{+}\hat{a}^{+}\hat{D}_{\alpha}\hat{S}_{\varsigma} = \mu\hat{a}^{+} - \nu^{*}\hat{a} + \alpha^{*}\hat{1}$$
$$= \hat{a}^{+}\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s) + \alpha^{*}\hat{1},$$

where

$$\hat{D}_{\alpha}{}^{+}\hat{a}\hat{D}_{\alpha} = \hat{a} + \alpha\hat{1},$$

$$\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha = \hat{a}^+ + \alpha^*\hat{1}.$$

((**Proof**))

$$\begin{aligned}\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}\hat{D}_\alpha\hat{S}_\varsigma &= \hat{S}_\varsigma^{\ +}(\hat{a}+\alpha\hat{1})\hat{S}_\varsigma \\ &= \hat{S}_\varsigma^{\ +}\hat{a}\hat{S}_\varsigma + \alpha\hat{1} \\ &= \mu\hat{a} - \nu\hat{a}^+ + \alpha\hat{1} \\ &= \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^+\sinh(s) + \alpha\hat{1}\end{aligned}$$

$$\begin{aligned}\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma &= \hat{S}_\varsigma^{\ +}(\hat{a}^+ + \alpha^*\hat{1})\hat{S}_\varsigma \\ &= \hat{S}_\varsigma^{\ +}\hat{a}^+\hat{S}_\varsigma + \alpha^*\hat{1} \\ &= \mu\hat{a}^+ - \nu^*\hat{a} + \alpha^*\hat{1} \\ &= \hat{a}^+\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s) + \alpha^*\hat{1}\end{aligned}$$

B-10 Eigenstate (I) $|\varsigma\rangle = \hat{S}_\varsigma|0\rangle$

We start with the vacuum state $|0\rangle$, satisfying

$$\hat{a}|0\rangle = 0,$$

or

$$\hat{S}_\varsigma\hat{a}\hat{S}_\varsigma^{\ +}\hat{S}_\varsigma|0\rangle = 0.$$

Using the relation

$$\begin{aligned}\hat{b} &= \hat{S}_\varsigma\hat{a}\hat{S}_\varsigma^{\ +} \\ &= \mu\hat{a} + \nu\hat{a}^+ \\ &= \hat{a}\cosh(s) + e^{i\vartheta}\hat{a}^+\sinh(s)\end{aligned}$$

we have

$$(\mu\hat{a} + \nu\hat{a}^+)\hat{S}_\varsigma|0\rangle = 0$$

Here, we define $\hat{S}_\varsigma|0\rangle = |\varsigma\rangle$ and

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+$$

$$\mu = \cosh(s), \qquad \nu = e^{i\vartheta}\sinh(s)$$

Then, we get

$$\hat{b}|\varsigma\rangle = 0$$

with

$$\mu^2 - |\nu|^2 = \cosh^2(s) - \sinh^2(s) = 1 \qquad \text{(Bogoliubov transformation)}$$

Thus, the squeezed vacuum state is an eigenstate of the operator

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+, \qquad \hat{b}|\varsigma\rangle = 0$$

with eigenvalue zero.

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+, \qquad \hat{b}^+ = \nu^*\hat{a} + \mu\hat{a}^+$$

or

$$\hat{a} = \mu\hat{b} - \nu\hat{b}^+, \qquad \hat{a}^+ = \mu\hat{b}^+ - \nu^*\hat{b} = -\nu^*\hat{b} + \mu\hat{b}^+$$

Note that

$$\left[\hat{b}, \hat{b}^+\right] = \hat{1}$$

B-11 Eigenstate (II): $|\alpha,\varsigma\rangle = \hat{D}_\alpha\hat{S}_\varsigma|0\rangle$

We start with the vacuum state $|0\rangle$, satisfying

$$\hat{a}|0\rangle = 0.$$

Multiplying by $\hat{D}_\alpha\hat{S}_\varsigma$ from the left and using the fact that this operator is unitary, we may write

$$\hat{D}_\alpha\hat{S}_\varsigma\hat{a}[\hat{S}_\varsigma^+\hat{D}_\alpha^+\hat{D}_\alpha\hat{S}_\varsigma]|0\rangle = 0,$$

When we use $\hat{D}_\alpha\hat{S}_\varsigma|0\rangle = |\alpha,\varsigma\rangle$, we have

$$\hat{D}_\alpha[\hat{S}_\varsigma \hat{a}\hat{S}_\varsigma^{\ +}]\hat{D}_\alpha^{\ +}\left|\alpha,\varsigma\right\rangle = 0\,.$$

or

$$\hat{D}_\alpha \hat{b}\hat{D}_\alpha^{\ +}\left|\alpha,\varsigma\right\rangle = 0$$

where

$$\begin{aligned}\hat{b} &= \hat{S}_\varsigma \hat{a}\hat{S}_\varsigma^{\ +}\\ &= \mu\hat{a} + \nu\hat{a}^+\\ &= \hat{a}\cosh(s) + e^{i\vartheta}\hat{a}^+\sinh(s)\end{aligned}$$

Thus, we get

$$[\hat{D}_\alpha \hat{a}\hat{D}_\alpha^{\ +}\mu + \hat{D}_\alpha \hat{a}^+\hat{D}_\alpha^{\ +}\nu]\left|\alpha,\varsigma\right\rangle = 0$$

or

$$[(\hat{a} - \alpha\hat{1})\mu + (\hat{a}^+ - \alpha^*\hat{1})\nu]\left|\alpha,\varsigma\right\rangle = 0$$

Using the Bogoliubov transformation
or

$$\hat{b}\left|\alpha,\varsigma\right\rangle = (\mu\hat{a} + \nu\hat{a}^+)\left|\alpha,\varsigma\right\rangle = (\mu\alpha + \nu\alpha^*)\left|\alpha,\varsigma\right\rangle = \gamma\left|\alpha,\varsigma\right\rangle$$

where

$$\gamma = \mu\alpha + \nu\alpha^* = \alpha\cosh(s) + \alpha^* e^{i\vartheta}\sinh(s)$$

$$\hat{D}_\alpha \hat{a}\hat{D}_\alpha^{\ +} = \hat{a} - \alpha\hat{1}$$

$$\hat{D}_\alpha \hat{a}^+\hat{D}_\alpha^{\ +} = \hat{a}^+ - \alpha^*\hat{1}$$

$\left|\alpha,\varsigma\right\rangle$ is the eigenket of $\hat{b}$ with the eigenvalue γ.

$$\hat{b}\left|\alpha,\varsigma\right\rangle = \gamma\left|\alpha,\varsigma\right\rangle$$

B-12 Bogoliubov transformation

$$\hat{S}_\varsigma^{\ +}\hat{a}\hat{S}_\varsigma = \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^+\sinh(s)$$
$$= \mu\hat{a} - \nu\hat{a}^+$$
$$= \hat{b}_1$$

Similarly, we get

$$\hat{S}_\varsigma^{\ +}\hat{a}^+\hat{S}_\varsigma = \hat{a}^+\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s)$$
$$= \mu\hat{a}^+ - \nu^*\hat{a}$$
$$= \hat{b}_1^+$$

The commutation relation;

$$\left[\hat{b},\hat{b}^+\right] = \hat{S}_\varsigma[\hat{a},\hat{a}^+]\hat{S}_\varsigma^{\ +}$$
$$= \hat{1}$$

$$\left[\hat{b}_1,\hat{b}_1^+\right] = (\mu^2 - |\nu|^2)\hat{S}_\varsigma^{\ +}[\hat{a},\hat{a}^+]\hat{S}_\varsigma$$
$$= \mu^2 - |\nu|^2$$
$$= \hat{1}$$

B-13

$$\langle\alpha,\varsigma|\hat{a}|\alpha,\varsigma\rangle = \langle 0|\hat{S}_\varsigma^{\ +}[\hat{D}_\alpha^{\ +}\hat{a}\hat{D}_\alpha]\hat{S}_\varsigma|0\rangle$$
$$= \langle 0|\hat{S}_\varsigma^{\ +}(\hat{a}+\alpha\hat{1})\hat{S}_\varsigma|0\rangle$$
$$= \langle 0|\hat{b}_1 + \alpha\hat{1}|0\rangle$$
$$= \alpha$$

$$\langle\alpha,\varsigma|\hat{a}^+|\alpha,\varsigma\rangle = \langle 0|\hat{S}_\varsigma^{\ +}[\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha]\hat{S}_\varsigma|0\rangle$$
$$= \langle 0|\hat{S}_\varsigma^{\ +}(\hat{a}^+ + \alpha^*\hat{1})\hat{S}_\varsigma|0\rangle$$
$$= \langle 0|\hat{b}_0 + \alpha^*\hat{1}|0\rangle$$
$$= \alpha^*$$

$$
\begin{aligned}
\langle\alpha,\varsigma|\hat{a}^2|\alpha,\varsigma\rangle &= \langle 0|\hat{S}_\varsigma^{\,+}[\hat{D}_\alpha^{\,+}\hat{a}^2\hat{D}_\alpha]\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{S}_\varsigma^{\,+}[\hat{D}_\alpha^{\,+}\hat{a}\hat{D}_\alpha\hat{D}_\alpha^{\,+}\hat{a}\hat{D}_\alpha]\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{S}_\varsigma^{\,+}(\hat{a}+\alpha\hat{1})^2\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{b}_1^{\,2}+2\alpha\hat{b}_1+\alpha^2\hat{1}|0\rangle \\
&= \alpha^2+\langle 0|\hat{b}_1^{\,2}|0\rangle \\
&= \alpha^2-\mu\nu \\
&= \alpha^2-\sinh(s)\cosh(s)e^{i\vartheta}
\end{aligned}
$$

where

$$
\langle 0|\hat{b}_1^{\,2}|0\rangle = -\mu\nu\langle 0|2\hat{n}+\hat{1}|0\rangle = -\mu\nu = -\sinh(s)\cosh(s)e^{i\vartheta}
$$

$$
\begin{aligned}
\langle\alpha,\varsigma|\hat{a}^{+2}|\alpha,\varsigma\rangle &= \langle 0|\hat{S}_\varsigma^{\,+}[\hat{D}_\alpha^{\,+}\hat{a}^{+2}\hat{D}_\alpha]\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{S}_\varsigma^{\,+}[\hat{D}_\alpha^{\,+}\hat{a}^{+}\hat{D}_\alpha\hat{D}_\alpha^{\,+}\hat{a}^{+}\hat{D}_\alpha]\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{S}_\varsigma^{\,+}(\hat{a}^{+}+\alpha^*\hat{1})^2\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{b}_1^{\,+2}+2\alpha^*\hat{b}_1^{\,+}+\alpha^{*2}\hat{1}|0\rangle \\
&= \alpha^{*2}+\langle 0|\hat{b}_1^{\,+2}|0\rangle \\
&= \alpha^{*2}-\langle 0|\mu\nu^*(2\hat{n}+\hat{1})|0\rangle \\
&= \alpha^{*2}-\mu\nu^* \\
&= \alpha^{*2}-\sinh(s)\cosh(s)e^{-i\vartheta}
\end{aligned}
$$

where

$$
\langle 0|\hat{b}_1^{\,+2}|0\rangle = -\mu\nu^*\langle 0|2\hat{n}+\hat{1}|0\rangle = -\mu\nu^* = -\sinh(s)\cosh(s)e^{-i\vartheta}
$$

The average of number operator:

$$
\begin{aligned}
\langle\alpha,\varsigma|\hat{a}^{+}\hat{a}|\alpha,\varsigma\rangle &= \langle 0|\hat{S}_\varsigma^{\,+}(\hat{D}_\alpha^{\,+}\hat{a}^{+}\hat{D}_\alpha)(\hat{D}_\alpha^{\,+}\hat{a}\hat{D}_\alpha)]\hat{S}_\varsigma|0\rangle \\
&= \langle 0|\hat{S}_\varsigma^{\,+}(\hat{a}^{+}+\alpha^*\hat{1})(\hat{a}+\alpha\hat{1})\hat{S}_\varsigma|0\rangle \\
&= \langle 0|(\hat{b}_1^{\,+}+\alpha^*\hat{1})(\hat{b}_1+\alpha\hat{1})|0\rangle \\
&= \langle 0|\hat{b}_1^{\,+}\hat{b}_1+|\alpha|^2\hat{1}|0\rangle \\
&= |\alpha|^2+|\nu|^2 \\
&= |\alpha|^2+\sinh^2(s)
\end{aligned}
$$

where

$$\langle 0|\hat{b}_1^+\hat{b}_1|0\rangle = |\nu|^2 = \sinh^2(s)$$

We also calculate

$$\begin{aligned}\langle \alpha,\varsigma|\hat{a}\hat{a}^+|\alpha,\varsigma\rangle &= \langle \alpha,\varsigma|\hat{a}^+\hat{a}+\hat{1}|\alpha,\varsigma\rangle \\ &= \langle \alpha,\varsigma|\hat{a}^+\hat{a}|\alpha,\varsigma\rangle + 1 \\ &= |\alpha|^2 + 1 + |\nu|^2 \\ &= |\alpha|^2 + \mu^2 \\ &= |\alpha|^2 + \cosh^2(s)\end{aligned}$$

B-13 S

$$\langle 0|\hat{b}_1|0\rangle = 0, \qquad \langle 0|\hat{b}_1^+|0\rangle = 0$$

$$\langle 0|\hat{b}_1\hat{b}_1^+|0\rangle = \mu^2 = \cosh^2(s)$$

$$\langle 0|\hat{b}_1^+\hat{b}_1|0\rangle = \langle 0|\hat{b}_1\hat{b}_1^+ - \hat{1}|0\rangle = \mu^2 - 1 = |\nu|^2 = \sinh^2(s)$$

$$\langle 0|\hat{b}_1^{+2}|0\rangle = -\mu\nu^*$$

$$\langle 0|\hat{b}^2|0\rangle = -\mu\nu$$

$$\begin{aligned}\langle 0|\hat{b}_1^{+2}b_1|0\rangle &= \langle 0|(\mu\hat{a}^+ - \nu^*\hat{a})^2(\mu\hat{a} - \nu\hat{a}^+)|0\rangle \\ &= \langle 0|(\mu\hat{a}^+ - \nu^*\hat{a})(\mu\hat{a}^+ - \nu^*\hat{a})(\mu\hat{a} - \nu\hat{a}^+)|0\rangle \\ &= \langle 0|(-\nu^*\hat{a})(\mu\hat{a}^+ - \nu^*\hat{a})(-\nu\hat{a}^+)|0\rangle \\ &= |\nu|^2\langle 0|\hat{a}(\mu\hat{a}^+ - \nu^*\hat{a})\hat{a}^+)|0\rangle \\ &= 0\end{aligned}$$

$$\langle 0|\hat{b}_1^+\hat{b}_1^2|0\rangle = \langle 0|\hat{b}_1^{+2}b_1|0\rangle^* = 0$$

B-14 Calculation of $(\Delta X)^2$

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^+), \qquad \hat{Y} = \frac{1}{2i}(\hat{a} - \hat{a}^+)$$

$$\hat{X}^2 = \frac{1}{4}(\hat{a}^+\hat{a}^+ + \hat{a}\hat{a} + 2\hat{n} + \hat{1})$$

$$\hat{Y}^2 = \frac{1}{4}(-\hat{a}^+\hat{a}^+ - \hat{a}\hat{a} + 2\hat{n} + \hat{1})$$

$$(\Delta X)^2 = \langle \alpha,\varsigma | \hat{X}^2 | \alpha,\varsigma \rangle - \langle \alpha,\varsigma | \hat{X} | \alpha,\varsigma \rangle^2$$

$$\langle \alpha,\varsigma | \hat{X} | \alpha,\varsigma \rangle = \frac{1}{2}\langle \alpha,\varsigma | \hat{a} + \hat{a}^+ | \alpha,\varsigma \rangle = \frac{1}{2}(\alpha + \alpha^*)$$

$$\begin{aligned}\langle \alpha,\varsigma | \hat{X}^2 | \alpha,\varsigma \rangle &= \frac{1}{4}\langle \alpha,\varsigma | \hat{a}^+\hat{a}^+ + \hat{a}\hat{a} + 2\hat{n} + \hat{1} | \alpha,\varsigma \rangle \\ &= \frac{1}{4}[\alpha^{*2} + \alpha^2 - (\mu\nu^* + \mu\nu) + 2|\alpha|^2 + 2|\nu|^2 + 1] \\ &= \frac{1}{4}[\alpha^2 + \alpha^{*2} - \sinh(s)\cosh(s)e^{-i\vartheta} - \sinh(s)\cosh(s)e^{i\vartheta} \\ &\quad +2|\alpha|^2 + 2\sinh^2(s) + 1]\end{aligned}$$

$$\begin{aligned}(\Delta X)^2 &= \frac{1}{4}[\alpha^{*2} + \alpha^2 - (\mu\nu^* + \mu\nu) + 2|\alpha|^2 + 2|\nu|^2 + 1] - \frac{1}{4}(\alpha^{*2} + \alpha^2 + 2|\alpha|^2)] \\ &= \frac{1}{4}(-\mu\nu - \mu\nu^* + 2|\nu|^2 + 1) \\ &= \frac{1}{4}(-\mu\nu - \mu\nu^* + \mu^2 + |\nu|^2) \\ &= \frac{1}{4}[s\,\mathrm{inh}^2(s) + \cosh^2(s) - 2\sinh(s)\cosh(s)\cos\vartheta]\end{aligned}$$

B-15 Calculation of $(\Delta Y)^2$

$$(\Delta Y)^2 = \langle \alpha,\varsigma | \hat{Y}^2 | \alpha,\varsigma \rangle - \langle \alpha,\varsigma | \hat{Y} | \alpha,\varsigma \rangle^2$$

$$\begin{aligned}\langle \alpha,\varsigma | \hat{Y}^2 | \alpha,\varsigma \rangle &= -\frac{1}{4}\langle \alpha,\varsigma | \hat{a}^+\hat{a}^+ + \hat{a}\hat{a} - 2\hat{n} - \hat{1} | \alpha,\varsigma \rangle \\ &= -\frac{1}{4}[\alpha^2 + \alpha^{*2} - \mu\nu - \mu\nu^* - 2(|\alpha|^2 + |\nu|^2) - 1]\end{aligned}$$

$$\langle \alpha,\varsigma | \hat{Y} | \alpha,\varsigma \rangle = \frac{1}{2i}\langle \alpha,\varsigma | \hat{a} - \hat{a}^+ | \alpha,\varsigma \rangle = \frac{1}{2i}(\alpha - \alpha^*)$$

$$\left(\Delta Y\right)^2 = -\frac{1}{4}[\alpha^2 + \alpha^{*2} - \mu\nu - \mu\nu^* - 2(|\alpha|^2 + |\nu|^2) - 1] + \frac{1}{4}(\alpha^2 + \alpha^{*2} - 2|\alpha|^2)$$
$$= \frac{1}{4}[(\mu\nu + \mu\nu^*) + 2|\nu|^2 + 1]$$
$$= \frac{1}{4}[(\mu\nu + \mu\nu^*) + \mu^2 + |\nu|^2]$$
$$= \frac{1}{4}[2\sinh(s)\cosh(s)\cos\vartheta + \sinh^2(s) + \cosh^2(s)]$$

$$(\Delta X)^2\left(\Delta Y\right)^2 = \frac{1}{16}[-(\mu\nu + \mu\nu^*) + \mu^2 + |\nu|^2][(\mu\nu + \mu\nu^*) + \mu^2 + |\nu|^2]$$
$$= \frac{1}{16}[(\mu^2 + |\nu|^2)^2 - \mu^2(\nu + \nu^*)^2]$$
$$= \frac{1}{16}[(\mu^4 + |\nu|^4 + 2\mu^2|\nu|^2) - \mu^2(\nu^2 + \nu^{*2} + 2|\nu|^2]$$
$$= \frac{1}{16}[(\mu^4 + |\nu|^4) - \mu^2(\nu^2 + \nu^{*2})]$$
$$= \frac{1}{16}(\mu^2 - \nu^2)(\mu^2 - \nu^{*2})$$
$$= \frac{1}{4}|\mu^2 - \nu^2|$$

For simplicity, we assume that $\vartheta = 0$

$$\left(\Delta X\right)^2 = \frac{1}{4}[\cosh(s) - \sinh(s)]^2 = \frac{1}{4}e^{-2s},$$

$$\left(\Delta Y\right)^2 = \frac{1}{4}[\cosh(s) + \sinh(s)]^2 = \frac{1}{4}e^{2s},$$

or

$$\Delta X = \frac{1}{2}e^{-s}, \qquad \Delta Y = \frac{1}{2}e^{s},$$

leading to

$$\Delta X \Delta Y = \frac{1}{4}.$$

B-16 Generalized quadrature

$$\hat{X}(\theta)=\hat{R}^{+}(\theta)\left(\frac{\hat{a}+\hat{a}^{+}}{2}\right)\hat{R}^{+}(\theta)=\frac{e^{i\theta}\hat{a}+e^{-i\theta}\hat{a}^{+}}{2}$$

$$\hat{Y}(\theta)=\hat{R}^{+}(\theta)\left(\frac{\hat{a}-\hat{a}^{+}}{2i}\right)\hat{R}(\theta)=\frac{e^{i\theta}\hat{a}-e^{-i\theta}\hat{a}^{+}}{2i}$$

$$\hat{R}(\theta)^{+}\hat{a}\hat{R}(\theta)=e^{i\theta}\hat{a},$$

$$\hat{R}(\theta)^{+}\hat{a}^{+}\hat{R}(\theta)=e^{-i\theta}\hat{a}^{+},$$

where

$$\hat{R}(\theta)=e^{-i\theta\hat{n}} \qquad \text{(rotation operator)}$$

C-1 Number fluctuation

$$\begin{aligned}
\langle\alpha,\varsigma|\hat{n}^{2}|\alpha,\varsigma\rangle &= \langle\alpha,\varsigma|\hat{a}^{+}\hat{a}\hat{a}^{+}\hat{a}|\alpha,\varsigma\rangle \\
&= \langle\alpha,\varsigma|\hat{a}^{+}(\hat{a}^{+}\hat{a}+\hat{1})\hat{a}|\alpha,\varsigma\rangle \\
&= \langle\alpha,\varsigma|\hat{a}^{+2}\hat{a}^{2}+\hat{n}|\alpha,\varsigma\rangle \\
&= \langle\alpha,\varsigma|\hat{a}^{+2}\hat{a}^{2}|\alpha,\varsigma\rangle+\langle\alpha,\varsigma|\hat{n}|\alpha,\varsigma\rangle \\
&= \langle 0|\hat{S}_{\varsigma}{}^{+}\hat{D}_{\alpha}{}^{+}(\hat{a}^{+2}\hat{a}^{2})\hat{D}_{\alpha}\hat{S}_{\varsigma}|0\rangle+\langle\alpha,\varsigma|\hat{n}|\alpha,\varsigma\rangle \\
&= \langle 0|\hat{S}_{\varsigma}{}^{+}(\hat{a}^{+}+\alpha^{*}\hat{1})^{2}(\hat{a}+\alpha\hat{1})\hat{S}_{\varsigma}|0\rangle+\langle\alpha,\varsigma|\hat{n}|\alpha,\varsigma\rangle
\end{aligned}$$

Note that

$$\begin{aligned}
&\langle 0|\hat{S}_{\varsigma}{}^{+}(\hat{a}^{+}+\alpha^{*}\hat{1})^{2}(\hat{a}+\alpha\hat{1})\hat{S}_{\varsigma}|0\rangle \\
&=\langle 0|(\hat{b}_{1}^{+}+\alpha^{*}\hat{1})^{2}(\hat{b}_{1}+\alpha\hat{1})^{2}|0\rangle \\
&=\langle 0|(\hat{b}_{1}^{+2}+2\alpha^{*}\hat{b}_{1}^{+}+\alpha^{*2}\hat{1})(\hat{b}_{1}^{2}+2\alpha\hat{b}_{1}+\alpha^{2}\hat{1})|0\rangle \\
&=\langle 0|\hat{b}_{1}^{+2}\hat{b}_{1}^{2}+2\alpha\hat{b}_{1}^{+2}\hat{b}_{1}+\alpha^{2}\hat{b}_{1}^{+2}+2\alpha^{*}\hat{b}_{1}^{+}\hat{b}_{1}^{2}+4|\alpha|^{2}\hat{b}_{1}^{+}\hat{b}_{1}+2\alpha^{2}\alpha^{*}\hat{b}_{1}^{+} \\
&\quad+\alpha^{*2}\hat{b}_{1}^{2}+2\alpha^{*2}\alpha\hat{b}_{1}+2|\alpha|^{4}\hat{1}|0\rangle \\
&=\langle 0|\hat{b}_{1}^{+2}\hat{b}_{1}^{2}|0\rangle+\alpha^{2}\langle 0|\hat{b}_{1}^{+2}|0\rangle+4|\alpha|^{2}\langle 0|\hat{b}_{1}^{+}\hat{b}_{1}|0\rangle \\
&\quad+\alpha^{*2}\langle 0|\hat{b}_{1}^{2}|0\rangle+|\alpha|^{4} \\
&=|\nu|^{2}[\mu^{2}+2|\nu|^{2}]-(\alpha^{2}\mu\nu^{*}+\alpha^{*2}\mu\nu)+4|\alpha|^{2}|\nu|^{2}+|\alpha|^{4}
\end{aligned}$$

where

$$\langle 0|\hat{b}_1^{+2}\hat{b}_1|0\rangle = 0, \qquad \langle 0|\hat{b}_1^{+}\hat{b}_1^{2}|0\rangle = 0$$

or

$$\begin{aligned}\langle 0|\hat{S}_\varsigma^{+}\hat{D}_\alpha^{+}(\hat{a}^{+2}\hat{a}^2)\hat{D}_\alpha\hat{S}_\varsigma|0\rangle &= \langle 0|\hat{b}_1^{+2}\hat{b}_1^{2} + \alpha^2\hat{b}_1^{+2} + 4|\alpha|^2\hat{b}_1^{+}\hat{b}_1 + \alpha^{*2}\hat{b}_1^{2} + |\alpha|^4\hat{1}|0\rangle \\ &= \langle 0|\hat{b}_1^{+2}\hat{b}_1^{2}|0\rangle + \alpha^2\langle 0|\hat{b}_1^{+2}|0\rangle + 4|\alpha|^2\langle 0|\hat{b}_1^{+}\hat{b}\ |0\rangle \\ &\ +\alpha^{*2}\langle 0|\hat{b}_1^{2}|0\rangle + |\alpha|^4 \\ &= |\nu|^2[\mu^2 + 2|\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 4|\alpha|^2|\nu|^2 + |\alpha|^4\end{aligned}$$

We also have

$$\langle\alpha,\varsigma|\hat{n}|\alpha,\varsigma\rangle = |\alpha|^2 + |\nu|^2$$

$$\langle\alpha,\varsigma|\hat{n}^2|\alpha,\varsigma\rangle = |\nu|^2[\mu^2 + 2|\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 4|\alpha|^2|\nu|^2 + |\alpha|^4 + |\alpha|^2 + |\nu|^2$$

$$\begin{aligned}(\Delta n)^2 &= \langle\alpha,\varsigma|\hat{n}^2|\alpha,\varsigma\rangle - \langle\alpha,\varsigma|\hat{n}|\alpha,\varsigma\rangle^2 \\ &= |\nu|^2[\mu^2 + 2|\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 4|\alpha|^2|\nu|^2 + |\alpha|^4 + |\alpha|^2 + |\nu|^2 - \left(|\alpha|^2 + |\nu|^2\right)^2 \\ &= |\nu|^2[\mu^2 + 2|\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 4|\alpha|^2|\nu|^2 + |\alpha|^4 + |\alpha|^2 + |\nu|^2 - \left(|\alpha|^4 + |\nu|^4 + 2|\alpha|^2|\nu|^2\right) \\ &= |\nu|^2[\mu^2 + 2|\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 2|\alpha|^2|\nu|^2 + |\alpha|^2 + |\nu|^2 - |\nu|^4 \\ &= |\nu|^2[\mu^2 + |\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 2|\alpha|^2|\nu|^2 + |\alpha|^2 + |\nu|^2\end{aligned}$$

When $\alpha = 0$, the photon number variance is

$$(\Delta n)^2 = |\nu|^2(\mu^2 + |\nu|^2 + 1) = 2|\nu|^2(|\nu|^2 + 1) = 2\langle n\rangle(\langle n\rangle + 1)$$

where

$$\langle n\rangle = |\nu|^2$$

When $\varsigma = 0$, the photon number variance is

$$(\Delta n)^2 = |\alpha|^2$$

where

$$\mu = 1\,, \qquad \nu = 0$$

In general,

$$\begin{aligned}
(\Delta n)^2 &= |\nu|^2[\mu^2 + |\nu|^2] - (\alpha^2\mu\nu^* + \alpha^{*2}\mu\nu) + 2|\alpha|^2|\nu|^2 + |\alpha|^2 + |\nu|^2 \\
&= \sinh^2(s)[\cosh^2(s) + \sinh^2(s)] - |\alpha|^2[e^{i(2\theta-\vartheta)} + e^{-i(2\theta-\vartheta)}]\cosh(s)\sinh(s) \\
&\quad +2|\alpha|^2\sinh^2(s) + |\alpha|^2 + \sinh^2(s) \\
&= |\alpha|^2[-2\cos(2\theta-\vartheta)\cosh(s)\sinh(s) + 2\sinh^2(s) + 1] \\
&\quad +2\sinh^2(s)[\sinh^2(s) + 1]] \\
&= |\alpha|^2\{2[\sin^2(\theta - \frac{\vartheta}{2}) - \cos^2(\theta - \frac{\vartheta}{2})]\cosh(s)\sinh(s) + 2\sinh^2(s) + 1]\} \\
&\quad +2\sinh^2(s)[\sinh^2(s) + 1]] \\
&= |\alpha|^2[e^{2s}\sin^2(\theta - \frac{\vartheta}{2}) + e^{-2s}\cos^2(\theta - \frac{\vartheta}{2})] + 2\sinh^2(s)[\sinh^2(s) + 1]]
\end{aligned}$$

B-18 Bogoliubov transformation

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^+\,, \qquad \hat{b}^+ = \nu^*\hat{a} + \mu\hat{a}^+$$

$$\hat{b}_1 = \mu\hat{a} - \nu\hat{a}^+\,, \qquad \hat{b}_1^+ = \mu\hat{a}^+ - \nu^*\hat{a}$$

$$\hat{a} = \mu\hat{b} - \nu\hat{b}^+\,, \qquad \hat{a}^+ = \mu\hat{b}^+ - \nu^*\hat{b}$$

$$\begin{pmatrix} \mu & \nu \\ \nu^* & \mu \end{pmatrix}^{-1} = \begin{pmatrix} \mu & -\nu \\ -\nu^* & \mu \end{pmatrix}$$

B-19

$$\hat{R} = e^{i\phi\hat{n}}$$

$$\hat{H}(\varsigma) = \frac{1}{2i}(\varsigma^*\hat{a}^2 - \varsigma\hat{a}^{+2})$$

Baker-Campbell-Hausdorff

$$\begin{aligned}
\hat{R}^+\hat{H}(\varsigma)\hat{R} &= \exp(-i\phi\hat{a}^+\hat{a})\frac{1}{2i}(\varsigma^*\hat{a}^2 - \varsigma\hat{a}^{+2})\exp(i\phi\hat{a}^+\hat{a}) \\
&= \exp(\hat{A})\hat{B}\exp(-\hat{A})
\end{aligned}$$

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

(a) $\hat{A} = -i\phi\hat{n}$, $\hat{B} = \hat{a}$

$$[\hat{A},\hat{B}] = (-i\phi)[\hat{n},\hat{a}] = i\phi\hat{a}$$

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{A}, i\phi\hat{a}] = \phi^2[\hat{n},\hat{a}] = -\phi^2\hat{a}$$

$$[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] = [-i\phi\hat{n}, -\phi^2\hat{a}] = -i\phi^3\hat{a}$$

$$[\hat{A},[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]] = [-i\phi\hat{n}, -i\phi^3\hat{a}] = \phi^4\hat{a}$$

$$\begin{aligned}\exp(\hat{A})\hat{B}\exp(-\hat{A}) &= \hat{a} + \frac{1}{1!}(i\phi)\hat{a}) + \frac{1}{2!}(i\phi)^2\hat{a} + \frac{1}{3!}(i\phi)^3\hat{a} + \frac{1}{4!}(i\phi)^4\hat{a} + ...\\ &= e^{i\phi}\hat{a}\end{aligned}$$

$$\hat{R}^+\hat{a}\hat{R} = e^{i\phi}\hat{a}$$

$$\begin{aligned}\exp(\hat{A})\hat{B}^2\exp(-\hat{A}) &= \exp(\hat{A})\hat{B}\exp(-\hat{A})\exp(\hat{A})\hat{B}\exp(-\hat{A})\\ &= (e^{i\phi}\hat{a})^2\\ &= e^{2i\phi}\hat{a}^2\end{aligned}$$

$$\hat{R}^+\hat{a}^+\hat{R} = e^{-i\phi}\hat{a}^+$$

(b) $\hat{A} = -i\phi\hat{n}$, $\hat{B} = \hat{a}^+$

$$[\hat{A},\hat{B}] = (-i\phi)[\hat{n},\hat{a}^+] = (-i\phi)\hat{a}^+$$

$$[\hat{A},[\hat{A},\hat{B}]] = [-i\phi\hat{n}, -i\phi\hat{a}^+] = (-i\phi)^2[\hat{n},\hat{a}^+] = (-i\phi)^2\hat{a}^+$$

$$[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] = [-i\phi\hat{n}, -\phi^2\hat{a}] = (-i\phi)^3\hat{a}$$

$$[\hat{A},[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]] = [-i\phi\hat{n}, (-i\phi)^3\hat{a}^+] = (-i\phi)^4[n,\hat{a}^+] = (-i\phi)^4\hat{a}^+$$

$$\begin{aligned}\exp(\hat{A})\hat{B}\exp(-\hat{A}) &= \hat{a}^+ + \frac{1}{1!}(-i\phi)\hat{a}^+ + \frac{1}{2!}(-i\phi)^2\hat{a}^+ + \frac{1}{3!}(-i\phi)^3\hat{a}^+ + \frac{1}{4!}(-i\phi)^4\hat{a}^+ + ...\\ &= e^{-i\phi}\hat{a}^+\end{aligned}$$

$$\exp(\hat{A})\hat{B}^2\exp(-\hat{A}) = \exp(\hat{A})\hat{B}\exp(-\hat{A})\exp(\hat{A})\hat{B}\exp(-\hat{A})$$
$$= (e^{-i\phi}\hat{a}^+)^2$$
$$= e^{-2i\phi}\hat{a}^{+2}$$

$$\hat{R}^+\hat{H}(\varsigma)\hat{R} = \exp(-i\phi\hat{a}^+\hat{a})\frac{1}{2i}(\varsigma^*\hat{a}^2 - \varsigma\hat{a}^{+2})\exp(i\phi\hat{a}^+\hat{a})$$
$$= \frac{1}{2i}(\varsigma^* e^{2i\phi}\hat{a}^2 - \varsigma e^{-2i\phi}\hat{a}^{+2})$$
$$= \frac{1}{2i}s(e^{-i\vartheta}e^{2i\phi}\hat{a}^2 - e^{i\vartheta}e^{-2i\phi}\hat{a}^{+2})$$
$$= H(\varsigma e^{-2i\phi})$$

where $\varsigma = se^{i\vartheta}$

Note that $\varsigma e^{-2i\phi} = se^{i(\vartheta - 2\phi)}$. When $\phi = \vartheta/2$, we get

$$\hat{R}^+\hat{H}(\varsigma)\hat{R} = H(s).$$

We have

$$\hat{H}(s) = \frac{1}{2i}s(\hat{a}^2 - \hat{a}^{+2}) = \frac{s}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})$$

$$[\hat{H}(s), \hat{x}] = \frac{s}{2\hbar}[(\hat{x}\hat{p} + \hat{p}\hat{x}), \hat{x}]$$
$$= \frac{s}{2\hbar}[\hat{p}, \hat{x}^2]$$
$$= -is\hat{x}$$

$$[\hat{H}(s), \hat{p}] = \frac{s}{2\hbar}[(\hat{x}\hat{p} + \hat{p}\hat{x}), \hat{p}]$$
$$= \frac{s}{2\hbar}[\hat{x}, \hat{p}^2]$$
$$= is\hat{p}$$

REFERENCES

C. Gerry and P. Knight, *Introductory Quantum Optics* (Cambridge, 2005).

U. Leonhardt, *Essential Quantum Optics from Quantum Measurements to Black Holes* (Cambridge, 2010).