Mirror-reflection operator

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In classical mechanics, mirror inversion can be defined as follows. First take any motion that satisfies the law of classical mechanics. Then reflect the motion into a mirror and imagine that the motion in the mirror is actually happening in front of your eyes, and check if the motion satisfies the same laws of classical mechanics. If it does, then classical mechanics is symmetric under the mirror inversion. Here we discuss the property of the mirror-reflection operator in quantum mechanics.

1. Reflection ($\hat{M}_y$) and Inversion ($\hat{\pi}$) operators

Suppose we redefine the operation $\hat{\pi}$. First you reflect in a mirror in the *x*-plane ($x = 0$) so that *x* goes to –*x*, *y* stays *y*, and *z* stays *z*. Then you turn the system 180° about the *x* axis so that *y* is made to go to –*y*, and *z* to –*z*. The whole thing is called an inversion. Every point is projected through the origin to the diametrically opposite position. All the co-ordinates of everything are reversed. We use the symbol $\hat{\pi}$ for this operation. It is a little more convenient than a simple reflection because it does not require that you specify which co-ordinate plane you used for the reflection. You need to specify only the point which is at the center of symmetry.

(i) Reflection (mirror)

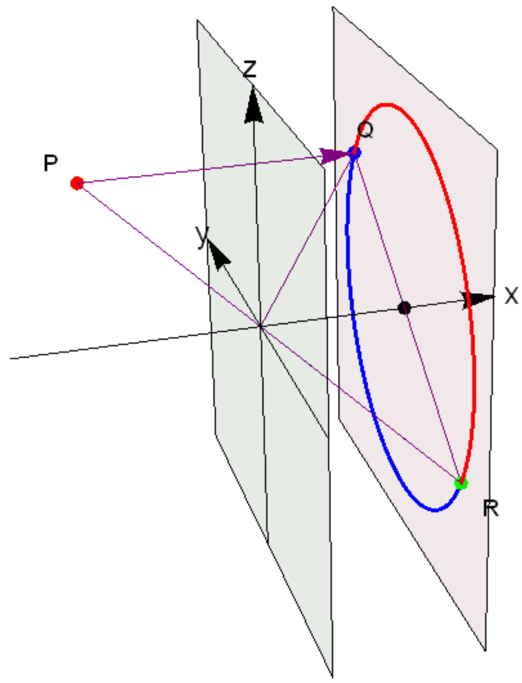
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}.$$

for the plane with $x = 0$.

(ii) Rotation around the *x* axis by 180°

$$\begin{pmatrix} x'' \\ y'' \\ x'' \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}.$$

Thus the parity (space inversion) is made of the reflection for the plane x = 0 and the rotation around the *x* axis by 180°.



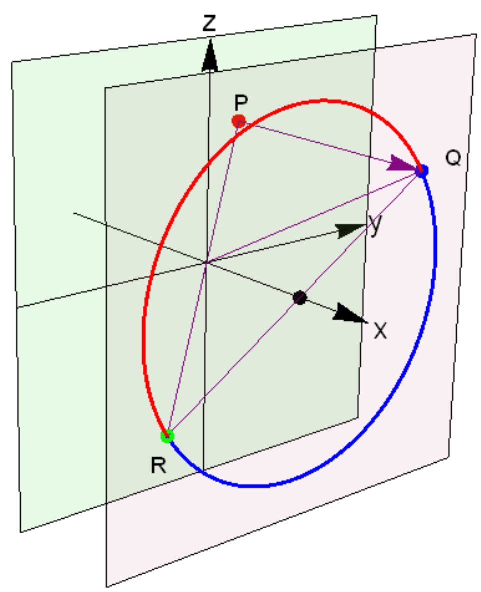


Fig. The operation of inversion (another method). Whatever is at the point P at (*x*, *y*, *z*) is moved to the point R at (-*x*, -*y*, -*z*). The inversion consists of the reflection M (the points P to Q) first and the rotation (the points Q to R) second.

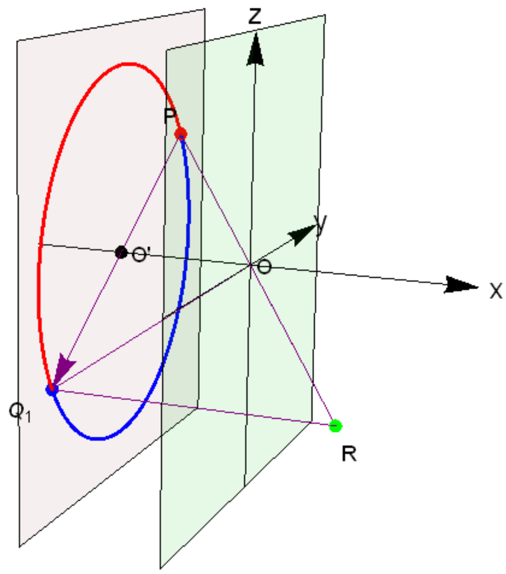


Fig. The operation of inversion. Whatever is at the point P at (x, y, z) is moved to the point R at (-x, -y, -z). The inversion consists of the rotation (the points P to Q_1) first and the reflection M (the points Q_1 to R) second.

2. Mirror reflection and space inversion (parity)

Here we consider the connection between mirror reflection and space inversion operators.

(i) Reflection (mirror); R_f (x = 0 plane)

If we make the reflection with respect to the y-z plane (x = 0 plane, denoted by the operator R_f), then we have

$$x \xrightarrow{Rf} -x\,,$$

$$y \xrightarrow{Rf} y\,,$$

$$z \xrightarrow{Rf} z\,.$$

(ii) π (parity)

If we now apply to this system a continuous rotation of 180° around the z-axis (the operator is denoted by $\hat{R}_x(\pi)$), we have

$$x \xrightarrow{R_f} (-x) \xrightarrow{R_y(\pi)} (-x)\,,$$

$$y \xrightarrow{R_f} y \xrightarrow{R_y(\pi)} (-y)\,,$$

$$z \xrightarrow{R_f} z \xrightarrow{R_y(\pi)} (-z)\,.$$

Then the space inverted state can be obtained by a continuous transformation from the mirror reflected state,

$$\hat{\pi} = \hat{R}_x(\pi)\hat{R}_f(x=0)\,.$$

3. Commutation relations and mirror reflection operator

We now consider the commutation relation between the rotation and space inversion. First we consider the operations, $\hat{R}_x(\pi)\hat{\pi}$ .and $\hat{\pi}\hat{R}_x(\pi)$.

For the operation $\hat{R}_x(\pi)\hat{\pi}$, we have

$$x \xrightarrow{\pi} (-x) \xrightarrow{R_x(\pi)} -x\,,$$

$$y \xrightarrow{\pi} (-y) \xrightarrow{R_x(\pi)} y\,,$$

$$z \xrightarrow{\pi} (-z) \xrightarrow{R_x(\pi)} z\,.$$

For the operation $\hat{\pi}\hat{R}_x(\pi)$, we have

$$x \xrightarrow{R_x(\pi)} x \xrightarrow{\pi} (-x)\,,$$

$$y \xrightarrow{R_x(\pi)} (-y) \xrightarrow{\pi} y\,,$$

$$z \xrightarrow{R_x(\pi)} (-z) \xrightarrow{\pi} z\,.$$

We see that the rotation operator $\hat{R}_x(\pi)$ and $\hat{\pi}$ commute. The mirror reflection operator $\hat{M}_x$ is given by

$$\hat{M}_x = \hat{R}_x(\pi)\hat{\pi} = \hat{\pi}\hat{R}_x(\pi).$$

Similarly we have

$$\hat{M}_y = \hat{R}_y(\pi)\hat{\pi} = \hat{\pi}\hat{R}_y(\pi), \qquad \hat{M}_z = \hat{R}_z(\pi)\hat{\pi} = \hat{\pi}\hat{R}_z(\pi).$$

where $\hat{R}_y(\pi)$ is the rotation operator of 180° around the *y*-axis. $\hat{R}_z(\pi)$ is the rotation operator of 180° around the *z*-axis.

Here we use the mirror reflection operator with respect to the *x*-*z* plane, which is defined by

$$\hat{M}_y = \hat{Y}\hat{\pi} = \hat{\pi}\hat{Y},$$

where $\hat{Y} = \hat{R}_y(\pi) = \exp(-\frac{i}{\hbar}\hat{J}_y\pi)$.

4. The rotation operator $\hat{Y}$

We now consider the rotation operator $\hat{R}_y(\pi) = \hat{Y}$, which is defined as

$$\hat{Y} = \hat{R}_y(\pi) = \exp(-\frac{i}{\hbar}\hat{J}_y\pi), \qquad \hat{Y}^{-1} = \hat{Y}^{+} = \exp(\frac{i}{\hbar}\hat{J}_y\pi).$$

We note that

$$\hat{J}_z\hat{Y} = -\hat{Y}\hat{J}_z, \qquad \hat{J}_\pm\hat{Y} = -\hat{Y}\hat{J}_\mp,$$

or

$$\hat{Y}^{-1}\hat{J}_z\hat{Y} = -\hat{J}_z, \qquad \hat{Y}^{-1}\hat{J}_\pm\hat{Y} = -\hat{J}_\mp.$$

This can be demonstrated by using the Baker-Hausdorff theorem. We also note that

$$\hat{Y}^2 = \exp(-\frac{i}{\hbar}\hat{J}_y 2\pi) = (-1)^{2j}.$$

When j is an integer,

$$\hat{Y}^2 = \hat{1}.$$

When j is an half integer,

$$\hat{Y}^2 = -\hat{1}$$

(i) The relation $\hat{J}_z\hat{Y} = -\hat{Y}\hat{J}_z$

$$\hat{J}_z\hat{Y}|j,m\rangle = -\hat{Y}\hat{J}_z|j,m\rangle = -m\hbar\hat{Y}|j,m\rangle$$

$\hat{Y}|j,m\rangle$ is the eigenket of $\hat{J}_z$ with the eigenvalue $-m\hbar$. In other words, we have

$$\hat{Y}|j,m\rangle = e^{i\alpha(j,m)}|j,-m\rangle,$$

$e^{i\alpha(j,m)}$ is the phase factor depending on the values of j and m.

(ii) The relation $\hat{J}_+\hat{Y} = -\hat{Y}\hat{J}_-$

$$\hat{J}_+\hat{Y}|j,m\rangle = e^{i\alpha(j,m)}\hat{J}_+|j,-m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j,-m+1\rangle,$$

Since $\hat{J}_+\hat{Y} = -\hat{Y}\hat{J}_-$, we have

$$\begin{aligned}\hat{J}_+\hat{Y}|j,m\rangle &= -\hat{Y}\hat{J}_-|j,m\rangle \\ &= -\hbar\sqrt{(j+m)(j-m+1)}\hat{Y}|j,m-1\rangle \\ &= -\hbar\sqrt{(j+m)(j-m+1)}e^{i\alpha(j,m-1)}|j,-m+1\rangle\end{aligned}$$

Thus we get

$$e^{i\alpha(j,m-1)} = -e^{i\alpha(j,m)}.$$

(iii) The relation $\hat{J}_-\hat{Y} = -\hat{Y}\hat{J}_+$

$$\hat{J}_-\hat{Y}|j,m\rangle = e^{i\alpha(j,m)}\hat{J}_-|j,-m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j,-m-1\rangle.$$

Since $\hat{J}_-\hat{Y} = -\hat{Y}\hat{J}_+$, we have

$$\begin{aligned}\hat{J}_-\hat{Y}\left|j,m\right\rangle &= -\hat{Y}\hat{J}_+\left|j,m\right\rangle \\ &= -\hbar\sqrt{(j-m)(j+m+1)}\hat{Y}\left|j,m+1\right\rangle \\ &= -\hbar\sqrt{(j-m)(j+m+1)}e^{i\alpha(j,m+1)}\left|j,-m-1\right\rangle\end{aligned}$$

Thus we get

$$e^{i\alpha(j,m+1)} = -e^{i\alpha(j,m)}.$$

(iv) The relation $\hat{Y}^2 = \exp(-\frac{i}{\hbar}\hat{J}_y 2\pi) = (-1)^{2j}$

$$\hat{Y}^2\left|j,m\right\rangle = e^{i\alpha(j,m)}\hat{Y}\left|j,-m\right\rangle = e^{i\alpha(j,m)}e^{i\alpha(j,-m)}\left|j,m\right\rangle = (-1)^{2j}\left|j,m\right\rangle,$$

or

$$e^{i\alpha(j,m)}e^{i\alpha(j,-m)} = (-1)^{2j}.$$

From the above consideration, we obtain

$$e^{i\alpha(j,m-1)} = -e^{i\alpha(j,m)}, \qquad e^{i\alpha(j,m+1)} = -e^{i\alpha(j,m)}, \tag{1}$$

$$e^{i\alpha(j,m)}e^{i\alpha(j,-m)} = (-1)^{2j} = (-1)^{j+m}(-1)^{j-m}. \tag{2}$$

From Eq.(1), we get

$$(-1)^{m-1}e^{i\alpha(j,m-1)} = (-1)^m e^{i\alpha(j,m)} = ... = (-1)^{-m}e^{i\alpha(j,-m)}.$$

Thus we have

$$e^{i\alpha(j,-m)} = (-1)^{2m}e^{i\alpha(j,m)}, \tag{3}$$

From Eqs.(2) and (3), we get

$$e^{2i\alpha(j,m)}(-1)^{2m} = (-1)^{2j}.$$

or

$$e^{2i\alpha(j,m)} = (-1)^{2(j-m)},$$

or

$$e^{i\alpha(j,m)} = (-1)^{j-m}.$$

In conclusion we get the formula

$$\hat{Y}\left|j,m\right\rangle = (-1)^{j-m}\left|j,-m\right\rangle$$

((**Note**))

(a) Commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y,$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y,$$

$$\hat{J}_+^{\,+} = \hat{J}_-, \qquad \hat{J}_-^{\,+} = \hat{J}_+,$$

$$[\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+, \qquad [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_-, \qquad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z.$$

(b)

$$\hat{J}_+\left|j,m\right\rangle = \hbar\sqrt{(j-m)(j+m+1)}\left|j,m+1\right\rangle,$$

$$\hat{J}_-\left|j,m\right\rangle = \hbar\sqrt{(j+m)(j-m+1)}\left|j,m-1\right\rangle.$$

(c) Baker-Hausdorff theorem

$$\exp(\hat{A}x)\hat{B}\exp(-\hat{A}x) = \hat{B} + \frac{x}{1!}[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{x^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

6. RHC and LHC photons

The RHC and LHC photons have the angular momentum

$$\hat{J}_z|R\rangle = \hbar|R\rangle, \qquad \hat{J}_z|L\rangle = -\hbar|L\rangle,$$

Here we choose

$$|R\rangle = |1,1\rangle, \qquad |L\rangle = |1,-1\rangle,$$

Then we get

$$\hat{Y}|R\rangle = \hat{Y}|1,1\rangle = (-1)^{1-1}|1,-1\rangle = |L\rangle,$$

and

$$\hat{Y}|L\rangle = \hat{Y}|1,-1\rangle = (-1)^{1+1}|1,1\rangle = |R\rangle.$$

Since

$$|x\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle), \qquad |y\rangle = \frac{1}{\sqrt{2}i}(|R\rangle - |L\rangle),$$

we get

$$\hat{Y}|x\rangle = \frac{1}{\sqrt{2}}\hat{Y}(|R\rangle + |L\rangle) = |x\rangle,$$

$$\hat{Y}|y\rangle = \frac{1}{\sqrt{2}i}\hat{Y}(|R\rangle - |L\rangle) = \frac{1}{\sqrt{2}i}\hat{Y}(|L\rangle - |R\rangle) = -|y\rangle.$$

$\hat{Y}$ is the rotation operator around the y-axis by π.

7. Parity operator and reflection operator

We use the parity operator $\hat{\pi}$,

$$\hat{\pi}|R\rangle = \hat{\pi}|1,1\rangle = (-1)^1|1,1\rangle = -|R\rangle,$$

$$\hat{\pi}|L\rangle = \hat{\pi}|1,-1\rangle = (-1)^1|1,-1\rangle = -|L\rangle.$$

From these relations, we have

$$\hat{\pi}|x\rangle = \frac{1}{\sqrt{2}}\hat{\pi}(|R\rangle + |L\rangle) = -\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) = -|x\rangle,$$

$$\hat{\pi}|y\rangle = \frac{1}{\sqrt{2}i}\hat{\pi}(|R\rangle - |L\rangle) = -\frac{1}{\sqrt{2}i}\hat{\pi}(|R\rangle - |L\rangle) = -|y\rangle.$$

The mirror reflection operator:

$$\hat{M}_y|x\rangle = \hat{Y}\hat{\pi}|x\rangle = -\hat{Y}|x\rangle = -|x\rangle, \qquad \hat{M}_y|y\rangle = \hat{Y}\hat{\pi}|y\rangle = -\hat{Y}|y\rangle = |y\rangle.$$

$|x\rangle$ is the eigenket of $\hat{M}_y$ with the eigenket (-1) and $|y\rangle$ is the eigenket of $\hat{M}_y$ with the eigenket (+1).

7. The eigenket and eigenvalue of the operator $\hat{M}_y$ for a positive integer *j*

The operator $\hat{M}_y$ is defined by

$$\hat{M}_y = \hat{Y}\hat{\pi}.$$

From the properties of $\hat{Y}$ and $\hat{\pi}$, we have

$$\hat{M}_y^{-1} = \hat{M}_y^{+},$$

since

$$\hat{M}_y^{+} = (\hat{Y}\hat{\pi})^{+} = \pi^{+}\hat{Y}^{+} = \pi\hat{Y}^{-1},$$

$$\hat{M}_y^{-i} = (\hat{Y}\hat{\pi})^{-1} = \hat{\pi}^{-1}\hat{Y}^{-1} = \hat{\pi}\hat{Y}^{-1}.$$

We also note that

$$\hat{M}_y^{2} = \hat{Y}\hat{\pi}\hat{Y}\hat{\pi} = \hat{Y}^2\hat{\pi}^2 = \exp(-\frac{i}{\hbar}\hat{J}_y 2\pi) = \hat{1},$$

or

$$\hat{M}_y{}^2 = \hat{1}.$$

We assume that $|\alpha\rangle$ is the eigenket of $\hat{M}_y$ with the eigenvalue η_α;

$$\hat{M}_y|\alpha\rangle = \eta_\alpha|\alpha\rangle,$$

Then we have

$$\hat{M}_y{}^2|\alpha\rangle = \eta_\alpha\hat{M}_y|\alpha\rangle = \eta_\alpha{}^2|\alpha\rangle .$$

Since $\hat{M}_y{}^2 = \hat{1}$, we get

$$\eta_\alpha{}^2 = 1.$$

When j is a positive integer, $\eta_\alpha{}^2 = 1$. So we have

$$\eta_\alpha = \pm 1.$$

The eigen values of $\hat{M}_y$ are real, and the eigenstates of $\hat{M}_y$ have either positive parity ($\eta_\alpha = 1$) or negative parity ($\eta_\alpha = -1$). They are sometimes said to be even or odd under reflection.

We start with

$$\hat{J}_z\hat{\pi} = \hat{\pi}\hat{J}_z .$$

Then we have

$$\hat{J}_z\hat{\pi}| j,m\rangle = \hat{\pi}\hat{J}_z| j,m\rangle = m\hbar\hat{\pi}| j,m\rangle .$$

$\hat{\pi}| j,m\rangle$ is the eigenket of $\hat{J}_z$ with the eigenvalue $m\hbar$. In fact we have

$$\hat{\pi}| j,m\rangle = (-1)^j| j,m\rangle = -| j,m\rangle ,$$

$|j,m\rangle$ is the eigenket of $\hat{\pi}$ with the eigenvalue (-1). Using this relation

$$\hat{M}_y|j,m\rangle = \hat{Y}\hat{\pi}|j,m\rangle = -\hat{Y}|j,m\rangle = -(-1)^{j-m}|j,-m\rangle,$$

and

$$\begin{aligned}\hat{M}_y{}^2|j,m\rangle &= -(-1)^{j-m}\hat{M}_y|j,-m\rangle \\ &= (-1)^{j-m}(-1)^{j+m}|j,m\rangle \\ &= (-1)^{2j}|j,m\rangle \\ &= |j,m\rangle\end{aligned}$$

From

$$\hat{M}_y|j,m\rangle = -(-1)^{j-m}|j,-m\rangle,$$

we have

$$\hat{M}_y|1,1\rangle = -|1,-1\rangle, \qquad \hat{M}_y|1,-1\rangle = -|1,1\rangle,$$

or

$$\hat{M}_y|R\rangle = -|L\rangle \qquad \hat{M}_y|L\rangle = -|R\rangle.$$

In other words, $|R\rangle$ and $|L\rangle$ are not the eigenkets of $\hat{M}_y$.

REFERENCES

M.L. Bellac, *Quantum Physics* (Cambridge, 2006)

F.S. Levin, *An Introduction to Quantum Theory* (Cambridge 2002).

R.P. Feynman, R.,B. Leighton, and M. Sands, *The Feynman Lectures in Physics*, 6th edition (Addison Wesley, Reading Massachusetts, 1977).

J.J. Sakurai, *Invariance Principles and Elementary Particles* (Princeton University Press, 1964).

J.J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, second edition (Addison-Wesley, New York, 2011).

D. Park, *Introduction to the Quantum Theory*, 3rd edition (McGraw-Hill, Inc., New York, 1974). p.394

U. Fano and A.R.P. Rau, *Symmetries in Quantum Physics* (Academic Press, 1996)).

G.H. Wagniére, *On Chirality and the Universal Asymmetry, Reflections on Image and Mirror Image* (Wiley-VCH, 2007).