Commutation relation related to the mirror-reflection operator based on Baker-Hausdorff lemma

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Here we discuss the property of the mirror-reflection operator, $\hat{M}_y$, which is defined by the product of rotation operator $\hat{Y}$ (with rotation angle $\theta = \pi$) and the parity operator, $\hat{\pi}$,

$$\hat{M}_y = \hat{Y}\hat{\pi} .$$

where $\hat{Y} = \exp(-\frac{i}{\hbar}\pi\hat{J}_y)$. To this end, we will evaluate a operator $\hat{M}_y^{-1}\hat{A}\hat{M}_y$, where $\hat{A}$ is an operator such as the position vector, momentum, and angular momentum, using the Baker-Hausdorff lemma. We are interested in the case of positive integer of the angular momentum

1. Baker-Hausdorff lemma

To calculate these operators, we use the Baker-Hausdorff lemma,

$$\exp(i\lambda\hat{G})\hat{A}\exp(-i\lambda\hat{G}) = \hat{A} + \frac{(i\lambda)}{1!}[\hat{G},\hat{A}] + \frac{(i\lambda)^2}{2!}[\hat{G},[\hat{G},\hat{A}]] + \frac{(i\lambda)^3}{3!}[\hat{G},[\hat{G},[\hat{G},\hat{A}]]] + \frac{(i\lambda)^4}{4!}[\hat{G},[\hat{G},[\hat{G},[\hat{G},\hat{A}]]]] + ...$$

where $\hat{G}$ is a Hermitian operator and λ is a real parameter. Using the commutation relations

$$[\hat{J}_x,\hat{J}_y] = i\hbar\hat{J}_z , \qquad [\hat{J}_y,\hat{J}_z] = i\hbar\hat{J}_x , \qquad [\hat{J}_z,\hat{J}_x] = i\hbar\hat{J}_y$$

we obtain

$$\exp(\frac{i\alpha}{\hbar}\hat{J}_z)\hat{J}_x\exp(-\frac{i\alpha}{\hbar}\hat{J}_z) = \hat{J}_x + \frac{(\frac{i\alpha}{\hbar})}{1!}[\hat{J}_z,\hat{J}_x] + \frac{(\frac{i\alpha}{\hbar})^2}{2!}[\hat{J}_z,[\hat{J}_z,\hat{J}_x]] + \frac{(\frac{i\alpha}{\hbar})^3}{3!}[\hat{J}_z,[\hat{J}_z,[\hat{J}_z,\hat{J}_x]]] + ...$$

$$= \hat{J}_x + \frac{(\frac{i\alpha}{\hbar})}{1!}(i\hbar\hat{J}_y) + \frac{(\frac{i\alpha}{\hbar})^2}{2!}[\hat{J}_z, i\hbar\hat{J}_y] + \frac{(\frac{i\alpha}{\hbar})^3}{3!}[\hat{J}_z,[\hat{J}_z, i\hbar\hat{J}_y]] + ...$$

$$= \hat{J}_x - \alpha\hat{J}_y + \frac{(\frac{i\alpha}{\hbar})^2}{2!}i\hbar(-i\hbar\hat{J}_x) + \frac{(\frac{i\alpha}{\hbar})^3}{3!}i\hbar(-i\hbar)[\hat{J}_z,\hat{J}_x] + ...$$

$$= \hat{J}_x - \alpha\hat{J}_y - \frac{\alpha^2}{2!}\hat{J}_x + \frac{(\frac{i\alpha}{\hbar})^3}{3!}\hbar^2(i\hbar)\hat{J}_y, + ...$$

$$= \hat{J}_x - \alpha\hat{J}_y - \frac{\alpha^2}{2!}\hat{J}_x + \frac{\alpha^3}{3!}\hat{J}_y, + ...$$

$$= \hat{J}_x(1 - -\frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + ..) - (\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + ...)\hat{J}_y,$$

$$= \hat{J}_x\cos\alpha - \hat{J}_y\sin\alpha$$

The results of the calculations which are obtained in a similar way, are summarized as follows.

$$\exp(\frac{i}{\hbar}\alpha\hat{J}_z)\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix}\exp(-\frac{i}{\hbar}\alpha\hat{J}_z) = \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix} = \Re_z(\alpha)\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix},$$

$$\exp(\frac{i}{\hbar}\alpha\hat{J}_x)\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix}\exp(-\frac{i}{\hbar}\alpha\hat{J}_x) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix} = \Re_x(\alpha)\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix},$$

$$\exp(\frac{i}{\hbar}\alpha\hat{J}_y)\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix}\exp(-\frac{i}{\hbar}\alpha\hat{J}_x) = \begin{pmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{pmatrix}\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix} = \Re_y(\alpha)\begin{pmatrix}\hat{J}_x\\ \hat{J}_y\\ \hat{J}_z\end{pmatrix},$$

where $\Re_x(\alpha)$, $\Re_y(\alpha)$, and $\Re_z(\alpha)$ are the rotation matrices.

2. Evaluation of $\hat{M}_y^{-1}\hat{A}\hat{M}_y$

Here we consider the orbital angular momentum $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ ($j = l$, a positive integer). The following commutation relations are obtained.

$$[\hat{x},\hat{L}_x] = 0\,, \quad [\hat{x},\hat{L}_y] = i\hbar\hat{z}\,, \quad [\hat{x},\hat{L}_z] = -i\hbar\hat{y}\,,$$

$$[\hat{y},\hat{L}_x] = -i\hbar\hat{z}\,, \quad [\hat{y},\hat{L}_y] = 0\,, \quad [\hat{y},\hat{L}_z] = i\hbar\hat{x}\,,$$

$$[\hat{z},\hat{L}_x] = i\hbar\hat{y}\,, \quad [\hat{z},\hat{L}_y] = i\hbar\hat{x}\,, \quad [\hat{z},\hat{L}_z] = 0\,,$$

$$[\hat{p}_x,\hat{L}_x] = 0\,, \quad [\hat{p}_x,\hat{L}_y] = i\hbar\hat{p}_z\,, \quad [\hat{p}_x,\hat{L}_z] = -i\hbar\hat{p}_y\,,$$

$$[\hat{p}_y,\hat{L}_x] = -i\hbar\hat{p}_z\,, \quad [\hat{p}_y,\hat{L}_y] = 0\,, \quad [\hat{p}_y,\hat{L}_z] = i\hbar\hat{p}_x\,,$$

$$[\hat{p}_z,\hat{L}_x] = i\hbar\hat{p}_y\,, \quad [\hat{p}_z,\hat{L}_y] = -i\hbar\hat{p}_x\,, \quad [\hat{p}_z,\hat{L}_z] = 0\,,$$

These commutation relations can be expressed as

$$[\hat{L}_y,\hat{A}_x] = -i\hbar\hat{A}_z\,, \quad [\hat{L}_y,\hat{A}_z] = i\hbar\hat{A}_x\,,$$

or

$$[\hat{A}_x,\hat{L}_y] = i\hbar\hat{A}_z\,, \quad [\hat{A}_z,\hat{L}_y] = -i\hbar\hat{A}_x\,,$$

where $\hat{\boldsymbol{A}} = \hat{\boldsymbol{r}}$, and $\hat{\boldsymbol{A}} = \hat{\boldsymbol{p}}$. Using these comuutation relations and the Baker-Hausdorff lemma, we have

$$\exp(\frac{i\alpha}{\hbar}\hat{L}_y)\hat{A}_x\exp(-\frac{i\alpha}{\hbar}\hat{L}_y) = \hat{A}_x + \frac{(\frac{i\alpha}{\hbar})}{1!}[\hat{L}_y,\hat{A}_x] + \frac{(\frac{i\alpha}{\hbar})^2}{2!}[\hat{L}_y,[\hat{L}_y,\hat{A}_x]] + \frac{(\frac{i\alpha}{\hbar})^3}{3!}[\hat{L}_y,[\hat{L}_y,[\hat{L}_y,\hat{A}_x]]] + ...$$

$$= \hat{A}_x + \frac{(\frac{i\alpha}{\hbar})}{1!}(-i\hbar\hat{A}_z) + \frac{(\frac{i\alpha}{\hbar})^2}{2!}[\hat{L}_y,-i\hbar\hat{A}_z] + \frac{(\frac{i\alpha}{\hbar})^3}{3!}[\hat{L}_y,[\hat{L}_y,-i\hbar\hat{A}_z]]] + ...$$

$$= \hat{A}_x + \alpha\hat{A}_z + \frac{(\frac{i\alpha}{\hbar})^2}{2!}(-i\hbar)i\hbar\hat{A}_x + \frac{(\frac{i\alpha}{\hbar})^3}{3!}(-i\hbar)(i\hbar)[\hat{L}_y,\hat{A}_x] + ...$$

$$= \hat{A}_x + \alpha\hat{A}_z + \frac{(\frac{i\alpha}{\hbar})^2}{2!}\hbar^2\hat{A}_x + \frac{(\frac{i\alpha}{\hbar})^3}{3!}\hbar^2(-i\hbar)\hat{A}_z + ...$$

$$= \hat{A}_x + \alpha\hat{A}_z - \frac{1}{2!}\alpha^2\hat{A}_x - \frac{\alpha^3}{3!}\hat{A}_z + ...$$

$$= \hat{A}_x(1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 ...) + \hat{A}_z(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^3}{5!} + ..)$$

$$= \hat{A}_x\cos\alpha + \hat{A}_z\sin\alpha$$

The results of the calculation are summarized as follows.

$$\exp(\frac{i}{\hbar}\alpha\hat{L}_z)\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix}\exp(-\frac{i}{\hbar}\alpha\hat{L}_z) = \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix} = \Re_z(\alpha)\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix},$$

$$\exp(\frac{i}{\hbar}\alpha\hat{L}_x)\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix}\exp(-\frac{i}{\hbar}\alpha\hat{L}_x) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix} = \Re_x(\alpha)\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix},$$

$$\exp(\frac{i}{\hbar}\alpha\hat{L}_y)\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix}\exp(-\frac{i}{\hbar}\alpha\hat{L}_y) = \begin{pmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{pmatrix}\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix} = \Re_y(\alpha)\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix}.$$

When $\alpha = \pi$, we assume that

$$\hat{Y} = \exp(-\frac{i}{\hbar}\alpha\hat{L}_y).$$

Then we get

$$Y^{-1}\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix}\hat{Y}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\hat{A}_x\\ \hat{A}_y\\ \hat{A}_z\end{pmatrix}.$$

(i) $\hat{\boldsymbol{A}}=\hat{\boldsymbol{r}}$

$$Y^{-1}\hat{x}\hat{Y}=-\hat{x}\,,\qquad Y^{-1}\hat{y}\hat{Y}=\hat{y}\,,\qquad Y^{-1}\hat{z}\hat{Y}=-\hat{z}\,.$$

(ii) $\hat{\boldsymbol{A}}=\hat{\boldsymbol{p}}$

$$Y^{-1}\hat{p}_x\hat{Y}=-\hat{p}_x\,,\qquad Y^{-1}\hat{p}_y\hat{Y}=\hat{p}_y\,,\qquad Y^{-1}\hat{p}_z\hat{Y}=-\hat{p}_z\,.$$

Using the above relations we have

(a) $\hat{L}_x=\hat{y}\hat{p}_z-\hat{z}p_y$

$$\begin{aligned}\hat{Y}^{-1}\hat{L}_x\hat{Y}&=\hat{Y}^{-1}(\hat{y}\hat{p}_z-\hat{z}p_y)\hat{Y}\\&=(\hat{Y}^{-1}\hat{y}\hat{Y})(\hat{Y}^{-1}\hat{p}_z\hat{Y})-(\hat{Y}^{-1}\hat{z}\hat{Y})(\hat{Y}^{-1}\hat{p}_y\hat{Y})\\&=-(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)\\&=-\hat{L}_x\end{aligned}$$

(b) $\hat{L}_y=\hat{z}\hat{p}_x-\hat{x}\hat{p}_z$

$$\begin{aligned}\hat{Y}^{-1}\hat{L}_y\hat{Y}&=\hat{Y}^{-1}(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)\hat{Y}\\&=(\hat{Y}^{-1}\hat{z}\hat{Y})(\hat{Y}^{-1}\hat{p}_x\hat{Y})-(\hat{Y}^{-1}\hat{x}\hat{Y})(\hat{Y}^{-1}\hat{p}_z\hat{Y})\\&=\hat{z}\hat{p}_x-\hat{x}\hat{p}_z\\&=\hat{L}_y\end{aligned}$$

(c) $\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x$

$$\begin{aligned}\hat{Y}^{-1}\hat{L}_z\hat{Y}&=\hat{Y}^{-1}(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\hat{Y}\\&=(\hat{Y}^{-1}\hat{x}\hat{Y})(\hat{Y}^{-1}\hat{p}_y\hat{Y})-(\hat{Y}^{-1}\hat{y}\hat{Y})(\hat{Y}^{-1}\hat{p}_x\hat{Y})\\&=-(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\\&=-\hat{L}_z\end{aligned}$$

3. Properties of mirror-reflection operator

Here we consider only the orbital angular momentum. The mirror-reflection operator $\hat{M}_y$ is defined as

$$\hat{M}_y = \hat{Y}\hat{\pi} = \hat{\pi}\hat{Y} , \qquad \hat{M}_y^{-1} = \left(\hat{Y}\hat{\pi}\right)^{-1} = \hat{\pi}^{-1}\hat{Y}^{-1} = \hat{\pi}\hat{Y}^{-1} ,$$

for the mirror reflection at $y = 0$ plane (the z-x plane). $\hat{\pi}$ is the parity operator, where

$$\hat{\pi}^{-1} = \hat{\pi} = \hat{\pi}^{+} ,$$

$$\hat{\pi}\hat{\boldsymbol{r}}\hat{\pi} = -\hat{\boldsymbol{r}} , \qquad \hat{\pi}\hat{\boldsymbol{p}}\hat{\pi} = -\hat{\boldsymbol{p}} , \qquad \hat{\pi}\hat{\boldsymbol{L}}\hat{\pi} = \hat{\boldsymbol{L}}$$

$$\hat{\pi}\hat{Y}\hat{\pi} = \hat{Y} .$$

We note that

$$\hat{M}_y^{\ 2} = \hat{Y}\hat{\pi}\hat{Y}\hat{\pi} = \hat{Y}^2 = \hat{1} .$$

Using the properties of $\hat{Y}$ and $\hat{\pi}$, we get

$$\hat{M}_y^{-1}\hat{x}\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{x}\hat{Y})\hat{\pi} = -\hat{\pi}\hat{x}\hat{\pi} = \hat{x} ,$$

$$\hat{M}_y^{-1}\hat{y}\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{y}\hat{Y})\hat{\pi} = \hat{\pi}\hat{y}\hat{\pi} = -\hat{y} ,$$

$$\hat{M}_y^{-1}\hat{z}\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{z}\hat{Y})\hat{\pi} = -\hat{\pi}\hat{z}\hat{\pi} = \hat{z} .$$

Similarly, for the momentum operator we have

$$\hat{M}_y^{-1}\hat{p}_x\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{p}_x\hat{Y})\hat{\pi} = -\hat{\pi}\hat{p}_x\hat{\pi} = \hat{p}_x ,$$

$$\hat{M}_y^{-1}\hat{p}_y\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{p}_y\hat{Y})\hat{\pi} = \hat{\pi}\hat{p}_y\hat{\pi} = -\hat{p}_y ,$$

$$\hat{M}_y^{-1}\hat{p}_z\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{p}_z\hat{Y})\hat{\pi} = -\hat{\pi}\hat{p}_z\hat{\pi} = \hat{p}_z .$$

For the orbital angular momentum,

$$\hat{M}_y{}^{-1}\hat{L}_x\hat{M}_y = \hat{\pi}(\hat{Y}^{-1}\hat{L}_x\hat{Y})\hat{\pi} = -\hat{L}_x,$$

$$\hat{M}_y{}^{-1}\hat{L}_y\hat{M}_y = \hat{L}_y,$$

$$\hat{M}_y{}^{-1}\hat{L}_z\hat{M}_y = -\hat{L}_z.$$

4. Selection rule for the matrix element

We define the matrix

$$A_{\beta\alpha} = \langle\beta|\hat{A}|\alpha\rangle.$$

We also assume that

$$\hat{M}_y|\alpha\rangle = \eta_\alpha|\alpha\rangle, \qquad \hat{M}_y|\beta\rangle = \eta_\beta|\beta\rangle,$$

and

$$\hat{A}' = \hat{M}_y\hat{A}\hat{M}_y{}^{-1} = \eta_A\hat{A}.$$

When $\eta_A = 1$, $\hat{A}$ is even and when $\eta_A = -1$, $\hat{A}$ is odd. We now calculate the matrix element

$$\begin{aligned} A_{\beta\alpha} &= \langle\beta|\hat{A}|\alpha\rangle \\ &= \langle\beta|\hat{M}_y{}^{-1}\hat{M}_y\hat{A}\hat{M}_y{}^{-1}\hat{M}_y|\alpha\rangle \\ &= \langle\beta|\hat{M}_y{}^{+}\hat{M}_y\hat{A}\hat{M}_y{}^{-1}\hat{M}_y|\alpha\rangle \\ &= \eta_\alpha\eta_\beta\eta_A\langle\beta|\hat{A}|\alpha\rangle \\ &= \eta_\alpha\eta_\beta\eta_A A_{\beta\alpha} \end{aligned}$$

When $\eta_\alpha\eta_\beta\eta_A = 1$, $A_{\beta\alpha} \neq 0$. When $\eta_\alpha\eta_\beta\eta_A = -1$, we have $A_{\beta\alpha} = 0$. Thus it is required that

$$\eta_\alpha\eta_\beta\eta_A = 1.$$

4. The condition for $\eta_\beta\eta_A$

$$\langle\beta|\alpha\rangle = \langle\beta|\hat{M}_y{}^{-1}\hat{M}_y\hat{M}_y{}^{-1}\hat{M}_y|\alpha\rangle$$
$$= \eta_\alpha\eta_\beta\langle\beta|\alpha\rangle$$

If $\eta_\alpha\eta_\beta = -1$, $\langle\beta|\alpha\rangle = 0$. (orthogonal to each other)

If $\eta_\alpha\eta_\beta = 1$, $\langle\beta|\alpha\rangle \neq 0$. (not orthogonal to each other).

APPENDIX-I Baker-Hausdorff lemma

This is a very useful tool to disentangle exponentials of certain operators often appear in quantum mechanics. Suppose that $\hat{A}$ and $\hat{B}$ are two operators such that

$$\hat{C} = [\hat{A}, \hat{B}],$$

where $\hat{C}$ commutes with both $\hat{A}$ and $\hat{B}$;

$$[\hat{C}, \hat{A}] = 0, \qquad [\hat{C}, \hat{B}] = 0.$$

Then the Baker-Hausdorff lemm holds:

$$\exp(\alpha\hat{A} + \beta\hat{B}) = \exp(\beta\hat{B})\exp(\alpha\hat{A})\exp(\frac{\alpha\beta}{2}\hat{C})$$

((**Proof**))

We consider

$$\hat{F}(\alpha) = \exp(\alpha\hat{A} + \beta\hat{B})$$

which is a function of α only. The derivative of $\hat{F}(\alpha)$ with respect to α can be calculated using the definition of $\hat{F}(\alpha)$ in terms of its Taylor series expansion;

$$\hat{F}(\alpha) = \sum_{n=0}^{\infty}\frac{1}{n!}(\alpha\hat{A} + \beta\hat{B})^n$$

$$\frac{d}{d\alpha}\hat{F}(\alpha) = \sum_{n=0}^{\infty}\frac{1}{n!}\{\hat{A}(\alpha\hat{A}+\beta\hat{B})^{n-1} + (\alpha\hat{A}+\beta\hat{B})\hat{A}(\alpha\hat{A}+\beta\hat{B})^{n-2}$$
$$+ (\alpha\hat{A}+\beta\hat{B})^{2}\hat{A}(\alpha\hat{A}+\beta\hat{B})^{n-3} + ... + (\alpha\hat{A}+\beta\hat{B})^{n-1}\hat{A}\}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\{n(\alpha\hat{A}+\beta\hat{B})^{n-1}\hat{A} + (n-1)\beta\hat{C}(\alpha\hat{A}+\beta\hat{B})^{n-2}$$
$$+ (n-2)\beta\hat{C}(\alpha\hat{A}+\beta\hat{B})^{n-2} + ... + \beta\hat{C}(\alpha\hat{A}+\beta\hat{B})^{n-2}\}$$
$$= \hat{F}A + \frac{\beta}{2}FC$$

Integrating this equation over a from 0 to α, we find that

$$\hat{F}(\alpha) = \hat{F}(\alpha = 0)\exp(\alpha\hat{A} + \frac{\alpha\beta}{2}\hat{C}),$$

$$\hat{F}(\alpha = 0) = \sum_{n=0}^{\infty}\frac{1}{n!}(\beta\hat{B})^{n} = \exp(\beta\hat{B})$$

where $\hat{C}$ commutes with $\hat{A}$. Then we obtain

$$\hat{F}(\alpha) = \exp(\beta\hat{B})\exp(\alpha\hat{A})\exp(\frac{\alpha\beta}{2}\hat{C})$$

APPENDIX II

K. Gottfried and T.-M. Yan, Quantum Mechanics: Fundamentas 2nd edition (Springer, 2003).

The Baker-Hausdorff lemma for arbitrary $\hat{A}$ and $\hat{B}$,

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A},\hat{B}] + \frac{1}{12}([\hat{A},[\hat{A},\hat{B}]] - [\ddot{B},[\hat{A},\hat{B}]]) + ...).$$

If $\hat{C} = [\hat{A},\hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$,

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A},\hat{B}]).$$

or

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-\frac{1}{2}[\hat{A},\hat{B}])$$

In order to prove this formula, we first can show that

$$[\hat{B},\exp(x\hat{A})] = \exp(x\hat{A})[\hat{B},\hat{A}]x \, .$$

Next we define

$$\hat{G}(x) = \exp(x\hat{A})\exp(x\hat{B})$$

And we can show that

$$\frac{d\hat{G}(x)}{dx} = (\hat{A}+\hat{B}+[\hat{A},\hat{B}]x)\hat{G}$$

Integrate this to obtain

$$\hat{G}(x) = \exp(x\hat{A})\exp(x\hat{B}) = \exp\{(\hat{A}+\hat{B})x + \frac{1}{2}[\hat{A},\hat{B}]x^2\}\hat{G}(x=0)$$

with

$$\hat{G}(x=0) = \hat{1}$$

Then we have for $x = 1$,

$$\hat{G}(x=1) = \exp(\hat{A})\exp(\hat{B}) = \exp\{(\hat{A}+\hat{B}) + \frac{1}{2}[\hat{A},\hat{B}]\}$$