Quarter-wave plate and half-wave plate

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1. Introduction

I had a good experience in doing experiment on the birefringence (double refraction) of calcite. When my daughter was a middle school student, I had good opportunity to assist her in doing a scientific project as a part of her research at the school. She chose the birefringence experiment by herself. To this end, I borrowed a large sample of calcite (Iceland Spa) from our technician (SUNY at Binghamton). My daughter and I had experiments on the double diffraction (birefringence) at home. We observed two rays emerging from the large sample of Iceland Spa. We also checked the direction of polarization vectors (direction of electric field) using my sunglass (a sort of polarizer) (just like Sir Lawrence Bragg doing the same experiment, using the polarizer; see **Fig.4**).

After that, I tried to understand the physics of birefringence in calcite. I read a lot of standard textbooks on optics. I learned from them that all members of the calcite group crystallize in the trigonal system, have perfect rhombohedral cleavage, and exhibit strong double refraction in transparent rhombohedrons. There is an optic axis in the calcite. The optic axis is a direction in which a ray of transmitted light suffers no birefringence. There are two kinds of rays propagating through the sample, the ordinary (o)-wave, and the extraordinary (e)-wave. The polarization vector of the o wave is perpendicular to the optic axis, while the polarization vector of the e-wave is along the optic axis. When one looks at the transmitted ray propagating along the optic axis, one can find only o-wave, where the polarization vector of the o-wave lies in the plane normal to the optic axis. The refraction index along the optic axis (n_e) is smaller than that along a direction perpendicular to the optic axis (n_o) for calcite (negative uniaxial crystal). Note that $n_e > n_o$ for positive uniaxial crystals such as quartz. Since the velocity v of light is related to the index of refraction as $v = c/n$. So, the direction (the optic axis) of the polarization vector for the e-wave is called as the fast axis, and the direction of the polarization vector for the o-wave is called the slow axis, since $n_o > n_e$. Thus, I understand that these properties are the origin of birefringence in calcite, although I do not fully understand the mechanism of fast and slow propagation of two rays.

Because of the complicated crystal structure (typically, rhombohedral), it is not so easy for one to understand the physics of the birefringence in calcite. In experiments, one makes use of the retarder (retardation plate), which plate is made by cutting a uniaxial birefringent crystal, so that the optic axis lies in the plane of the input surface of the plate. A polarized input beam propagates in a direction normal to the top surface of the retarder and propagates through the crystal.

The retarder is used for the quarter-wave plate and the half-wave plate. Here we discuss the role of the quarter-wave plate and the half-wave plate, using quantum mechanics (the projection operator, the principle of superposition). Thanks to the use of such a plate, we are free from examining the location of the optic axis in the calcite bulk sample used in the experiment. Experimentally, quarter wave plate is used to generate the right-hand circularly polarized light $|R\rangle$ and the left-hand circularly polarized light $|L\rangle$ from the linearly polarized lights with $|x\rangle$ and $|y\rangle$.

On the other hand, the half-wave plate is used to change the linearly polarization vectors of wave from $|x\rangle$ to $|y\rangle$, or from $|y\rangle$ to $|x\rangle$.

2. Role of retarder: quarter-wave plate and half-wave plate

The use of the quarter-wave plate and half-wave plate makes it possible to change the photon polarization state. Such changes in photon polarized state can be explained by using the concept of projection operators in quantum mechanics. Experimentally, noticeable phenomena occur when the angle between the optic axis ($|y'\rangle$) [see **Fig.1**] and the incident linearly polarized light (such as $|y\rangle$) is 45°. By using the quarter-wave plate, a circularly polarized wave with the polarization $|R\rangle$ can be generated from the linearly polarized photon ($|y\rangle$),.

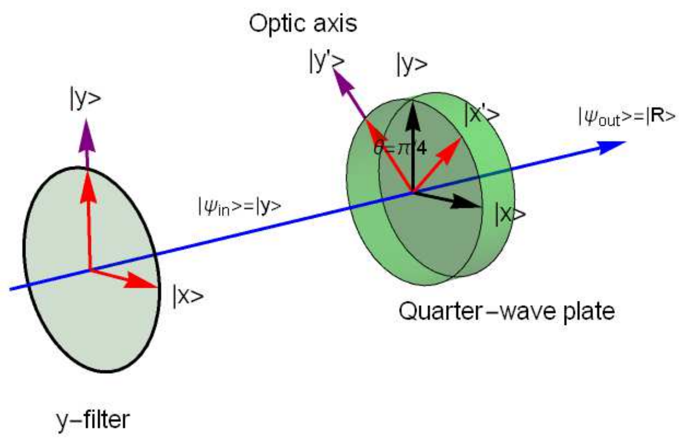


Fig.1 *y*-filter and quarter-wave plate (retarder) with an optic axis along the direction $|y'\rangle$. The input photon state is $|\psi_{in}\rangle = |y\rangle$. θ is the angle between $|y'\rangle$ and $|y\rangle$ on the top flat surface of the retarder. When $\theta = 45°$, the output photon state is $|\psi_{out}\rangle = |R\rangle$.

The half-wave plate shifts the path phase by $\lambda / 2$, but their main function is to rotate the linearly polarized light by twice the angle between the fast axis and a polarization vector. when the angle between the optic axis ($|y'\rangle$) [see **Fig. 2**] and the incident linearly polarized light (such as $|y\rangle$) is 45°, the linearly polarized wave ($|y\rangle$) is changed into another linearly polarized wave ($|x\rangle$).

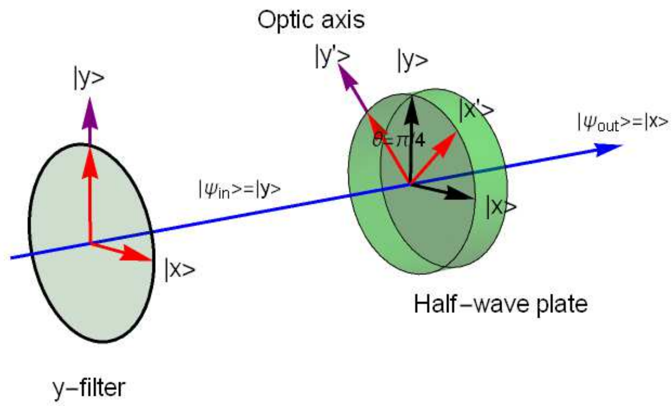


Fig.2 y-filter and half-wave plate (retarder) with an optic axis along the direction $\left|y'\right\rangle$. The input photon state is $\left|\psi_{in}\right\rangle = \left|y\right\rangle$. θ is the angle between $\left|y'\right\rangle$ and $\left|y\right\rangle$ on the top flat surface of the retarder. When $\theta = 45°$, the output photon state $\left|\psi_{out}\right\rangle = \left|x\right\rangle$.

((**Definition**))

$\left|R\right\rangle$: Right-hand circularly (RHC) polarized photon

$\left|L\right\rangle$: Left-hand circularly (LHC) polarized photon

$\left|x\right\rangle, \left|y\right\rangle$: Linearly polarized photons

3. Experimental demonstrations in Web sites

Before discussing on the role of the quarter-wave plate and half-wave plate in terms of quantum mechanics, it is very important for one to be familiar with the experimental demonstrations. Thanks to Web sites, one can see many instructive experimental demonstrations. During the preparation of this lecture note (Phys.421, Quantum Mechanics 1, Fall 2020), I watched an excellent videos (MIT Video Demonstrations in Lasers and Optics) on the quarter-wave plate, half-wave plate, and optical isolation. Prof. Shaoul Ezekiel (**Fig.3**) presented experimental demonstrations on the polarization of light, using a He-Ne laser (632.8 nm, red), polarization filter, quarter-wave plate, half-wave plate, and mirror. After watching videos repeatedly, I realize that the experimental demonstrations by Prof. Ezekiel are so useful to our understanding the properties of the polarization of light. So, in this note I will use several figures from the copy of the video by Prof. Ezekiel. Recently I notice that Prof. Ezekiel passed away (1935 – 2015).

Fig.3 Photograph of Prof. Shaoul Ezekiel (1935 – 2015), MIT.
https://www.rle.mit.edu/sezekiel/wp-content/themes/sezekiel_theme/images/sezekiel_header.jpg

I also find a very interesting video where Sir Lawrence Bragg (Nobel Prize laureate in physics, 1915) was checking the direction of the polarization vectors of two light rays (ordinary wave, *o*-wave, and extraordinary *e*-wave) in front of a large calcite crystal. I think that just like Sir Lawrence Bragg, students need to enjoy doing or watching experiments and learn about the beauty of Nature, before they try to understand (or learn) the physics (mainly quantum mechanics).

Fig.4 Sir Lawrence Bragg (1890 – 1971) who was doing an experiment of examining the direction of polarization vectors of *o*-wave and *e*-wave in a large calcite crystal, using the polaroid filter. *Reflection, Refraction and Polarization of light* (Sir Lawrence Bragg at the Royal Institution of Great Britain).
https://www.youtube.com/watch?v=7_zrpCkOBhU

I also find an amazing picture of calcite at the Wikipedia: birefringence. **Figure 5** clearly shows that the incident beam on the surface of retarder splits into the *o*-wave and *e*-wave. The polarization vector of the *e*-wave is parallel the optic axis, while the polarization vector of the *o*-wave is perpendicular to the optic axis.

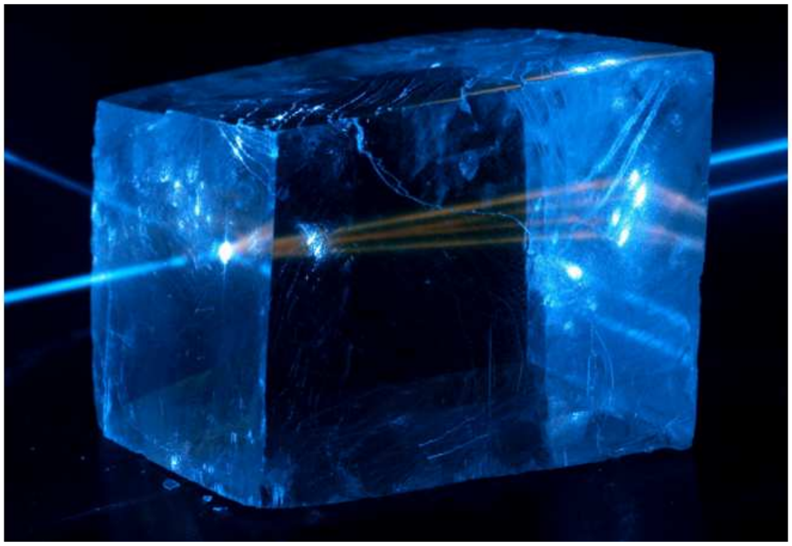


Fig.5 Birefringence in calcite crystal as laser beam splits into two rays (o-wave and e-wave) while travelling from left to light. (Wikipedia).
https://en.wikipedia.org/wiki/Birefringence#/media/File:Fluorescence_in_calcite.jpg

4. Retarder (retardation plate) for birefringence

A retarder plate is made by cutting a uniaxial birefringent crystal (such as calcite (a negative uniaxial crystal, so that the optic axis lies in the plane of the input surface of the plate. As a polarized input beam propagates in a direction normal to the top surface of the retarder. As the beam propagates through the crystal. The retarder plate modifies the polarization of the incident light because of the different propagation velocities, characterized by the two indices n_o and n_e. In other words, the input light beam can be resolved into an *o*-ray (ordinary wave) and *e*-ray (extra-ordinary wave). The polarization vector of the *e*-wave is parallel to the optic axis, while the polarization vector of the *o*-wave is perpendicular to the optic axis. These two polarization vectors lie in the top surface or retarder. If the retarder plate has a thickness *d*, the optical path difference between the two orthogonally polarized waves is

$$\pm(n_e - n_o)d = N\lambda ,$$

where *d* is the thickness and λ is the wavelength. The number *N* is called the retardation and is expressed in fractions of a wavelength. For example, $N = 1/4$ corresponds to a quarter-wave retardation and $N = 1/2$ corresponds to a half-wave retardation. The phase difference between the two orthogonally polarized waves generated by propagating through the retarder plate is simply 2π times the retardation.

The detail of the retarder and the optic axis is shown in **Fig, 6 - 9**. In **Fig.7**, we show how the retarder is constructed from the successive rotation of Euler angles (see the APPENDIX for detail).

As shown in **Figs.8 and 9** that the polarization vectors [$|x\rangle$, $|y\rangle$, $|x'\rangle$, $|y'\rangle$] lie on the retarder top flat surface.

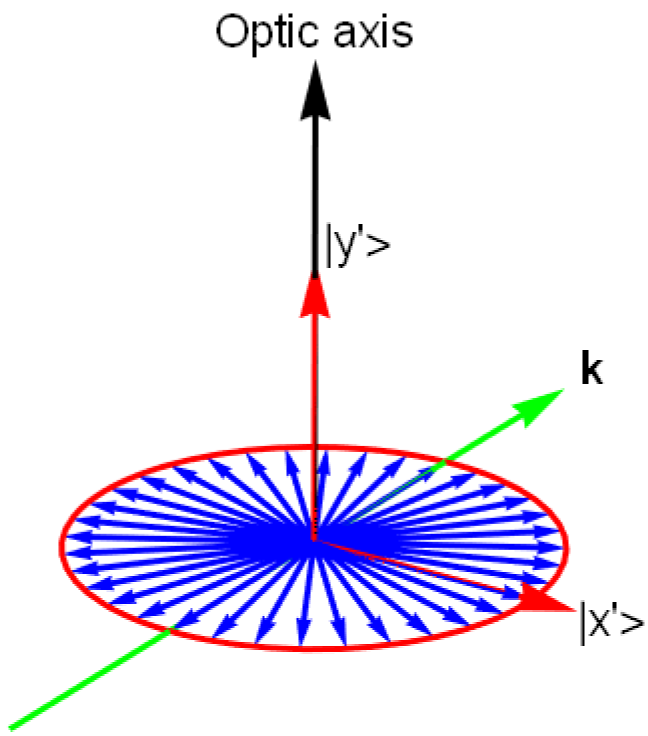


Fig.6 Definition of the retarder plane and the optic axis. The optic axis is perpendicular to the retarder plane. The polarization vector for the *o*-wave ($|x'\rangle$ lies in the retarder plane normal to the optic axis. The polarization vector for the *e*-wave ($|y'\rangle$ is parallel to the optic axis. The wave propagates along the wave vector ***k*** which is perpendicular to $|x'\rangle$ and $|y'\rangle$. The velocity of *e*-wave with the polarization vector $|y'\rangle$ is $v_e = c/n_e$ (fast axis), while the velocity of *o*-wave with the polarization vector $|x'\rangle$ is $v_o = c/n_o$ (slow axis). $n_0 = 1.6583$ and $n_e = 1.4864$ for calcite (negative uniaxial crystal).

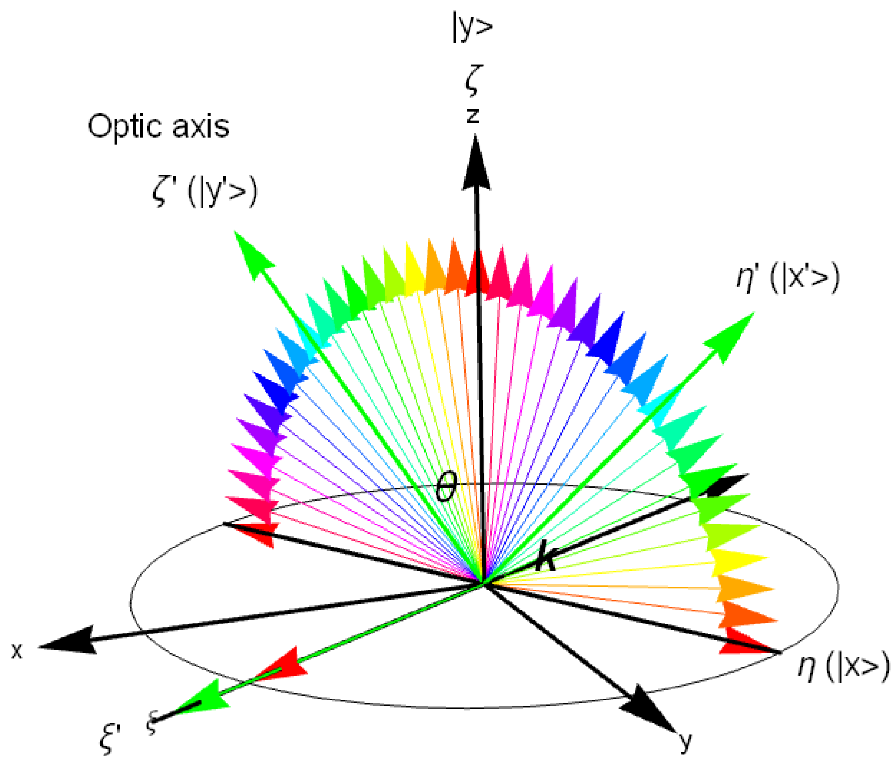


Fig.7 Retarder (retardation plate), which is constructed from the successive rotation of Euler angles (see the APPENDIX for detail). The polarization vectors [$|x\rangle$, $|y\rangle$, $|x'\rangle$, $|y'\rangle$] lie on the retarder (retardation plate).

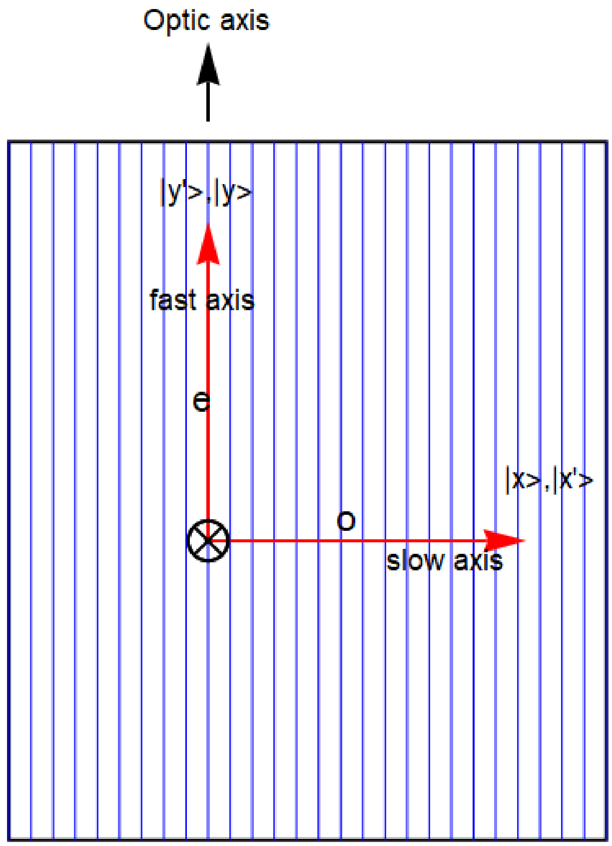


Fig.8 Retarder before rotation. $|x'\rangle = |x\rangle$. $|y'\rangle = |y\rangle$. The direction of the wave vector $\boldsymbol{k}$ is into the page.

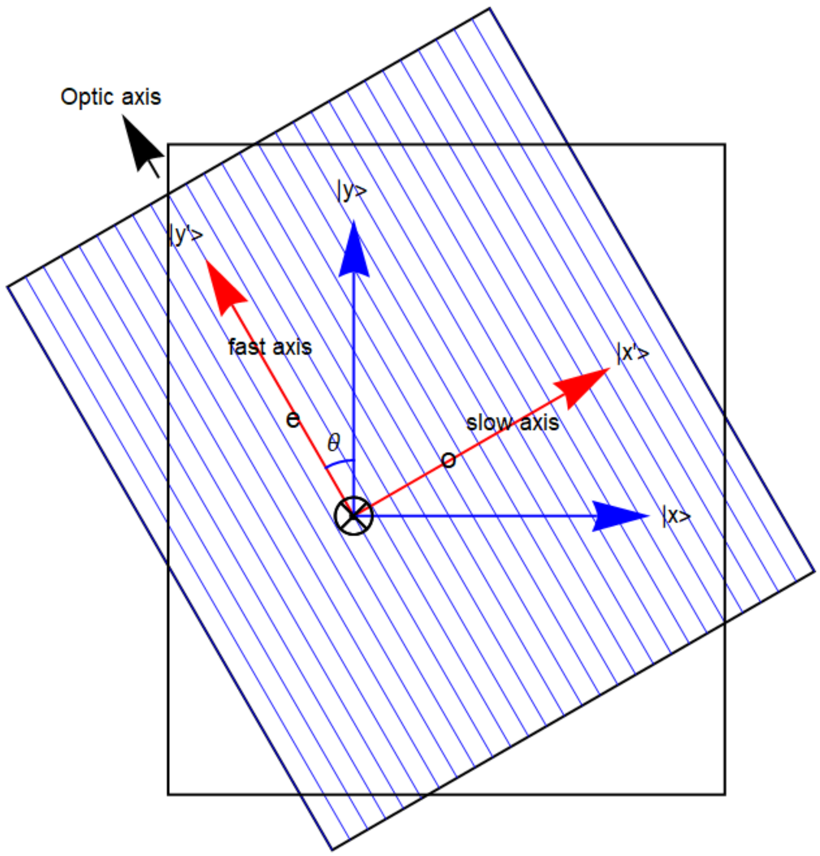
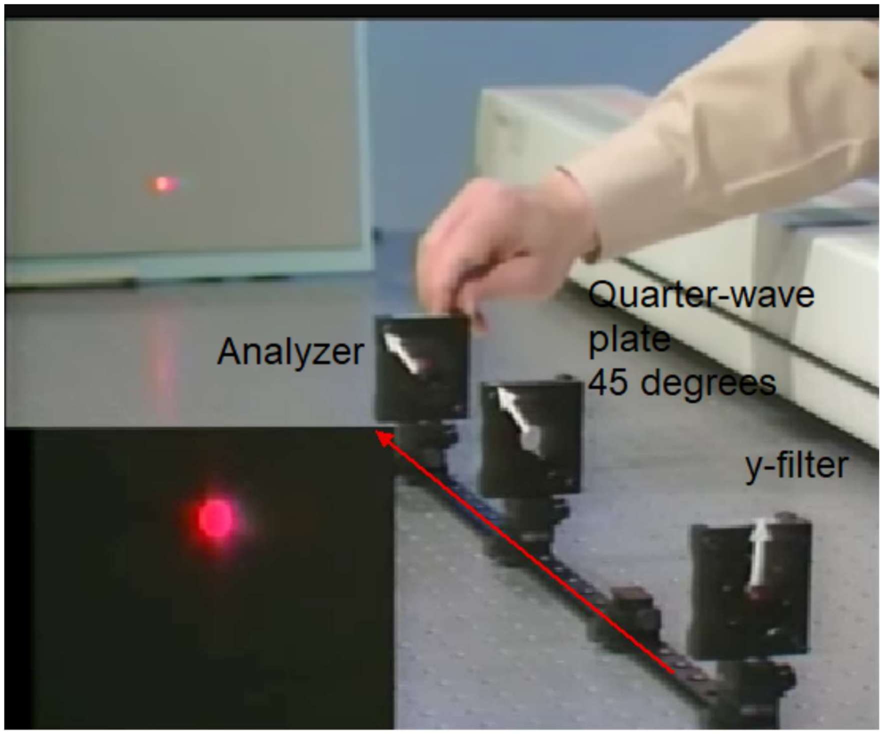


Fig.9 The retarder after the rotation. The optic axis is rotated around the origin by the angle θ. The directions of the fast axis and slow axis are shown for calcite (negative uniaxial crystal, $n_o > n_e$). The direction of propagating waves is into the page.

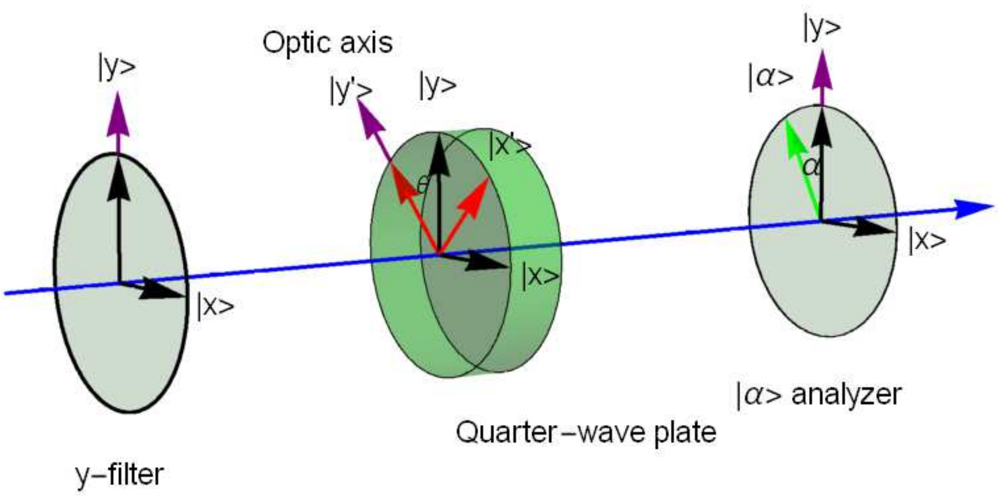
5. Experimental demonstrations by Prof. Ezekiel

(a) Quarter-wave plate

Here is the copy of the video demonstration for the quarter-wave plate (Prof. Shaoul Ezekiel). The experimental set-up is shown in **Fig.10 (a) and (b).** When the angle for the quarter-wave plate is $\theta = 45°$, the input light beam with the polarization vector $|y\rangle$ is changed into the right-hand circular light beam with the polarization vector $|R\rangle$. It is also experimentally demonstrated that that the output observed intensity by the analyzer is independent of the rotation angle α of the analyzer.



(a)



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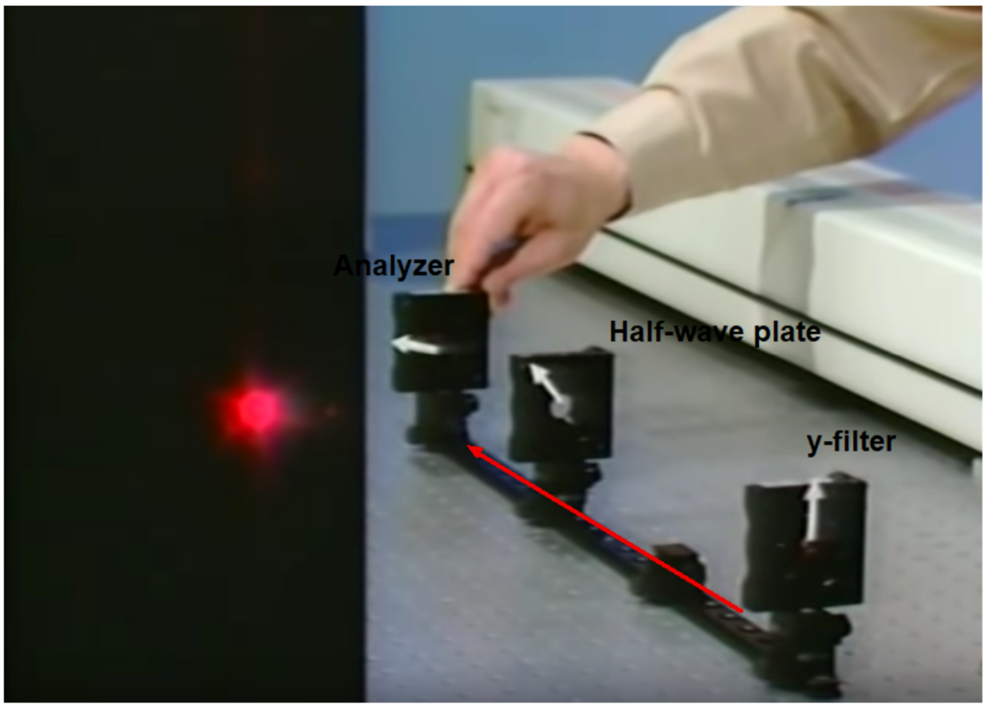
(b)

Fig.10(a) (b) Experiment of quarter-wave plate (by Prof. Ezekiel). The *y*-filter – quarter-wave plate – analyzer. When $\theta = 45°$ for the quarter-wave plate, one can generate the circular polarized wave. The intensity after the analyzer is independent of the polarization of the analyzer. In other words, even if one rotates the analyzer, the intensity does not change.

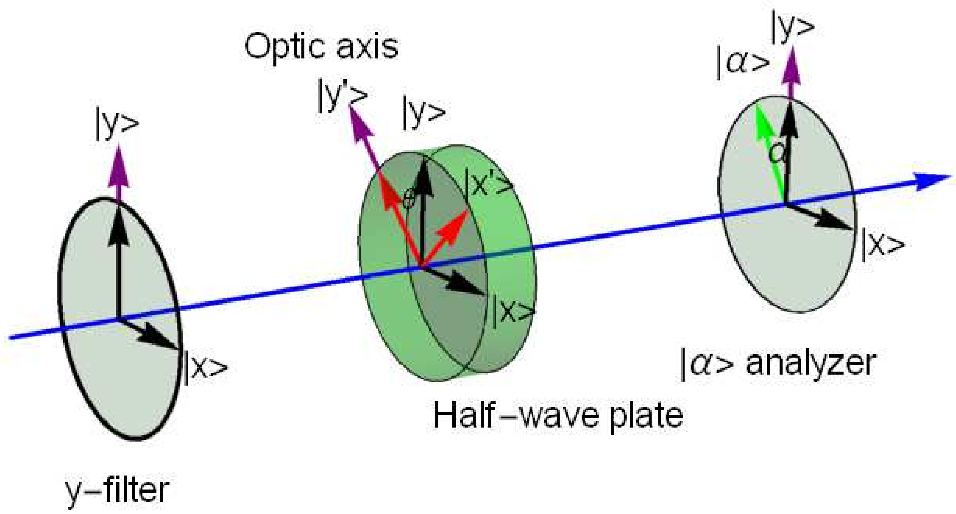
[Quarter-wave plate/MIT Video Demonstrations in lasers and optics]
https://www.youtube.com/watch?v=7_zrpCkOBhU

(b) Half-wave plate

Here is the copy of the video demonstration for the half-wave plate (Prof. Ezekiel). The experimental set-up is shown in **Fig.11(a) and (b).** When the angle is $\theta = 45°$, the input light beam with the polarization vector $|y\rangle$ is changed into the linearly polarized light beam with $|x\rangle$ by the half-wave plate. When the analyzer is set up with $\alpha = 2\theta = 90°$, we can see the light beam with $|x\rangle$.



(a)



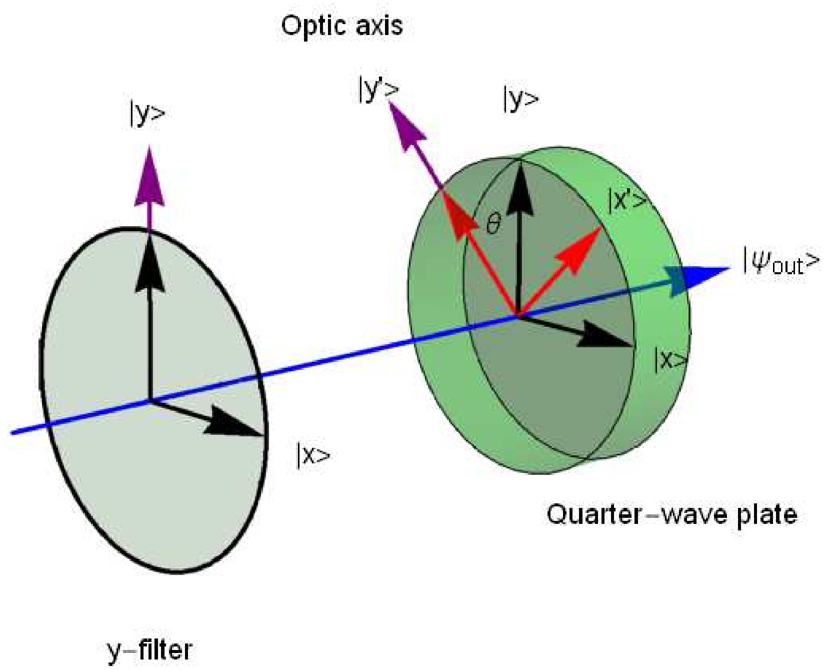
(b)

Fig.11 (a) (b) Experiment of half-wave plate (by Prof. Ezekiel). Experimental set-up. *y*-filter, half-wave plate, and $\left|\alpha\right\rangle$-filter. The input photon state is $\left|y\right\rangle$. When $\theta = 45°$ for the half-wave plate, one can find the change of state from $\left|y\right\rangle$ to $\left|x\right\rangle$ (with $2\theta = 90°$).

Optics: Half-wave plate | MIT Video Demonstrations in Lasers and Optics
https://www.youtube.com/watch?v=_sUVXHfUVsY

3. Projection operator of quarter-wave plate (quantum mechanics)

Here we discuss the role of the quarter-wave and half-wave plates for the photon polarization in quantum mechanics.



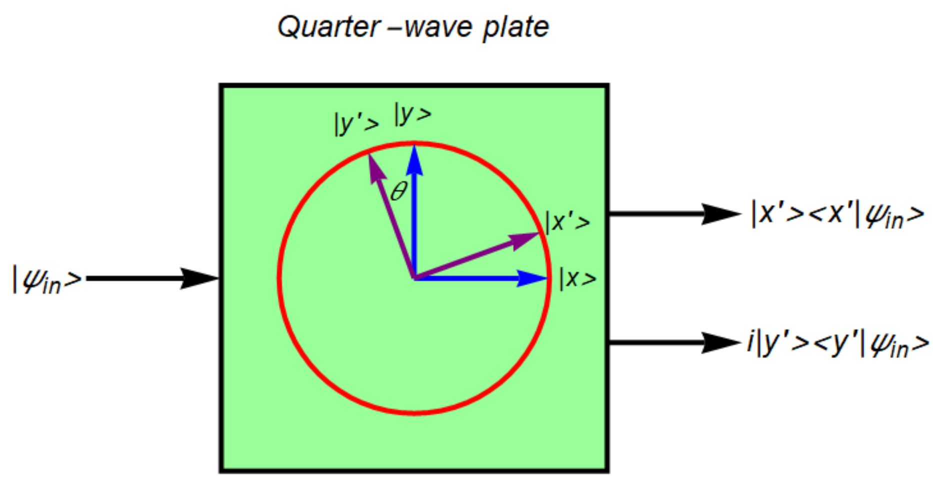


Fig.12 Projection operator for the quarter-wave plate. The input photon state $|\psi_{in}\rangle = |x\rangle$. The output photon state is given by $|\psi_{out}\rangle = |x'\rangle\langle x'|\psi_{in}\rangle + i|y'\rangle\langle y'|\psi_{in}\rangle$ using the projection operator for the quarter-wave plate.

We start with the $|x'\rangle$ and $|y'\rangle$ as

$$|x'\rangle = \hat{U}_\theta|x\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle, \quad |y'\rangle = \hat{U}_\theta|y\rangle = -\sin\theta|x\rangle + \cos\theta|y\rangle,$$

where $\hat{U}_\theta$ is the unitary operator defined by

$$\hat{U}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Case (a): Input state, $|\psi_{in}\rangle = |x\rangle$

Suppose that the input photon state is given by $|\psi_{in}\rangle = |x\rangle$. Thus, we have the output photon state as

$$\begin{aligned}
|\psi_{out}\rangle &= |x'\rangle\langle x'|\psi_{in}\rangle + i|y'\rangle\langle y'|\psi_{in}\rangle \\
&= \cos\theta|x'\rangle - i\sin\theta|y'\rangle \\
&= \cos\theta(\cos\theta|x\rangle + \sin\theta|y\rangle) - i\sin\theta(-\sin\theta|x\rangle + \cos\theta|y\rangle) \\
&= (\cos^2\theta + i\sin^2\theta)|x\rangle + \sin\theta\cos\theta(1-i)|y\rangle
\end{aligned}$$

For simplicity, we introduce an operator $\hat{A}_Q(\theta)$ defined by

$$\begin{aligned}
\hat{A}_Q(\theta) &= |x'\rangle\langle x'| + i|y'\rangle\langle y'| \\
&= \hat{U}_\theta(|x\rangle\langle x| + i|y\rangle\langle y|)\hat{U}_\theta^+ \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & i\cos^2\theta + \sin^2\theta \end{pmatrix}
\end{aligned}$$

where the Hermitian conjugate of $\hat{U}_\theta$ is

$$\hat{U}_\theta^+ = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus, we have the output photon state as

$$\hat{A}_Q(\theta)|x\rangle = \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & i\cos^2\theta + \sin^2\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + i\sin^2\theta \\ (1-i)\sin\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(1+\cos 2\theta + i\frac{1}{2}(1-\cos 2\theta) \\ (1-i)\frac{1}{2}\sin 2\theta \end{pmatrix}$$

where

$$\hat{A}_Q(\theta+\pi)|x\rangle = \hat{A}_Q(\theta)|x\rangle.$$

which has a periodicity of π. Using the above formula, we have

$$\hat{A}_Q(\theta = -\frac{\pi}{4})|x\rangle = -\frac{1}{2}\begin{pmatrix} 1-i \\ -1+i \end{pmatrix} = -e^{-i\pi/4}|L\rangle,$$

$$\hat{A}_Q(\theta = 0)|x\rangle = |x\rangle,$$

$$\hat{A}_Q(\theta = \frac{\pi}{4})|x\rangle = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} = e^{i\pi/4}|L\rangle,$$

$$\hat{A}_Q(\theta = \frac{\pi}{2})|x\rangle = i|x\rangle,$$

$$\hat{A}_Q(\theta = \frac{3\pi}{4})|x\rangle = e^{i\pi/4}|R\rangle,$$

$$\hat{A}_Q(\theta = \pi)|x\rangle = |x\rangle,$$

$$\hat{A}_Q(\theta = \frac{5\pi}{4})|x\rangle = e^{i\pi/4}|L\rangle,$$

$$\hat{A}_Q(\theta = \frac{3\pi}{2})|x\rangle = i|x\rangle,$$

$$\hat{A}_Q(\theta = \frac{7\pi}{4})|x\rangle = e^{i\pi/4}|R\rangle,$$

$$\hat{A}_Q(\theta = 2\pi)|x\rangle = |x\rangle,$$

where

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad |L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

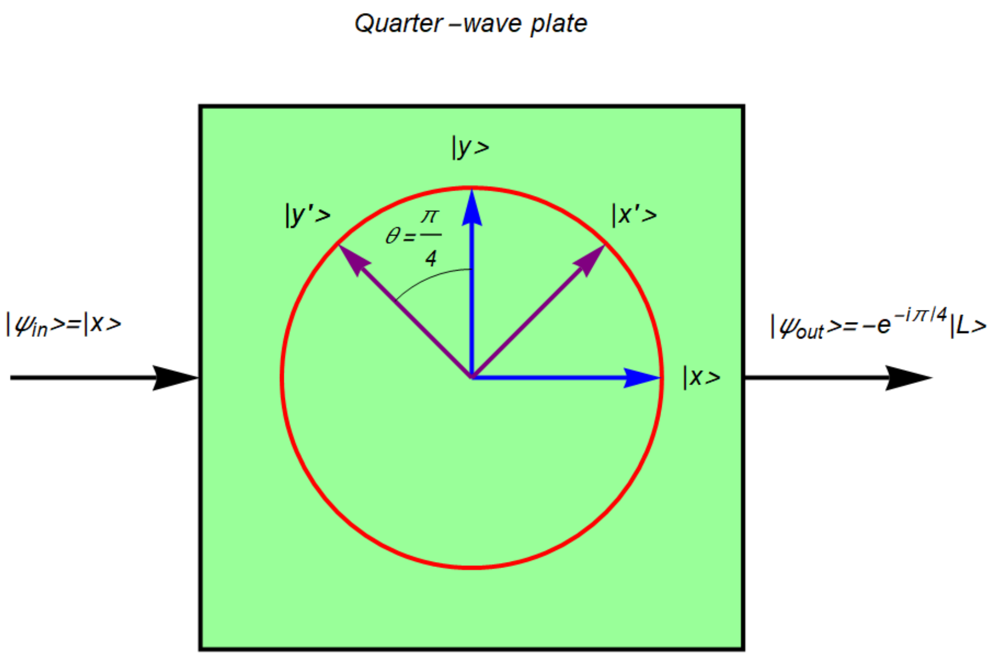


Fig.13 Quarter-wave plate with $\theta = 45°$. The input photon state is $|\psi_{in}\rangle = |x\rangle$. The output photon state is $|\psi_{out}\rangle = -e^{-i\pi/4}|L\rangle$.

We note that when $\theta = 45°$ and , we have

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}[|x'\rangle - i|y'\rangle] = |L'\rangle = e^{i\pi/4}|L\rangle \qquad \textbf{(Feynman)}$$

See the comment of Feynman on the phase factor for $|L'\rangle = e^{i\pi/4}|L\rangle$ in **Section 11.4** (The polarization states of photon under the rotation).

Case (b) $|\psi_{in}\rangle = |y\rangle$

When the input photon state is given by $|\psi_{in}\rangle = |y\rangle$, we have the output photon state as

$$\hat{A}_Q(\theta)|y\rangle = \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & i\cos^2\theta + \sin^2\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} (1-i)\sin\theta\cos\theta \\ i\cos^2\theta + \sin^2\theta \end{pmatrix}$$

where

$$\hat{A}_Q(\theta+\pi)|y\rangle = \hat{A}_Q(\theta)|y\rangle$$

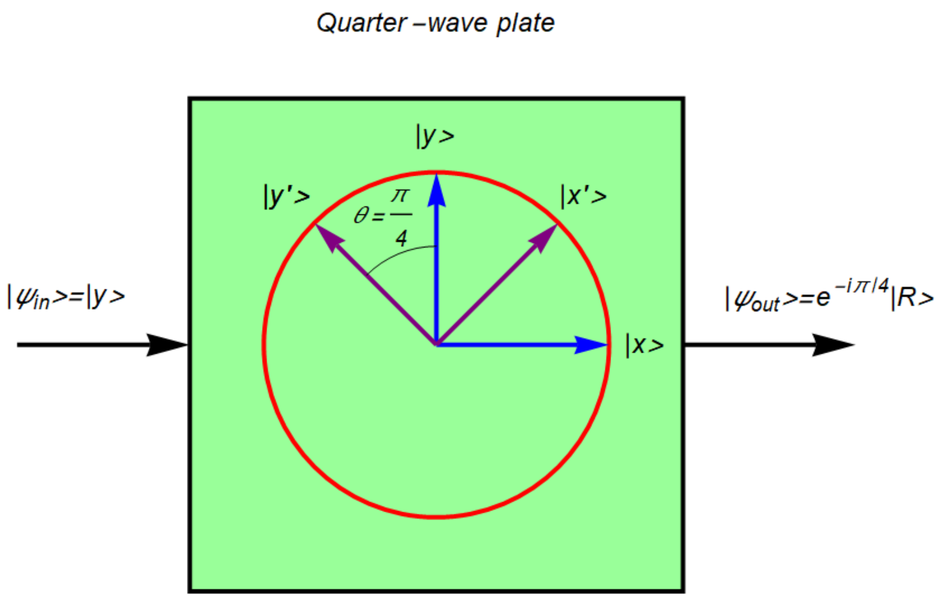


Fig.14 Quarter-wave plate with $\theta = 45°$. The input photon state $|\psi_{in}\rangle = |y\rangle$. The output photon state $|\psi_{out}\rangle = |R'\rangle = e^{-i\pi/4}|R\rangle$.

We note that

$$\hat{A}_Q(\theta = 0)|y\rangle = i|y\rangle,$$

$$\hat{A}_Q(\theta = \frac{\pi}{4})|y\rangle = e^{-i\pi/4}|R\rangle,$$

$$\hat{A}_Q(\theta = \frac{\pi}{2})|y\rangle = |y\rangle,$$

$$\hat{A}_Q(\theta = \frac{3\pi}{4})|y\rangle = e^{i\pi/4}|R\rangle,$$

$$\hat{A}_Q(\theta = \pi)|y\rangle = i|y\rangle,$$

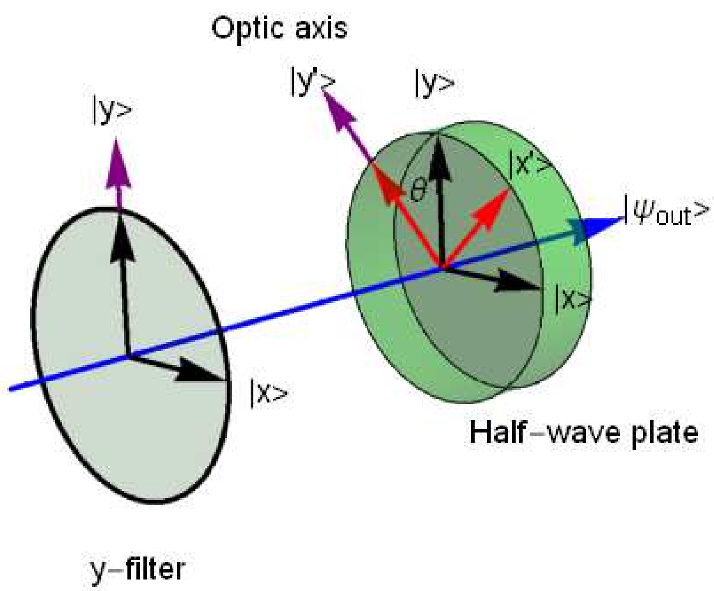
$$\hat{A}_Q(\theta = \frac{5\pi}{4})|y\rangle = e^{-i\pi/4}|R\rangle,$$

$$\hat{A}_Q(\theta = \frac{3\pi}{2})|y\rangle = |y\rangle,$$

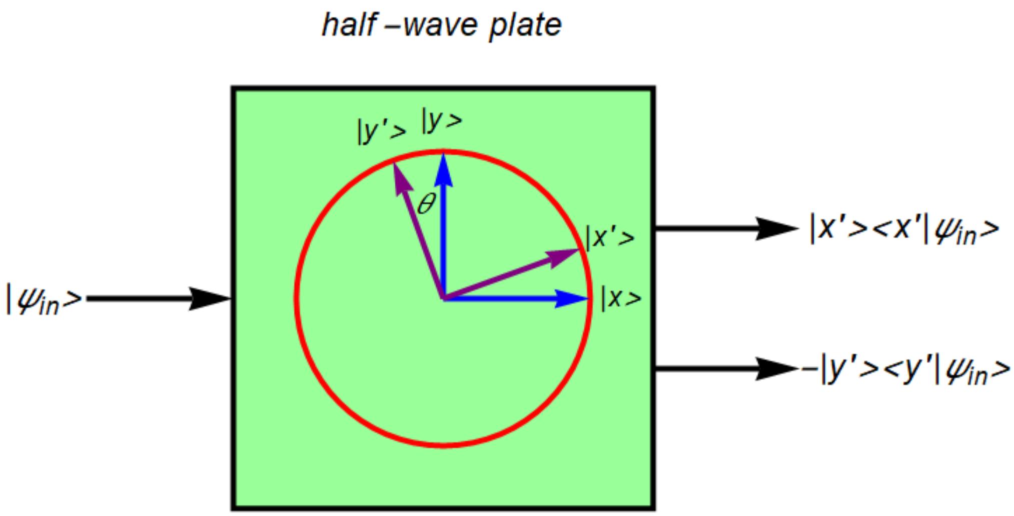
$$\hat{A}_Q(\theta = \frac{7\pi}{4})|y\rangle = e^{i\pi/4}|R\rangle,$$

$$\hat{A}_Q(\theta = 2\pi)|y\rangle = i|y\rangle.$$

5. Projection operator for the Halfwave plate (quantum mechanics)



(a)



(b)

Fig.15 (a) (b) Projection operator for half-wave plate. The initial photon state $|\psi_{in}\rangle$ and the output photon state. $|\psi_{out}\rangle = |x'\rangle\langle x'|\psi_{in}\rangle - |y'\rangle\langle y'|\psi_{in}\rangle$.

For the half-wave plate, we define the projection operator by

$$\begin{aligned}
\hat{A}_H(\theta) &= |x'\rangle\langle x'| - |y'\rangle\langle y'| \\
&= \hat{U}_\theta(|x\rangle\langle x| - i|y\rangle\langle y|)\hat{U}_\theta^{+} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\cos^2\theta + \sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}
\end{aligned}$$

which is different from the operator for the quarter-wave plate.

(b) $|\psi_i\rangle = |x\rangle$

Suppose that the input photon state is $|\psi_{in}\rangle = |x\rangle$. Then, we get the output photon state as

$$|\psi_{out}\rangle = \hat{A}_H(\theta)|x\rangle = \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}.$$

The probability amplitude after the α-filter as

$$\langle \alpha | \psi_{out} \rangle = \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$$
$$= \cos(2\theta - \alpha)$$

We note that

$$|\psi_{out}\rangle = \hat{A}_H(\theta = 0)|x\rangle = |x\rangle,$$

$$|\psi_{out}\rangle = \hat{A}_H(\theta = \pi/4)|x\rangle = |y\rangle,$$

$$|\psi_{out}\rangle = \hat{A}_H(\theta = \pi/2)|x\rangle = -|x\rangle,$$

$$|\psi_{out}\rangle = \hat{A}_H(\theta = 3\pi/4)|x\rangle = -|y\rangle,$$

$$|\psi_{out}\rangle = \hat{A}_H(\theta = \pi)|x\rangle = |x\rangle.$$

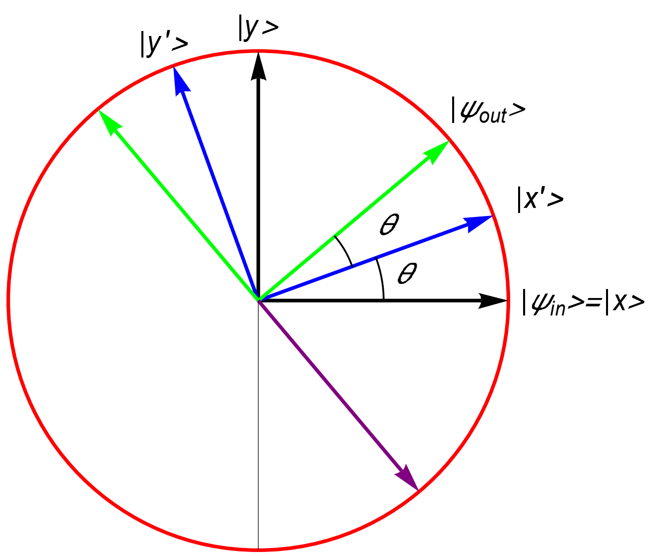


Fig.16 The relation between the input photon state $|\psi_{in}\rangle = |x\rangle$ and the input photon state $|\psi_{out}\rangle = \hat{A}_H(\theta)|x\rangle$ for the half-wave plate.

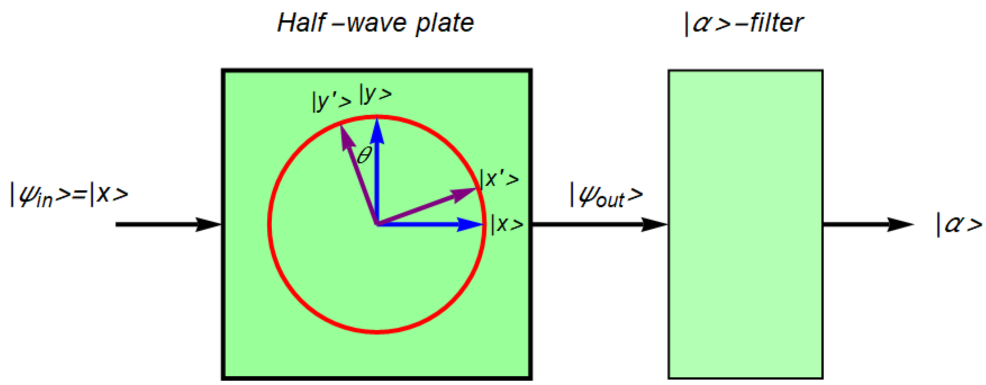


Fig.17 Half-wave plate. The input photon state $|\psi_{in}\rangle = |x\rangle$. The probability amplitude is $\langle\alpha|\psi_{out}\rangle$. The probability: $P = |\langle\alpha|\psi_{out}\rangle|^2 = \cos^2(2\theta - \alpha)$.

(b) $|\psi_i\rangle = |y\rangle$

For the input photon state $|\psi_{in}\rangle = |y\rangle$, the output photon state is given by

$$|\psi_{out}\rangle = \hat{A}_H(\theta)|y\rangle = \begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}.$$

The probability amplitude:

$$\begin{aligned} \langle\alpha|\psi_{out}\rangle &= \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix} \\ &= \sin(2\theta - \alpha) \end{aligned}$$

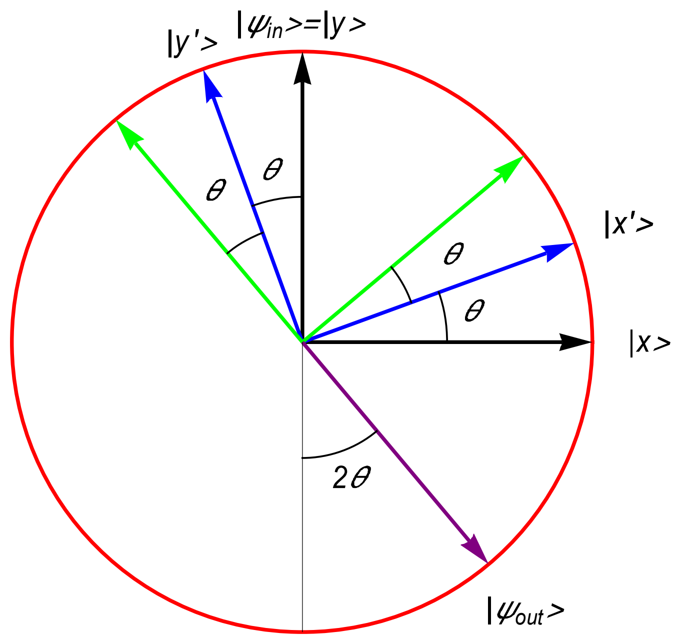


Fig.18 The relation between the input photon state $|\psi_{in}\rangle = |y\rangle$ and the input photon state $|\psi_{out}\rangle = \hat{A}_H(\theta)|y\rangle$ for the half-wave plate.

The probability amplitude:

$$P = |\langle\alpha|\psi_{out}\rangle|^2 = \sin^2(2\theta - \alpha).$$

The output photon state:

$$|\psi_{out}\rangle = \hat{A}_H(\theta)|y\rangle = \begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}.$$

For typical values of θ, we have

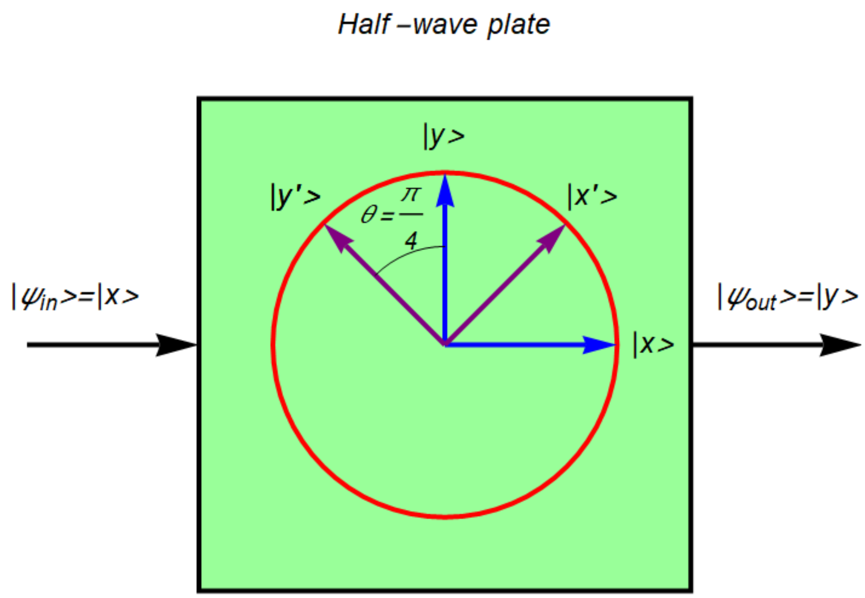
$$|\psi_{out}\rangle = \hat{A}_H(\theta = 0)|y\rangle = -|y\rangle,$$

$$|\psi_{out}\rangle = \hat{A}_H(\theta = \frac{\pi}{4})|y\rangle = |x\rangle,$$

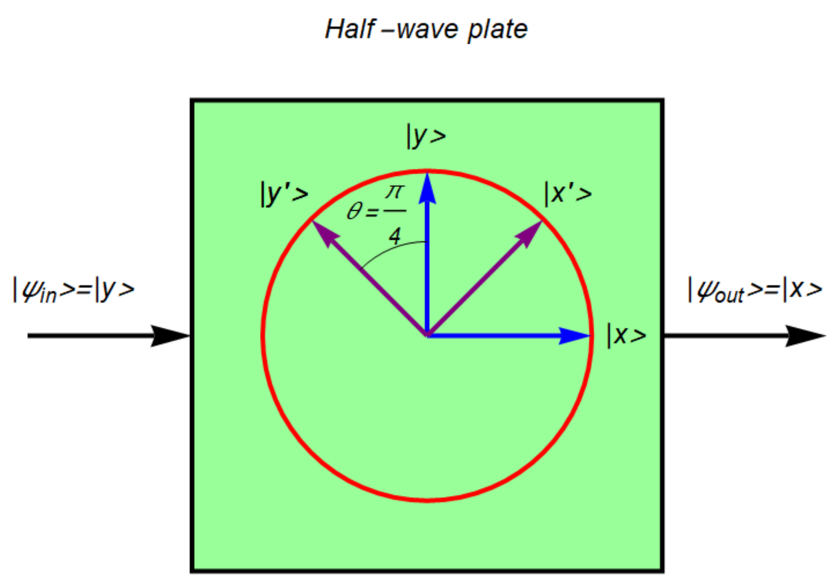
$$|\psi_{out}\rangle = \hat{A}_H(\theta = \frac{\pi}{2})|y\rangle = |y\rangle,$$

$$\left|\psi_{out}\right\rangle = \hat{A}_H(\theta = \frac{3\pi}{4})\left|y\right\rangle = -\left|x\right\rangle,$$

$$\left|\psi_{out}\right\rangle = \hat{A}_H(\theta = \pi)\left|y\right\rangle = -\left|y\right\rangle.$$



(a)



(b)

Fig.19 The output photon states for half-wave plate with $\theta = 45°$. (a) $|\psi_{in}\rangle = |x\rangle$ and $|\psi_{out}\rangle = |y\rangle$. (b) $|\psi_{in}\rangle = |y\rangle$ and $|\psi_{out}\rangle = |x\rangle$.

6. Summary: Projection operators for quarter-wave plate and half-wave plate

Here we summarize the above results for the projection operators for quarter-wave plate and half-wave plate, when the input states are given as $|\psi_{in}\rangle = |x\rangle$ or $|y\rangle$.

(a) $|\psi_{in}\rangle = |x\rangle$

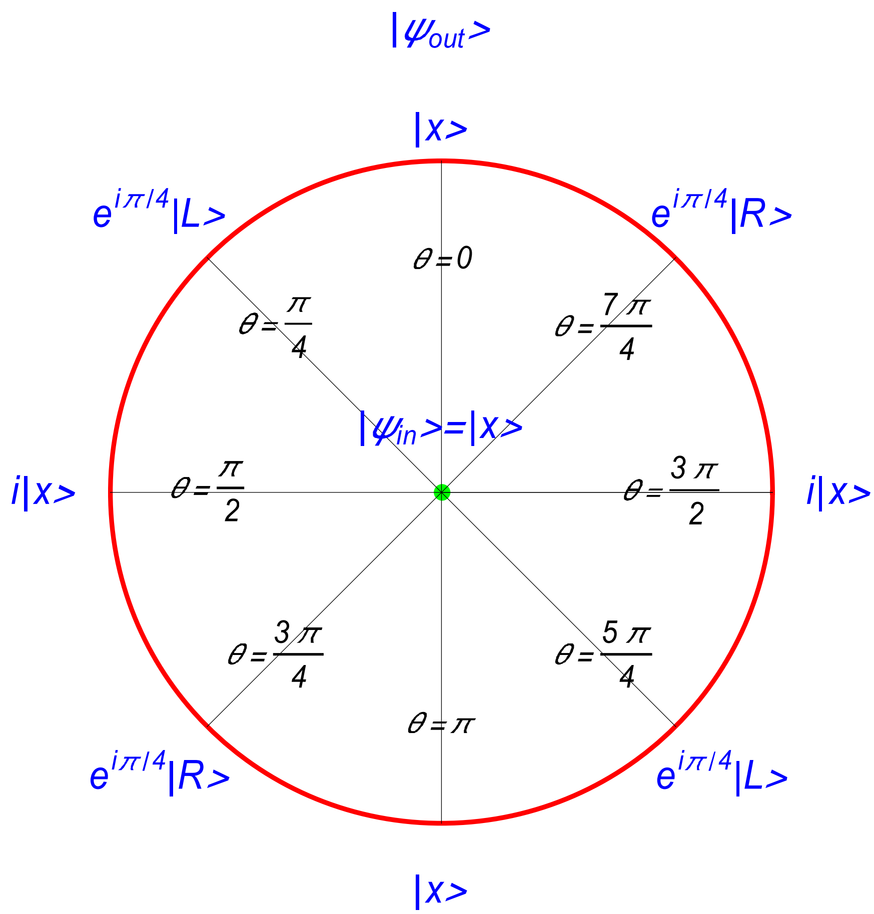


Fig.20 Quarter-wave plate. θ is an angle between the y axis (the vertical axis) and the optic axis. $|\psi_{in}\rangle = |x\rangle$. $|\psi_{out}\rangle = \hat{A}_Q(\theta)|x\rangle$.

(b) $|\psi_{in}\rangle = |y\rangle$

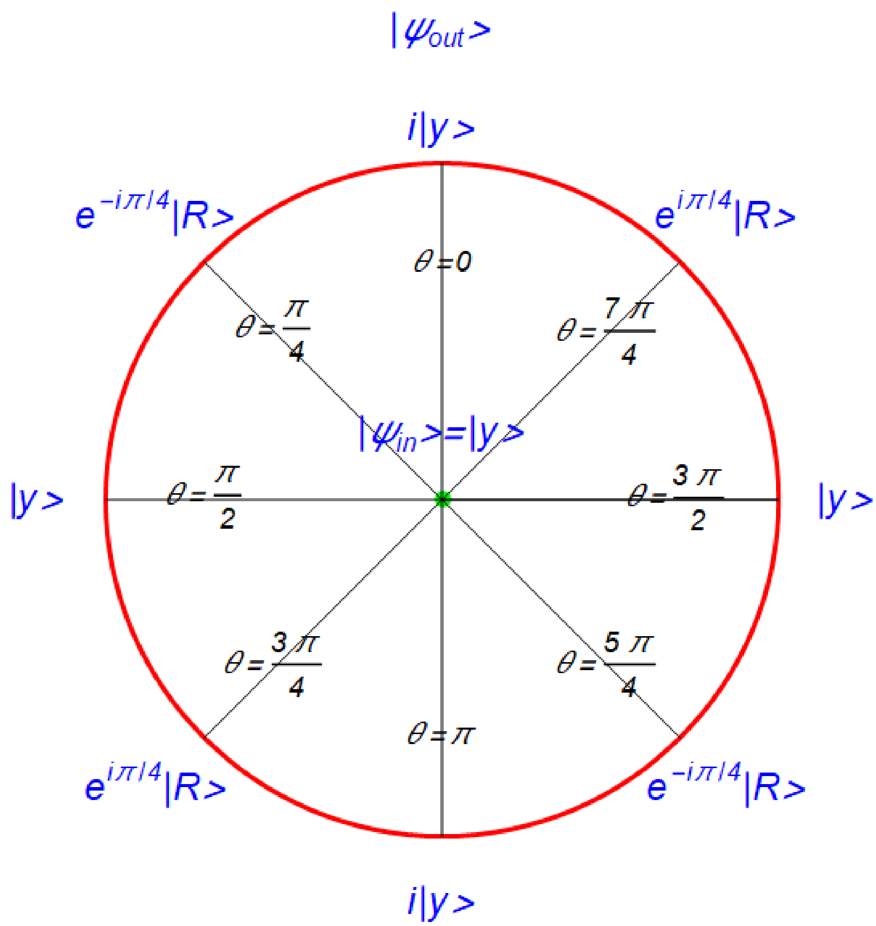


Fig.21 Quarter wave plate. θ is an angle between the y axis (the vertical axis) and the optic axis. $|\psi_{in}\rangle = |y\rangle$. $|\psi_{out}\rangle = \hat{A}_Q(\theta)|y\rangle$.

7. Summary: Quantum mechanics: Projection operators for half-wave plate

Here we summarize the above results for the projection operators for half-wave plate, when the input states are given as $|\psi_{in}\rangle = |x\rangle$ or $|y\rangle$.

(a) $|\psi_{in}\rangle = |x\rangle$

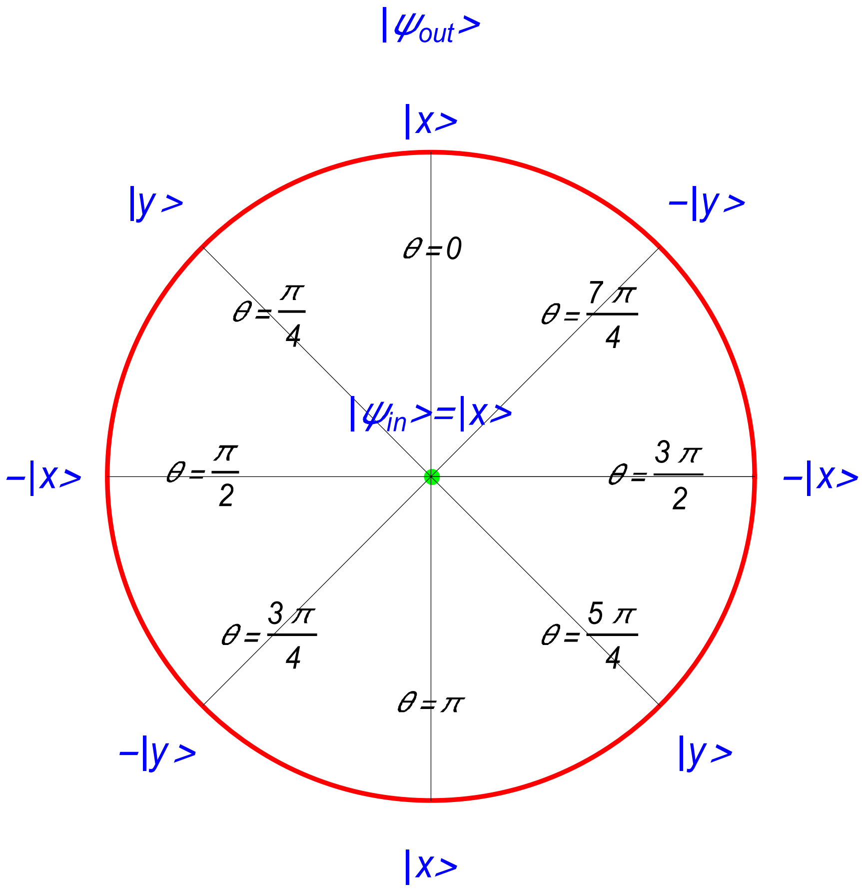


Fig.22 Half-wave plate. θ is an angle between the y axis (the vertical axis) and the optic axis. $|\psi_{in}\rangle = |x\rangle$. $|\psi_{out}\rangle = \hat{A}_H(\theta)|x\rangle$.

(b) $|\psi_{in}\rangle = |y\rangle$

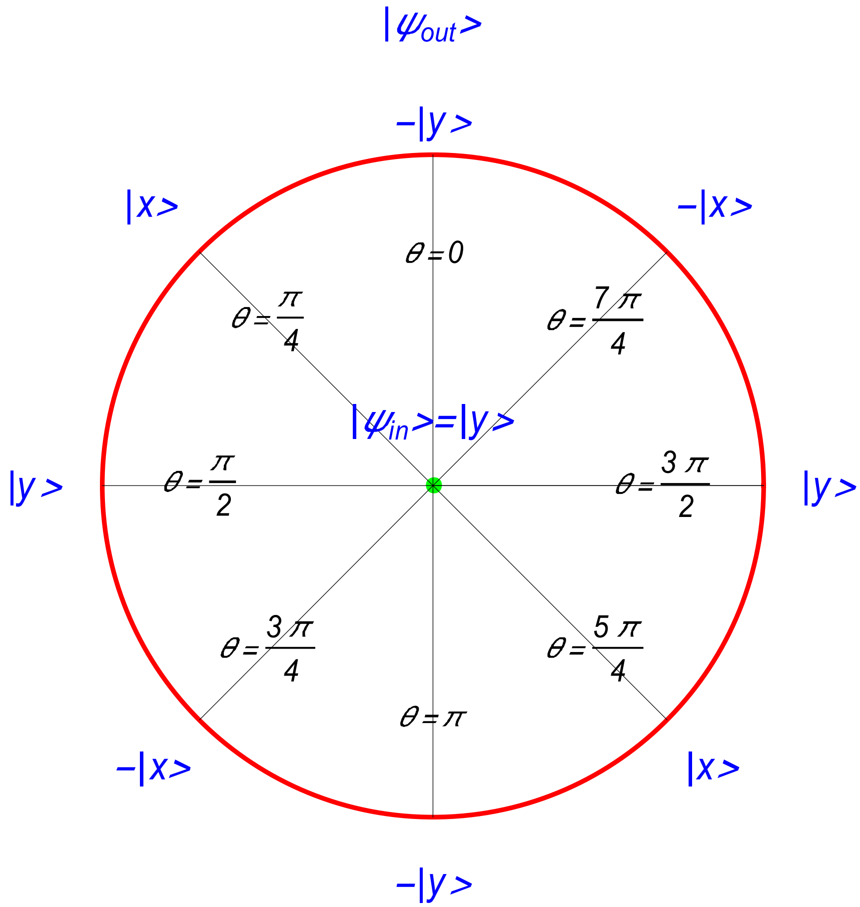


Fig.23 Half-wave plate. θ is an angle between the y axis (the vertical axis) and the optic axis. $|\psi_{in}\rangle = |y\rangle$. $|\psi_{out}\rangle = \hat{A}_H(\theta)|y\rangle$.

7. Intensity of analyzer after quarter-wave plate

We now consider the probability of finding the system in the state

$$|\alpha\rangle = -\sin\alpha|x\rangle + \cos\alpha|y\rangle,$$

when $|\psi_{in}\rangle = |y\rangle$ and the quarter-wave plate with the angle θ.

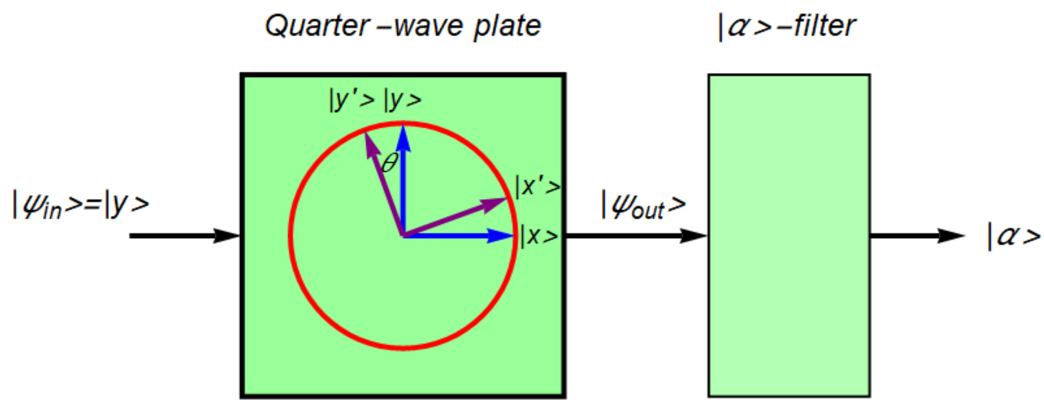


Fig.24 Probability $P=\left|\langle\alpha|\psi_{out}\rangle\right|^2=\left|\langle\alpha|\hat{A}_Q(\theta)|y\rangle\right|^2$. $|\alpha\rangle=-\sin\alpha|x\rangle+\cos\alpha|y\rangle$.

The probability is given by

$$P=\left|\langle\alpha|\psi_{out}\rangle\right|^2=\left|\langle\alpha|\hat{A}_Q(\theta)|y\rangle\right|^2.$$

When $|\psi_{in}\rangle=|y\rangle$, we have

$$\begin{aligned}|\psi_{out}\rangle&=|x'\rangle\langle x'|\psi_{in}\rangle+i|y'\rangle\langle y'|\psi_{in}\rangle\\&=\sin\theta|x'\rangle+i\cos\theta|y'\rangle\\&=(1-i)\sin\theta\cos\theta|x\rangle+(\sin^2\theta+i\cos^2\theta)|y\rangle\end{aligned}$$

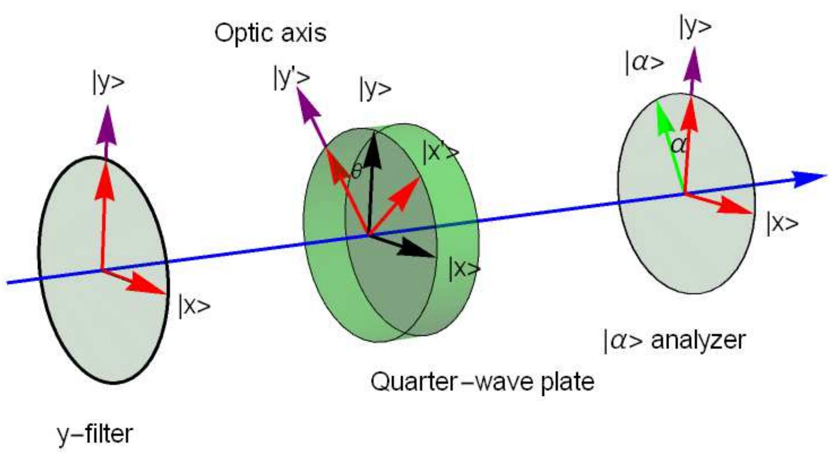


Fig.25 Experimental configuration of the y-filter, quarter-wave plate and $|\alpha\rangle$-analyzer. $|\psi_{in}\rangle = |y\rangle$. Determination of the intensity (probability) $|\langle\alpha|\psi_{out}\rangle|^2$ by the analyzer, where the angle θ (angle between the optic axis and the z axis). The angle α is measured from the z-axis.

We now consider the case of $|\psi_{in}\rangle = |y\rangle$. Using the analyzer after the quarter-wave plate, one can evaluate the probability $|\langle\alpha|\psi_{out}\rangle|^2$ as a function of the angle α when θ is changed as a parameter. The results are as follows.

(i) For $0 \le \theta < 45°$, The maximum of the probability shifts to side of increasing α, while the maximum value decreases from 1 to 0.5, with increasing α.

(ii) For $\theta = 45°$, the probability remains constant (=0.5), independent of the angle α. This means that the state of wave changes from the initial state $|\psi_{in}\rangle = |y\rangle$ to the final state $|\psi_{out}\rangle = |R\rangle$ (right-hand circularly polarized photon).

(iii) For $45° \le \theta < 90°$, the minimum of probability shifts to side of increasing α, while the minimum value decreases from 0.5 to 0, with increasing α.

(iv) For $\theta = 90°$, the probability becomes zero at $\alpha = 90°$. This means that the state remains unchanged; $|\psi_{in}\rangle = |y\rangle$ and $|\psi_{out}\rangle = |y\rangle$.

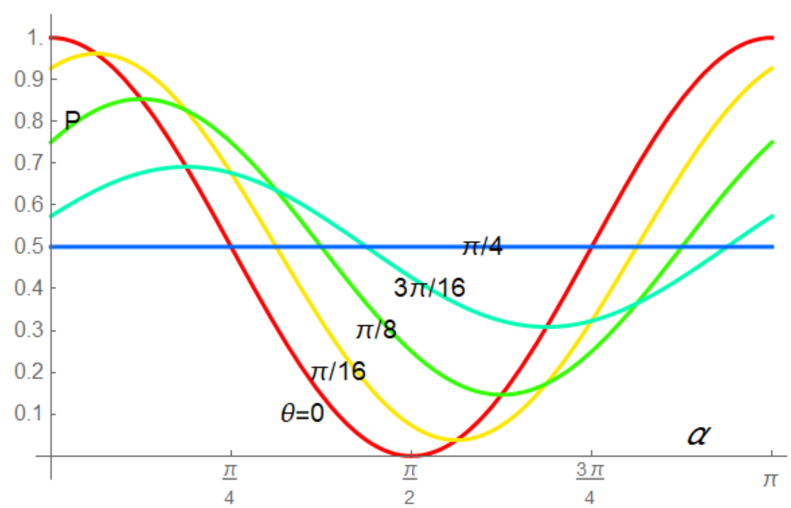


Fig.26 The quarter-wave plate. The probability $P = |\langle\alpha|\psi_{out}\rangle|^2$ aa a function of α. θ is the angle between the optic axis and $|y\rangle$. $0 \le \theta \le 45°$. Circularly polarized wave emerges for $\theta = 45°$. The probability is 1/2, independent of α.

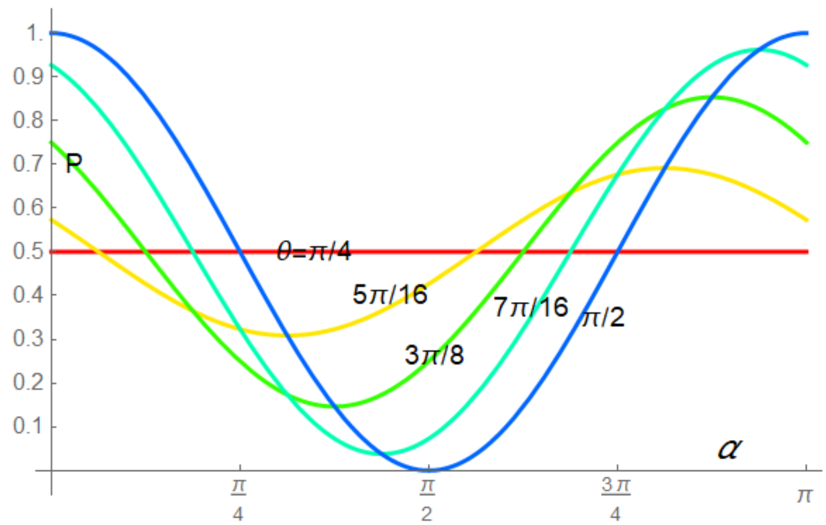


Fig.27 The quarter-wave plate. The probability $P=\left|\langle\alpha|\psi_{out}\rangle\right|^2$ as a function of angle α. θ is the angle between the optic axis and $|y\rangle$. $45\le\theta\le 90°$.

8. Intensity of analyzer after half-wave plate

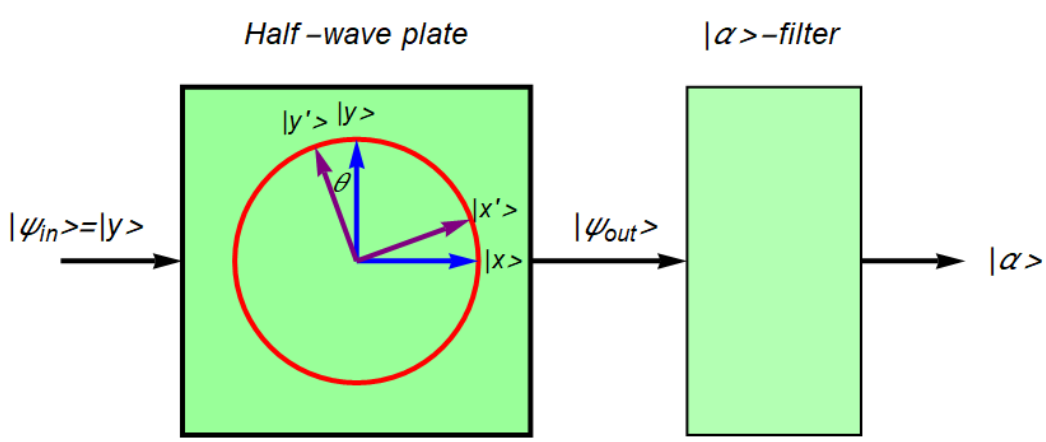


Fig.28 Probability for the half-wave plate with $|\alpha\rangle$ filter and $|\psi_{in}\rangle=|y\rangle$. The probability $P=\left|\langle\alpha|\psi_{out}\rangle\right|^2=\sin^2(2\theta-\alpha)$. $|\alpha\rangle=-\sin\alpha|x\rangle+\cos\alpha|y\rangle$.

When the input photon state is given as $|\psi_{in}\rangle=|y\rangle$, the output photon state is

$$|\psi_{out}\rangle=\hat{A}_H(\theta)|y\rangle=\begin{pmatrix}\sin(2\theta)\\ -\cos(2\theta)\end{pmatrix}.$$

The probability amplitude:

$$\langle\alpha|\psi_{out}\rangle = \begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\begin{pmatrix}\sin(2\theta) \\ -\cos(2\theta)\end{pmatrix}$$
$$= \sin(2\theta - \alpha)$$

Then, the probability is obtained as

$$P = |\langle\alpha|\psi_{out}\rangle|^2 = \sin^2(2\theta - \alpha).$$

Using the analyzer after the half-wave plate, one can evaluate the probability $|\langle\alpha|\psi_{out}\rangle|^2$ as a function of the angle α when θ is changed as a parameter.

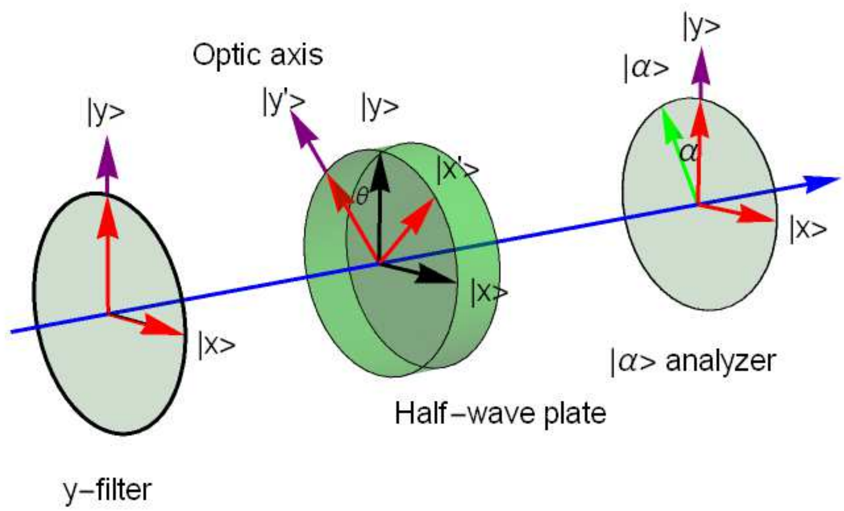


Fig.29 Experimental configuration of the y-filter, half-wave plate and $|\alpha\rangle$ -analyzer. $|\psi_{in}\rangle = |y\rangle$. Determination of intensity (probability) $|\langle\alpha|\psi_{out}\rangle|^2$ by the analyzer, where the angle θ (angle between the optic axis and the z axis).

The results are as follows.

(i) For $\theta = 0°$, the probability takes a peak at $\alpha = 0°$ as is predicted. This means that the state of wave undergoes no change: the initial state $|\psi_{in}\rangle = |y\rangle$ and the final state $|\psi_{out}\rangle = |y\rangle$.

(ii) For $0 \le \theta < 45°$, The maximum of the probability shifts to side of increasing α, with the maximum value kept constant.

(iii) For $\theta = 45°$, the probability takes a peak at $\alpha = 90°$. This means that the state of wave changes from the initial state $|\psi_{in}\rangle = |y\rangle$ to the final state $|\psi_{out}\rangle = |x\rangle$. We note that the peak probability of the analyzer remains unchanged, and equal to unity.

(iv) For $\theta = 90°$, the intensity becomes zero at $\alpha = 90°$. This means that the state remains unchanged; $|\psi_{in}\rangle = |y\rangle$ and $|\psi_{out}\rangle = |y\rangle$.

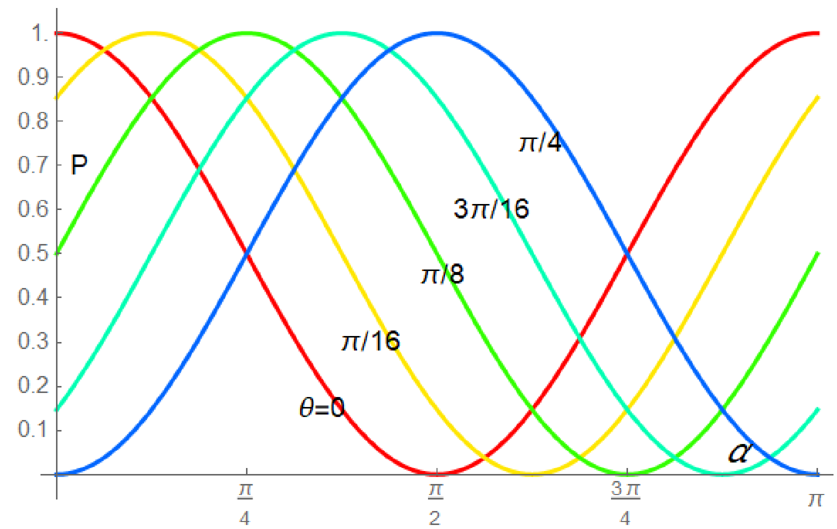


Fig.30 The half-wave plate. Probability $|\langle \alpha | \psi_{out} \rangle|^2$ observed by the analyzer as a function of the angle α, where the angle θ (angle between the optic axis and the z axis). $0 \le \theta \le 45°$.

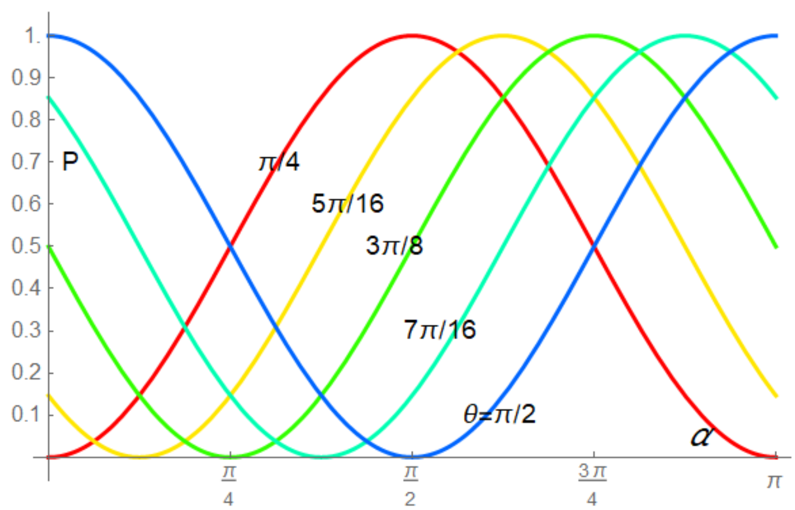


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Fig.31 Half-wave plate. Probability $|\langle\alpha|\psi_{out}\rangle|^2$ observed by the analyzer as a function of the angle α, where the angle θ (angle between the optic axis and the z axis). $45 \le \theta \le 90°$.

9. Optical isolator

This figure shows the experimental demonstration for the optical isolation by Prof. Shaol

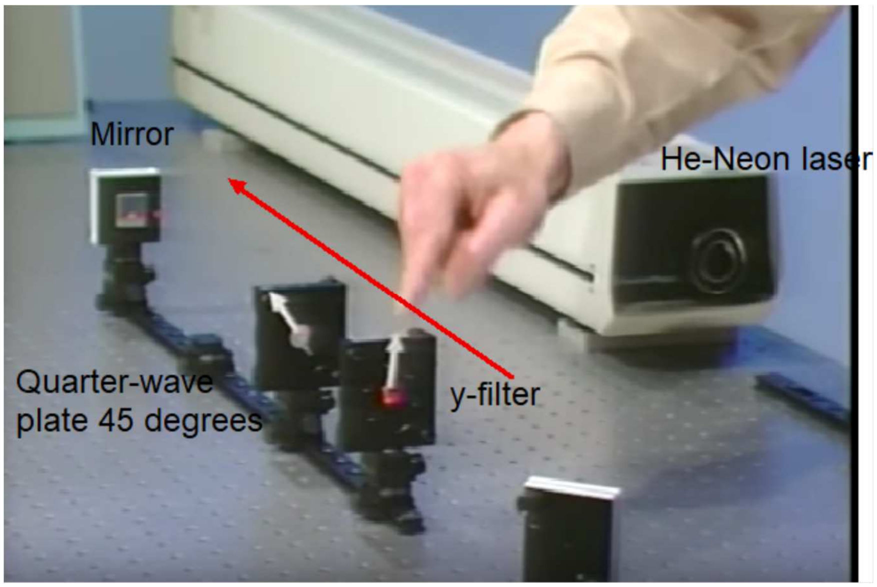

Fig.32 Experimental demonstration by Prof. Shaoul Ezekiel. Optical isolation with y-filter – quarter-wave plate ($\theta = 45°$) – mirror. $|y\rangle \to |R\rangle \to |L\rangle \to |x\rangle$.

Optical isolators are a combination of a linear polarizer, a quarter-wave retarder plate, and a mirror. Incident light is linearly polarized with by the polarizer and converted to circular polarization by the quarter-wave plate. If any portion of the emerging beam is reflected back into the isolator, the quarter wave plate produces a beam, which is linearly polarized perpendicular to the input beam. This beam is blocked by the linear polarizer and not returned to the input side of the system.

One sends the beam through a polarizer first, then through a quarter-wave plate, with the waveplate's axis being oriented at 45° against the polarization direction. Any light reflected back after the waveplate will do a double pass through it, so that it effectively sees a half-wave plate. Its polarization direction is rotated by 90°, so that it will be blocked by the polarizer and thus cannot get back to the laser source.

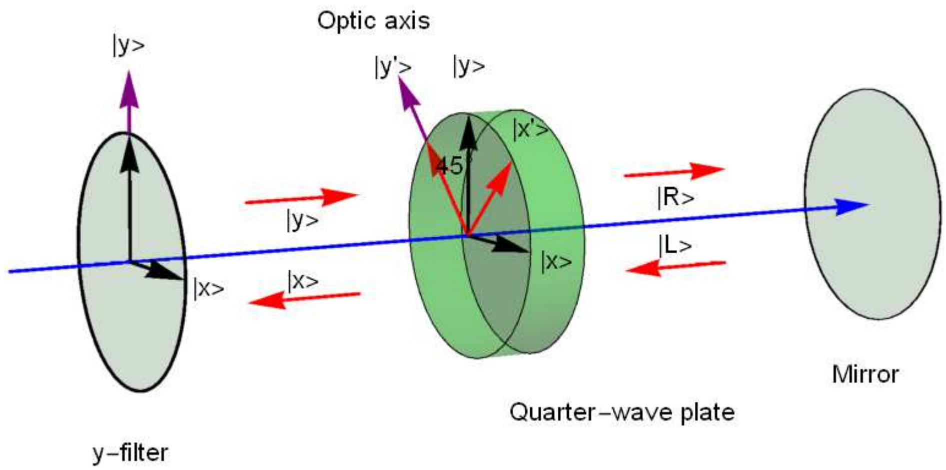

Fig.33 Optical isolation. $|y\rangle \to |R\rangle$ (reflection by mirror)$\to |L\rangle \to |x\rangle$.

$\hat{A}_Q(\theta = \frac{\pi}{4})|y\rangle = e^{-i\pi/4}|R\rangle$, (Quarter-wave plate with 45° on one-way trip)

$\hat{A}_H(\theta = \frac{\pi}{2})|R\rangle = -|L\rangle$, (Reflection on mirror, as a half-wave plate)

$\hat{A}_Q(\theta = -\frac{\pi}{4})|L\rangle = e^{i\pi/4}|x\rangle$ (Quarter-wave plate with 45° on round trip)

((**Note**))

Quarter-wave plate:

$\hat{A}_Q(\theta = \frac{\pi}{4})|L\rangle = e^{-i\pi/4}|y\rangle$, $\hat{A}_Q(\theta = -\frac{\pi}{4})|R\rangle = -e^{-i\pi/4}|y\rangle$.

Reflection on mirror (half-wave plate):

$\hat{A}_H(\theta = \frac{\pi}{2})|R\rangle = -|L\rangle$, $\hat{A}_H(\theta = \frac{\pi}{2})|L\rangle = -|R\rangle$,

$\hat{A}_H(\theta = \frac{\pi}{2})|x\rangle = -|x\rangle$, $\hat{A}_H(\theta = \frac{\pi}{2})|y\rangle = |x\rangle$.

10. Conclusion

A retarder is a birefringent material that alters (retards) the polarization state or phase of light traveling through it. A retarder has a fast (extraordinary) and slow (ordinary) axis. As polarized light passes through a retarder, the light passing through the fast axis travels more quickly through the wave retarder than through the slow axis. In the case of a quarter-wave plate, the wave plate retards the velocity of one of the polarization components one quarter of a wave out of phase from the other polarization component. Polarized light passing through a quarter wave retarder thus becomes circularly polarized. The action of the quarter wave is sometimes referred to as twisting or rotating the polarized light. Note that depending on which polarization component is retarded, one will have either a left-handed or right-handed circular polarizer. On the other hand, a half-wave plate (two quarter wave plates combined) will not create circularly polarized light, but, instead, rotate the polarization vector of linearly polarized light by 90º for $\theta = 45°$.

Video: MIT Video Demonstrations in Lasers and Optics (Prof. Shaoul Ezekiel)

Optics: Quarter-wave plate | MIT Video Demonstrations in Lasers and Optics.
https://www.youtube.com/watch?v=EBVNbRN805o

Optics: Half-wave plate | MIT Video Demonstrations in Lasers and Optics.
https://www.youtube.com/watch?v=_sUVXHfUVsY

Optics: Optical isolation: | MIT Video Demonstrations in Lasers and Optics.
https://www.youtube.com/watch?v=G9kl6-lRHNs

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APPENDIX Euler rotations: Euler angles and the directions of polarization vectors

Rotation-1

Rotation around the z axis by an angle φ

$x - y - z$ to $\xi - \eta - \varsigma$

where $\varsigma = z$

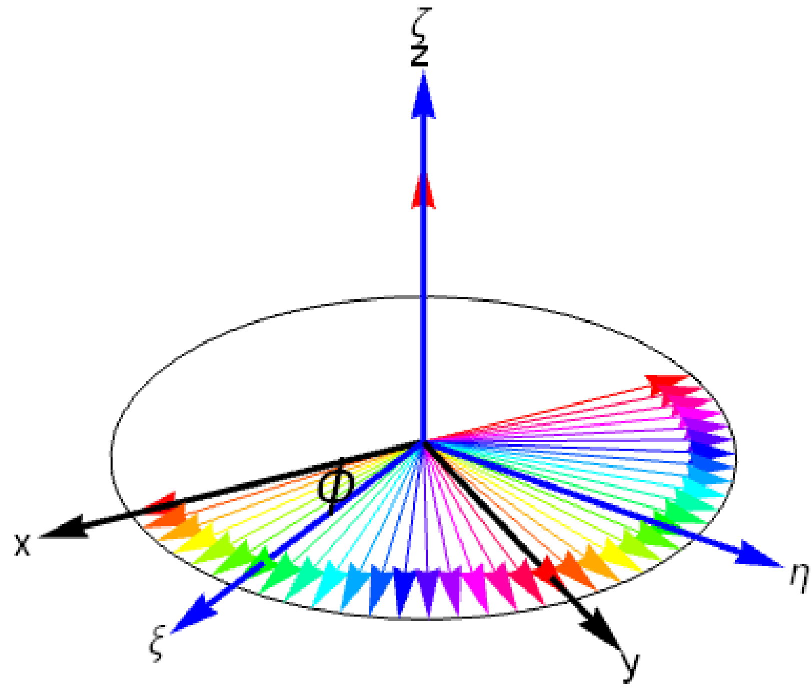

Fig.1A z (ζ): optic axis. $\xi = |x'\rangle$ (polarization vector of o-wave). $\zeta = |y'\rangle$ (polarization vector of e-wave). o: ordinary wave. e: extraordinary wave. The direction of both the waves is parallel to the η axis.

((Rotation-2))

Rotation around ξ ($= \xi'$) axis by an angle θ

$\xi - \eta - \zeta$ to $\xi' - \eta' - \zeta'$

where $\xi' = \xi$.

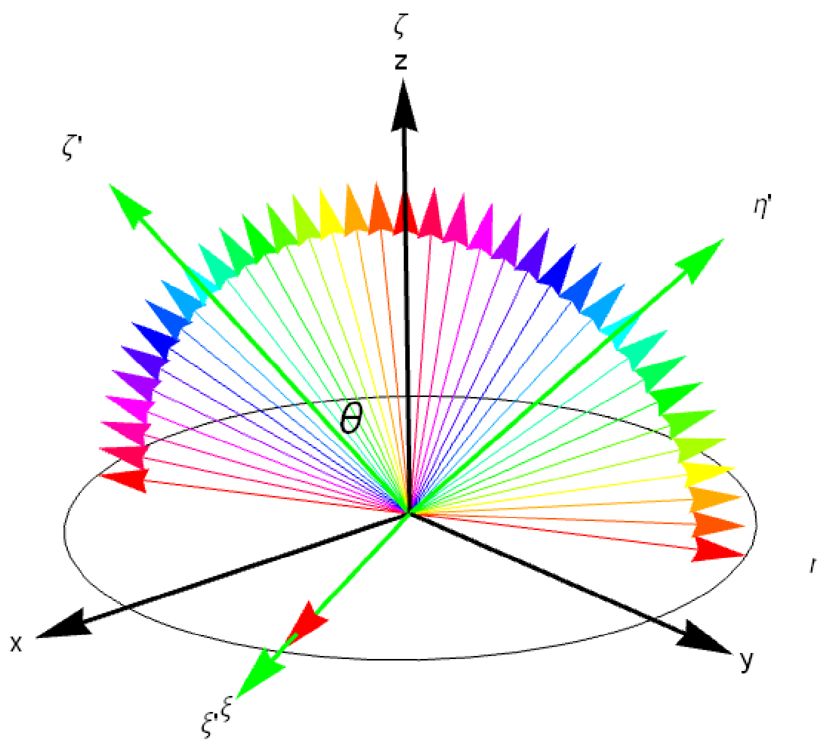

Fig.1B z (ζ): optic axis. $\xi = |x'\rangle$ (polarization vector of o-wave). $\eta' = |y'\rangle$ (polarization vector of e-wave). o:ordinary wave. e: extraordinary wave. The direction of both the waves is parallel to the ζ' axis.

Rotation-3

Rotation around z axis by an angle ψ

$\xi' - \eta' - \zeta'$ to $x' - y' - z'$

where $\zeta' = z'$

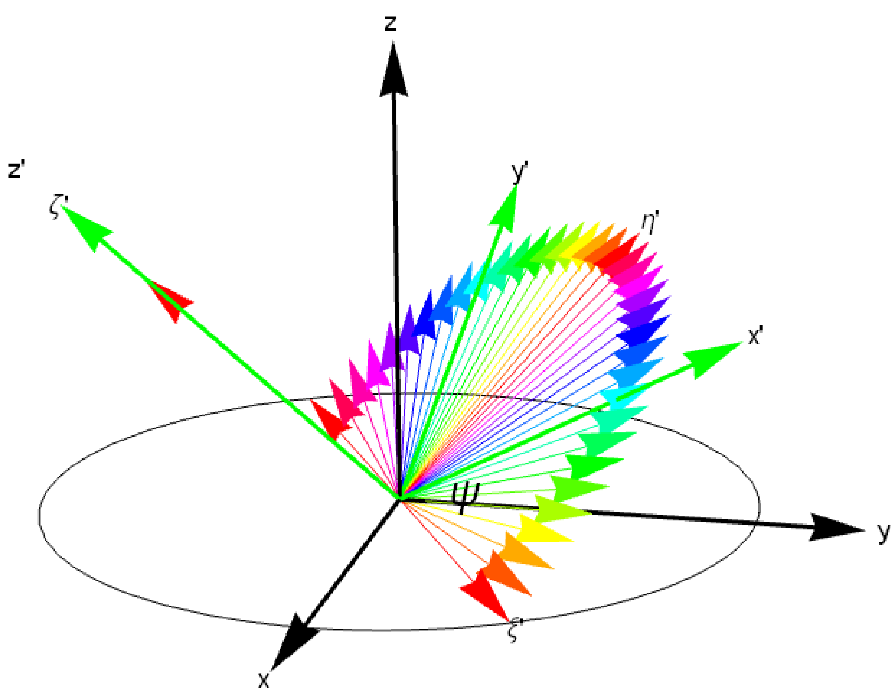

Fig.1C