Thought experiments of Dirac *n*-polarizers and Stern-Gerlach (spin 1/2): understanding the principle of superposition

Masatsugu Sei Suzuki and Itsuko S. Suzuki
Department of Physics, SUNY at Binghamton
(Date: September 26, 2020)

Here we discuss the principle of superposition in quantum mechanics on the two well-known Gedanken (thought) experiments; Dirac *n*-polarizers experiment for photon, and Stern-Gerlach experiment for spin 1/2 particle (such as electron or proton). For both cases, we calculate the probability of the processes and find the maximum probability with the use of ContourPlot and ContourPlot3D in Mathematica.

During the calculation, I read many standard textbooks of quantum mechanics and checked whether similar discussions have been already done. As far as I know, however, there is no further discussion on the Dirac *n* polarizers (n = 3, and 4), which lead to the probability maximum, partly because of somewhat complicated calculations. This note was written while Phys.421 (Quantum Mechanics I) in the Fall 2020 has been taught by me. This note and the associated power point (PPT) were prepared for teaching this course by hybrid system of in-person and on-line.

1. Introduction

In his famous book (1931), Dirac introduced the principle of quantum superposition with a 3-polarizer experiment on polarized photons to understand the principle of superposition in quantum mechanics. This experiment is often called as the Dirac polarizers. This problem has been extensively discussed in many authors) including Baym, and Feynman) of standard textbooks of quantum mechanics.

Here we discuss the principle of the superposition in quantum mechanics from the examples of the Dirac polarizers, where n (=1, 2, 3, …) polarizers are inserted between x- and y-filters. The angles of n polarizers are given by θ_1, θ_2, …., θ_n. We use the Mathematica programs (ContourPlot, ContourPlot3D) to determine the values of angles of polarizers which leads to the maximum probability. We consider the probability of finding the system in the $|y\rangle$ state (the final state), after the photon in the initial state $|x\rangle$ pass the n-filters. We are interested in the values of the angles θ_1, θ_2, …., and θ_n when the probability becomes maximum. We will also show that when $\Delta\theta = \theta_i - \theta_{i-1}$ (>0) with $\theta_1 + \theta_2 + \theta_3 + ... + \theta_n = \frac{\pi}{4} n$, the probability takes a maximum. It is amazing that in the limit of $n \to \infty$, the probability tends to unity.

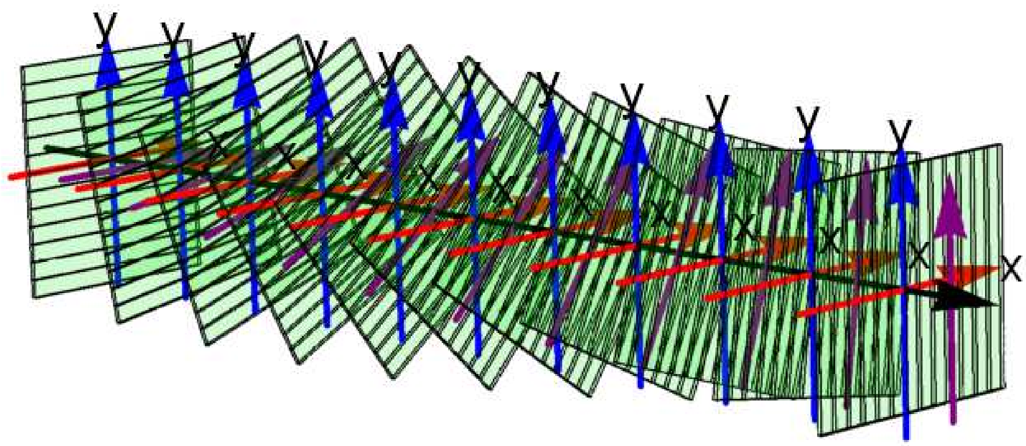


Fig.1 Dirac polarizers ($n = 9$), which lead to the maximum probability. See a detail below.

We also discuss the SG experiments for spin 1/2, where there are two SG sets between the two SGz's (one is used as the initial state $|+z\rangle$ and the other is used as the final state $|\pm z\rangle$). The magnetic field in the two SG sets inserted between two SGz's sets, is directed in the z-x plane. We will determine the angles of the magnetic field directions to obtain the maximum probability. We will show that the result for the Dirac polarizers with n= 1 is very similar to that of the SG experiments.

2. Dirac polarizers (n = 0)

First, they pass through the x-polarizer and next, move in the y-polarizer.

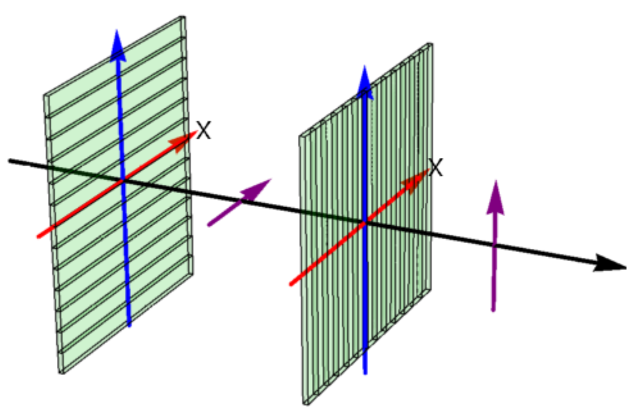


Fig.2 x- and y-filters. $P = |\langle y | x \rangle|^2 = 0$.

Experimentally, it is obvious that no photon is observed after the second polarizer. In quantum mechanics, the probability for finding photons after the second polarizer is expressed by

$$P = \left|\langle y \,|\, x \rangle\right|^2 = 0$$

since $\langle y | x \rangle = 0$.

2. Dirac polarizers (*n* = 1)

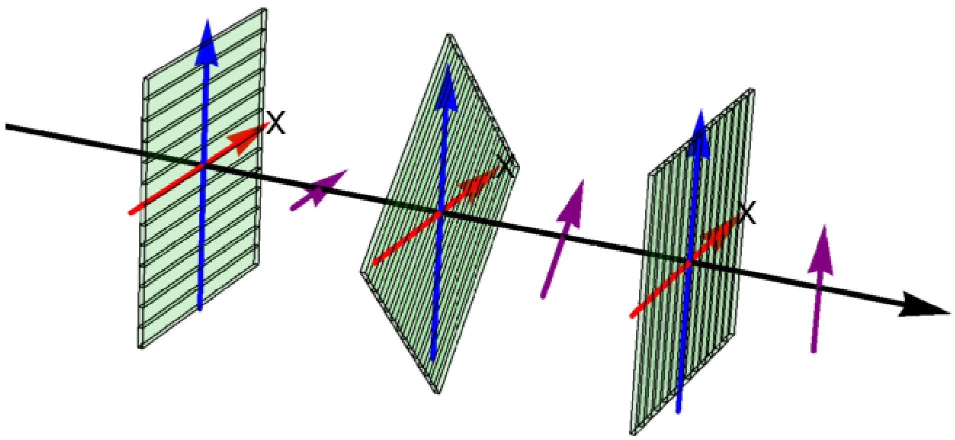


Fig.3 Dirac polarizer (n = 1). $\theta -$ filter between the x-filter and the y-filter. $|x\rangle \rightarrow \left|\theta = \frac{\pi}{4}\right\rangle \rightarrow |y\rangle$.

If we insert a third polarizer at some non-zero angle, θ, between the crossed polarizers (x- and y filters), the light *reappears* although we have inserted an absorbing object that can only reject all photons polarized perpendicular to its axis!. In this space, we can choose as basis states the states of horizontal and vertical linear polarization, which we denote $|x\rangle$ and $|y\rangle$. We denote $|\theta\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle$| as the state of a photon polarized linearly along a direction at an angle θ with the horizontal axis ($0 \le \theta < \pi$). This state is a linear superposition of polarization of light comes from the fact that the photon is a "spin one" particle. It is a point-like massless particle that carries an intrinsic angular momentum whose projection on the direction of propagation is either $|x\rangle$ or $|y\rangle$. For the particular value θ, we have

$$|\theta\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle.$$

The probability for a horizontally polarized photon to get through a polarizer at an angle θ is

$$P_1 = |\langle\theta|x\rangle|^2 = \cos^2\theta.$$

If this latter polarizer is the y-filter, the probability of finding that a photon crosses the y-filter

$$P_2 = |\langle y|\theta\rangle|^2 = \sin^2\theta.$$

Thus, the resultant probability of finding that a [photon crosses the entire set up is given by a product of P_1 and P_2 as

$$P(\theta) = P_1 P_2 = \sin^2\theta\cos^2\theta = \frac{1}{4}[\sin(2\theta)]^2.$$

Conveniently we use the projection operator $\hat{P}(\theta) = |\theta\rangle\langle\theta|$,

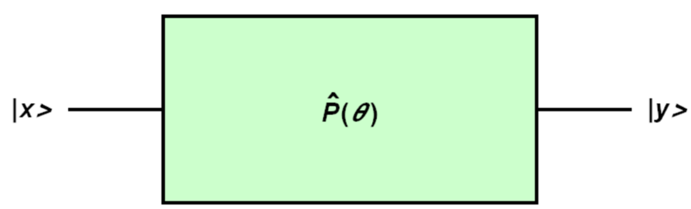


Fig.4 Projection operator $\hat{P}(\theta) = |\theta\rangle\langle\theta|$.

$$\langle\theta|x\rangle\langle y|\theta\rangle = \langle y|\theta\rangle\langle\theta|x\rangle = \langle y|\hat{P}(\theta)|x\rangle,$$

or

$$\langle y|\hat{P}(\theta)|x\rangle = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta),$$

and

$$P(\theta) = \left|\langle y|\hat{P}(\theta)|x\rangle\right|^2 = \frac{1}{4}\sin^2(2\theta)$$
.

We make a plot of the probability $P(\theta)$ as a function of the angle θ. We find that $P(\theta)$ has a maximum ($P_{\max}$ = 1/4 = 0.25) at the angle $\theta = \frac{\pi}{4}$.

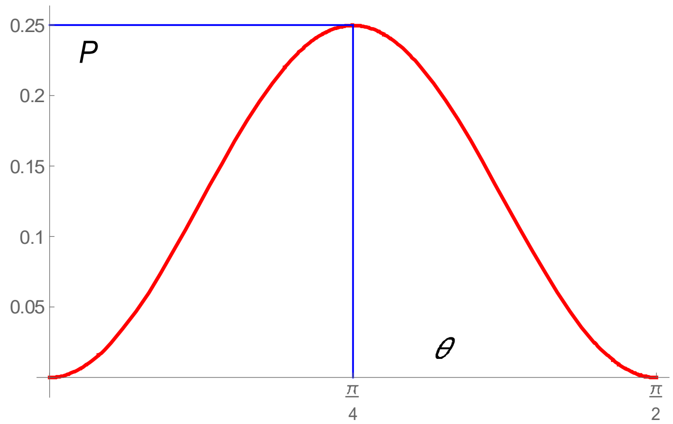


Fig.5 Probability P as a function of θ for the Dirac polarizers with n = 1.

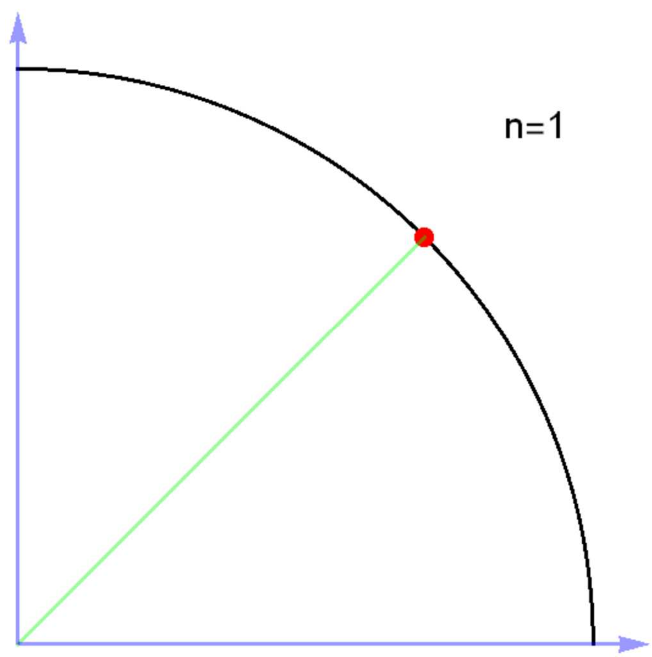


Fig.6 Dirac polarizer (n = 1). $\theta = \frac{\pi}{4}$ for the maximum probability.

2. Dirac polarizers (n = 2)

We now consider the Dirac polarizers (n = 2). There are two polarizers with angle α and β between the x- and y- filters. What are the values of the angles when the probability takes maximum?

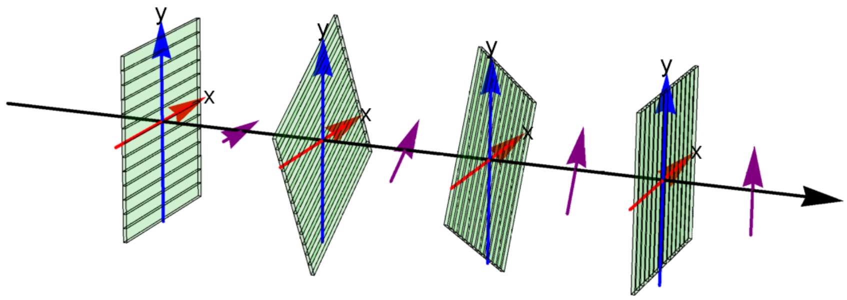


Fig.7 Two filters (α - filter and β - filter) between the x-filter and y-filter. α and β are the angles from the x axis in the x-y plane.

From the definition of the probability amplitudes, we have

$$\langle\alpha|x\rangle\langle\beta|\alpha\rangle\langle y|\beta\rangle = \langle y|\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|x\rangle$$
$$= \langle y|\hat{P}(\beta)\hat{P}(\alpha)|x\rangle$$

Using these ket vectors,

$$\hat{P}(\alpha)|x\rangle = \begin{pmatrix} \cos^2\alpha \\ \sin\alpha\cos\alpha \end{pmatrix}, \qquad \hat{P}(\beta)|y\rangle = \begin{pmatrix} \sin\beta\cos\beta \\ \sin^2\beta \end{pmatrix}$$

we get

$$\langle y|\hat{P}(\beta)\hat{P}(\alpha)|x\rangle = \begin{pmatrix} \sin\beta\cos\beta & \sin^2\beta \end{pmatrix}\begin{pmatrix} \cos^2\alpha \\ \sin\alpha\cos\alpha \end{pmatrix}$$
$$= \cos\alpha\sin\beta\begin{pmatrix} \cos\beta & \sin\beta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$
$$= \cos\alpha\sin\beta\cos(\alpha-\beta)$$

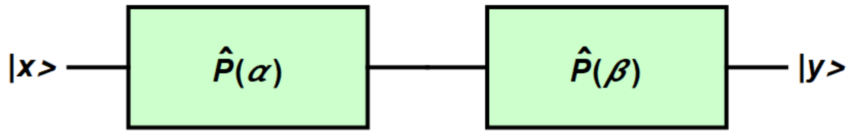


Fig.8 Transition process $(|x\rangle \rightarrow |\alpha\rangle \rightarrow |\beta\rangle \rightarrow |y\rangle)$.

Projection operator. $\hat{P}(\beta)\hat{P}(\alpha) = |\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|$.

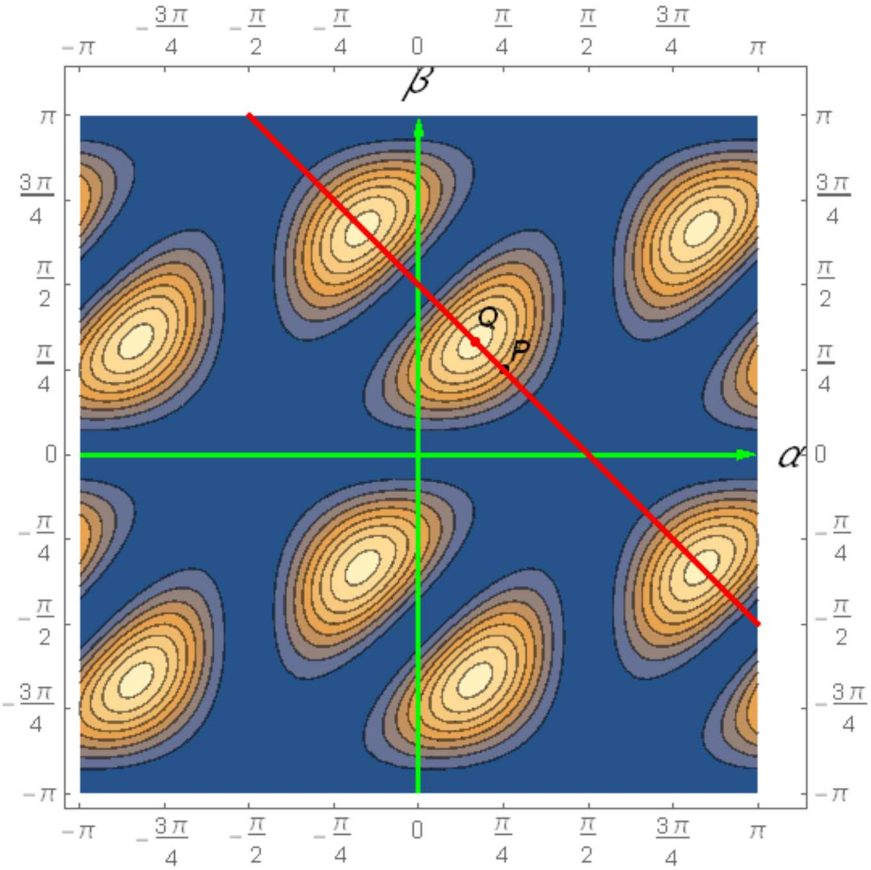
Probability amplitude: $\langle y|\hat{P}(\beta)\hat{P}(\alpha)|x\rangle$

Thus, the probability is obtained as

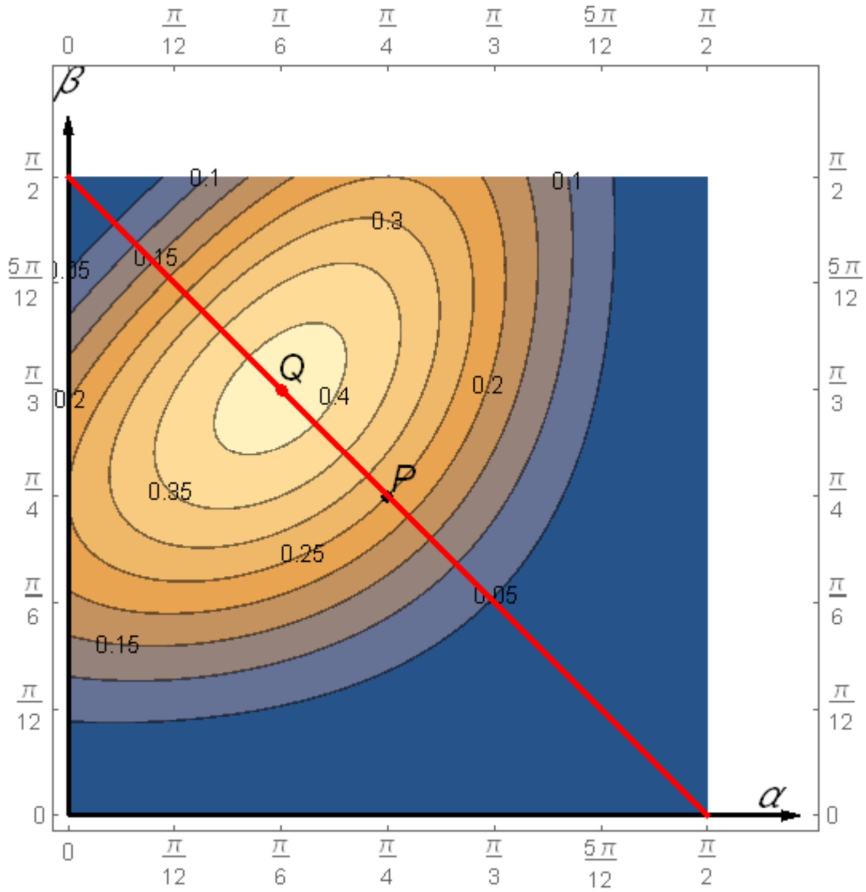
$$P(\alpha,\beta) = [\cos(\alpha-\beta)\cos\alpha\sin\beta]^2.$$

We use the ContourPlot of the Mathematica in the (α, β) plane, in order to determine the values of α and β for maximum in $P(\alpha, \beta)$.

((**ContourPlot**))



(a)



(b)

Fig.9(a) and (b)

ContourPlot of $P(\alpha,\beta)$ in the (α,β) plane. P =1/4 at the point P: $(\alpha=\frac{\pi}{4},\beta=\frac{\pi}{4})$. P takes a maximum (=0.421875) at point Q: $(\alpha=\frac{\pi}{6},\beta=\frac{\pi}{3})$, where the probability becomes maximum. The red line is denoted by $\alpha+\beta=\frac{\pi}{2}$. Both the points P and Q lie on this straight line.

Using the above results of the ContourPlot, we find Point Q: $(\alpha=\frac{\pi}{6},\beta=\frac{\pi}{3})$. The probability is 1/4 at the point Q. We also show the Plot3D of $P(\alpha,\beta)$ in the (α,β) plane. We get the same result.

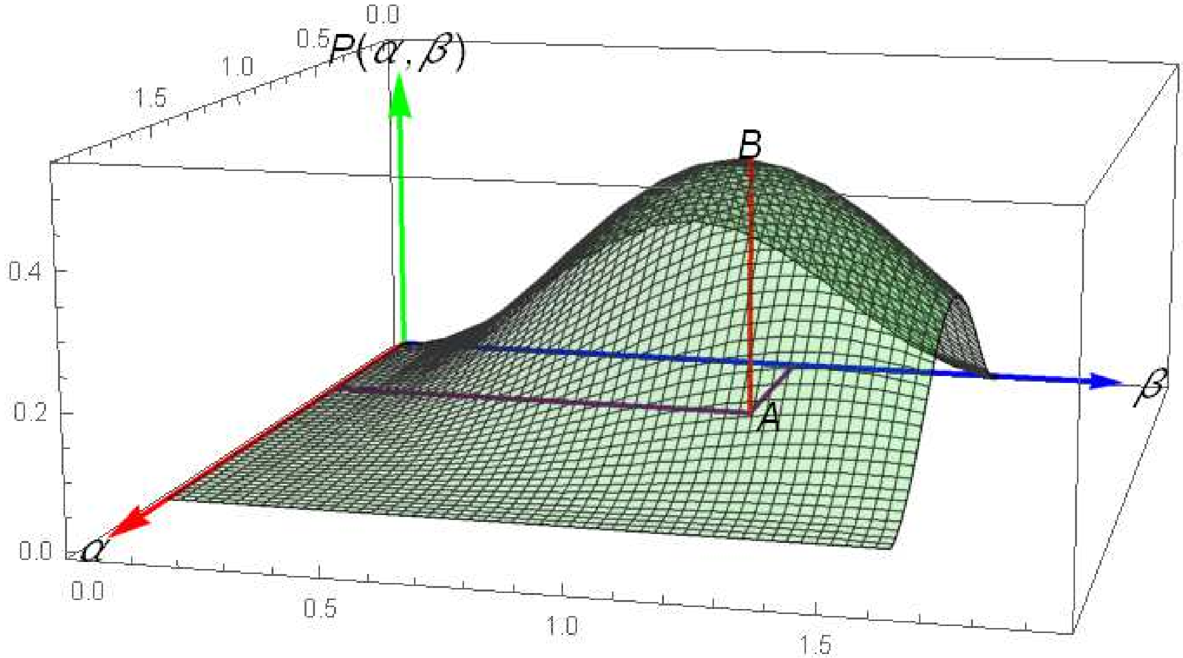


Fig.10 Plot3D of $P(\alpha,\beta)$ in the (α,β) plane. The point A: ($\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$, P =0). The point B: ($\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$, $P_{\max}$ = 0.421875).

It is found that the points P and Q on the same straight line

$$\alpha + \beta = \frac{\pi}{2}.$$

If we use this condition, we get a simple expression as

$$f(\alpha) = P(\alpha, \beta = \frac{\pi}{2} - \alpha) = \sin^2(2\alpha)\cos^4\alpha .$$

We make a plot of $f(\alpha)$ as a function of α.

$$f(\alpha) = P(\alpha, \beta = \frac{\pi}{2} - \alpha) = \sin^2(2\alpha)\cos^4\alpha .$$

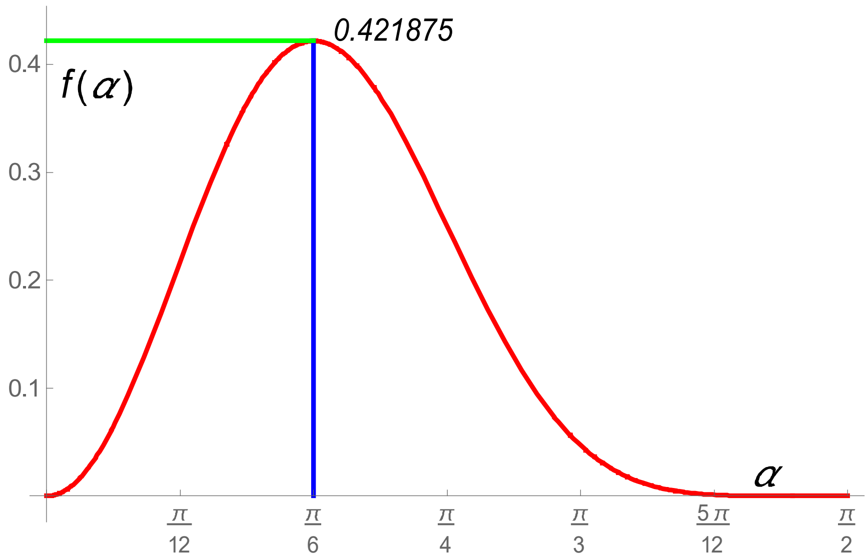


Fig.11 Plot of $f(\alpha)=P(\alpha,\beta=\frac{\pi}{2}-\alpha)$ as a function of α. It takes a maximum (P_{max} = 0.421875) at $\alpha=\frac{\pi}{6}$.

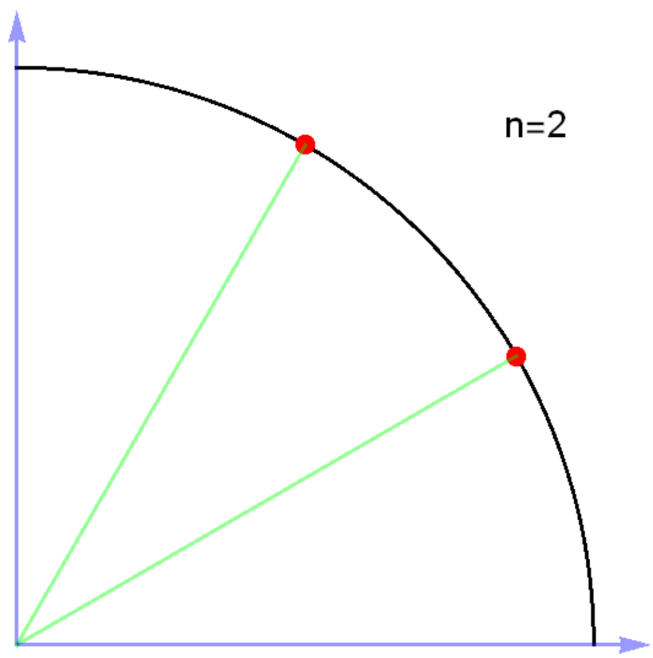


Fig.12 Dirac polarizer (n = 2). $\alpha=\frac{\pi}{6}$ and $\beta=\frac{\pi}{3}$ for the maximum probability. $(|x\rangle \to \left|\alpha=\frac{\pi}{6}\right\rangle \to \left|\beta=\frac{\pi}{3}\right\rangle \to |y\rangle)$.

3. Dirac polarizers (n = 3)

We now consider the Dirac polarizers (n = 3). There are three polarizers with angle α, β, and γ between the x- and y- filters. What are the values of the angles when the probability takes maximum?

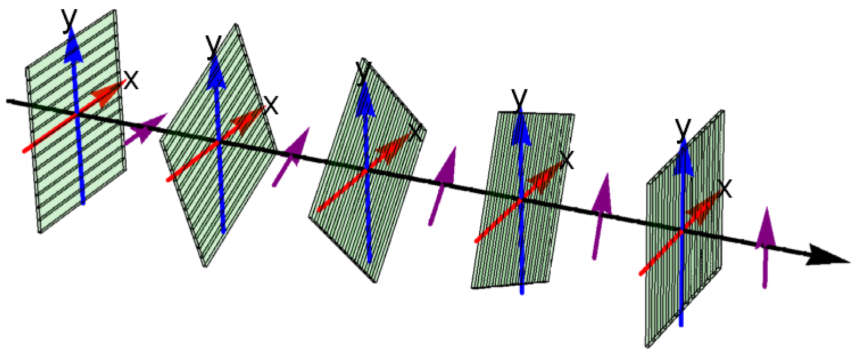


Fig.13 Three filters (α - filter, β - filter, and γ-filter) between the x-filter and y-filter.

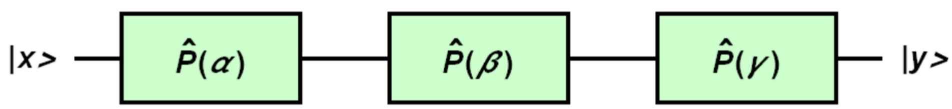


Fig.14 Transition process ($|x\rangle \to |\alpha\rangle \to |\beta\rangle \to |\gamma\rangle \to |y\rangle$).

Projection operator. $\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha) = |\gamma\rangle\langle\gamma|\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|$. Probability amplitude: $\langle y|\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha)|x\rangle$

The probability amplitude is

$$\begin{aligned}
\langle\alpha|x\rangle\langle\beta|\alpha\rangle\langle\gamma|\beta\rangle\langle y|\gamma\rangle &= \langle y|\gamma\rangle\langle\gamma|\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|x\rangle \\
&= \langle y|\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha)|x\rangle \\
&= \langle y|\hat{P}(\gamma))\ \hat{P}(\beta)\ (\hat{P}(\alpha)|x\rangle
\end{aligned}$$

or

$$
\begin{aligned}
&\begin{pmatrix}\sin\gamma\cos\gamma & \sin^2\gamma\end{pmatrix}\begin{pmatrix}\cos^2\beta & \sin\beta\cos\beta\\ \sin\beta\cos\beta & \sin^2\beta\end{pmatrix}\begin{pmatrix}\cos^2\alpha\\ \sin\alpha\cos\alpha\end{pmatrix}\\
&=\sin\gamma\cos\alpha\begin{pmatrix}\cos\gamma & \sin\gamma\end{pmatrix}\begin{pmatrix}\cos^2\beta & \sin\beta\cos\beta\\ \sin\beta\cos\beta & \sin^2\beta\end{pmatrix}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\\
&=\sin\gamma\cos\alpha\cos(\alpha-\beta)\begin{pmatrix}\cos\gamma & \sin\gamma\end{pmatrix}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\\
&=\sin\gamma\cos\alpha\cos(\alpha-\beta)\cos(\gamma-\alpha)
\end{aligned}
$$

Thus, the probability is

$$
\begin{aligned}
P(\alpha,\beta,\gamma) &= \left|\langle y|\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha)|x\rangle\right|^2\\
&=[\cos(\alpha-\beta)\cos(\beta-\gamma)\sin\gamma\cos\alpha]^2
\end{aligned}.
$$

It takes a maximum is $P_{\max} = 0.53079$ at $\alpha = \dfrac{\pi}{8}$, $\beta = \dfrac{\pi}{4}$, and $\gamma = \dfrac{3\pi}{8}$.

((**ContourPlot3D**))

Using the CountourPlot3D, we determine the angles α, β, and γ, yielding the maximum probability.

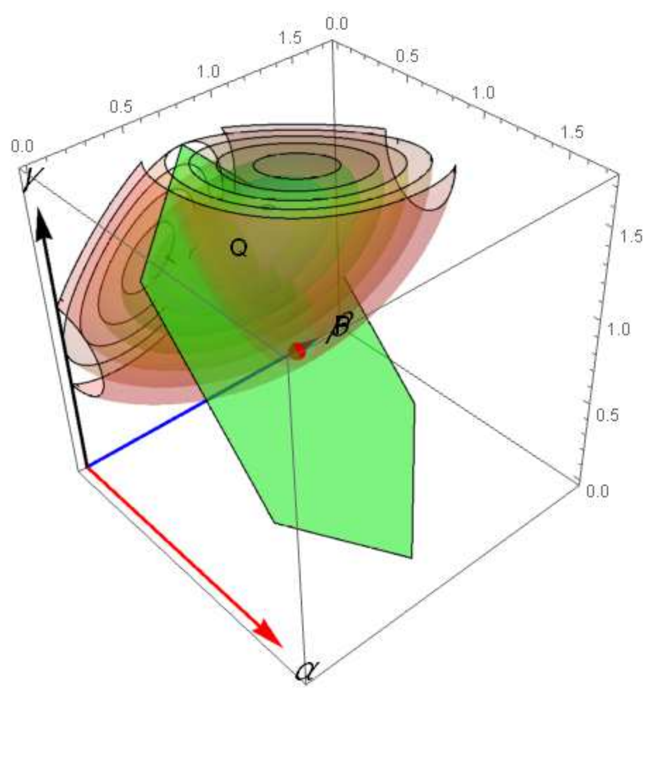


Fig.15(a), (b) (c)

ContourPlot3D. The probability takes a maximum at the point Q with $\alpha = \frac{\pi}{8}$, $\beta = \frac{\pi}{4}$, and $\gamma = \frac{3\pi}{8}$. The surface (denoted by green) is expressed by $\alpha + \beta + \gamma = \frac{3\pi}{4}$.

$\alpha + \beta + \gamma = \frac{3\pi}{4}$. $\gamma = \frac{3\pi}{8}$ and $\beta = \frac{\pi}{4}$, which leads to $\alpha = \frac{\pi}{8}$.

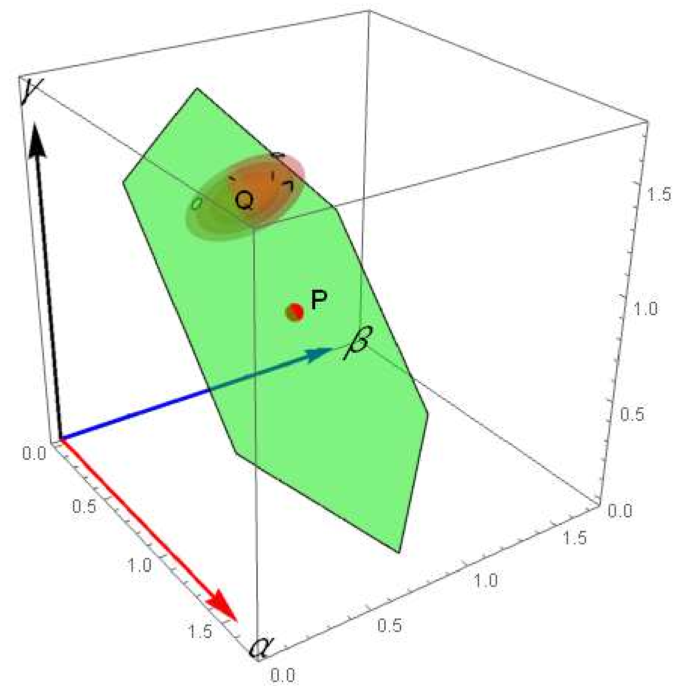


Fig.15(b) ContourPlot3D. Points Q $(\alpha = \frac{\pi}{8}, \beta = \frac{\pi}{4}, \gamma = \frac{3\pi}{4})$ and the point P $(\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{4}, \gamma = \frac{\pi}{4})$.

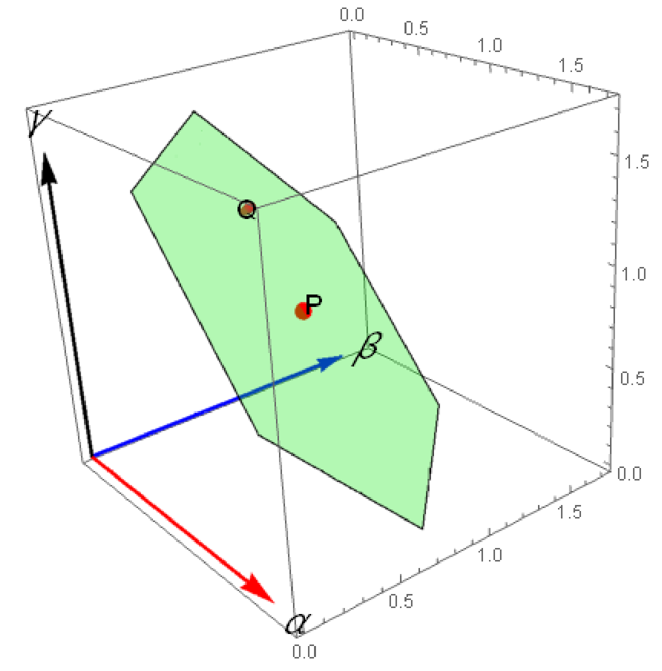


Fig.15(c) The probability takes a maximum at the point Q $(\alpha = \frac{\pi}{8}, \beta = \frac{\pi}{4}, \gamma = \frac{3\pi}{4})$.

$$\alpha = \frac{\pi}{8}, \quad \beta = \frac{\pi}{4}, \gamma = \frac{3\pi}{8},$$

and

$$\alpha + \beta + \gamma = \frac{3\pi}{4}.$$

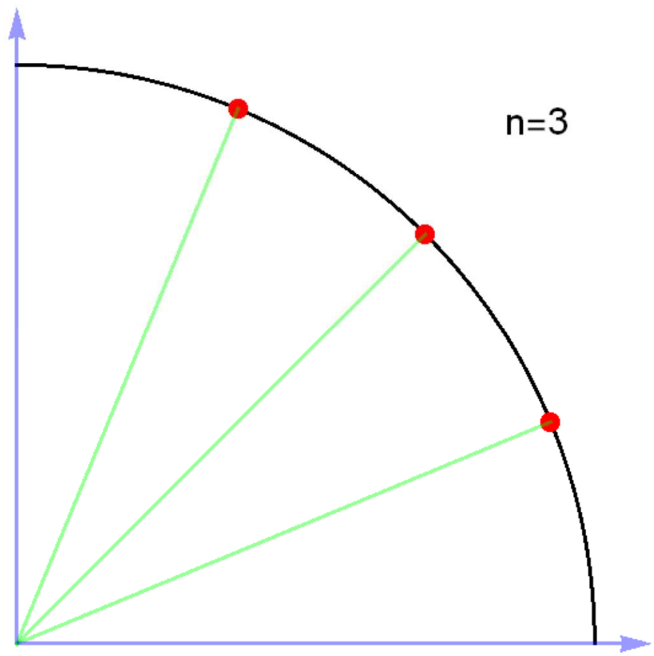


Fig.16 Dirac polarizer (n = 3). $\alpha = \frac{\pi}{8}$, $\beta = \frac{\pi}{4}$, $\gamma = \frac{\pi}{4}$ for the maximum probability ($P_{\max}$ = 0.53079). ($|x\rangle \to \left|\alpha = \frac{\pi}{8}\right\rangle \to \left|\beta = \frac{\pi}{4}\right\rangle \to \left|\gamma = \frac{3\pi}{8}\right\rangle \to |y\rangle$)

It is found that the points P and Q lie on the same plane

$$\alpha + \beta + \gamma = \frac{3\pi}{4}.$$

((**ContourPlot**))

$$\begin{aligned} g(\alpha,\beta) &= [\cos(\alpha-\beta)\cos(\beta-\gamma)\sin\gamma\cos\alpha]^2 \\ &= [\cos(\alpha-\beta)\cos(\alpha+2\beta-\frac{3\pi}{4})\sin(\frac{3\pi}{4}-\alpha-\beta)\cos\alpha]^2 \end{aligned}$$

What is the ContourPlot in this case?

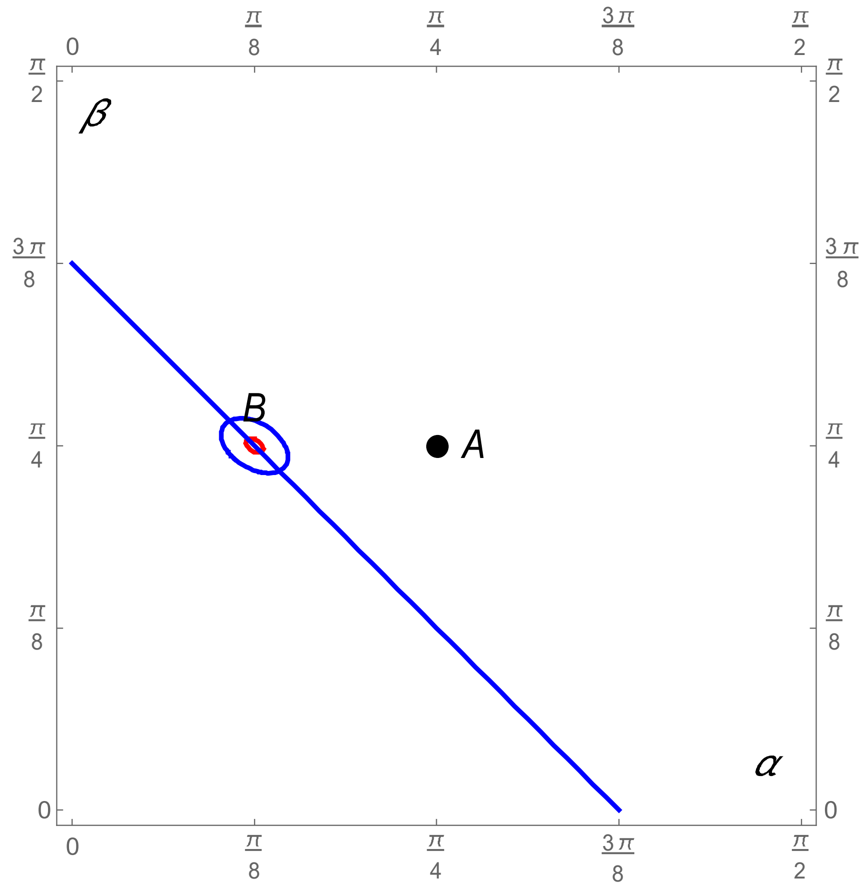


Fig.17 Straight line (denoted by blue line): $\alpha+\beta=\dfrac{3\pi}{8}$. Point A ($\alpha=\beta=\dfrac{\pi}{4}$). Point B ($\alpha=\dfrac{\pi}{8}$, $\beta=\dfrac{\pi}{4}$). The probability takes a maximum (0.53079) at the point B.

4. Dirac polarizers (n = 4)

How about the Dirac polarizers (n = 4)? There are four polarizers with angles α, β, γ, and δ between the x- filter and y-filter.

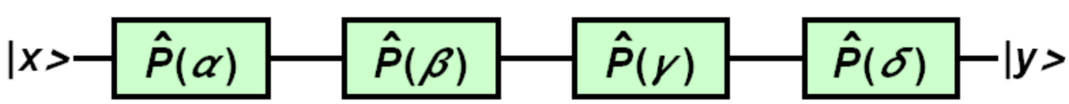


Fig.17 Transition process ($|x\rangle \to |\alpha\rangle \to |\beta\rangle \to |\gamma\rangle \to |\delta\rangle \to |y\rangle$).

Projection operator. $\hat{P}(\delta)\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha)=|\delta\rangle\langle\delta|\gamma\rangle\langle\gamma|\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|$.

The probability amplitude is

$$
\begin{aligned}
\langle\alpha|x\rangle\langle\beta|\alpha\rangle\langle\gamma|\beta\rangle\langle y|\gamma\rangle &= \langle y|\delta\rangle\langle\delta|\gamma\rangle\langle\gamma|\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|x\rangle \\
&= \langle y|\hat{P}(\delta)\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha)|x\rangle \\
&= (\langle y|\hat{P}(\delta)[\hat{P}(\gamma))\ \hat{P}(\beta)]\ (\hat{P}(\alpha)|x\rangle) \\
&= \cos\alpha\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\delta)\sin\delta
\end{aligned}
$$

Thus, the probability is

$$
\begin{aligned}
P(\alpha,\beta,\gamma,\delta) &= \left|\langle y|\hat{P}(\delta)\hat{P}(\gamma)\hat{P}(\beta)\hat{P}(\alpha)|x\rangle\right|^2 \\
&= [\cos\alpha\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\delta)\sin\delta]^2
\end{aligned}
.
$$

Here we assume that

$$
\alpha+\beta+\gamma+\delta = 4\frac{\pi}{4} = \pi \ .
$$

We use the ContourPlot3D to determine the values of α, β, γ, and δ when the probability takes a maximum.

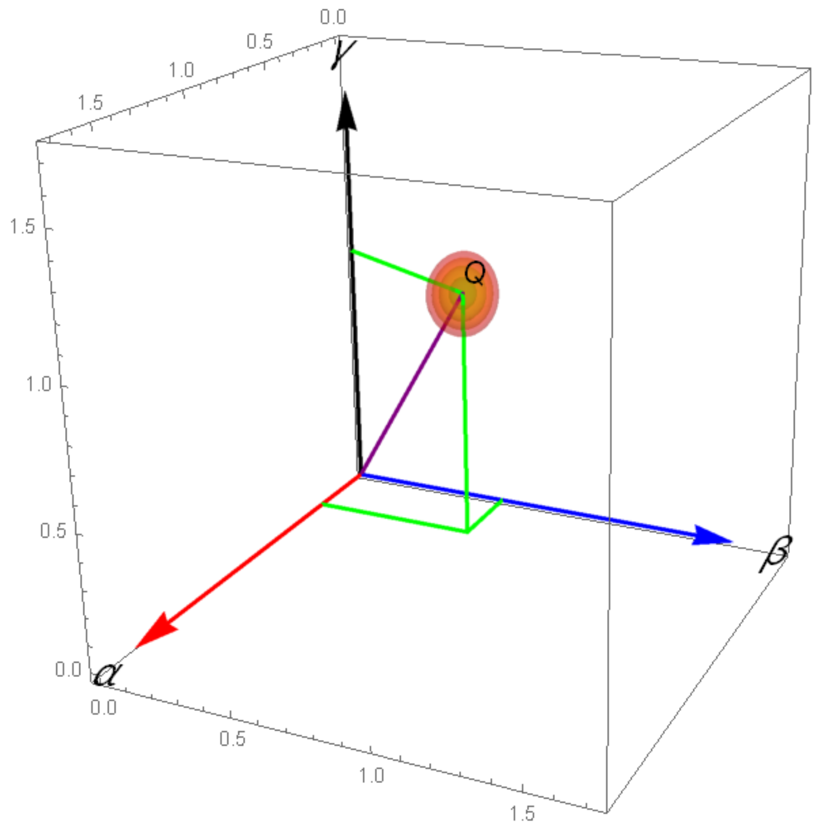


Fig.18 ContourPlot3D for $P(\alpha,\beta,\gamma)$, where $\alpha+\beta+\gamma+\delta=4\frac{\pi}{4}=\pi$. Maximum probability. $P_{\max}=0.605429$. The point Q ($\alpha=\frac{\pi}{10}$, $\beta=\frac{\pi}{5}$, $\gamma=\frac{3\pi}{10}$) which leads to $\delta=\frac{2\pi}{5}$.

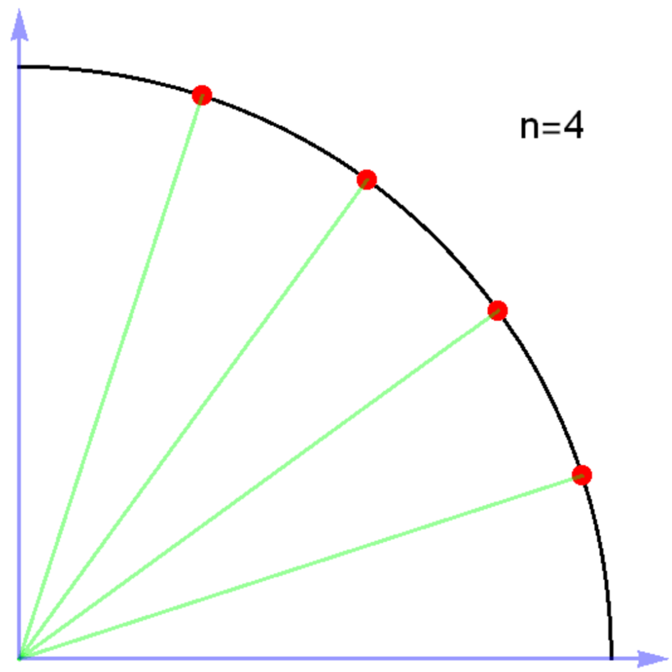


Fig.19 Dirac polarizers (n = 4).

$$(|x\rangle \rightarrow \left|\alpha = \frac{\pi}{10}\right\rangle \rightarrow \left|\beta = \frac{\pi}{5}\right\rangle \rightarrow \left|\gamma = \frac{3\pi}{10}\right\rangle \rightarrow \left|\delta = \frac{2\pi}{5}\right\rangle \rightarrow |y\rangle)$$

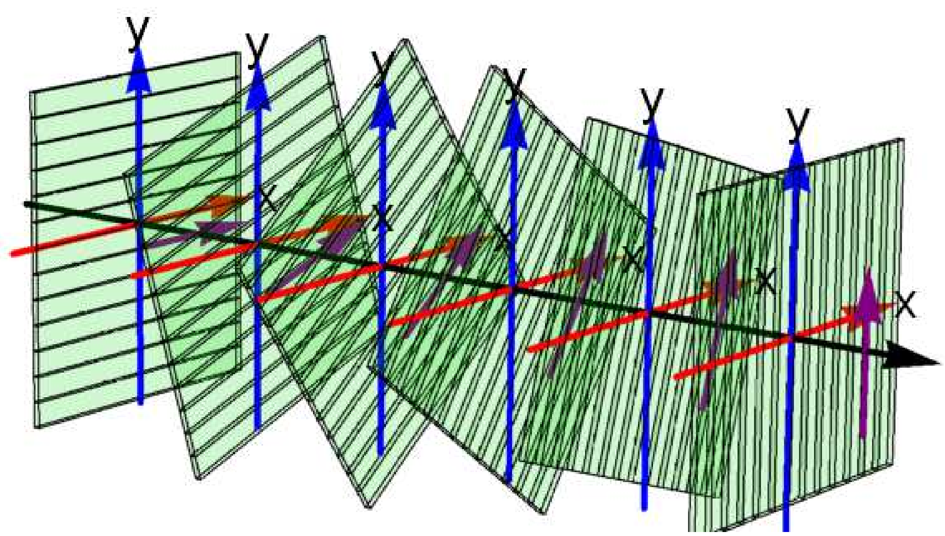


Fig.20 Dirac polarizers (n = 4), which lead to the maximum probability.

5. Dirac polarizers (*n*)

From the above discussion, it is guessed that the probability

$$P(\theta_1,\theta_2,...,\theta_n)=\left|\langle y|\hat{P}_{\theta_n}\hat{P}_{\theta_{n-1}}\cdots\hat{P}_{\theta_4}\hat{P}_{\theta_3}\hat{P}_{\theta_2}\hat{P}_{\theta_1}|x\rangle\right|^2$$

takes a maximum when

$$\Delta\theta=\theta_i-\theta_{i-1}\ (>0)$$

$$\theta_1+\theta_2+\theta_3+......+\theta_n=\frac{\pi}{4}n\ .$$

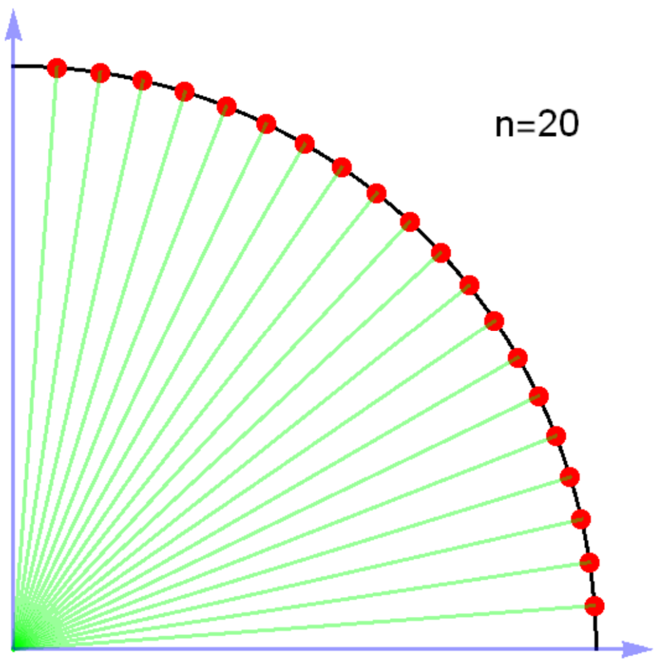


Fig.21 Experimental configuration of Dirac polarizers with n = 20.

The probability is assumed to be expressed by

$$f(n)=\cos^{2n}[\frac{\pi}{2(n+1)}]\ \sin^2[\frac{n\pi}{2(n+1)}]$$

where $\Delta\theta=\dfrac{\pi}{2(n+1)}$

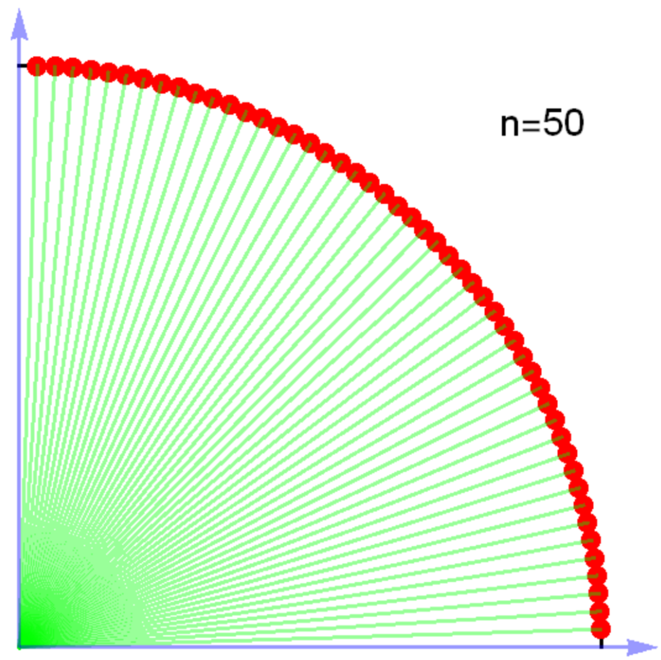


Fig.22 Dirac polarizers with $n = 50$.

We calculate the maximum probability $f(n)$ as a function of *n*. The result is shown below.

Table 1: *n* vs maximum probability $f(n)$.

n	$f(n)$

1	0.25
2	0.421875
3	0.53079
4	0.605429
5	0.659668
6	0.700834
7	0.733133
8	0.759148
9	0.780546
10	0.798456
11	0.813665
12	0.826741
13	0.838103
14	0.848068
15	0.856877
16	0.864721
17	0.87175
18	0.878085
19	0.883824
20	0.889047

100	0.975865
200	0.987799
300	0.991836
400	0.993866
500	0.995087
600	0.995903
700	0.996486
800	0.996924
900	0.997265
1000	0.997538

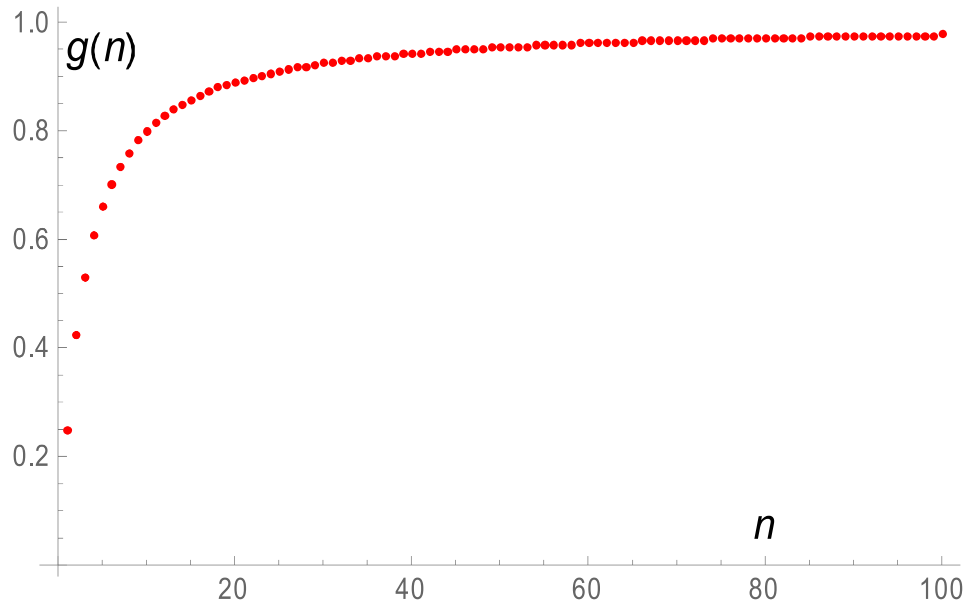


Fig.23 Plot of $f(n)$ as a function of n. $f(n)$ tends to 1 in the limit of $n \to \infty$.

6. Dirac three polarizers experimente ((Basdevant, youtube))

When you go to the web sites, you may find very nice videos on the Dirac three polarizers demonstrations. I found an interesting article on these topics in the book written by Basdevant, Lectures on Quantum mechanics). The copy of a part of the article is reproduced here.

"This is even clearer if we cross the polarizer and analyzer at a right angle $\theta = 90°$. Nothing gets through. States of orthogonal polarizations are incompatible; there is a zero probability that a photon in the horizontal polarization state can be found in the vertical polarization state. Now, we can observe an amazing phenomenon. If we insert a third polarizer at some non-zero angle, say 45°, between the crossed polarizers, the light *reappears* (Fig.) although we have inserted an absorbing object that can only reject all photons polarized perpendicular to its axis! (Actually, is it really the only thing it can do? No! It's not a triviality to say that it is also able to let photons pass if their polarization is parallel to its axis.)

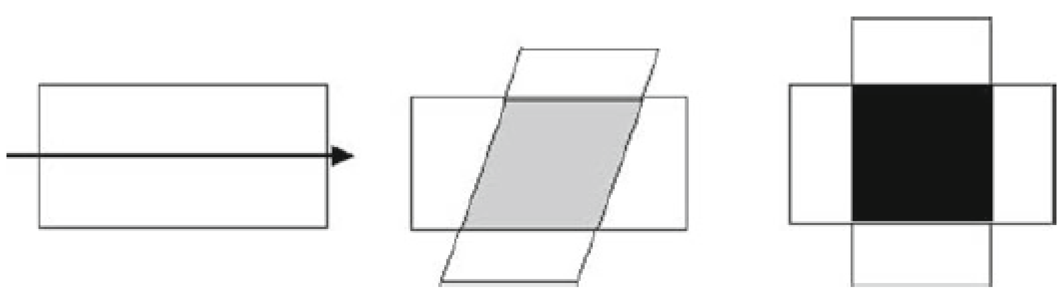


Fig.24 Outgoing light from a horizontal polarizer (*left*). Intensity across an analyzer at an angle θ (*middle*); extinction if the analyzer is at 90° of the first one (*right*)

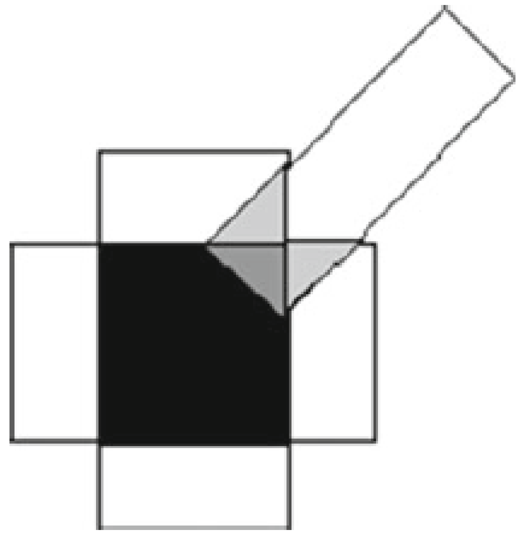


Fig.25 Reappearance of light emerging from two crossed polarizers if a third polarizer at some angle is inserted between them.

((**Youtube**))
Dirac Three Polarizers Experiments
https://www.informationphilosopher.com/solutions/experiments/dirac_3-polarizers/

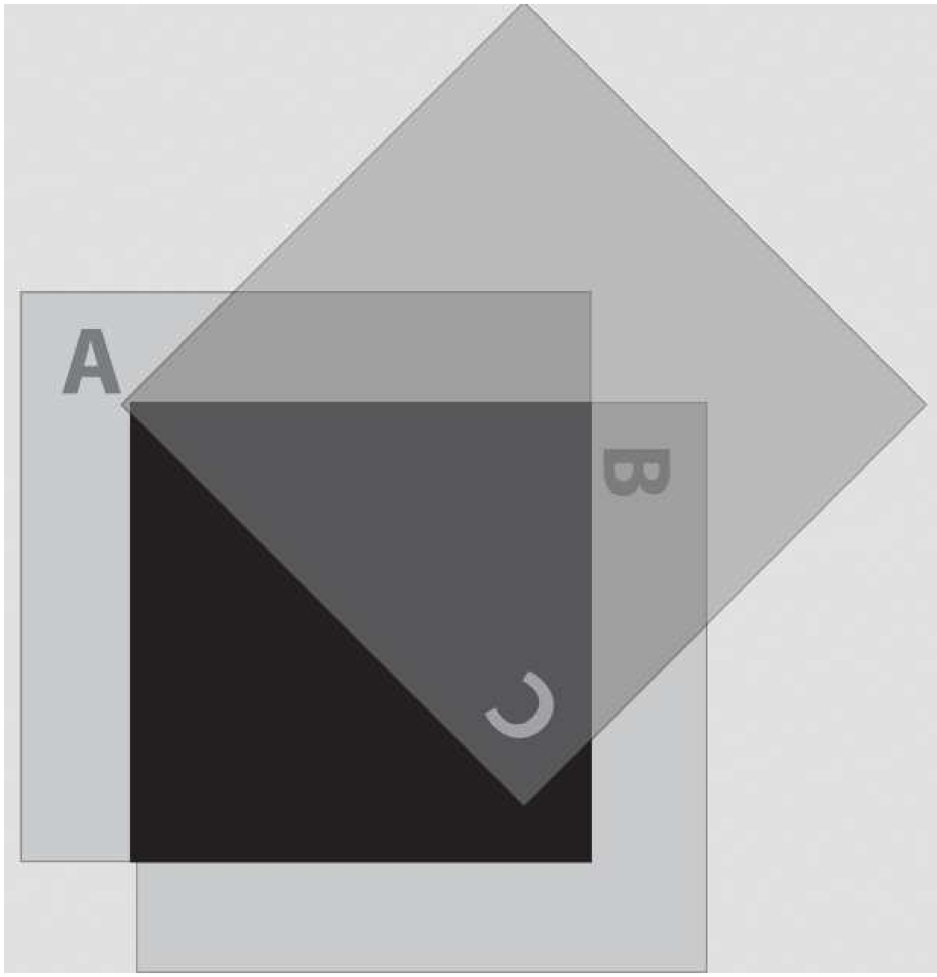


Fig.26 A filter with $|x\rangle$, B filter with $|y\rangle$, and C filter with $\left|\theta=\frac{\pi}{4}\right\rangle$. $|x\rangle \to \left|\theta=\frac{\pi}{4}\right\rangle \to |y\rangle$, which is the same result from our result (see **Fig.3**).

((**REFERENCES**))

Three polarizers-scientist experiments

https://www.youtube.com/watch?v=-Yh-U8Ro-P0

7. Stern-Gerlach experiments for spin 1/2

Like Dirac polarizers experiment, we will do similar experiments using the Stern-Gerlach experiments for spin 1/2. The magnetic field in the SG experiments is applied in the (x, z) plane, The unit vector for the magnetic field direction is denoted by the vector $\boldsymbol{n}$. We use the conventional notations for the spin 1/2 system in the quantum mechanics.

The projection operators:

$$\hat{P}(+\boldsymbol{n}) = |+\boldsymbol{n}\rangle\langle+\boldsymbol{n}| = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$\hat{P}(-\boldsymbol{n}) = |-\boldsymbol{n}\rangle\langle-\boldsymbol{n}| = \begin{pmatrix} \sin^2\frac{\theta}{2} & -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix},$$

$$\hat{P}(+\boldsymbol{n})|+z\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{pmatrix}, \qquad \hat{P}(+\boldsymbol{n})|-z\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin^2\frac{\theta}{2} \end{pmatrix},$$

$$\hat{P}(-\boldsymbol{n})|+z\rangle = \begin{pmatrix} \sin^2\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{pmatrix}, \qquad \hat{P}(-\boldsymbol{n})|-z\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \cos^2\frac{\theta}{2} \end{pmatrix}.$$

where θ is the angle from the *z* axis in the *z*-*x* plane.
The eigenkets:

$$|+\boldsymbol{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix},$$

$$|-\boldsymbol{n}\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} \qquad \text{(conventional form)}$$

For simplicity we use the qubits (quantum bits),

$|+z\rangle = |0\rangle$, $\quad |-z\rangle = |1\rangle$, $\quad$ (from the Bloch sphere).

8. Example-1

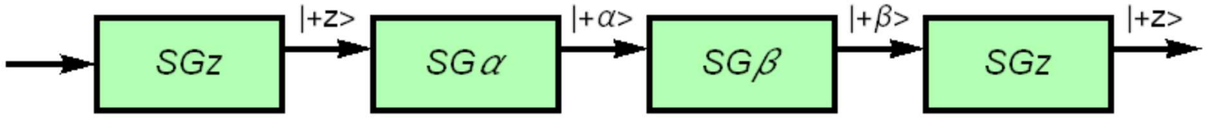


Fig.27 SG experiment-1. $|+z\rangle \rightarrow |+\alpha\rangle \rightarrow |+\beta\rangle] \rightarrow |+z\rangle$. α and β are angles from the z axis in the *z*-*x* plane.

The probability amplitude is

$$\langle +z|\hat{P}(+\beta)\hat{P}(+\alpha)|+z\rangle = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos(\frac{\alpha-\beta}{2}).$$

The probability is given by

$$P = \left|\langle +z|\hat{P}(+\beta)\hat{P}(+\alpha)|+z\rangle\right|^2 = \cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2(\frac{\alpha-\beta}{2}),$$

which is a function of α and β. We use the ContourPlot of the Mathematica to find the maximum probability. From **Fig.28** of the ContourPlot, we find $P_{\max} = 1$ (maximum probability) when

$$\alpha = \beta = 0 \qquad |+0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

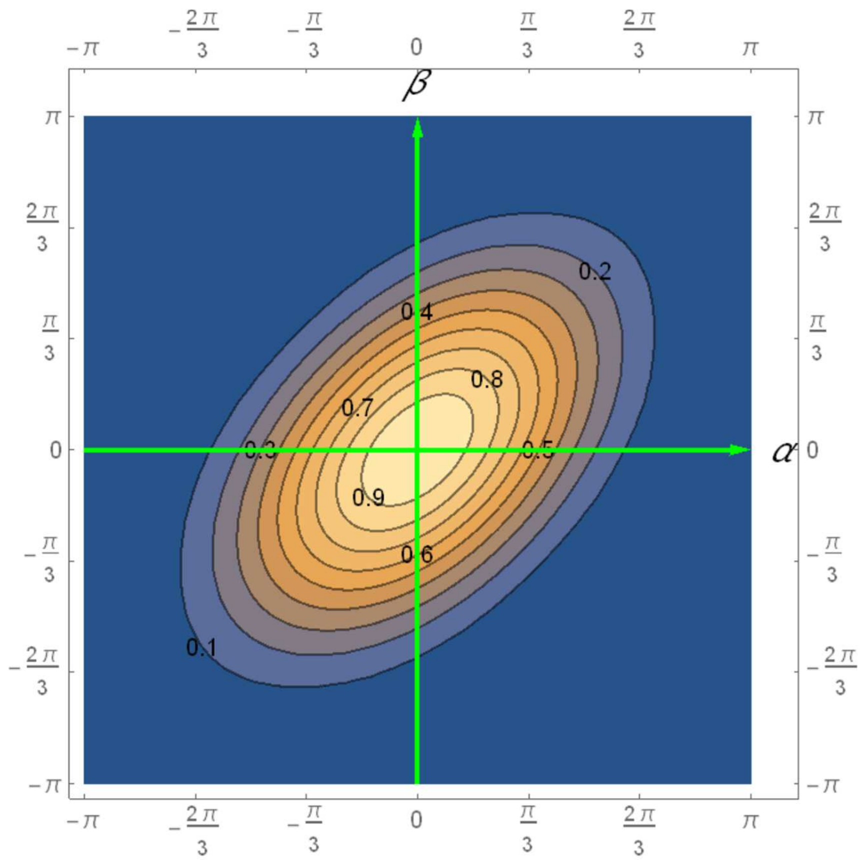


Fig.28 ContourPlot in the α-β plane. $P_{\max} = 1$. $|+z\rangle \to [|+0\rangle \to |+0\rangle] \to |+z\rangle$, where $|+0\rangle = |+z\rangle$.

9. Example-2

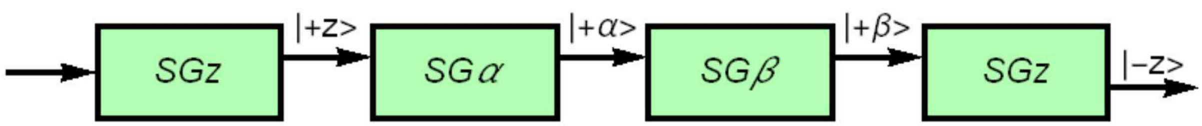


Fig.29 SG experiment-2. $|+z\rangle \to |+\alpha\rangle \to |+\beta\rangle] \to |-z\rangle$.

The probability amplitude is

$$\langle -z|\hat{P}(+\beta)\hat{P}(+\alpha)|+z\rangle = \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\left(\frac{\alpha-\beta}{2}\right).$$

The probability is given by

$$P = \left| \langle -z | \hat{P}(+\beta)\hat{P}(+\alpha) | +z \rangle \right|^2 = \cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\cos^2\left(\frac{\alpha-\beta}{2}\right),$$

which is a function of α and β. We use the ContourPlot of the Mathematica to find the maximum probability. From **Fig.30** of the ContourPlot, we find the maximum probability $P_{\max}$ = 0.421875, when

$$\alpha = \frac{\pi}{3},\ \ \beta = \frac{2\pi}{3} \qquad \left| +\frac{\pi}{3} \right\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \qquad \left| +\frac{2\pi}{3} \right\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix},$$

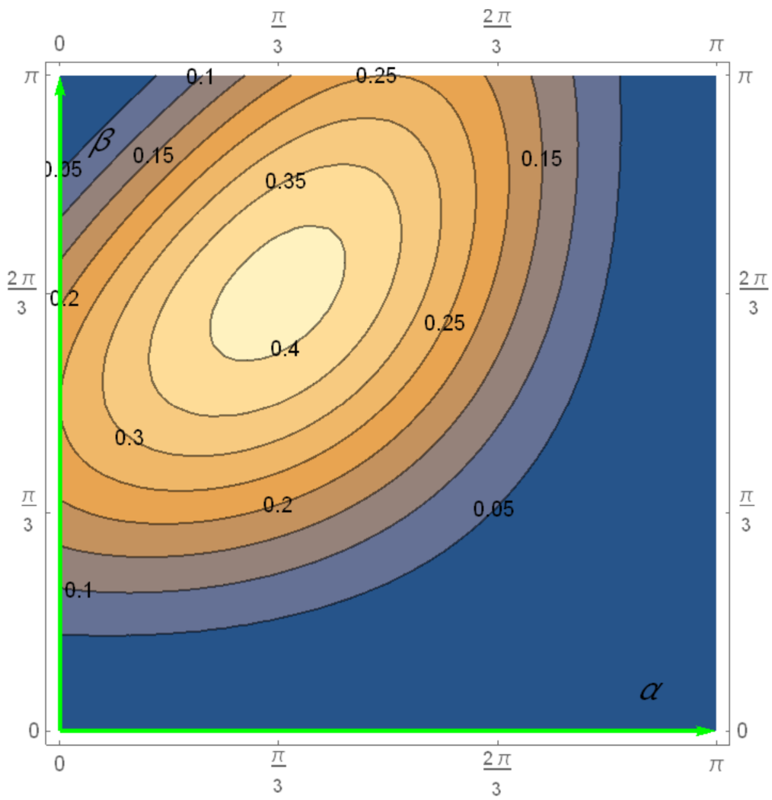


Fig.30 ContourPlot in the α-β plane. $P_{\max} = 0.421875$ (maximum probability).

$$|+z\rangle \to [\left| +\frac{\pi}{3} \right\rangle \to \left| +\frac{2\pi}{3} \right\rangle] \to |-z\rangle .$$

10. Example-3

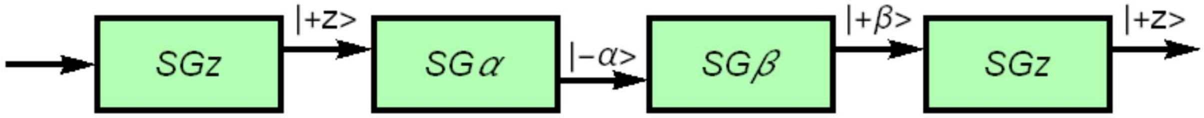


Fig.31 SG experiment-3. $|+z\rangle \rightarrow |-\alpha\rangle \rightarrow |+\beta\rangle] \rightarrow |+z\rangle$.

The probability amplitude:

$$\langle +z|\hat{P}(+\beta)\hat{P}(-\alpha)|+z\rangle = \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin(\frac{\alpha-\beta}{2}).$$

The result is obtained as follows.

$$\alpha = \pi,\ \ \beta = 0 \qquad |-\pi\rangle = -\begin{pmatrix}1\\0\end{pmatrix}, \qquad |+0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$P_{\max} = 1$.

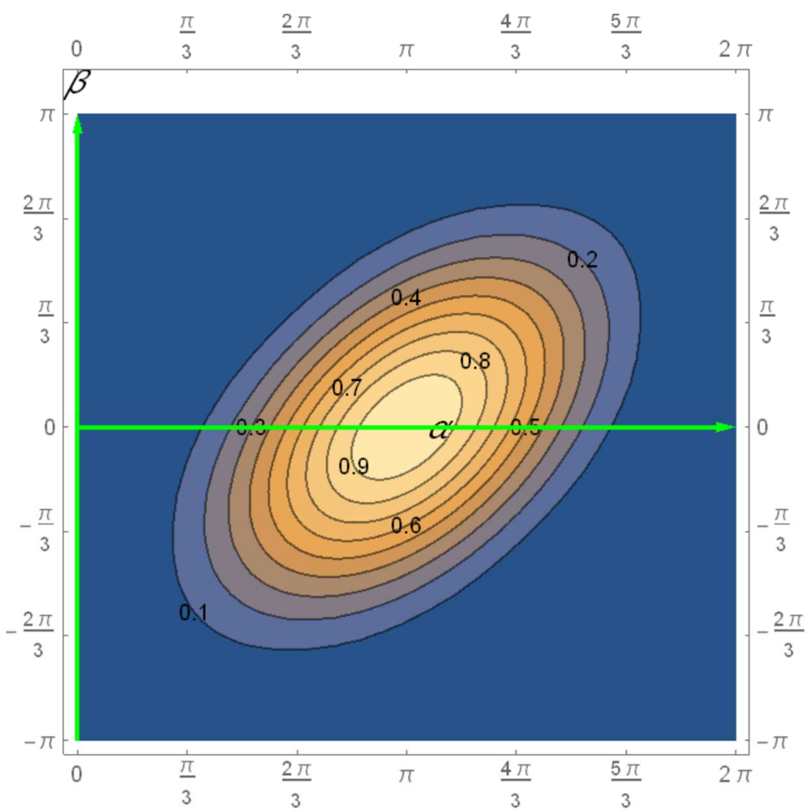


Fig.32 ContourPlot in the α-β plane. $P_{\max} = 1$ (maximum probability).
$|+z\rangle \to [|-\pi\rangle \to |+0\rangle] \to |+z\rangle$.

11. Example-4

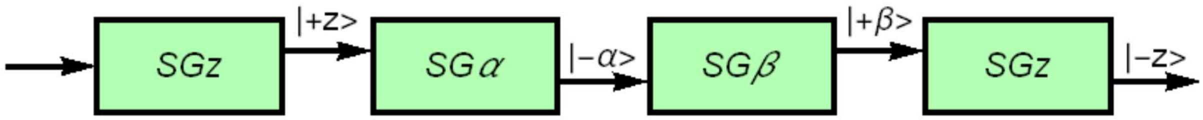


Fig.33 SG experiment-4. $|+z\rangle \to |-\alpha\rangle \to |+\beta\rangle] \to |-z\rangle$

The probability amplitude:

$$\langle -z|\hat{P}(+\beta)\hat{P}(-\alpha)|+z\rangle = \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\left(\frac{\alpha-\beta}{2}\right).$$

The result of ContourPlot for the case obtained as follows.

$$\alpha = \frac{4\pi}{3}, \quad \beta = \frac{2\pi}{3}. \qquad \left|-\frac{4\pi}{3}\right\rangle = -\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \qquad \left|+\frac{2\pi}{3}\right\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$P_{\max} = 0.421875.$

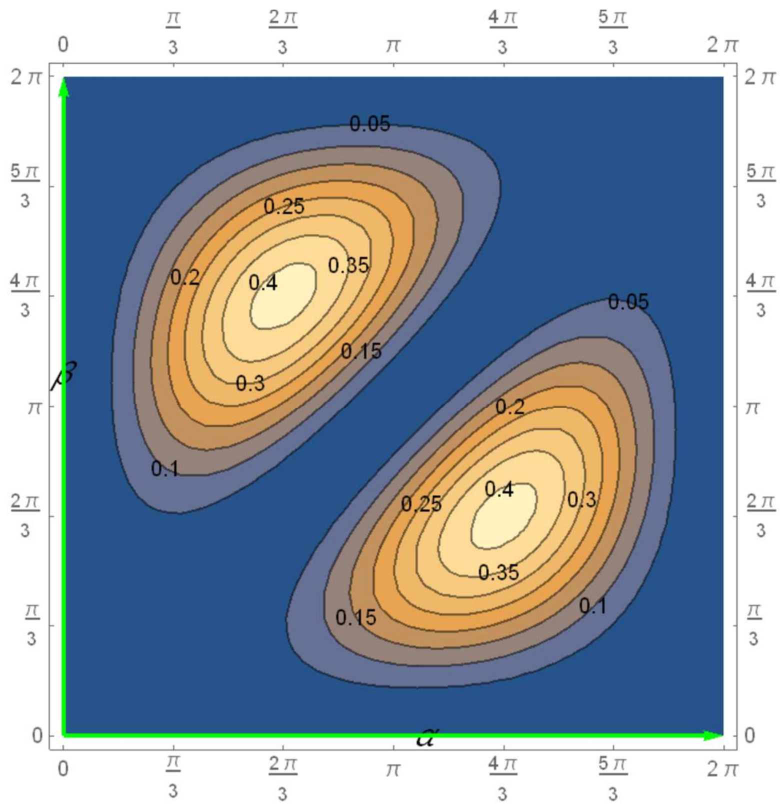


Fig.34 ContourPlot in the α-β plane. $P_{\max} = 0.421875$ (maximum probability).
$|+z\rangle \to [\left|-\frac{4\pi}{3}\right\rangle \to \left|+\frac{2\pi}{3}\right\rangle] \to |-z\rangle$.

12. Example-5

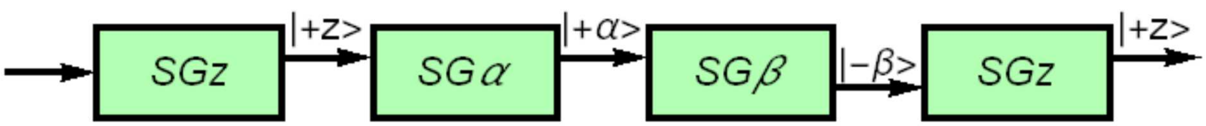


Fig.35 SG experiment-5. $|+z\rangle \to |+\alpha\rangle \to |-\beta\rangle] \to |+z\rangle$.

The probability amplitude:

$$\langle +z|\hat{P}(-\beta)\hat{P}(+\alpha)|+z\rangle = -\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\left(\frac{\alpha-\beta}{2}\right).$$

The result is obtained as follows.

$\alpha = 0, \;\; \beta = \pi \qquad\qquad |+0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \qquad |-\pi\rangle = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -|0\rangle$

$P_{\max} = 1.$

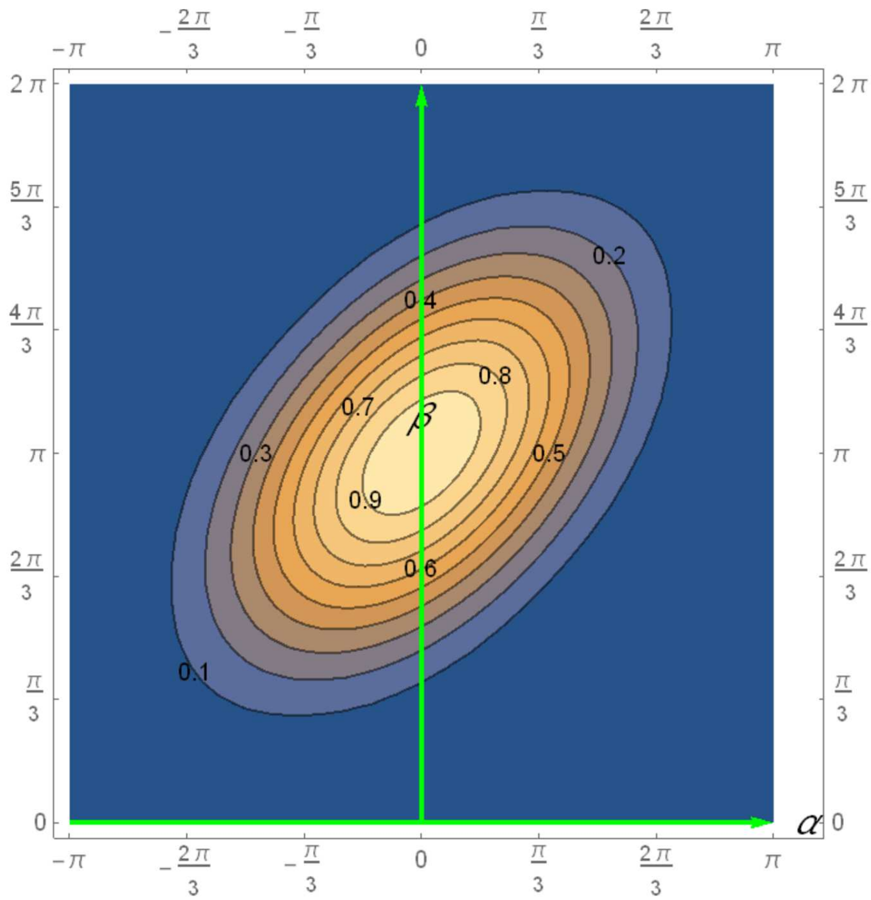


Fig.36 ContourPlot in the α-β plane. $P_{\max} = 1$ (maximum probability).
$|+z\rangle \to [|+0\rangle \to |-\pi\rangle] \to |+z\rangle$.

13. Example-6

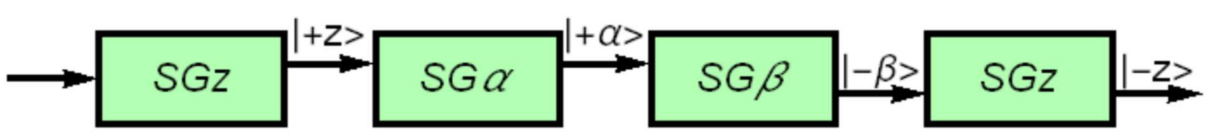


Fig.37 SG experiments-6. $|+z\rangle \to |+\alpha\rangle \to |-\beta\rangle] \to |-z\rangle$.

The probability amplitude:

$$\langle -z|\hat{P}(-\beta)\hat{P}(+\alpha)|+z\rangle = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\left(\frac{\alpha-\beta}{2}\right).$$

The result is as follows.

$$\alpha = \frac{\pi}{3},\ \ \beta = -\frac{\pi}{3} \qquad \left|+\frac{\pi}{3}\right\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \qquad \left|-(-\frac{\pi}{3}\right\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$P_{\max} = 0.421875.$

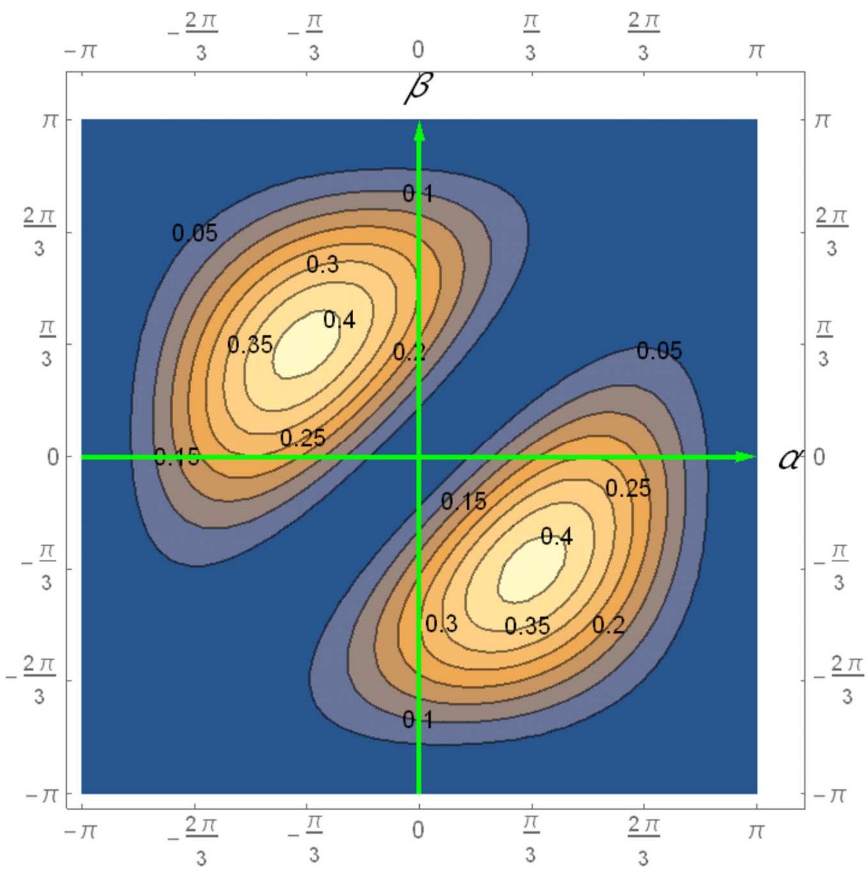


Fig.38 ContourPlot in the α-β plane. $P_{\max} = 0.421875$ (maximum probability).

$|+z\rangle \rightarrow [\left|+\frac{\pi}{3}\right\rangle \rightarrow \left|-(-\frac{\pi}{3})\right\rangle] \rightarrow |-z\rangle$.

14. Example-7

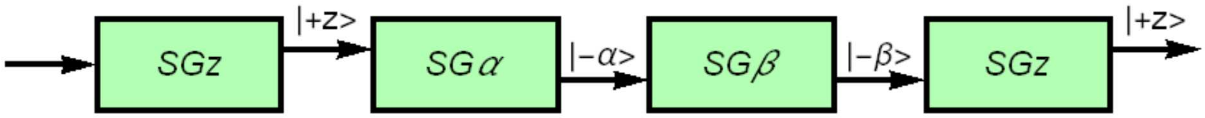


Fig.39 SG experiment-7. $|+z\rangle \rightarrow |-\alpha\rangle \rightarrow |-\beta\rangle] \rightarrow |+z\rangle$.

The probability amplitude:

$$\langle +z|\hat{P}(-\beta)\hat{P}(-\alpha)|+z\rangle = \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\left(\frac{\alpha-\beta}{2}\right).$$

The result is as follows.

$\alpha = \pi, \;\; \beta = \pi \;\; (P_{\max} = 1).$

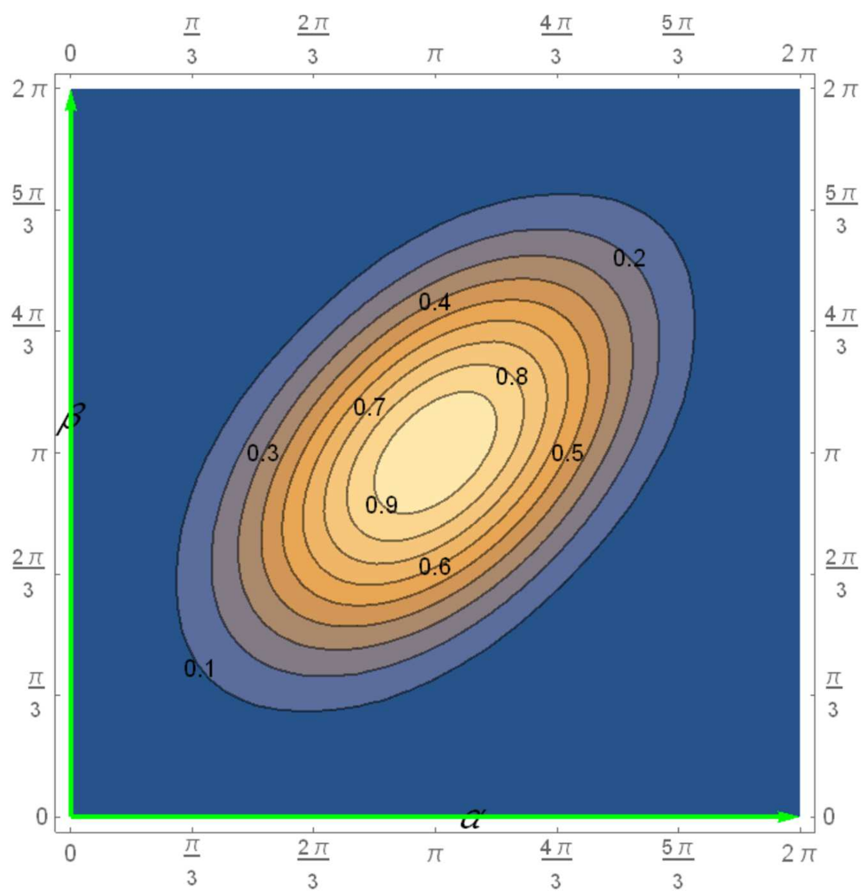


Fig40. ContourPlot in the α-β plane. $P_{\max} = 1$ (maximum probability).
$|+z\rangle \rightarrow [|-\pi\rangle \rightarrow |-\pi\rangle] \rightarrow |+z\rangle$.

15. Example-8

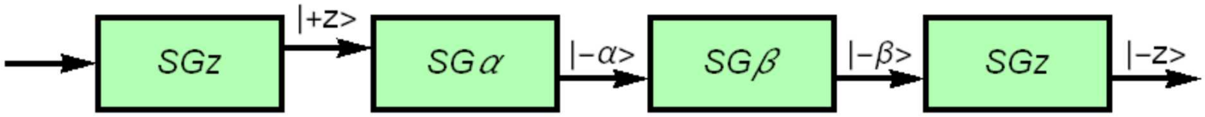


Fig.41 SG experiment-8. $|+z\rangle \rightarrow |-\alpha\rangle \rightarrow |-\beta\rangle] \rightarrow |-z\rangle$.

The probability amplitude:

$$\langle -z|\hat{P}(-\beta)\hat{P}(-\alpha)|+z\rangle = -\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\left(\frac{\alpha-\beta}{2}\right).$$

The result is as follows.

$$\alpha = \frac{2\pi}{3},\ \ \beta = \frac{\pi}{3}, \qquad \left|-\frac{2\pi}{3}\right\rangle = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}, \qquad \left|-\frac{\pi}{3}\right\rangle = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix},$$

$P_{\max}$ = 0.421875 (maximum probability).

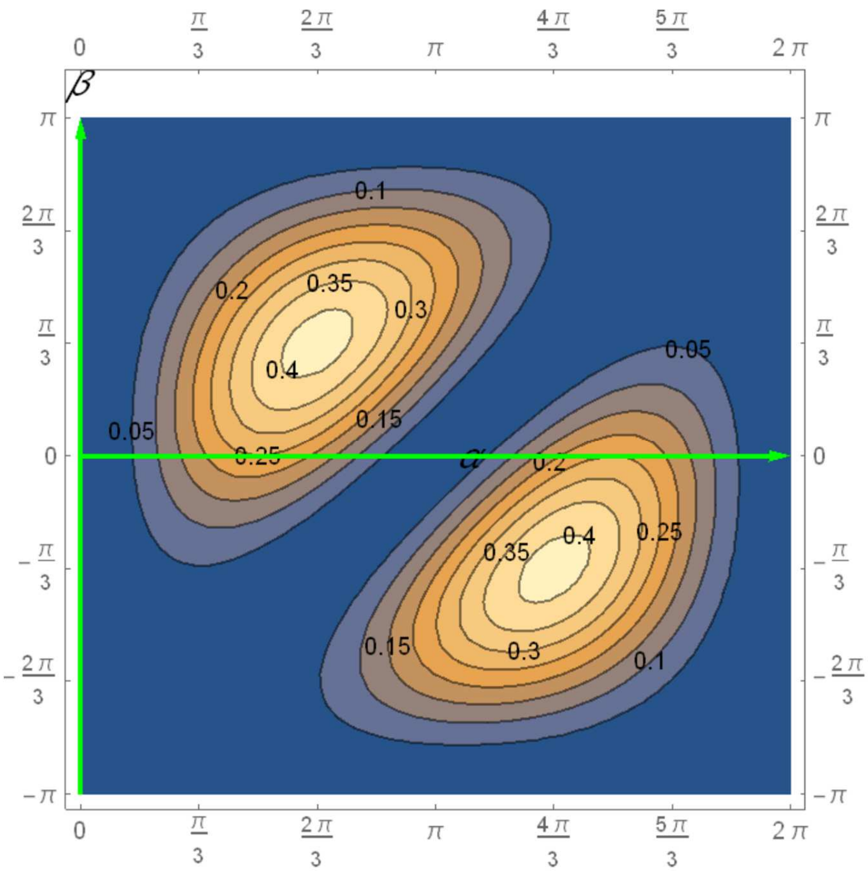


Fig.42 ContourPlot in the α-β plane. $P_{\max} = 0.421875$. $|+z\rangle \to [\left|-\frac{2\pi}{3}\right\rangle \to \left|-\frac{\pi}{3}\right\rangle] \to |-z\rangle$

16. "Bloch circle"

Here we make a plot of the states

$$|+\boldsymbol{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad |-\boldsymbol{n}\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

where $\phi = 0$. For convenience, we use the notations of qubits (quantum bits),

$$|+z\rangle = |0\rangle, \qquad |-z\rangle = |1\rangle .$$

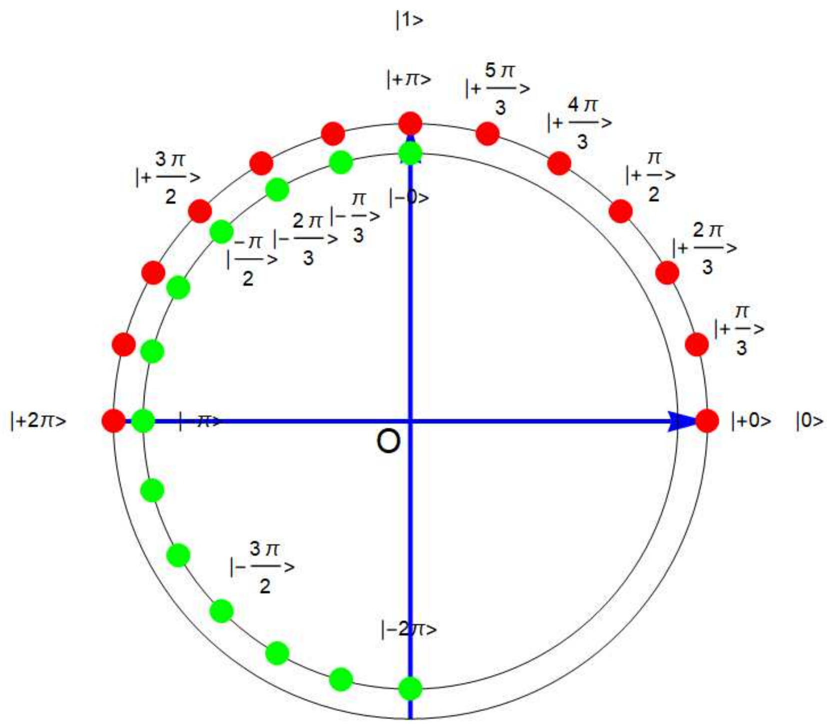


Fig.43 Bloch-circle (analogy from the Bloch-sphere, qubits). $|+z\rangle = |0\rangle$. $|-z\rangle = |1\rangle$. The states $|+\mathbf{n}\rangle$ (red closed circles) and $|-\mathbf{n}\rangle$ (green closed circles). Note that the angle θ for the state $|+\theta\rangle$ is twice larger than that of angle in the $\{|0\rangle, |1\rangle\}$ plane. $|\pm\mathbf{n}\rangle$ has periodicity of 4π instead of 2π for spin 1/2.

17. Three SG sets

Here we consider the case of three SG sets between two SGz's sets. For simplicity, the following change of states occurs such as $|+z\rangle \to |+\alpha\rangle \to |+\beta\rangle \to |+\gamma\rangle \to |-z\rangle$.

Fig.44 SG experiment. $|+z\rangle \to |+\alpha\rangle \to |+\beta\rangle \to |+\gamma\rangle \to |-z\rangle$. All angles are ones from the z axis in the z-x plane.

The probability amplitude for this process is

$$\langle -z|\hat{P}(+\gamma)\hat{P}(+\beta)\hat{P}(+\alpha)|+z\rangle$$

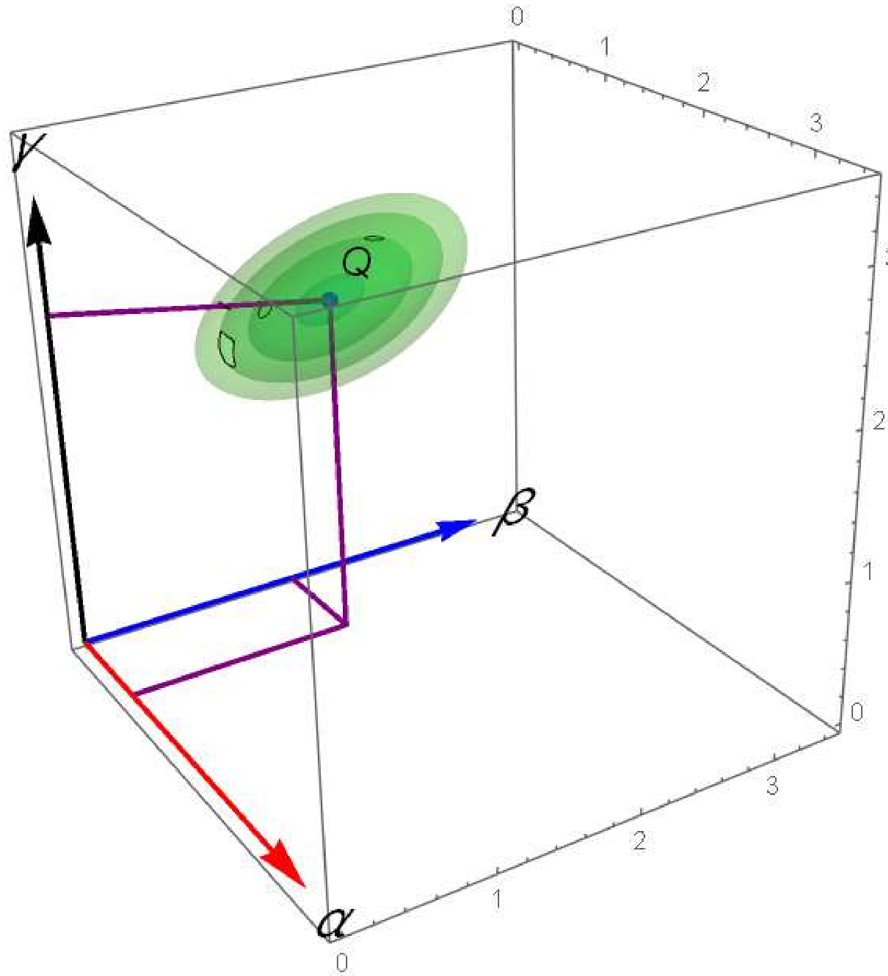


Fig.45 ContourPlot3D in the (α,β,γ) space. $P_{\max}=0.53079$ at the point Q $\left(\alpha=\frac{\pi}{4},\beta=\frac{2\pi}{4},\gamma=\frac{3\pi}{4}\right)$.

The result $\left(\alpha=\frac{\pi}{4},\beta=\frac{2\pi}{4},\gamma=\frac{3\pi}{4}\right)$ for the SG experiment is similar to $\left(\alpha=\frac{\pi}{8},\beta=\frac{2\pi}{8},\gamma=\frac{3\pi}{8}\right)$ for the Dirac polarizers with n = 3, except for the angles (the angle for the Dirac polarizers are half of those for the SG experiment),

18. Summary: Superposition and indeterminacy ((Dirac))

With the help of the ContourPlot and ContourPlot3D of the Mathematica, we get the maximum probability for the Dirac n polarizers experiment and SG experiments where the magnetic field is applied in the z-x plane.We find that all results are those which are reasonably predicted (as common sense) without any calculations.

((Dirac))

The significance of the principle of superposition in quantum mechanics, in particular for photon polarization was discussed by Dirac (see the book of Dirac). Here is the summary of the discussion by Dirac.

(a) The nature of the relationships which the superposition principle requires to exist between the states of any system is of a kind that cannot be explained in terms of familiar physical concepts. One cannot in the classical sense picture a system being partly in each of two states and see the equivalence of this to the system being completely in some other state. There is an entirely new idea involved, to which one must get accustomed and in terms of which one must proceed to build up an exact mathematical theory, without having any detailed classical picture.

(b) The intermediate character of the state formed by superposition thus expresses itself through the probability of a particular result for an observation being intermediate between the corresponding probabilities for the original states, not through the result itself being intermediate between the corresponding results of the original states.

(c) It is important to remember that the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in the classical theory, as is shown by the fact that the quantum superposition principle demands indeterminancy in the results of observations, in order to be capable of a sensible physical interpretation. The analogies are thus liable to be misleading.

REFERENCES

P.A.M. Dirac, The Principle of Quantum Mechanics, 4th edition (Oxford, 1958).

R.P. Feynman, R.B. Leighton, and M. Sands, The Feynman Lectures on Physics, New Millennium edition, vol.3 (Basic Book, 2010).

J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics 2nd edition (Cambridge, 2017).

G. Baym, Lectures on Quantum Mechanics (Westview, 1990).

J.L. Basdevant, Lectures on Quantum Mechanics (Springer, 2007).

J.S. Townsend, A Modern Approach to Quantum Mechanics, 2nd edition (University Science Books, 2012).

A. Goswami, Quantum Mechanics, 2nd edition (Wm.C. Brown, 1997).s