Properties of angular momentum

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Here we consider the general formalism of angular momentum. We will discuss the various properties of the angular momentum operator including the commutation relations. The eigenvalues and eigenkets of the angular momentum are determined. The matrix elements of the angular momentum operators are evaluated for angular momentum with integers as well as half-integers.

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1. Commutation relations

The commutation relations for the orbital angular momentum

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

Generalization: definition of an angular momentum. The origin of the above relations lies in the geometric properties of rotations in three-dimensional space.

Now we define an angular momentum $\hat{J}_i$ ($i = x, y, z$) as any set of three observables satisfying

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, \qquad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, \qquad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

$$\hat{\boldsymbol{J}}^2 = \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hat{J}_z{}^2$$

$$[\hat{\boldsymbol{J}}^2, \hat{J}_x] = \hat{0}, \quad [\hat{\boldsymbol{J}}^2, \hat{J}_y] = \hat{0}, \quad [\hat{\boldsymbol{J}}^2, \hat{J}_z] = \hat{0}$$

((**Proof**))

$$\begin{aligned}
[\hat{\boldsymbol{J}}^2, \hat{J}_x] &= [\hat{J}_x{}^2, \hat{J}_x] + [\hat{J}_y{}^2, \hat{J}_x] + [\hat{J}_z{}^2, \hat{J}_x] \\
&= [\hat{J}_y{}^2, \hat{J}_x] + [\hat{J}_z{}^2, \hat{J}_x] \\
&= -2i\hbar(\hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y) + 2i\hbar(\hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y) \\
&= \hat{0}
\end{aligned}$$

since

$$[\hat{J}_y{}^2, \hat{J}_x] = \hat{J}_y\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y\hat{J}_y$$
$$= \hat{J}_y(\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y) + (\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y)\hat{J}_y$$
$$= -\hat{J}_y[\hat{J}_x, \hat{J}_y] - [\hat{J}_x, \hat{J}_y]\hat{J}_y$$
$$= -2i\hbar(\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y)$$

$$[\hat{J}_z{}^2, \hat{J}_x] = \hat{J}_z\hat{J}_z\hat{J}_x - \hat{J}_x\hat{J}_z\hat{J}_z$$
$$= \hat{J}_z(\hat{J}_z\hat{J}_x - \hat{J}_x\hat{J}_z) + (\hat{J}_z\hat{J}_x - \hat{J}_x\hat{J}_z)\hat{J}_z$$
$$= \hat{J}_z[\hat{J}_z, \hat{J}_x] + [\hat{J}_z, \hat{J}_x]\hat{J}_z$$
$$= 2i\hbar(\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y)$$

2. General theory of angular momentum

(a) $\hat{J}_+$ and $\hat{J}_-$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

where

$$\hat{J}_+{}^+ = \hat{J}_-, \qquad \hat{J}_-{}^+ = \hat{J}_+$$

$$[\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+, \qquad [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_-, \qquad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z$$

$$[\hat{\boldsymbol{J}}^2, \hat{J}_+] = [\hat{\boldsymbol{J}}^2, \hat{J}_-] = [\hat{\boldsymbol{J}}^2, \hat{J}_z] = \hat{0}$$

$$\hat{J}_+\hat{J}_- = \hat{\boldsymbol{J}}^2 - \hat{J}_z{}^2 + \hbar\hat{J}_z$$

$$\hat{J}_-\hat{J}_+ = \hat{\boldsymbol{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z$$

Thus we have

$$\hat{\boldsymbol{J}}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z{}^2$$

Formula:

$$[\boldsymbol{a}\cdot\hat{\boldsymbol{J}}, \boldsymbol{b}\cdot\hat{\boldsymbol{J}}] = i\hbar(\boldsymbol{a}\times\boldsymbol{b})\cdot\hat{\boldsymbol{J}}$$

((**Proof**))

$$I = [\boldsymbol{a} \cdot \hat{\boldsymbol{J}}, \boldsymbol{b} \cdot \hat{\boldsymbol{J}}] = \sum_{i,j} a_i b_j [\hat{J}_i, \hat{J}_j]$$

Since

$$[\hat{J}_i, \hat{J}_j] = i\hbar \varepsilon_{ijk} \hat{J}_k$$

then

$$I = \sum_{i,j} a_i b_j i\hbar \varepsilon_{ijk} \hat{J}_k = i\hbar (\boldsymbol{a} \times \boldsymbol{b}) \cdot \hat{\boldsymbol{J}}$$

(b) Notation for the eigenvalues of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$

For any ket $|\psi\rangle$

$$\begin{aligned}\langle \psi | \hat{\boldsymbol{J}}^2 | \psi \rangle &= \langle \psi | \hat{J}_x{}^2 | \psi \rangle + \langle \psi | \hat{J}_y{}^2 | \psi \rangle + \langle \psi | \hat{J}_z{}^2 | \psi \rangle \\ &= \langle \psi | \hat{J}_x{}^+ \hat{J}_x | \psi \rangle + \langle \psi | \hat{J}_y{}^+ \hat{J}_y | \psi \rangle + \langle \psi | \hat{J}_z{}^+ \hat{J}_z | \psi \rangle \geq 0\end{aligned}$$

For an eigenket $|\psi_\alpha\rangle$

$$\hat{\boldsymbol{J}}^2 |\psi_\alpha\rangle = \alpha |\psi_\alpha\rangle$$

$$\langle \psi_\alpha | \hat{\boldsymbol{J}}^2 | \psi_\alpha \rangle = \alpha \langle \psi_\alpha | \psi_\alpha \rangle = \alpha \geq 0$$

We shall write

$$\hat{\boldsymbol{J}}^2 |\psi_\alpha\rangle = \hbar^2 j(j+1) |\psi_\alpha\rangle = \lambda \hbar^2 |\psi_\alpha\rangle$$

where

$$\lambda = j(j+1) \geq 0$$

((**Note**)) **Levi-Civita symbol**

In three dimensions, the Levi-Civita symbol ε_{ijk} is defined as follows:

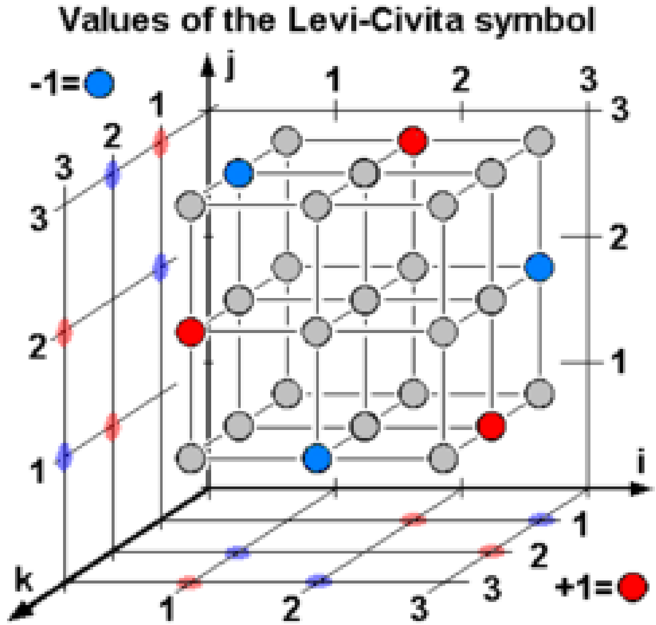
ε_{ijk} is 1 if (i, j, k) is an even permutation of (1,2,3).

ε_{ijk} is -1 if it is an odd permutation,

ε_{ijk} is 0 if any index is repeated.

$\varepsilon_{123} = 1$, $\varepsilon_{132} = -1$, $\varepsilon_{213} = -1$,

$$\varepsilon_{231} = 1, \qquad \varepsilon_{312} = 1, \qquad \varepsilon_{321} = -1.$$



3. Eigenvalue equations for $\hat{J}^2$ and $\hat{J}_z$

$$|\psi_\alpha\rangle = |j,m\rangle$$

$|j.m\rangle$ is the simultaneous eigenket of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, since $[\hat{\boldsymbol{J}}^2, \hat{J}_z] = \hat{0}$

$$\hat{\boldsymbol{J}}^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$$

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

We discuss the eigenvalues of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$

Lemma 1 (Properties of eigenvalues of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$)

j and *m* satisfy the inequality

$-j \le m \le j$

((**Proof**))

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle \ge 0,$$

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle \geq 0 .$$

We find

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle = \langle j,m|\hat{\boldsymbol{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z|j,m\rangle = \hbar^2[j(j+1) - m(m+1)] \geq 0 ,$$

and

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = \langle j,m|\hat{\boldsymbol{J}}^2 - \hat{J}_z{}^2 + \hbar\hat{J}_z|j,m\rangle = \hbar^2[j(j+1) - m(m-1)] \geq 0 .$$

Then we have

$$[j(j+1) - m(m+1)] = (j-m)(j+m+1) \geq 0 ,$$

$$[j(j+1) - m(m-1)] = (j-m+1)(j+m) \geq 0 .$$

Then

$$-(j+1) \leq m \leq j ,$$

and

$$-j \leq m \leq j+1 .$$

If $-j \leq m \leq j$, these two conditions are satisfied simultaneously.

Lemma II (Properties of the ket vector of $\hat{J}_-|j,m\rangle$)

(i) If $m = -j$, $\hat{J}_-|j,m\rangle = \hat{0}$

(ii) If $m > -j$, $\hat{J}_-|j,m\rangle$ is a non-null eigenket of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$ with the eigenvalues $j(j+1)\hbar^2$ and $(m-1)\hbar$.

[**Proof of (i)**)]
Since

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = \hbar^2[(j+m)(j-m+1)] = 0$$

for $m = -j$, we get

$$\hat{J}_-|j,m\rangle = \hat{0}$$

for *m*=-*j*.

Conversely, if

$$\hat{J}_-|j,m\rangle = \hat{0}$$

then we have

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = \hbar^2[(j+m)(j-m+1)] = 0$$

Then we have $j = -m$.

[**Proof of (ii)**]
Since

$$[\hat{\boldsymbol{J}}^2, \hat{J}_-] = \hat{0}$$

$$[\hat{\boldsymbol{J}}^2, \hat{J}_-]|j,m\rangle = \hat{0}$$

or

$$\hat{\boldsymbol{J}}^2\hat{J}_-|j,m\rangle = \hat{J}_-\hat{\boldsymbol{J}}^2|j,m\rangle = \hbar^2 j(j+1)\hat{J}_-|j,m\rangle$$

So $\hat{J}_-|j,m\rangle$ is an eigenket of $\hat{\boldsymbol{J}}^2$ with the eigenvalue $\hbar^2 j(j+1)$.

Moreover,

$$[\hat{J}_z, \hat{J}_-]|j,m\rangle = -\hbar\hat{J}_-|j,m\rangle$$

or

$$\hat{J}_z\hat{J}_-|j,m\rangle = \hat{J}_-\hat{J}_z|j,m\rangle - \hbar\hat{J}_-|j,m\rangle = \hbar(m-1)\hat{J}_-|j,m\rangle$$

So $\hat{J}_-|j,m\rangle$ is an eigenket of $\hat{J}_z$ with the eigenvalue $\hbar(m-1)$.

Lemma III (Properties of the ket vector of $\hat{J}_+|j,m\rangle$)

(i) If $m = j$,

$$\hat{J}_+|j,m\rangle = \hat{0}.$$

(ii) If $m<j$, $\hat{J}_+|j,m\rangle$ is a non-null eigenket of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$ with the eigenvalues $j(j+1)\hbar^2$ and $(m+1)\hbar$.

[**Proof of (i)**]

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle = \hbar^2[(j-m)(j+m+1)] \geq 0$$

If $m=j$, then

$$\hat{J}_+|j,m\rangle = \hat{0}.$$

[**Proof of (ii)**]

$$[\hat{\boldsymbol{J}}^2, \hat{J}_+] = \hat{0},$$

or

$$[\hat{\boldsymbol{J}}^2, \hat{J}_+]|j,m\rangle = \hat{0},$$

or

$$\langle j,m|\hat{\boldsymbol{J}}^2\hat{J}_+|j,m\rangle = \hbar^2 j(j+1)\hat{J}_+|j,m\rangle.$$

$\hat{J}_+|j,m\rangle$ is an eigenket of $\hat{J}^2$ with an eigenvalues $\hbar^2 j(j+1)$.

$$[\hat{J}_z, \hat{J}_+]|j,m\rangle = \hbar\hat{J}_+|j,m\rangle$$

or

$$\hat{J}_z\hat{J}_+|j,m\rangle = \hat{J}_+\hat{J}_z|j,m\rangle + \hbar\hat{J}_+|j,m\rangle = \hbar(m+1)|j,m\rangle$$

$\hat{J}_+|j,m\rangle$ is an eigenket of $\hat{J}_z$ with an eigenvalues $\hbar(m+1)$.

4. Determination of the spectrum of $\hat{J}^2$ and $\hat{J}_z$.

There exists a positive or zero integer p such that

$$m - p = -j \qquad (1)$$

$|j,m\rangle$: [eigenvalues $\hbar m$, $\hbar^2 j(j+1)$]

$\hat{J}_{-}|j,m\rangle$: [$\hbar(m$-1), $\hbar^2 j(j$+1)]

$(\hat{J}_{-})^2|j,m\rangle$: [$\hbar(m$-2), $\hbar^2 j(j$+1)]

....

$(\hat{J}_{-})^p|j,m\rangle$: [$\hbar(m$-p), $\hbar^2 j(j$+1)]

$$(\hat{J}_{-})^{p+1}|j,m\rangle = \hat{0}$$

There exists a positive or zero integer such that

$$m + q = j, \qquad (2)$$

$|j,m\rangle$: [eigenvalues $\hbar$m, $\hbar^2 j(j$+1)]

$\hat{J}_{+}|j,m\rangle$: [$\hbar(m$+1), $\hbar^2 j(j$+1)]

$(\hat{J}_{+})^2|j,m\rangle$: [$\hbar(m$+2), $\hbar^2 j(j$+1)]

....

$(\hat{J}_{+})^q|j,m\rangle$: [$\hbar(m+q)$, $\hbar^2 j(j$+1)]

$$(\hat{J}_{+})^{q+1}|j,m\rangle = \hat{0}$$

Combining Eqs.(1) and (2),

$$p + q = 2j$$

Since *p* and *q* are integers, *j* is therefore an integer or a half-integer.

j = 0, 1/2, 1, 3/2, 2, ...

m = -j, -j+1,.........., j-1, j.

(i) If *j* is an integer, then *m* is an integer.
(ii) If *j* is a half-integer, then *m* is a half-integer.

5. $|j,m\rangle$ representation

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle = \langle j,m|\hat{\boldsymbol{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z|j,m\rangle = \hbar^2[(j-m)(j+m+1)]$$

Since

$$\hat{J}_+|j,m\rangle = \alpha|j,m+1\rangle$$

we have

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle = |\alpha|^2 = \hbar^2[(j-m)(j+m+1)].$$

Thus

$$\hat{J}_+|j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j,m+1\rangle.$$

Similarly

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = \langle j,m|\hat{\boldsymbol{J}}^2 - \hat{J}_z{}^2 + \hbar\hat{J}_z|j,m\rangle = \hbar^2[(j+m)(j-m+1)].$$

Since

$$\hat{J}_-|j,m\rangle = \beta|j,m-1\rangle,$$

we have

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = |\beta|^2 = \hbar^2[(j+m)(j-m+1)],$$

or

$$\hat{J}_-|j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j,m-1\rangle.$$

In summary:

$$\hat{\boldsymbol{J}}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$$

$$\hat{J}_z|j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_+|j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j,m+1\rangle$$

$$\hat{J}_-|j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j,m-1\rangle$$

6. Matrix element of the angular momentum

Using Mathematica, you determine the matrix elements of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$

$$\hat{\boldsymbol{J}}^2 |j.m\rangle = \hbar^2 j(j+1)|j.m\rangle$$

$$\hat{J}_z |j,m\rangle = \hbar m|j,m\rangle$$

$$\hat{J}_+ |j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j,m+1\rangle$$

$$\hat{J}_- |j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j,m-1\rangle$$

Since

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$$

$$\hat{J}_- = \hat{J}_x - i\hat{J}_y$$

we get

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$$

$$\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-) = -\frac{i}{2}(\hat{J}_+ - \hat{J}_-)$$

$$\hat{J}_x |j,m\rangle = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)|j,m\rangle$$
$$= \frac{\hbar}{2}(\sqrt{(j-m)(j+m+1)}|j,m+1\rangle + \sqrt{(j+m)(j-m+1)}|j,m-1\rangle)$$

$$\hat{J}_y |j,m\rangle = -\frac{i}{2}(\hat{J}_+ - \hat{J}_-)|j,m\rangle$$
$$= -\frac{i\hbar}{2}(\sqrt{(j-m)(j+m+1)}|j,m+1\rangle - \sqrt{(j+m)(j-m+1)}|j,m-1\rangle)$$

$$\hat{J}_z |j,m\rangle = \hbar m|j,m\rangle$$

Thus the matrix elements are expressed by

$$\langle j,m'|\hat{J}_x|j,m\rangle = \frac{\hbar}{2}(\sqrt{(j-m)(j+m+1)}\langle j,m'|j,m+1\rangle + \sqrt{(j+m)(j-m+1)}\langle j,m'|j,m-1\rangle)$$
$$= \frac{\hbar}{2}(\sqrt{(j-m)(j+m+1)}\delta_{m',m+1} + \sqrt{(j+m)(j-m+1)}\delta_{m',m-1})$$

$$\langle j,m'|\hat{J}_y|j,m\rangle = -\frac{i\hbar}{2}(\sqrt{(j-m)(j+m+1)}\langle j,m'|j,m+1\rangle - \sqrt{(j+m)(j-m+1)}\langle j,m'|j,m-1\rangle)$$
$$= -\frac{i\hbar}{2}(\sqrt{(j-m)(j+m+1)}\delta_{m',m+1} - \sqrt{(j+m)(j-m+1)}\delta_{m',m-1})$$

$$\langle j.m'|\hat{J}_z|j,m\rangle = m\hbar\langle j,m|j,m'\rangle = \hbar m\delta_{m,m'}$$

Using this formula, we can get the matrix of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ for each j (= 1/2, 1, 3/2, 2, 5/2, ...) and *m* (=*j*, *j*-1, *j*-2,...., -*j*+1, -*j*).

7. Effect of fluctuation in the direction of *J*

We now consider the fluctuation of the angular momentum,

$$(\Delta J_x)^2 = \langle j,m|(\hat{J}_x - \langle \hat{J}_x\rangle \hat{1})^2|j,m\rangle = \langle \hat{J}_x{}^2\rangle - \langle \hat{J}_x\rangle^2 = \langle \hat{J}_x{}^2\rangle$$

$$(\Delta J_y)^2 = \langle j,m|(\hat{J}_y - \langle \hat{J}_y\rangle \hat{1})^2|j,m\rangle = \langle \hat{J}_y{}^2\rangle - \langle \hat{J}_y\rangle^2 = \langle \hat{J}_y{}^2\rangle$$

Then we have

$$\begin{aligned}\langle \hat{\boldsymbol{J}}^2\rangle &= \hbar^2 j(j+1)\\ &= \langle \hat{J}_x{}^2\rangle + \langle \hat{J}_y{}^2\rangle + \langle \hat{J}_z{}^2\rangle\\ &= (\Delta J_x)^2 + (\Delta J_x)^2 + \langle \hat{J}_z{}^2\rangle\\ &= (\Delta J_x)^2 + (\Delta J_x)^2 + \hbar^2 m^2\end{aligned}$$

or

$$(\Delta J_x)^2 + (\Delta J_x)^2 = \hbar^2[j(j+1) - m^2]$$

The fluctuation in the component of the angular momentum which is normal to the *z* axis will be a minimum when *m* = *j*. We therefore obtain

$$[(\Delta J_x)^2 + (\Delta J_x)^2]_{\min} = j\hbar^2$$

This means in a rough manner of speaking that the minimum angle between the direction of the ***J*** vector and the z axis is given by

$$\sin\theta_{\min} = \frac{\sqrt{j}}{\sqrt{j(j+1)}} = \frac{1}{\sqrt{j+1}}$$

or

$$\cos\theta_{\min} = \frac{j}{\sqrt{j(j+1)}}$$

((D. **Bohm, quantum theory, p.319**))

It is not correct to imagine that the angular momentum points in some definite direction which we do not happen to be able to measure with complete precision. Instead, whenever $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$ have definite values, one should imagine that the entire cone of directions corresponding to those values of J_x and J_y consistent with the given j^2 and m are covered simultaneously because important physical consequence may follow from the effects of interference of wave functions corresponding to different components of angular momentum.

8. Vector representation of allowed angular momentum

We consider a case which j is some fixed number (j = 1/2, 1, 3/2, 1,...). Then the total angular momentum may be represented by a vector of length

$$\hbar\sqrt{j(j+1)}$$

The component m in the z direction is

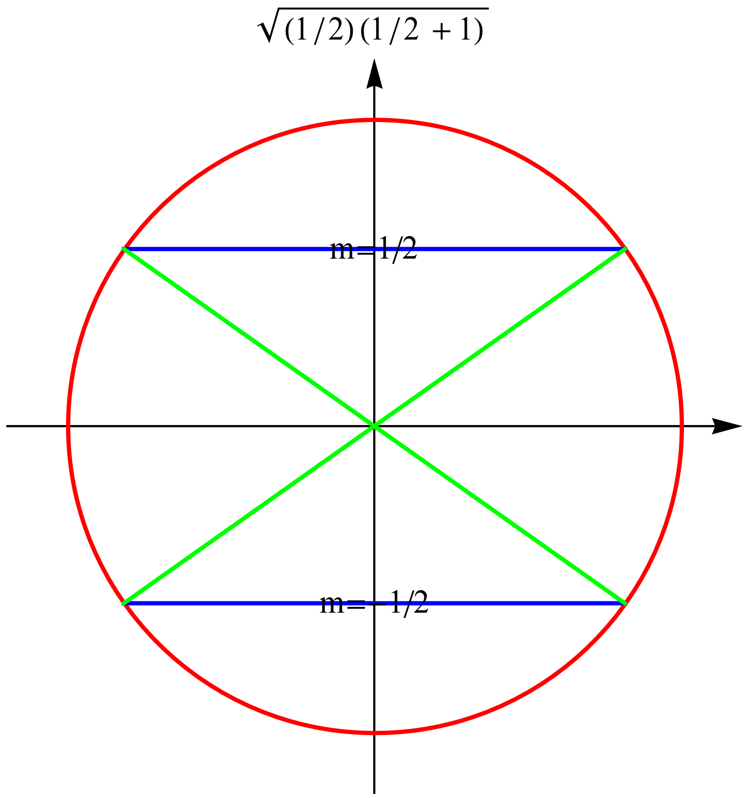
$$m = j, j - 1, j - 2, ..., -j + 1, -j$$

The vector ***J*** should be thought of as covering a cone, with vector angle given by

$$\cos\theta_m = \frac{m}{\sqrt{j(j+1)}}$$

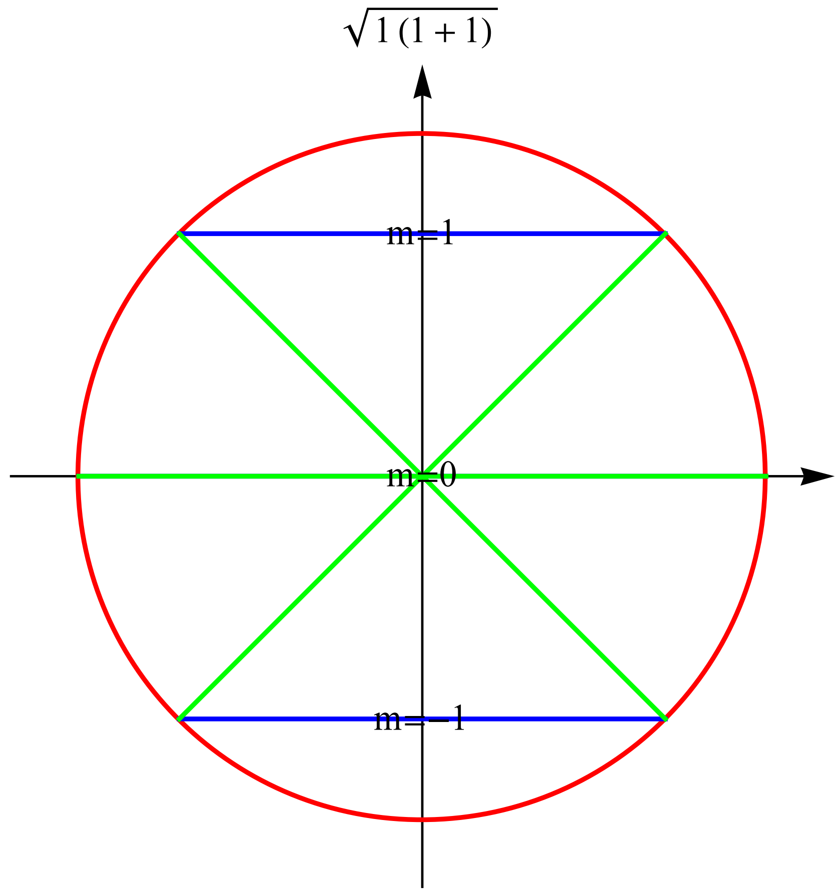
(a)

j = 1/2
m = 1/2, -1/2



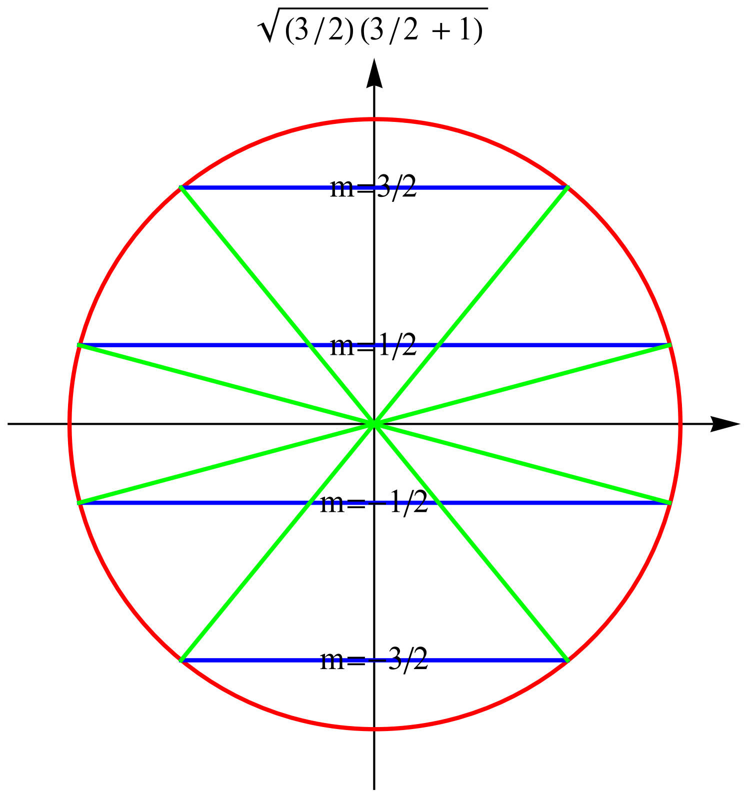
(b)

$j = 1$
$m = 1, 0, -1$



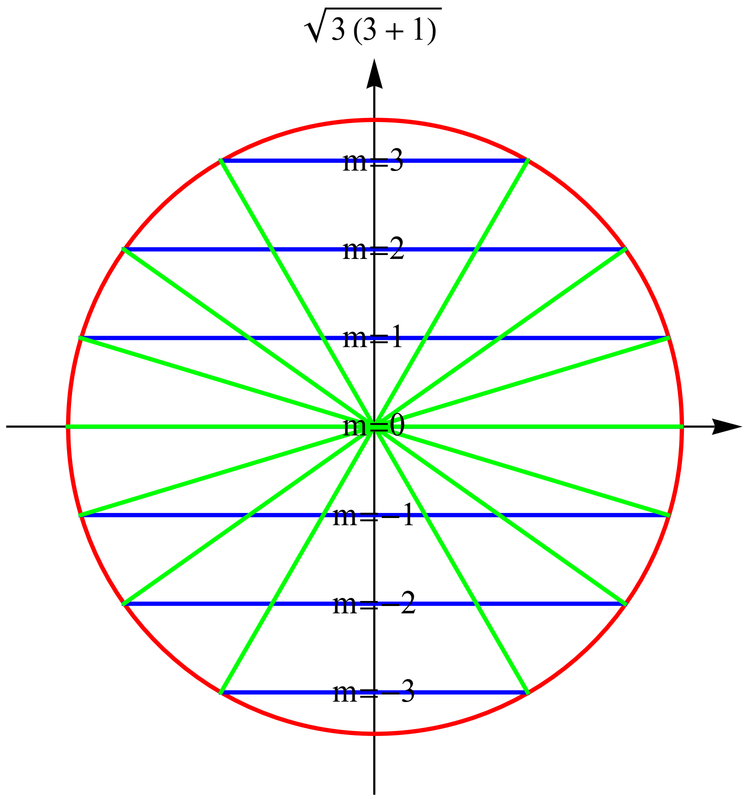
(c)

j = 3/2
m = 3/2, 1/2, -1/2, -3/2



(d)

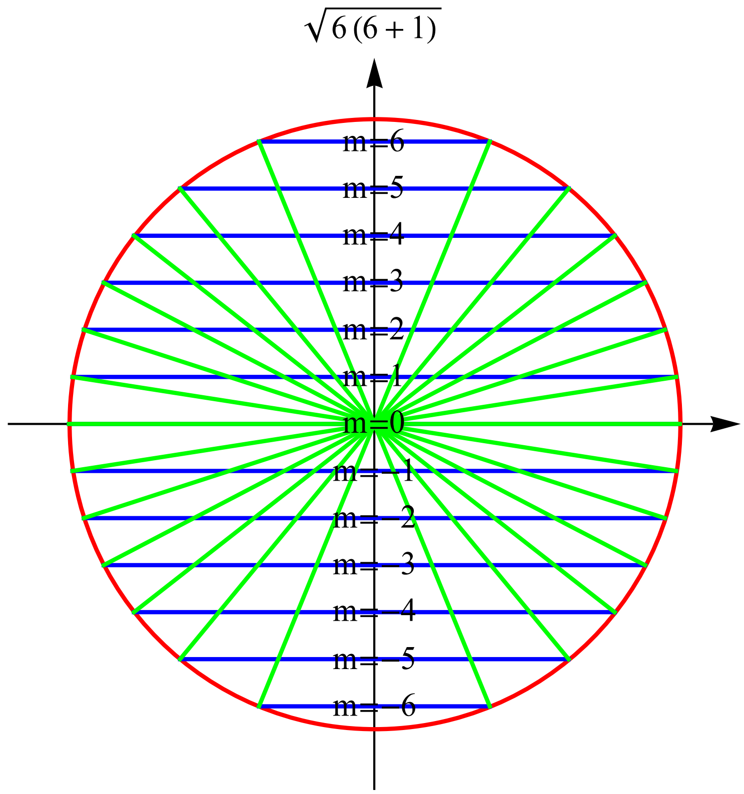
$j = 3$
$m = 3, 2, 1, 0, -1, -2, -3$



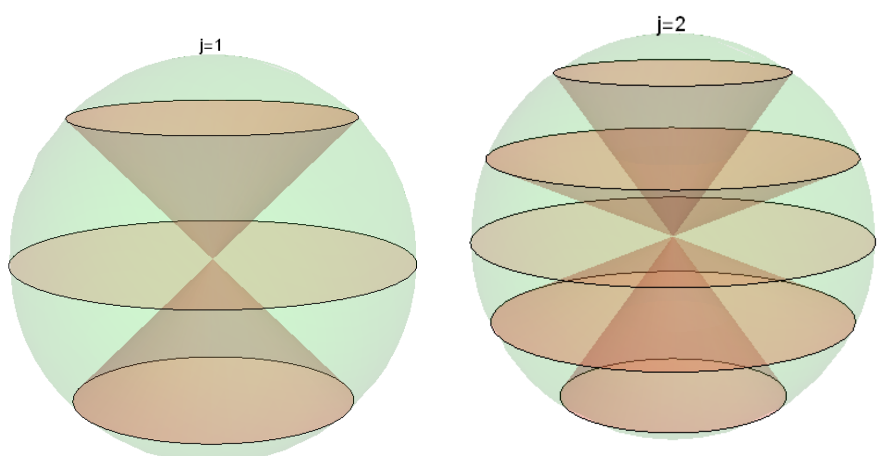
(e)

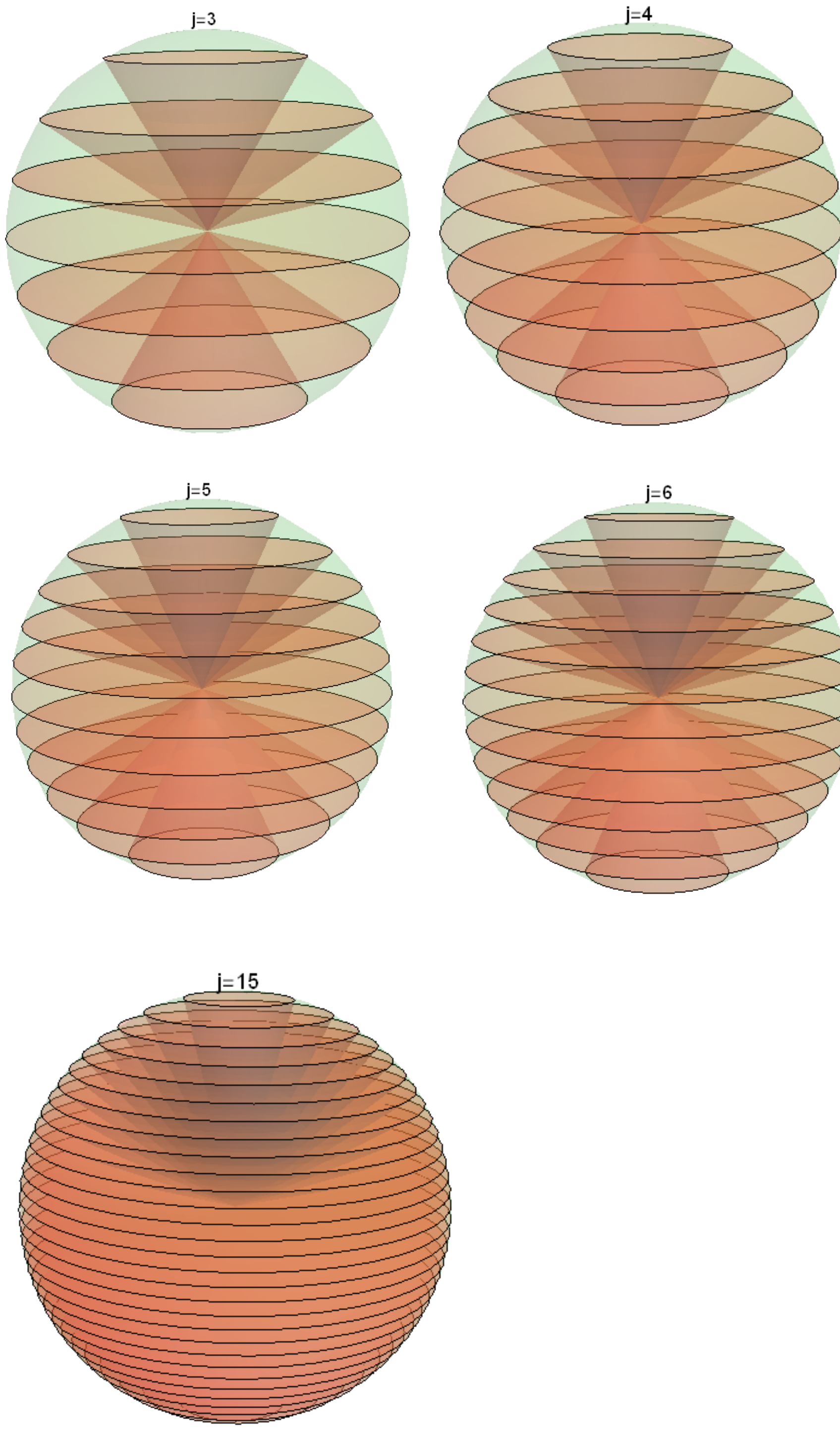
$j = 6$

$m = 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6$

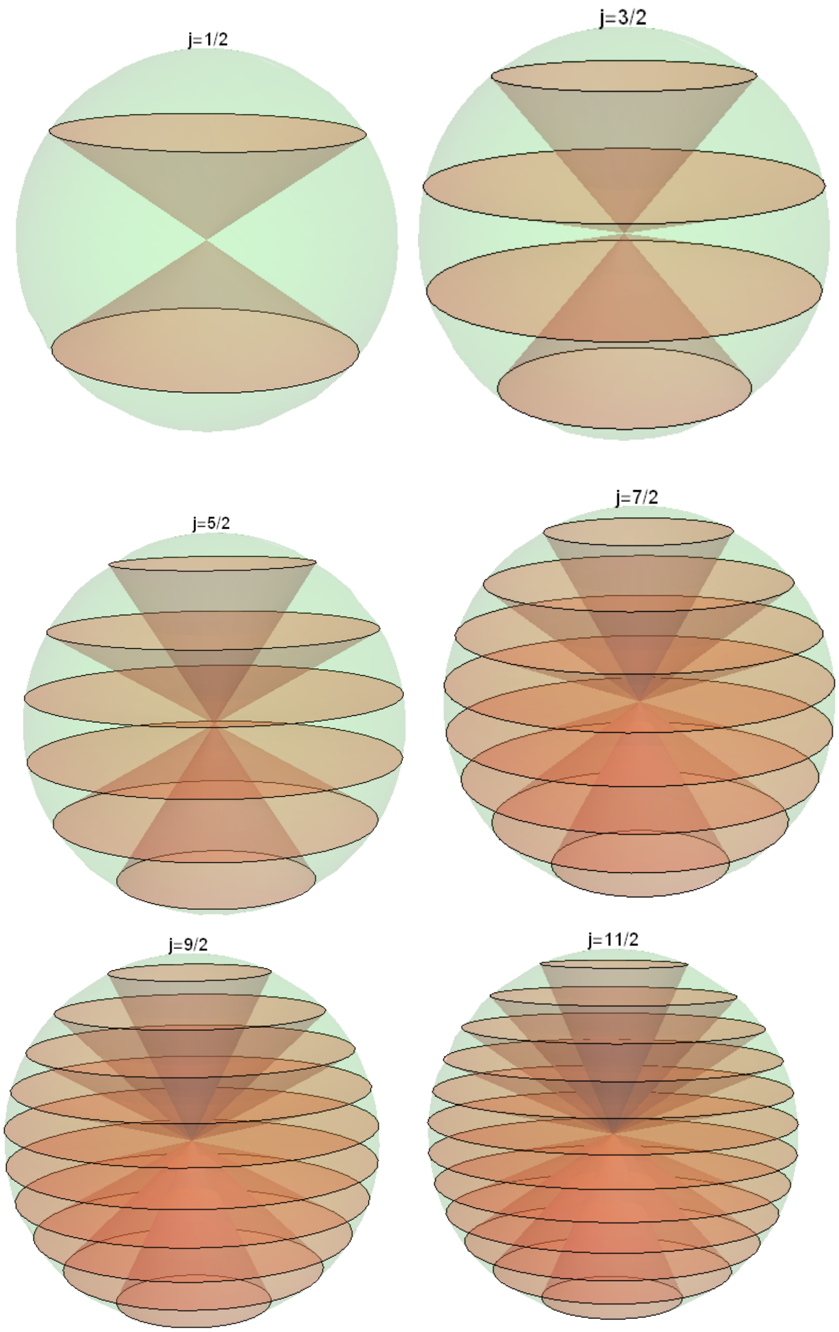


(a) j = integers





(b) j = half integer



REFERENCES

J.J. Sakurai, *Modern Quantum Mechanics*, Revised Edition (Addison-Wesley, Reading Massachusetts, 1994).

C. Cohen-Tannoudji and B. Diu, and F. Laloe, *Quantum Mechanics*, vol.1 and vol. 2 (John Wiley & Sons, New York, 1977).

A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, 1957).

M.E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, 1957, New York, Dover Publications, New York, 1957).
L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics* (Addison-Wesley, Reading, 1981).
D.M. Brink and G.R. Satchler, *Angular Momentum*, 2^{nd} edition (Clarendon Press, Oxford, 1968).
D.H. McIntyre, *Quantum Mechanics: A paradigms Approach* (Pearson, 2012).
V. Devanthan, Angular Momentum Techniques in Quantum Mechanics (Kluwer Academic).