Commuting Observables

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The commutator of two operators is defined between the products of the two operators taken in alternate orders;

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}.$$

If the commutator is equal to zero, we say that the operator or observables commute. If its is not zero, we say that they do not commute. Whether or not two observables commute has important ramifications in analyzing a quantum system and in making measurements of the two observables represented by those observables.

1. Simultaneous eigenkets

Suppose that $\hat{A}$ and $\hat{B}$ are Hermitian operators, and $\hat{A}$ and $\hat{B}$ commute.

$$\hat{A}\hat{B}=\hat{B}\hat{A}.$$

We consider the nondegenerate case for simplicity. $\left|a\right\rangle$ is the eigenket of $\hat{A}$ with an eigenvalue *a*.

$$\hat{A}\left|a\right\rangle=a\left|a\right\rangle.$$

Then we get

$$\hat{A}(\hat{B}\left|a\right\rangle)=\hat{B}\hat{A}\left|a\right\rangle=a(\hat{B}\left|a\right\rangle.$$

This means that $\hat{B}\left|a\right\rangle$ is the eigenket of $\hat{A}$ with the eigenvalue a;

$$\hat{B}\left|a\right\rangle=b\left|a\right\rangle.$$

The eigenket $\left|a\right\rangle$ is the eigenket of $\hat{B}$ with the eigenvalue *b*. Thus $\left|a\right\rangle$ can be rewritten as

$$\left|a,b\right\rangle,$$

which is the simultaneous eigenket of $\hat{A}$ and $\hat{B}$.

We may use $\left|a,b\right\rangle$ to characterize the simultaneous eigenket.

$$\hat{A}|a,b\rangle = a|a,b\rangle,$$

$$\hat{B}|a,b\rangle = b|a,b\rangle.$$

Then

$$\hat{A}\hat{B}|a,b\rangle = b\hat{A}|a,b\rangle = ab|a,b\rangle,$$

$$\hat{B}\hat{A}|a,b\rangle = a\hat{B}|a,b\rangle = ab|a,b\rangle,$$

or

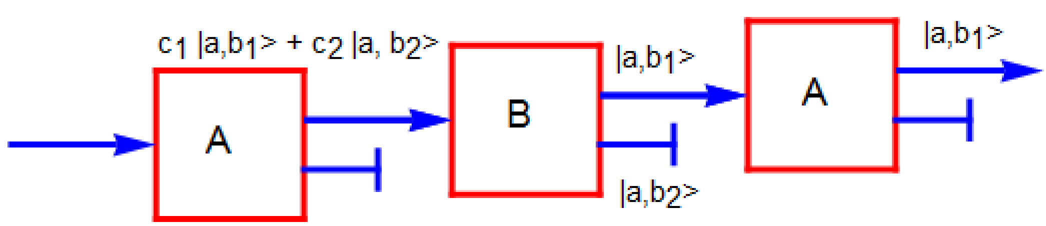
$$[\hat{A},\hat{B}]|a,b\rangle = 0.$$

But since any state $|\psi\rangle$ can be written as a superposition of the complete eigenstates $|a,b\rangle$, then

$$[\hat{A},\hat{B}]|\psi\rangle = 0, \qquad \text{and hence} \qquad [\hat{A},\hat{B}] = 0.$$

2. Measurements

We now consider the measurements of $\hat{A}$ and $\hat{B}$ when they are compatible. Suppose we measure $\hat{A}$ first and obtain result *a*. Subsequently we measure $\hat{B}$ and get result b_1. Finally we measure *A* again. It follows from our measurement formalism that third measurement always gives *a* with certainty, that is, the second (*B*) measurement does not destroy the previous information obtained in the first (*A*) measurement.



After the first (A) measurement, which yields *a*, the system is thrown into

$$c_1|a,b_1\rangle + c_2|a,b_2\rangle.$$

The second (B) measurement may select just one of the terms in the linear combination, say,

$$|a, b_1\rangle .$$

But the third (A) measurement applied to it still yields *a*. The state is described by

$$|a, b_1\rangle .$$

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3. Example-1: simultaneous eigenkets
Sakurai Modern Quantum Mechanics
((Sakurai problem 1-23))

Consider a 3D ket space. If a certain set of orthonormal kets – say, $|1\rangle$, $|2\rangle$, and $|3\rangle$ - are used as the base kets, the operators $\hat{A}$ and $\hat{B}$ are represented by

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix},$$

with *a* and *b* both real.

(a) Obviously $\hat{A}$ exhibits a degenerate spectrum. Does $\hat{B}$ also exhibit a degenerate spectrum?

(b) Show that $\hat{A}$ and $\hat{B}$ commute.

(c) Find a new set of orthonormal kets that are simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$. Specify the eigenvalues of $\hat{A}$ and $\hat{B}$ for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

((**Solution**))

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix},$$

or

$$\hat{A}|1\rangle = a|1\rangle, \qquad \hat{A}|2\rangle = -a|2\rangle, \qquad \hat{A}|3\rangle = -a|3\rangle,$$

$$\hat{B}|1\rangle = b|1\rangle, \qquad \hat{B}|2\rangle = ib|3\rangle, \qquad \hat{B}|3\rangle = -ib|2\rangle,$$

$$[\hat{A}, \hat{B}] = 0 .$$

The eigenkets of $\hat{B}$ should be the eigenkets of $\hat{A}$, and vice versa.
The operators $\hat{A}$ and $\hat{B}$ are the Hermitian operators.

Eigenkets of $\hat{A}$:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(eigenvalue: } a\text{)}$$

$$|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{(eigenvalue: -}a\text{)}$$

$$|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{(eigenvalue: -}a\text{)}$$

Eigenkets of $\hat{B}$:

$$|\psi_{b3}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle , \qquad \text{(eigenvalue: } b\text{)}$$

$$\hat{B}|2\rangle = ib|3\rangle , \qquad \hat{B}|3\rangle = -ib|2\rangle .$$

In the subspace spanned by $|2\rangle$ and $|3\rangle$, we consider the eigenvalue problem

$$\hat{B}_{sub} = \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix},$$

$$\hat{B}_{sub}|\phi\rangle = \lambda|\phi\rangle ,$$

with

$$|\phi\rangle = \begin{pmatrix} C_2 \\ C_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = \lambda\begin{pmatrix} C_2 \\ C_3 \end{pmatrix},$$

or

$$\begin{pmatrix} -\lambda & -ib \\ ib & -\lambda \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = 0,$$

$$M = \begin{pmatrix} -\lambda & -ib \\ ib & -\lambda \end{pmatrix}.$$

Det[M]=0:

$$\begin{vmatrix} -\lambda & -ib \\ ib & -\lambda \end{vmatrix} = \lambda^2 - b^2 = 0.$$

(i) $\lambda = b$

$$\begin{pmatrix} -b & -ib \\ ib & -b \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = 0,$$

$$C_3 = iC_2,$$

$$|C_2|^2 + |C_3|^3 = 1.$$

Then we have $C_2 = -\frac{i}{\sqrt{2}}$ and $C_3 = \frac{1}{\sqrt{2}}$,

or

$$|\psi_{b2}\rangle = \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{i}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle . \qquad \text{(eigenvalue } b\text{)}$$

(ii) λ = -b

$$\begin{pmatrix} b & -ib \\ ib & b \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = 0 ,$$

$$C_3 = iC_2 ,$$

$$|C_2|^2 + |C_3|^3 = 1 .$$

Then we have $C_2 = -\frac{i}{\sqrt{2}}$ and $C_3 = \frac{1}{\sqrt{2}}$,

or

$$|\psi_{b1}\rangle = \begin{pmatrix} 0 \\ \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{i}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle , \qquad \text{(eigenvalue -}b\text{)}$$

Since any combinations of $|2\rangle$ and $|3\rangle$ are the eigenkets of $\hat{A}$ with an eigenvalue (-a).
Then

$$\hat{A}|\psi_{b1}\rangle = -a|\psi_{b1}\rangle , \qquad \text{and} \qquad \hat{A}|\psi_{b2}\rangle = -a|\psi_{b2}\rangle .$$

In conclusion, $|\psi_{b1}\rangle$, $|\psi_{b2}\rangle$, and $|\psi_{b3}\rangle$ are the simultaneous eigenkets of $\hat{A}$ and $\hat{B}$.

((**Mathematica**))

```
Clear["Global`*"]; conjugateRule = {Complex[re_, im_] :> Complex[re, -im]};
Unprotect[SuperStar]; SuperStar /: exp_* := exp /. conjugateRule;
Protect[SuperStar];
```

```
A = ( a  0  0
      0 -a  0
      0  0 -a ); B = ( b  0    0
                       0  0  -ⅈ b
                       0 ⅈ b   0 );
```

(a)

```
eq1 = Eigensystem[A]
```

```
{{-a, -a, a}, {{0, 0, 1}, {0, 1, 0}, {1, 0, 0}}}
```

```
ψa1 = Normalize[eq1[[2, 1]]]
```

```
{0, 0, 1}
```

```
ψa2 = Normalize[eq1[[2, 2]]]
```

```
{0, 1, 0}
```

```
ψa3 = Normalize[eq1[[2, 3]]]
```

```
{1, 0, 0}
```

```
{ψa1*.ψa2, ψa2*.ψa3, ψa3*.ψa1}
```

```
{0, 0, 0}
```

```
Orthogonalize[{ψa1, ψa2}]
```

```
{{0, 0, 1}, {0, 1, 0}}
```

Any combination of ψa1 and ψa2 belongs to the eigenvalue of A with the eigenvalue (-a)

```
eq2 = Eigensystem[B];

ψb1 = Normalize[eq2[[2, 1]]]; ψb2 = Normalize[eq2[[2, 2]]];
ψb3 = Normalize[eq2[[2, 3]]];

{ψb1*.ψb2, ψb2*.ψb3, ψb3*.ψb1}
```

{0, 0, 0}

```
{eq2[[1, 1]], eq2[[1, 2]], eq2[[1, 3]]}
```

{-b, b, b}

```
{ψb1, ψb2, ψb3}
```

$$\left\{\left\{0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{0, -\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \{1, 0, 0\}\right\}$$

(b)

```
A.B - B.A // Simplify
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

(c) The simultaneous eigenvectors of A and B

Summary

$\psi b1 = \frac{-i}{\sqrt{2}} \psi a2 + \frac{1}{\sqrt{2}} \psi a1$ (A : -a, B : -b)

$\psi b2 = \frac{i}{\sqrt{2}} \psi a2 + \frac{1}{\sqrt{2}} \psi a1$ (A, -a, B : b)

$\psi b3 = \psi a3$ (A : a, B : b)

UT = {ψb1, ψb2, ψb3}

$\left\{\left\{0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{0, -\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \{1, 0, 0\}\right\}$

U = Transpose[UT]

$\left\{\{0, 0, 1\}, \left\{\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}\right\}$

UH = UT*

$\left\{\left\{0, -\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \{1, 0, 0\}\right\}$

UH.A.U

{{-a, 0, 0}, {0, -a, 0}, {0, 0, a}}

UH.B.U

{{-b, 0, 0}, {0, b, 0}, {0, 0, b}}

---

4. Example-2: Simultaneous eigenkets

Cohen-Tannoudji
Problem 11 (p.206)

Consider a physical system whose 3D (dimensional) state space is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$. In the basis of these three vectors, taken in this order, the two operators $\hat{H}$ and $\hat{B}$ are defined by

$$\hat{H} = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \hat{B} = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $E_0 = \hbar\omega_0$.

(a) Are $\hat{H}$ and $\hat{B}$ Hermitian?

(b) Show that $\hat{H}$ and $\hat{B}$ commute. Give a basis of eigenvectors common to $\hat{H}$ and $\hat{B}$.

((**Solution**))

Since

$$[\hat{H}, \hat{B}] = 0,$$

$\hat{H}$ and $\hat{B}$ have simultaneous eigenkets.

For $\hat{H}$,

$$\hat{H}|u_1\rangle = \hbar\omega_0|u_1\rangle, \qquad \hat{H}|u_2\rangle = -\hbar\omega_0|u_2\rangle, \qquad \hat{H}|u_3\rangle = -\hbar\omega_0|u_3\rangle,$$

where

$$|u_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |u_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |u_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that the states $|u_2\rangle$, and $|u_3\rangle$ are degenerate states with the eigenvalue , but are independent;

$$\langle u_1|u_2\rangle = 0.$$

Any combination of $|u_2\rangle$, and $|u_3\rangle$ is also an eigenket of $\hat{H}$ with the eigenvalue $-\hbar\omega_0$.

For $\hat{B}$

$$\hat{B}|u_1\rangle = b|u_1\rangle, \quad \hat{B}|u_2\rangle = b|u_3\rangle, \qquad \hat{B}|u_3\rangle = b|u_2\rangle.$$

Then $|u_1\rangle$ is the eigenket of $\hat{B}$ with the eigenvalue *b*. However, $|u_2\rangle$ and $|u_2\rangle$ are not the eigenkets of $\hat{B}$. The matrix representation of $\hat{B}$ under the basis of $|u_2\rangle$ and $|u_2\rangle$ of the subspace is expressed by

$$\hat{B}_{sub} = b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = b\hat{\sigma}_x .$$

The eigenkets of $\hat{B}_{sub}$ are

$$|u_2'\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}[|u_2\rangle + |u_3\rangle], \qquad \text{for the eigenvalue } b,$$

$$|u_3'\rangle |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}[|u_2\rangle - |u_3\rangle], \qquad \text{for the eigenvalue } -b,$$

These eigenkets are also the eigenkets of $\hat{H}$.

In summary

(i) $|u_1\rangle$

$$\hat{H}|u_1\rangle = \hbar\omega_0|u_1\rangle, \qquad \hat{B}|u_1\rangle = b|u_1\rangle .$$

(ii) $|u_2'\rangle$

$$\hat{H}|u_2'\rangle = -\hbar\omega_0|u_2'\rangle, \qquad \hat{B}|u_2'\rangle = b|u_2'\rangle .$$

(iii) $|u_3'\rangle$

$$\hat{H}|u_3'\rangle = -\hbar\omega_0|u_3'\rangle, \qquad \hat{B}|u_3'\rangle = -b|u_3'\rangle .$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};;
```

$$A = \begin{pmatrix} E0 & 0 & 0 \\ 0 & -E0 & 0 \\ 0 & 0 & -E0 \end{pmatrix};$$

$$B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{pmatrix};$$

(b)

```
A.B - B.A // Simplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
eq1 = Eigensystem[A]
```

```
{{-E0, -E0, E0}, {{0, 0, 1}, {0, 1, 0}, {1, 0, 0}}}
```

```
ψH1 = Normalize[eq1[[2, 3]]]
```

```
{1, 0, 0}
```

```
ψH2 = Normalize[eq1[[2, 2]]]
```

```
{0, 1, 0}
```

```
ψH3 = Normalize[eq1[[2, 1]]]
```

{0, 0, 1}

```
eq2 = Eigensystem[B]
```

{{-b, b, b}, {{0, -1, 1}, {0, 1, 1}, {1, 0, 0}}}

```
ψb1 = Normalize[eq2[[2, 3]]]
```

{1, 0, 0}

```
ψb2 = Normalize[eq2[[2, 2]]]
```

$\left\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

```
ψb3 = -Normalize[eq2[[2, 1]]]
```

$\left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$

```
{ψb1*.ψb2, ψb2*.ψb3, ψb3*.ψb1}
```

{0, 0, 0}

$\psi b1 = \psi H1$ (A : E0, B : b)

$\psi b2 = \frac{1}{\sqrt{2}}\psi H1 + \frac{1}{\sqrt{2}}\psi H2$ (A, -E0, B : b)

$\psi b3 = \frac{1}{\sqrt{2}}\psi H1 - \frac{1}{\sqrt{2}}\psi H2$ (A : -E0, B : -b)

```
UT = {ψb1, ψb2, ψb3}; U = Transpose[UT]; UH = UT*;
U // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
UH // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
UH.U
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

UH.A.U
{{E0, 0, 0}, {0, -E0, 0}, {0, 0, -E0}}

UH.B.U // Simplify
{{b, 0, 0}, {0, b, 0}, {0, 0, -b}}
```

---

5. Example-3 Simultaneous eigenkets

Shankar 1-8-10 p.46

By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonarized. Find the eigenvectors common to both and verify that under a unitary transformation to this basis, both matrices are diagonalized.

$$\hat{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Since $\hat{A}$ is degenerate and $\hat{B}$ is not, you must be prudent in deciding which matrix dictates the choice of basis.

((**Solution**))

For $\hat{A}$,

$$|\psi_{a1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{(eigenvalue, 2)} \qquad \hat{A}|\psi_{a1}\rangle = 2|\psi_{a1}\rangle,$$

$$|\psi_{a2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{(eigenvalue, 0)} \qquad \hat{A}|\psi_{a2}\rangle = 0|\psi_{a2}\rangle,$$

$$|\psi_{a3}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{(eigenvalue, 0)} \qquad \hat{A}|\psi_{a3}\rangle = 0|\psi_{a3}\rangle.$$

For $\hat{B}$,

$$|\psi_{b1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{(eigenvalue, 3)} \qquad \hat{B}|\psi_{a1}\rangle = 3|\psi_{a1}\rangle,$$

$$|\psi_{b2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \text{(eigenvalue, 2)} \qquad \hat{B}|\psi_{b2}\rangle = 2|\psi_{b2}\rangle,$$

$$|\psi_{b3}\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \text{(eigenvalue, -1)} \qquad \hat{B}|\psi_{b3}\rangle = -|\psi_{b3}\rangle.$$

Then $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ are the simultaneous eigenkets of $\hat{A}$ and $\hat{B}$, where

$$|\psi_1\rangle = |\psi_{a1}\rangle = |\psi_{b1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{A}|\psi_1\rangle = 2|\psi_1\rangle, \qquad \hat{B}|\psi_1\rangle = 3|\psi_1\rangle,$$

$$|\psi_2\rangle = |\psi_{b2}\rangle = \frac{1}{\sqrt{3}}[2|\psi_{a2}\rangle - |\psi_{a3}\rangle], \qquad \hat{A}|\psi_2\rangle = 0|\psi_2\rangle \qquad \hat{B}|\psi_2\rangle = 2|\psi_2\rangle,$$

$$|\psi_3\rangle = |\psi_{b3}\rangle = \frac{1}{\sqrt{3}}[2|\psi_{a2}\rangle + |\psi_{a3}\rangle], \qquad ,\hat{A}|\psi_2\rangle = 0|\psi_2\rangle, \qquad \hat{B}|\psi_3\rangle = -|\psi_3\rangle.$$

((**Mathematica**))

```
Clear["Global`*"]; exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};;
```

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix};\ B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix};$$

(a)

```
A.B - B.A // Simplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
eq1 = Eigensystem[A]
```

```
{{2, 0, 0}, {{1, 0, 1}, {-1, 0, 1}, {0, 1, 0}}}
```

```
ψa1 = Normalize[eq1[[2, 1]]]
```

$$\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
ψa2 = Normalize[eq1[[2, 2]]]
```

$$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
ψa3 = Normalize[eq1[[2, 3]]]
```

$\{0, 1, 0\}$

```
{ψa1*.ψa2, ψa2*.ψa3, ψa3*.ψa1}
```

$\{0, 0, 0\}$

```
eq2 = Eigensystem[B]
```

$\{\{3, 2, -1\}, \{\{1, 0, 1\}, \{-1, -1, 1\}, \{-1, 2, 1\}\}\}$

```
ψb1 = Normalize[eq2[[2, 1]]]
```

$$\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
ψb2 = Normalize[eq2[[2, 2]]]
```

$$\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$$

```
ψb3 = Normalize[eq2[[2, 3]]]
```

$$\left\{-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right\}$$

```
{ψb1*.ψb2, ψb2*.ψb3, ψb3*.ψb1}
```

$\{0, 0, 0\}$

Summary

$\psi a1 = \psi b1 = \left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$ (A: eigenvalue 2, B: eigenvalue 3)

$\psi a2 = \left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$ (A: 0), $\psi b2 = \left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ (B; 2)

$\psi a3=\{0, 1, 0\}$ (A: 0), $\psi b3 = \left\{-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ (B: -1)

Note that

$\psi b2 = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,\psi a2 - \psi a3\right)$, $\psi b3 = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,\psi a2 + \psi a3\right)$ are the eigenvectors of A with the eigenvalue 0 since any combination of $\psi a2$ and $\psi a3$ is an eigenvector of A with the eigenvalue 0.

```
UT = {ψb1, ψb2, ψb3}
```

$$\left\{\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right\}\right\}$$

```
U = Transpose[UT]
```

$$\left\{\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right\}, \left\{0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right\}\right\}$$

```
UH = UT*
```

$$\left\{\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right\}\right\}$$

```
UH.U
```

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

```
UH.A.U
```

{{2, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```
UH.B.U // Simplify
```

{{3, 0, 0}, {0, 2, 0}, {0, 0, −1}}

REFERENCES

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