Eigenvalue problem for *S* = 1 (or *J* = 1)

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Here we show how to solve the eigenvalue for S = 1 in conventional ways.

We determine the eigenstates of $\hat{S}_x$ and $\hat{S}_y$ for a spin-1 particle in terms of the eigenstates $\left| j=1, m \right\rangle$ (*m* = 1, 0, -1) of $\hat{S}_z$

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(i) Eigenvalue and eigenkets of $\hat{S}_x$

$$\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left|1,1\right\rangle_x = \hat{U}_x \left|1,1\right\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix},$$

$$\left|1,0\right\rangle_x = \hat{U}_x \left|1,0\right\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix},$$

$$\left|1,-1\right\rangle_x = \hat{U}_x \left|1,-1\right\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

where $\hat{U}_x$ is a unitary operator.

Eigenvalue problem

$$\hat{S}_x|\psi\rangle = \lambda\hbar|\psi\rangle$$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix} = \lambda\begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix}$$

or

$$\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For nontrivial solution, the determinant should be zero,

$$\begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = 0$$

or

$$\lambda(\lambda-1)(\lambda+1) = 0$$

Note that

$$\hat{S}_x|1,1\rangle_x = \hbar|1,1\rangle_x \qquad (\lambda = 1)$$

$$\hat{J}_x|1,0\rangle_x = 0|1,0\rangle_x \qquad (\lambda = 0)$$

$$\hat{J}_x|1,-1\rangle_x = -\hbar|1,-1\rangle_x \qquad (\lambda = -1)$$

(a) $\hat{S}_x|1,1\rangle_x = \hbar|1,1\rangle_x$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{11} = \frac{1}{\sqrt{2}}U_{21}, \qquad U_{31} = \frac{1}{\sqrt{2}}U_{21}$$

with the normalization condition

$$|U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 = 1$$

So we get $|U_{21}| = \frac{1}{\sqrt{2}}$. Here we choose $U_{21} = \frac{1}{\sqrt{2}}$

$$U_{11} = \frac{1}{2}, \qquad U_{31} = \frac{1}{2}$$

Finally we obtain the eigenket $|1,1\rangle_x$,

$$|1,1\rangle_x = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}(|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle)$$

(b) $\hat{S}_x|1,0\rangle_x = 0|1,0\rangle_x = 0$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Then we have

$$U_{22} = 0, \qquad U_{12} + U_{32} = 0$$

with the normalization condition

$$|U_{12}|^2 + |U_{22}|^2 + |U_{32}|^2 = 1$$

So we have $U_{12} = \frac{1}{\sqrt{2}}, \qquad U_{32} = -\frac{1}{\sqrt{2}}$

In summary we get

$$|1,0\rangle_x = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} = \frac{1}{2}(\sqrt{2}|1,1\rangle - \sqrt{2}|1,-1\rangle)$$

(c) $\hat{S}_x |1,-1\rangle_x = -\hbar |1,-1\rangle_x$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = -\begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{33} + \frac{1}{\sqrt{2}} U_{23} = 0, \qquad U_{13} + \frac{1}{\sqrt{2}} U_{23} = 0$$

with the normalization condition

$$|U_{13}|^2 + |U_{23}|^2 + |U_{33}|^2 = 1$$

So we get $|U_{23}| = \frac{1}{\sqrt{2}}$. Here we choose $U_{23} = -\frac{1}{\sqrt{2}}$

$$U_{13} = \frac{1}{2}, \qquad U_{33} = \frac{1}{2}$$

$$|1,-1\rangle_x = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}(|1,1\rangle - \sqrt{2}|1,0\rangle + |1,-1\rangle)$$

The unitary operator is obtained as

$$\hat{U}_x = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\hat{U}_x^{\ +} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\hat{U}_x^{\ +}\hat{J}_x\hat{U}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We now calculate the rotation operator $\hat{R}_y(\alpha) = \exp(-\frac{i}{\hbar}\hat{J}_x\alpha)$

$$\exp(-\frac{i}{\hbar}\hat{J}_x\alpha) = \hat{U}_x \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \hat{U}_x^{+}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\frac{\alpha}{2} & -\frac{i}{\sqrt{2}}\sin\alpha & -\sin^2\frac{\alpha}{2} \\ -\frac{i}{\sqrt{2}}\sin\alpha & \cos\alpha & -\frac{i}{\sqrt{2}}\sin\alpha \\ -\sin^2\frac{\alpha}{2} & -\frac{i}{\sqrt{2}}\sin\alpha & \cos^2\frac{\alpha}{2} \end{pmatrix}$$

(ii) Eigenvalues and eigenkets of $\hat{S}_y$

$$\hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$|1,1\rangle_y = \hat{U}_y|1,1\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix},$$

$$|1,0\rangle_y = \hat{U}_y|1,0\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix},$$

$$|1,-1\rangle_y = \hat{U}_y|1,-1\rangle = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

where $\hat{U}_y$ is a unitary operator. Note that

$$\hat{S}_y|1,1\rangle_y = \hbar|1,1\rangle_y,$$

$$\hat{S}_y|1,0\rangle_y = 0|1,0\rangle_y,$$

$$\hat{S}_y|1,-1\rangle_y = -1|1,-1\rangle_y,$$

(a) $$\hat{S}_y|1,1\rangle_y = \hbar|1,1\rangle_y$$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}, \text{ or } \qquad \begin{pmatrix} -1 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -1 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -1 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{11} = -\frac{i}{\sqrt{2}}U_{21}, \qquad U_{31} = \frac{i}{\sqrt{2}}U_{21}$$

with the normalization condition

$$|U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 = 1$$

So we get $|U_{21}| = \frac{1}{\sqrt{2}}$. Here we choose $U_{21} = \frac{i}{\sqrt{2}}$

$$U_{11} = \frac{1}{2}, \qquad U_{31} = -\frac{1}{2}$$

Finally we obtain the eigenket $|1,1\rangle_y$,

$$|1,1\rangle_y = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix} = \frac{1}{2}(|1,1\rangle + i\sqrt{2}|1,0\rangle - |1,-1\rangle)$$

(b) $$\hat{S}_y|1,0\rangle_y = 0|1,0\rangle_y = 0$$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Then we have

$$U_{22} = 0, \qquad U_{12} = U_{32}$$

with the normalization condition

$$|U_{12}|^2 + |U_{22}|^2 + |U_{32}|^2 = 1$$

So we have $U_{12} = \frac{1}{\sqrt{2}}, \qquad U_{32} = \frac{1}{\sqrt{2}}$

In summary we get

$$|1,0\rangle_y = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} = \frac{1}{2}(\sqrt{2}|1,1\rangle + \sqrt{2}|1,-1\rangle)$$

(c) $S_y|1,-1\rangle_y = -\hbar|1,-1\rangle_y$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = -\begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 1 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 1 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then we have

$$U_{13} - \frac{i}{\sqrt{2}}U_{23} = 0, \qquad U_{33} + \frac{i}{\sqrt{2}}U_{23} = 0$$

with the normalization condition

$$|U_{13}|^2 + |U_{23}|^2 + |U_{33}|^2 = 1$$

So we get $|U_{23}| = \frac{1}{\sqrt{2}}$. Here we choose $U_{23} = -\frac{i}{\sqrt{2}}$

$$U_{13} = \frac{1}{2}, \qquad U_{33} = -\frac{1}{2}$$

$$|1,-1\rangle_y = \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix} = \frac{1}{2}(|1,1\rangle - i\sqrt{2}|1,0\rangle - |1,-1\rangle)$$

The unitary operator is obtained as

$$\hat{U}_y = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}$$

$$\hat{U}_y^{+} = \frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix}$$

$$\hat{U}_y^{+}\hat{J}_y\hat{U}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We now calculate the rotation operator $\hat{R}_y(\alpha) = \exp(-\frac{i}{\hbar}\hat{J}_y\alpha)$

$$\exp(-\frac{i}{\hbar}\hat{J}_y\theta) = \hat{U}_y \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \hat{U}_y{}^{+}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}\sin\theta & \sin^2\frac{\theta}{2} \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \sin^2\frac{\theta}{2} & \frac{1}{\sqrt{2}}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix}$$

((**Mathematica**))

Matrices j = 1

```
Clear["Global`*"]; j = 1; exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
Jx[j_, n_, m_] := ħ/2 Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
   ħ/2 Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jy[j_, n_, m_] := -ħ/2 i Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
   ħ/2 i Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jz[j_, n_, m_] := ħ m KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

```
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
Jz // MatrixForm
```

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

Eigenvalues and eigenkets of Jx

```
eq1 = Eigensystem[Jx]
```

$$\left\{\{-\hbar, \hbar, 0\}, \left\{\left\{1, -\sqrt{2}, 1\right\}, \left\{1, \sqrt{2}, 1\right\}, \{-1, 0, 1\}\right\}\right\}$$

```
ψ1x = Normalize[eq1[[2, 2]]]; ψ1x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

```
ψ2x = -Normalize[eq1[[2, 3]]]; ψ2x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψ3x = Normalize[eq1[[2, 1]]]; ψ3x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

UxT = {ψ1x, ψ2x, ψ3x}; Ux = Transpose[UxT]; Ux // MatrixForm

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

UxH = UxT*; UxH // MatrixForm

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

UxH.Ux

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

UxH.Jx.Ux // Simplify

{{ℏ, 0, 0}, {0, 0, 0}, {0, 0, -ℏ}}

Ux.Jx.Ux // Simplify

{{ℏ, 0, 0}, {0, 0, 0}, {0, 0, -ℏ}}

MatrixExp$\left[\frac{-i}{\hbar} Jx\, \alpha\right]$ **// TrigFactor // MatrixForm**

$$\begin{pmatrix} \mathrm{Cos}\left[\frac{\alpha}{2}\right]^2 & -\frac{i\,\mathrm{Sin}[\alpha]}{\sqrt{2}} & -\mathrm{Sin}\left[\frac{\alpha}{2}\right]^2 \\ -\frac{i\,\mathrm{Sin}[\alpha]}{\sqrt{2}} & \mathrm{Cos}[\alpha] & -\frac{i\,\mathrm{Sin}[\alpha]}{\sqrt{2}} \\ -\mathrm{Sin}\left[\frac{\alpha}{2}\right]^2 & -\frac{i\,\mathrm{Sin}[\alpha]}{\sqrt{2}} & \mathrm{Cos}\left[\frac{\alpha}{2}\right]^2 \end{pmatrix}$$

```
Ux.(Exp[-ⅈ α]  0  0
    0          1  0
    0          0  Exp[ⅈ α]).UxH // ExpToTrig // TrigFactor // MatrixForm
```

$$\begin{pmatrix} \text{Cos}\left[\frac{\alpha}{2}\right]^2 & -\frac{i\,\text{Sin}[\alpha]}{\sqrt{2}} & -\text{Sin}\left[\frac{\alpha}{2}\right]^2 \\ -\frac{i\,\text{Sin}[\alpha]}{\sqrt{2}} & \text{Cos}[\alpha] & -\frac{i\,\text{Sin}[\alpha]}{\sqrt{2}} \\ -\text{Sin}\left[\frac{\alpha}{2}\right]^2 & -\frac{i\,\text{Sin}[\alpha]}{\sqrt{2}} & \text{Cos}\left[\frac{\alpha}{2}\right]^2 \end{pmatrix}$$

Eigenvalues and eigenkets of Jy

```
eq2 = Eigensystem[Jy]
```

$$\left\{\{-\hbar, \hbar, 0\}, \left\{\left\{-1, i\sqrt{2}, 1\right\}, \left\{-1, -i\sqrt{2}, 1\right\}, \{1, 0, 1\}\right\}\right\}$$

```
ψ1y = -Normalize[eq2[[2, 2]]]; ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

```
ψ2y = Normalize[eq2[[2, 3]]]; ψ2y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψ3y = - Normalize[eq2[[2, 1]]]; ψ3y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

```
UyT = {ψ1y, ψ2y, ψ3y}; Uy = Transpose[UyT]; Uy // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
UyH = UyT*; UyH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
UyH.Uy // Simplify
```

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

$$\text{Uy}.\begin{pmatrix} \text{Exp}[-i\,\theta] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{Exp}[i\,\theta] \end{pmatrix}.\text{UyH} \,//\, \text{ExpToTrig} \,//\, \text{TrigFactor} \,//\, \text{MatrixForm}$$

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{\text{Sin}[\theta]}{\sqrt{2}} & \text{Sin}\left[\frac{\theta}{2}\right]^2 \\ \frac{\text{Sin}[\theta]}{\sqrt{2}} & \text{Cos}[\theta] & -\frac{\text{Sin}[\theta]}{\sqrt{2}} \\ \text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{\text{Sin}[\theta]}{\sqrt{2}} & \text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

$$\text{MatrixExp}\left[\frac{-i}{\hbar}\,\text{Jy}\,\theta\right] \,//\, \text{Simplify} \,//\, \text{MatrixForm}$$

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{\text{Sin}[\theta]}{\sqrt{2}} & \text{Sin}\left[\frac{\theta}{2}\right]^2 \\ \frac{\text{Sin}[\theta]}{\sqrt{2}} & \text{Cos}[\theta] & -\frac{\text{Sin}[\theta]}{\sqrt{2}} \\ \text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{\text{Sin}[\theta]}{\sqrt{2}} & \text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

APPENDIX-II

The matrix of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ for J = 4 and 9/2, where the matrix of $\hat{J}_z$ has a diagonal form.

(a) $J = 4$

$\hat{J}_x =$

$$\begin{pmatrix} 0 & \sqrt{2}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\,\hbar & 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & \sqrt{5}\,\hbar & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{5}\,\hbar & 0 & \sqrt{5}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5}\,\hbar & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 & \sqrt{2}\,\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\,\hbar & 0 \end{pmatrix}$$

$\hat{J}_y =$

$$\begin{pmatrix} 0 & -i\sqrt{2}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{2}\,\hbar & 0 & -i\sqrt{\frac{7}{2}}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{7}{2}}\,\hbar & 0 & -\frac{3i\hbar}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3i\hbar}{\sqrt{2}} & 0 & -i\sqrt{5}\,\hbar & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{5}\,\hbar & 0 & -i\sqrt{5}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{5}\,\hbar & 0 & -\frac{3i\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3i\hbar}{\sqrt{2}} & 0 & -i\sqrt{\frac{7}{2}}\,\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{7}{2}}\,\hbar & 0 & -i\sqrt{2}\,\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}\,\hbar & 0 \end{pmatrix}$$

$\hat{J}_z =$

$$\begin{pmatrix} 4\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4\hbar \end{pmatrix}$$

(b) $J = 9/2$

$\hat{J}_x =$

$$\begin{pmatrix} 0 & \frac{3\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3\hbar}{2} & 0 & 2\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & \sqrt{6}\,\hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6}\,\hbar & 0 & \frac{5\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5\hbar}{2} & 0 & \sqrt{6}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6}\,\hbar & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\hbar & 0 & \frac{3\hbar}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\hbar}{2} & 0 \end{pmatrix}$$

$\hat{J}_y =$

$$\begin{pmatrix} 0 & -\frac{3 i \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3 i \hbar}{2} & 0 & -2 i \hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 i \hbar & 0 & -\frac{1}{2} i \sqrt{21}\, \hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} i \sqrt{21}\, \hbar & 0 & -i \sqrt{6}\, \hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \sqrt{6}\, \hbar & 0 & -\frac{5 i \hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5 i \hbar}{2} & 0 & -i \sqrt{6}\, \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \sqrt{6}\, \hbar & 0 & -\frac{1}{2} i \sqrt{21}\, \hbar & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} i \sqrt{21}\, \hbar & 0 & -2 i \hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 i \hbar & 0 & -\frac{3 i \hbar}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3 i \hbar}{2} & 0 \end{pmatrix}$$

$\hat{J}_z =$

$$\begin{pmatrix} \frac{9 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\hbar}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3 \hbar}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5 \hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7 \hbar}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9 \hbar}{2} \end{pmatrix}$$