

Matrix representation of angular momentum with J
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Here we summarize the matrix representation of the angular momentum with

$j = 1/2, 1, 3/2, 2, 5/2, 3,$ and so on.

We also show the eigenkets and the corresponding unitary operators. The Mathematica programs are very useful for the derivation of these forms. We solve the eigenvalue problem for the angular momentum (\hat{J}_x , \hat{J}_y , and \hat{J}_z). The normalized eigenkets are obtained. The corresponding unitary operators are also discussed.

1. $j = 1/2$

2x2 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2} \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \hbar \hat{\sigma}_x,$$

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \frac{1}{2} \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2} \hbar \hat{\sigma}_y,$$

$$\hat{J}_z = \hbar \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2} \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \hbar \hat{\sigma}_z$$

under the basis of $\{|+z\rangle, |-z\rangle\}$, where $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are the Pauli matrices.

where

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

(a) **The eigenkets of \hat{J}_x :**

$$|+x\rangle = \hat{U}_x|+z\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-x\rangle = \hat{U}_x|-z\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with the unitary operator \hat{U}_x ,

$$\hat{U}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(b) The eigenkets of \hat{J}_y :

$$|+y\rangle = \hat{U}_y|+z\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-y\rangle = \hat{U}_y|-z\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

with

$$\hat{U}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \hat{U}_y^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

2. $j = 1$

3x3 matrices

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

where

$$|1,1\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) **The eigenkets of \hat{J}_x :**

$$|1,1\rangle_x = \hat{U}_x |1,1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad |1,0\rangle_x = \hat{U}_x |1,0\rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$|1,-1\rangle_x = \hat{U}_x |1,-1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

with the unitary operator,

$$\hat{U}_x = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad \hat{U}_x^\dagger = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

(b) **The eigenkets of \hat{J}_y :**

$$|1,1\rangle_y = \hat{U}_y |1,1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}, \quad |1,0\rangle_y = \hat{U}_y |1,0\rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$|1,-1\rangle_y = \hat{U}_y |1,-1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}$$

with the unitary operator,

$$\hat{U}_y = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}, \quad \hat{U}_y^+ = \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

3. $j = 3/2$

4 x 4 matrices

$$\hat{J}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$\hat{J}_y = \frac{i\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$\hat{J}_z = \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left| \frac{3}{2}, \frac{1}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\left| \frac{3}{2}, -\frac{1}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(a) The eigenket of \hat{J}_x ;

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle_x = \hat{U}_x \left| \frac{3}{2}, \frac{3}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix},$$

$$\left| \frac{3}{2}, \frac{1}{2} \right\rangle_x = \hat{U}_x \left| \frac{3}{2}, \frac{1}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{pmatrix}$$

$$\left| \frac{3}{2}, -\frac{1}{2} \right\rangle_x = \hat{U}_x \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{pmatrix},$$

$$\left| \frac{3}{2}, -\frac{3}{2} \right\rangle_x = \hat{U}_x \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{pmatrix}$$

with the unitary operator

$$\hat{U}_x = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \\ \sqrt{3} & 1 & -1 & -\sqrt{3} \\ \sqrt{3} & -1 & -1 & \sqrt{3} \\ 1 & -\sqrt{3} & \sqrt{3} & -1 \end{pmatrix}$$

(b) **The eigenket of \hat{J}_y ;**

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle_y = \hat{U}_y \left| \frac{3}{2}, \frac{3}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ i\sqrt{3} \\ -\sqrt{3} \\ -i \end{pmatrix},$$

$$\left| \frac{3}{2}, \frac{1}{2} \right\rangle_y = \hat{U}_y \left| \frac{3}{2}, \frac{1}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} \\ i \\ 1 \\ i\sqrt{3} \end{pmatrix}$$

$$\left| \frac{3}{2}, -\frac{1}{2} \right\rangle_y = \hat{U}_y \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} \\ -i \\ 1 \\ -i\sqrt{3} \end{pmatrix},$$

$$\left| \frac{3}{2}, -\frac{3}{2} \right\rangle_y = \hat{U}_y \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -i\sqrt{3} \\ -\sqrt{3} \\ i \end{pmatrix}$$

with

$$\hat{U}_y = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \\ i\sqrt{3} & i & -i & -i\sqrt{3} \\ -\sqrt{3} & 1 & 1 & -\sqrt{3} \\ -i & i\sqrt{3} & -i\sqrt{3} & i \end{pmatrix}$$

4. $j = 2$

5x5 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

with

$$|2,2\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2,1\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2,0\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|2,-1\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2,-2\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) **Eigenvectors of \hat{J}_x**

$$|2,2\rangle_x = \hat{U}_x |2,2\rangle_z = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \\ \sqrt{6} \\ 2 \\ 1 \end{pmatrix}, \quad |2,1\rangle_x = \hat{U}_x |2,1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$$|2,0\rangle_x = \hat{U}_x |2,0\rangle_z = \frac{1}{4} \begin{pmatrix} \sqrt{6} \\ 0 \\ -2 \\ 0 \\ \sqrt{6} \end{pmatrix}, \quad |2,-1\rangle_x = \hat{U}_x |2,-1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$|2,-2\rangle_x = \hat{U}_x |2,-2\rangle_z = \frac{1}{4} \begin{pmatrix} 1 \\ -2 \\ \sqrt{6} \\ -2 \\ 1 \end{pmatrix}$$

with the unitary operator,

$$\hat{U}_x = \frac{1}{4} \begin{pmatrix} 1 & 2 & \sqrt{6} & 2 & 1 \\ 2 & 2 & 0 & -2 & -2 \\ \sqrt{6} & 0 & -2 & 0 & \sqrt{6} \\ 2 & -2 & 0 & 2 & -2 \\ 1 & -2 & \sqrt{6} & -2 & 1 \end{pmatrix}$$

(b) **Eigenvectors of \hat{J}_y**

$$|2,2\rangle_y = \hat{U}_y |2,2\rangle_z = \frac{1}{4} \begin{pmatrix} 1 \\ 2i \\ -\sqrt{6} \\ -2i \\ 1 \end{pmatrix}, \quad |2,1\rangle_y = \hat{U}_y |2,1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \\ i \\ -1 \end{pmatrix},$$

$$|2,0\rangle_y = \hat{U}_y |2,0\rangle_z = \frac{1}{4} \begin{pmatrix} \sqrt{6} \\ 0 \\ 2 \\ 0 \\ \sqrt{6} \end{pmatrix}, \quad |2,-1\rangle_y = \hat{U}_y |2,-1\rangle_z = \frac{1}{2} \begin{pmatrix} 1 \\ -i \\ 0 \\ -i \\ -1 \end{pmatrix},$$

$$|2,-2\rangle_y = \hat{U}_y |2,-2\rangle_z = \frac{1}{4} \begin{pmatrix} 1 \\ -2i \\ -\sqrt{6} \\ 2i \\ 1 \end{pmatrix},$$

with the unitary operator,

$$\hat{U}_y = \frac{1}{4} \begin{pmatrix} 1 & 2 & \sqrt{6} & 2 & 1 \\ 2i & 2i & 0 & -2i & -2i \\ -\sqrt{6} & 0 & 2 & 0 & -\sqrt{6} \\ -2i & 2i & 0 & -2i & 2i \\ 1 & -2 & \sqrt{6} & -2 & 1 \end{pmatrix}$$

$$\hat{U}_y^\dagger = \frac{1}{4} \begin{pmatrix} 1 & -2i & -\sqrt{6} & 2i & 1 \\ 2 & -2i & 0 & -2i & -2 \\ \sqrt{6} & 0 & 2 & 0 & \sqrt{6} \\ 2 & 2i & 0 & 2i & -2 \\ 1 & 2i & -\sqrt{6} & -2i & 1 \end{pmatrix}$$

5. $j = 5/2$

6x6 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & \frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{5}}{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \frac{\sqrt{5}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 \end{pmatrix},$$

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & \frac{-i\sqrt{5}}{2} & 0 & 0 & 0 & 0 \\ \frac{i\sqrt{5}}{2} & 0 & -i\sqrt{2} & 0 & 0 & 0 \\ 0 & i\sqrt{2} & 0 & -\frac{3i}{2} & 0 & 0 \\ 0 & 0 & \frac{3i}{2} & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & \frac{-i\sqrt{5}}{2} \\ 0 & 0 & 0 & 0 & \frac{i\sqrt{5}}{2} & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar \begin{pmatrix} \frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \end{pmatrix}.$$

with

$$\left| \frac{5}{2}, \frac{5}{2} \right\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left| \frac{5}{2}, \frac{3}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left| \frac{5}{2}, \frac{1}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left| \frac{5}{2}, -\frac{1}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \left| \frac{5}{2}, -\frac{3}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \left| \frac{5}{2}, -\frac{5}{2} \right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(a) **Eigenvectors of \hat{J}_x**

$$\left| \frac{5}{2}, \frac{5}{2} \right\rangle_x = \hat{U}_x \left| \frac{5}{2}, \frac{5}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{5} \\ \sqrt{10} \\ \sqrt{10} \\ \sqrt{5} \\ 1 \end{pmatrix}$$

$$\left| \frac{5}{2}, \frac{3}{2} \right\rangle_x = \hat{U}_x \left| \frac{5}{2}, \frac{3}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{5} \\ 3 \\ \sqrt{2} \\ -\sqrt{2} \\ -3 \\ -\sqrt{5} \end{pmatrix}$$

$$\left| \frac{5}{2}, \frac{1}{2} \right\rangle_x = \hat{U}_x \left| \frac{5}{2}, \frac{1}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{10} \\ \sqrt{2} \\ -2 \\ -2 \\ \sqrt{2} \\ \sqrt{10} \end{pmatrix}$$

$$\left| \frac{5}{2}, -\frac{1}{2} \right\rangle_x = \hat{U}_x \left| \frac{5}{2}, -\frac{1}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{10} \\ -\sqrt{2} \\ -2 \\ 2 \\ \sqrt{2} \\ -\sqrt{10} \end{pmatrix},$$

$$\left| \frac{5}{2}, -\frac{3}{2} \right\rangle_x = \hat{U}_x \left| \frac{5}{2}, -\frac{3}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{5} \\ -3 \\ \sqrt{2} \\ \sqrt{2} \\ -3 \\ \sqrt{5} \end{pmatrix},$$

$$\left| \frac{5}{2}, -\frac{5}{2} \right\rangle_x = \hat{U}_x \left| \frac{5}{2}, -\frac{5}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{5} \\ \sqrt{10} \\ -\sqrt{10} \\ \sqrt{5} \\ -1 \end{pmatrix},$$

with the unitary operator given by

$$\hat{U}_x = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 & \sqrt{5} & \sqrt{10} & \sqrt{10} & \sqrt{5} & 1 \\ \sqrt{5} & 3 & \sqrt{2} & -\sqrt{2} & -3 & -\sqrt{5} \\ \sqrt{10} & \sqrt{2} & -2 & -2 & \sqrt{2} & \sqrt{10} \\ \sqrt{10} & -\sqrt{2} & -2 & 2 & \sqrt{2} & -\sqrt{10} \\ \sqrt{5} & -3 & \sqrt{2} & \sqrt{2} & -3 & \sqrt{5} \\ 1 & -\sqrt{5} & \sqrt{10} & -\sqrt{10} & \sqrt{5} & -1 \end{pmatrix}.$$

(b) Eigenvectors of \hat{J}_y

$$\left| \frac{5}{2}, \frac{5}{2} \right\rangle_y = \hat{U}_y \left| \frac{5}{2}, \frac{5}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 \\ i\sqrt{5} \\ -\sqrt{10} \\ -i\sqrt{10} \\ \sqrt{5} \\ i \end{pmatrix},$$

$$\left| \frac{5}{2}, \frac{3}{2} \right\rangle_y = \hat{U}_y \left| \frac{5}{2}, \frac{3}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{5} \\ i3 \\ -\sqrt{2} \\ i\sqrt{2} \\ -i3 \\ -i\sqrt{5} \end{pmatrix},$$

$$\left| \frac{5}{2}, \frac{1}{2} \right\rangle_y = \hat{U}_y \left| \frac{5}{2}, \frac{1}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{10} \\ i\sqrt{2} \\ 2 \\ i2 \\ \sqrt{2} \\ i\sqrt{10} \end{pmatrix},$$

$$\left| \frac{5}{2}, -\frac{1}{2} \right\rangle_y = \hat{U}_y \left| \frac{5}{2}, -\frac{1}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{10} \\ -i\sqrt{2} \\ 2 \\ -i2 \\ \sqrt{2} \\ -i\sqrt{10} \end{pmatrix},$$

$$\left| \frac{5}{2}, -\frac{3}{2} \right\rangle_y = \hat{U}_y \left| \frac{5}{2}, -\frac{3}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} \sqrt{5} \\ -i3 \\ -\sqrt{2} \\ -i\sqrt{2} \\ -3 \\ i\sqrt{5} \end{pmatrix},$$

$$\left| \frac{5}{2}, -\frac{5}{2} \right\rangle_y = \hat{U}_y \left| \frac{5}{2}, -\frac{5}{2} \right\rangle_z = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 \\ -i\sqrt{5} \\ -\sqrt{10} \\ i\sqrt{10} \\ \sqrt{5} \\ -i \end{pmatrix},$$

with the unitary operator given by

$$\hat{U}_y = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 & \sqrt{5} & \sqrt{10} & \sqrt{10} & \sqrt{5} & 1 \\ i\sqrt{5} & 3i & i\sqrt{2} & -i\sqrt{2} & -i3 & -i\sqrt{5} \\ -\sqrt{10} & -\sqrt{2} & 2 & 2 & -\sqrt{2} & -\sqrt{10} \\ -i\sqrt{10} & i\sqrt{2} & i2 & -i2 & -i\sqrt{2} & i\sqrt{10} \\ \sqrt{5} & -3 & \sqrt{2} & \sqrt{2} & -3 & \sqrt{5} \\ i & -i\sqrt{5} & i\sqrt{10} & -i\sqrt{10} & i\sqrt{5} & -i \end{pmatrix}.$$

6. $j = 3$

7x7 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{pmatrix}$$

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{5}{2}} & 0 & -i\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{3} & 0 & -i\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{3} & 0 & -i\sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{\frac{5}{2}} & 0 & -i\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{3}{2}} & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

(a) **Eigenvectors of \hat{J}_x**

$$|3,3\rangle_x = \hat{U}_x |3,3\rangle_z = \frac{1}{8} \begin{pmatrix} 1 \\ \sqrt{6} \\ \sqrt{15} \\ 2\sqrt{5} \\ \sqrt{15} \\ \sqrt{6} \\ 1 \end{pmatrix},$$

$$|3,2\rangle_x = \hat{U}_x |3,2\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{6} \\ 4 \\ \sqrt{10} \\ 0 \\ -\sqrt{10} \\ -4 \\ -\sqrt{6} \end{pmatrix},$$

$$|3,1\rangle_x = \hat{U}_x |3,1\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{15} \\ \sqrt{10} \\ -1 \\ -2\sqrt{3} \\ -1 \\ \sqrt{10} \\ \sqrt{15} \end{pmatrix},$$

$$|3,0\rangle_x = \hat{U}_x |3,0\rangle_z = \frac{1}{4} \begin{pmatrix} \sqrt{5} \\ 0 \\ -\sqrt{3} \\ 0 \\ \sqrt{3} \\ 0 \\ -\sqrt{5} \end{pmatrix},$$

$$|3,-1\rangle_x = \hat{U}_x |3,-1\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{15} \\ -\sqrt{10} \\ -1 \\ 2\sqrt{3} \\ -1 \\ -\sqrt{10} \\ \sqrt{15} \end{pmatrix},$$

$$|3,-2\rangle_x = \hat{U}_x |3,-2\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{6} \\ -4 \\ \sqrt{10} \\ 0 \\ -\sqrt{10} \\ 4 \\ -\sqrt{6} \end{pmatrix},$$

$$|3,-3\rangle_x = \hat{U}_x |3,-3\rangle_z = \frac{1}{8} \begin{pmatrix} 1 \\ -\sqrt{6} \\ \sqrt{15} \\ -2\sqrt{5} \\ \sqrt{15} \\ -\sqrt{6} \\ 1 \end{pmatrix},$$

(b) **Eigenvectors of \hat{J}_y**

$$|3,3\rangle_y = \hat{U}_y |3,3\rangle_z = \frac{1}{8} \begin{pmatrix} 1 \\ i\sqrt{6} \\ -\sqrt{15} \\ -i2\sqrt{5} \\ \sqrt{15} \\ i\sqrt{6} \\ -1 \end{pmatrix},$$

$$|3,2\rangle_y = \hat{U}_y |3,2\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{6} \\ i4 \\ -\sqrt{10} \\ 0 \\ -\sqrt{10} \\ -i4 \\ \sqrt{6} \end{pmatrix},$$

$$|3,1\rangle_y = \hat{U}_y |3,1\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{15} \\ i\sqrt{10} \\ 1 \\ i2\sqrt{3} \\ -1 \\ i\sqrt{10} \\ -\sqrt{15} \end{pmatrix},$$

$$|3,0\rangle_y = \hat{U}_y |3,0\rangle_z = \frac{1}{4} \begin{pmatrix} \sqrt{5} \\ 0 \\ \sqrt{3} \\ 0 \\ \sqrt{3} \\ 0 \\ \sqrt{5} \end{pmatrix},$$

$$|3,-1\rangle_y = \hat{U}_y |3,-1\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{15} \\ -i\sqrt{10} \\ 1 \\ -i2\sqrt{3} \\ -1 \\ -i\sqrt{10} \\ -\sqrt{15} \end{pmatrix},$$

$$|3,-2\rangle_y = \hat{U}_y |3,-2\rangle_z = \frac{1}{8} \begin{pmatrix} \sqrt{6} \\ -i4 \\ -\sqrt{10} \\ 0 \\ -\sqrt{10} \\ i4 \\ \sqrt{6} \end{pmatrix},$$

$$|3,-3\rangle_y = \hat{U}_y |3,-3\rangle_z = \frac{1}{8} \begin{pmatrix} 1 \\ -i\sqrt{6} \\ -\sqrt{15} \\ i2\sqrt{5} \\ \sqrt{15} \\ -i\sqrt{6} \\ -1 \end{pmatrix}.$$

7. $j = 7/2$

8x8 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & \frac{\sqrt{7}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{7}}{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \frac{\sqrt{15}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{15}}{2} & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & \frac{\sqrt{15}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \frac{\sqrt{7}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{2} & 0 \end{pmatrix}$$

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & \frac{-i\sqrt{7}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i\sqrt{7}}{2} & 0 & -i\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{3} & 0 & \frac{-i\sqrt{15}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i\sqrt{15}}{2} & 0 & -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i & 0 & \frac{-i\sqrt{15}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i\sqrt{15}}{2} & 0 & -i\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{3} & 0 & \frac{-i\sqrt{7}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{i\sqrt{7}}{2} & 0 \end{pmatrix}$$

$$\hat{J}_z = \hbar \begin{pmatrix} \frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{2} \end{pmatrix}$$

6. Eigenvalue problems for $j = 1/2$

We make a program for the matrix elements in Mathematica.

((Mathematica-1))

Matrices $j = 1/2$

```

Clear["Global`*"]; j = 1/2;
exp_* := exp /. {Complex[re_, im_] :=> Complex[re, -im]};
Jx[j_, n_, m_] :=  $\frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jy[j_, n_, m_] :=  $-\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jz[j_, n_, m_] :=  $\hbar m$  KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jx // MatrixForm

$$\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

Jy // MatrixForm

$$\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}$$

Jz // MatrixForm

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

eq1 = Eigensystem[Jx]

$$\left\{ \left\{ -\frac{\hbar}{2}, \frac{\hbar}{2} \right\}, \{ \{-1, 1\}, \{1, 1\} \} \right\}$$


```

```
 $\psi_{1x} = \text{Normalize}[\text{eq1}[[2, 2]]]; \psi_{1x} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
 $\psi_{2x} = -\text{Normalize}[\text{eq1}[[2, 1]]]; \psi_{2x} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
 $UxT = \{\psi_{1x}, \psi_{2x}\}; Ux = \text{Transpose}[UxT]; Ux // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
 $\text{eq2} = \text{Eigensystem}[Jy]$ 
```

$$\left\{ \left\{ -\frac{\hbar}{2}, \frac{\hbar}{2} \right\}, \left\{ \{i, 1\}, \{-i, 1\} \right\} \right\}$$

```
 $\psi_{1y} = i \text{Normalize}[\text{eq2}[[2, 2]]]; \psi_{1y} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

```
 $\psi_{2y} = -i \text{Normalize}[\text{eq2}[[2, 1]]]; \psi_{2y} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

```
 $UyT = \{\psi_{1y}, \psi_{2y}\}; Uy = \text{Transpose}[UyT]; Uy // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

7. Eigenvalue problems for $j = 1$ (Mathematica)

Matrices $j = 1$

```

Clear["Global`*"]; j = 1; exp_* := exp /. {Complex[re_, im_] => Complex[re, -im]};
Jx[j_, n_, m_] :=  $\frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jy[j_, n_, m_] :=  $-\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jz[j_, n_, m_] :=  $\hbar m$  KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jx // MatrixForm

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

Jy // MatrixForm

$$\begin{pmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

Jz // MatrixForm

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

eq1 = Eigensystem[Jx]
{{0, - $\hbar$ ,  $\hbar$ }, {{-1, 0, 1}, {1,  $-\sqrt{2}$ , 1}, {1,  $\sqrt{2}$ , 1}}}

```

$\psi_{1x} = \text{Normalize}[\text{eq1}[[2, 3]]]; \psi_{1x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$\psi_{2x} = -\text{Normalize}[\text{eq1}[[2, 1]]]; \psi_{2x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\psi_{3x} = \text{Normalize}[\text{eq1}[[2, 2]]]; \psi_{3x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$U_{xT} = \{\psi_{1x}, \psi_{2x}, \psi_{3x}\}; U_x = \text{Transpose}[U_{xT}]; U_x // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$U_{xH} = U_{xT}^*; U_{xH} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$U_{xH}.U_x$

$\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}\}$

$\text{eq2} = \text{Eigensystem}[\text{Jy}]$

$\{\{0, -\hbar, \hbar\}, \{\{1, 0, 1\}, \{-1, i\sqrt{2}, 1\}, \{-1, -i\sqrt{2}, 1\}\}\}$

```
 $\psi_{1y} = -\text{Normalize}[\text{eq2}[[2, 3]]]; \psi_{1y} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

```
 $\psi_{2y} = \text{Normalize}[\text{eq2}[[2, 1]]]; \psi_{2y} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
 $\psi_{3y} = -\text{Normalize}[\text{eq2}[[2, 2]]]; \psi_{3y} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

```
 $\text{UyT} = \{\psi_{1y}, \psi_{2y}, \psi_{3y}\}; \text{Uy} = \text{Transpose}[\text{UyT}]; \text{Uy} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
 $\text{UyH} = \text{UyT}^*; \text{UyH} // \text{MatrixForm}$ 
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
 $\text{UyH.Uy} // \text{Simplify}$ 
```

```
 $\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}\}$ 
```

8. Eigenvalue problems for $j = 3/2$ (Mathematica)

Matrices $j = 3/2$

```

Clear["Global`*"]; j = 3/2;
exp_* := exp /. {Complex[re_, im_] :=> Complex[re, -im]};
Jx[j_, n_, m_] :=  $\frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jy[j_, n_, m_] :=  $-\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jz[j_, n_, m_] :=  $\hbar m$  KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jx // MatrixForm

```

$$\begin{pmatrix} 0 & \frac{\sqrt{3}\hbar}{2} & 0 & 0 \\ \frac{\sqrt{3}\hbar}{2} & 0 & \hbar & 0 \\ 0 & \hbar & 0 & \frac{\sqrt{3}\hbar}{2} \\ 0 & 0 & \frac{\sqrt{3}\hbar}{2} & 0 \end{pmatrix}$$

Jy // MatrixForm

$$\begin{pmatrix} 0 & -\frac{1}{2} i \sqrt{3} \hbar & 0 & 0 \\ \frac{1}{2} i \sqrt{3} \hbar & 0 & -i \hbar & 0 \\ 0 & i \hbar & 0 & -\frac{1}{2} i \sqrt{3} \hbar \\ 0 & 0 & \frac{1}{2} i \sqrt{3} \hbar & 0 \end{pmatrix}$$

Jz // MatrixForm

$$\begin{pmatrix} \frac{3\hbar}{2} & 0 & 0 & 0 \\ 0 & \frac{\hbar}{2} & 0 & 0 \\ 0 & 0 & -\frac{\hbar}{2} & 0 \\ 0 & 0 & 0 & -\frac{3\hbar}{2} \end{pmatrix}$$

eq1 = Eigensystem[Jx]

$$\left\{ \left\{ -\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2} \right\}, \left\{ \{-1, \sqrt{3}, -\sqrt{3}, 1\}, \right. \right. \\ \left. \left. \left\{ 1, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 1 \right\}, \left\{ -1, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 1 \right\}, \left\{ 1, \sqrt{3}, \sqrt{3}, 1 \right\} \right\} \right\}$$

$\psi_{1x} = \text{Normalize}[eq1[[2, 2]]] // \text{Simplify}; \psi_{1x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \end{pmatrix}$$

$\psi_{2x} = -\text{Normalize}[eq1[[2, 4]]] // \text{Simplify}; \psi_{2x} // \text{MatrixForm}$

$$\begin{pmatrix} -\frac{1}{2\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

$\psi_{3x} = \text{Normalize}[eq1[[2, 3]]] // \text{Simplify}; \psi_{3x} // \text{MatrixForm}$

$$\begin{pmatrix} -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \end{pmatrix}$$

```
ψ4x = -Normalize[eq1[[2, 1]]] // Simplify;
ψ4x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

```
UxT = {ψ1x, ψ2x, ψ3x, ψ4x}; Ux = Transpose[UxT];
Ux // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

```
UxH = UxT*; UxH.Ux
```

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

```
eq2 = Eigensystem[Jy]
```

```
{{{-\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}}, {{-i, -\sqrt{3}, i\sqrt{3}, 1},
{\frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{i}{\sqrt{3}}, 1}}, {-i, \frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 1}}, {i, -\sqrt{3}, -i\sqrt{3}, 1}}}
```

```
ψ1y = -i Normalize[eq2[[2, 2]]] // Simplify;
ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ -\frac{1}{2} i \sqrt{\frac{3}{2}} \end{pmatrix}$$

```

 $\psi_{2y} = i \text{ Normalize}[eq2[[2, 4]]] // \text{Simplify};$ 
 $\psi_{2y} // \text{MatrixForm}$ 

```

$$\begin{pmatrix} -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2} i \sqrt{\frac{3}{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```

 $\psi_{3y} = -i \text{ Normalize}[eq2[[2, 3]]] // \text{Simplify};$ 
 $\psi_{3y} // \text{MatrixForm}$ 

```

$$\begin{pmatrix} -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2} i \sqrt{\frac{3}{2}} \end{pmatrix}$$

```

ψ4y = i Normalize[eq2[[2, 1]]] // Simplify;
ψ4y // MatrixForm

```

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{1}{2} i \sqrt{\frac{3}{2}} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```

UyT = {ψ1y, ψ2y, ψ3y, ψ4y}; Uy = Transpose[UyT];
Uy // MatrixForm

```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{i}{2\sqrt{2}} & -\frac{1}{2} i \sqrt{\frac{3}{2}} & -\frac{i}{2\sqrt{2}} & -\frac{1}{2} i \sqrt{\frac{3}{2}} \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2} i \sqrt{\frac{3}{2}} & \frac{i}{2\sqrt{2}} & -\frac{1}{2} i \sqrt{\frac{3}{2}} & \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```

UyH = UyT*; UyH.Uy

```

```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```

9. Eigenvalue problems for $j = 2$

Matrices $j = 2$

```

Clear["Global`*"]; j = 2;
Jx[j_, n_, m_] :=  $\frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jy[j_, n_, m_] :=  $-\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
 $\frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
Jz[j_, n_, m_] :=  $\hbar m$  KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];

```

Jx // MatrixForm

$$\begin{pmatrix} 0 & \hbar & 0 & 0 & 0 \\ \hbar & 0 & \sqrt{\frac{3}{2}} \hbar & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} \hbar & 0 & \sqrt{\frac{3}{2}} \hbar & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} \hbar & 0 & \hbar \\ 0 & 0 & 0 & \hbar & 0 \end{pmatrix}$$

Jy // MatrixForm

$$\begin{pmatrix} 0 & -i \hbar & 0 & 0 & 0 \\ i \hbar & 0 & -i \sqrt{\frac{3}{2}} \hbar & 0 & 0 \\ 0 & i \sqrt{\frac{3}{2}} \hbar & 0 & -i \sqrt{\frac{3}{2}} \hbar & 0 \\ 0 & 0 & i \sqrt{\frac{3}{2}} \hbar & 0 & -i \hbar \\ 0 & 0 & 0 & i \hbar & 0 \end{pmatrix}$$

Jz // MatrixForm

$$\begin{pmatrix} 2\hbar & 0 & 0 & 0 & 0 \\ 0 & \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar & 0 \\ 0 & 0 & 0 & 0 & -2\hbar \end{pmatrix}$$

eq1 = Eigensystem[Jx]

$$\left\{ \{0, -2\hbar, -\hbar, \hbar, 2\hbar\}, \left\{ \left\{ 1, 0, -\sqrt{\frac{2}{3}}, 0, 1 \right\}, \left\{ 1, -2, \sqrt{6}, -2, 1 \right\}, \right. \right. \\ \left. \left. \left\{ -1, 1, 0, -1, 1 \right\}, \left\{ -1, -1, 0, 1, 1 \right\}, \left\{ 1, 2, \sqrt{6}, 2, 1 \right\} \right\} \right\}$$

$\psi_{1x} = \text{Normalize}[eq1[[2, 5]]] // \text{Simplify}; \psi_{1x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

$\psi_{2x} = -\text{Normalize}[eq1[[2, 4]]] // \text{Simplify}; \psi_{2x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$\psi_{3x} = \text{Normalize}[eq1[[2, 1]]] // \text{Simplify}; \psi_{3x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \\ \frac{\sqrt{\frac{3}{2}}}{2} \end{pmatrix}$$

$\psi_{4x} = -\text{Normalize}[eq1[[2, 3]]] // \text{Simplify}; \psi_{4x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

`ψ5x = Normalize[eq1[[2, 2]]] // Simplify; ψ5x // MatrixForm`

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

`UxT = {ψ1x, ψ2x, ψ3x, ψ4x, ψ5x}; Ux = Transpose[UxT]; Ux // MatrixForm`

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

`eq2 = Eigensystem[Jy]`

$$\left\{ \{0, -2\hbar, -\hbar, \hbar, 2\hbar\}, \left\{ \left\{ 1, 0, \sqrt{\frac{2}{3}}, 0, 1 \right\}, \left\{ 1, -2i, -\sqrt{6}, 2i, 1 \right\}, \right. \right. \\ \left. \left. \left\{ -1, i, 0, i, 1 \right\}, \left\{ -1, -i, 0, -i, 1 \right\}, \left\{ 1, 2i, -\sqrt{6}, -2i, 1 \right\} \right\} \right\}$$

`ψ1y = Normalize[eq2[[2, 5]]] // Simplify; ψ1y // MatrixForm`

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ -\frac{i}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{i}{2} \\ \frac{1}{4} \end{pmatrix}$$

`UyT = {ψ1y, ψ2y, ψ3y, ψ4y, ψ5y}; Uy = Transpose[UyT]; UyH = UyT*;
Uy // MatrixForm`

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{i}{2} & \frac{i}{2} & 0 & -\frac{i}{2} & -\frac{i}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2} & \frac{i}{2} & 0 & -\frac{i}{2} & \frac{i}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

10. Eigenvalue problems for $j = 5/2$

Matrices $j = 5/2$

```
Clear["Global`*"]; j = 5/2;
```

```
Jx[j_, n_, m_] := - $\frac{\hbar}{2}$   $\sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
```

```
 $\frac{\hbar}{2}$   $\sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
```

```
Jy[j_, n_, m_] := - $\frac{\hbar}{2}$  i  $\sqrt{(j-m)(j+m+1)}$  KroneckerDelta[n, m+1] +
```

```
 $\frac{\hbar}{2}$  i  $\sqrt{(j+m)(j-m+1)}$  KroneckerDelta[n, m-1];
```

```
Jz[j_, n_, m_] :=  $\hbar$  m KroneckerDelta[n, m];
```

```
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

```
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

```
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

Jx // MatrixForm

$$\begin{pmatrix} 0 & \frac{\sqrt{5}\hbar}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{5}\hbar}{2} & 0 & \sqrt{2}\hbar & 0 & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 & \frac{3\hbar}{2} & 0 & 0 \\ 0 & 0 & \frac{3\hbar}{2} & 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & 0 & \sqrt{2}\hbar & 0 & \frac{\sqrt{5}\hbar}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}\hbar}{2} & 0 \end{pmatrix}$$

Jy // MatrixForm

$$\begin{pmatrix} 0 & -\frac{1}{2}i\sqrt{5}\hbar & 0 & 0 & 0 & 0 \\ \frac{1}{2}i\sqrt{5}\hbar & 0 & -i\sqrt{2}\hbar & 0 & 0 & 0 \\ 0 & i\sqrt{2}\hbar & 0 & -\frac{3i\hbar}{2} & 0 & 0 \\ 0 & 0 & \frac{3i\hbar}{2} & 0 & -i\sqrt{2}\hbar & 0 \\ 0 & 0 & 0 & i\sqrt{2}\hbar & 0 & -\frac{1}{2}i\sqrt{5}\hbar \\ 0 & 0 & 0 & 0 & \frac{1}{2}i\sqrt{5}\hbar & 0 \end{pmatrix}$$

Jz // MatrixForm

$$\begin{pmatrix} \frac{5\hbar}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\hbar}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3\hbar}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5\hbar}{2} \end{pmatrix}$$

eq1 = Eigensystem[Jx]

$$\left\{ \left\{ -\frac{5\hbar}{2}, -\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}, \frac{5\hbar}{2} \right\}, \right.$$

$$\left\{ \{-1, \sqrt{5}, -\sqrt{10}, \sqrt{10}, -\sqrt{5}, 1\}, \left\{ 1, -\frac{3}{\sqrt{5}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, -\frac{3}{\sqrt{5}}, 1 \right\}, \right.$$

$$\left\{ -1, \frac{1}{\sqrt{5}}, \sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, -\frac{1}{\sqrt{5}}, 1 \right\}, \left\{ 1, \frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, \frac{1}{\sqrt{5}}, 1 \right\}, \right.$$

$$\left. \left\{ -1, -\frac{3}{\sqrt{5}}, -\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \frac{3}{\sqrt{5}}, 1 \right\}, \left\{ 1, \sqrt{5}, \sqrt{10}, \sqrt{10}, \sqrt{5}, 1 \right\} \right\}$$

$\psi_{1x} = \text{Normalize}[\text{eq1}[[2, 6]]]; \psi_{1x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{1}{4\sqrt{2}} \end{pmatrix}$$

$\psi_{2x} = -\text{Normalize}[\text{eq1}[[2, 5]]]; \psi_{2x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{3}{4\sqrt{2}} \\ \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{3}{4\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{2}}}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ \frac{1}{4} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} \\ \frac{\sqrt{5}}{4} \end{pmatrix}$$

`$\psi_{4x} = -\text{Normalize}[\text{eq1}[[2, 3]]]; \psi_{4x} // \text{MatrixForm}$`

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ -\frac{1}{4} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \\ -\frac{\sqrt{5}}{4} \end{pmatrix}$$

$\psi_{5x} = \text{Normalize}[\text{eq1}[[2, 2]]]; \psi_{5x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{5}}{2} \\ 4 \\ -\frac{3}{4\sqrt{2}} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4\sqrt{2}} \\ \frac{\sqrt{5}}{2} \\ 4 \end{pmatrix}$$

$\psi_{6x} = -\text{Normalize}[\text{eq1}[[2, 1]]]; \psi_{6x} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ -\frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{4} \\ -\frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{2} \\ \frac{1}{4\sqrt{2}} \end{pmatrix}$$

$UxT = \{\psi1x, \psi2x, \psi3x, \psi4x, \psi5x, \psi6x\}$; $Ux = \text{Transpose}[UxT]$; $UxH = UxT^*$;

$Ux // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{1}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{3}{4\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & -\frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{4} & -\frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{1}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{1}{4\sqrt{2}} \end{pmatrix}$$

$UxH // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{1}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{3}{4\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & -\frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{4} & -\frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{1}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{1}{4\sqrt{2}} \end{pmatrix}$$

$UxH.Ux$

$\{ \{1, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0\},$
 $\{0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 1\} \}$

eq2 = Eigensystem[Jy]

$$\left\{ \left\{ -\frac{5\hbar}{2}, -\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}, \frac{5\hbar}{2} \right\}, \right.$$

$$\left\{ \left\{ i, \sqrt{5}, -i\sqrt{10}, -\sqrt{10}, i\sqrt{5}, 1 \right\}, \left\{ -i, -\frac{3}{\sqrt{5}}, i\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, \frac{3i}{\sqrt{5}}, 1 \right\}, \right.$$

$$\left\{ \left\{ i, \frac{1}{\sqrt{5}}, i\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \frac{i}{\sqrt{5}}, 1 \right\}, \left\{ -i, \frac{1}{\sqrt{5}}, -i\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, -\frac{i}{\sqrt{5}}, 1 \right\}, \right.$$

$$\left. \left\{ \left\{ i, -\frac{3}{\sqrt{5}}, -i\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, -\frac{3i}{\sqrt{5}}, 1 \right\}, \left\{ -i, \sqrt{5}, i\sqrt{10}, -\sqrt{10}, -i\sqrt{5}, 1 \right\} \right\} \right\}$$

$\psi_{1y} = i \text{ Normalize}[eq2[[2, 6]]]; \psi_{1y} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ \frac{1}{4} i \sqrt{\frac{5}{2}} \\ -\frac{\sqrt{5}}{4} \\ -\frac{i\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{i}{4\sqrt{2}} \end{pmatrix}$$

$\psi_{2y} = -i \text{Normalize}[\text{eq2}[[2, 5]]]; \psi_{2y} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{5}}{2} \\ \frac{4}{3i} \\ \frac{1}{4\sqrt{2}} \\ -\frac{1}{4} \\ \frac{i}{4} \\ -\frac{3}{4\sqrt{2}} \\ -\frac{1}{4} i \sqrt{\frac{5}{2}} \end{pmatrix}$$

$\psi_{3y} = i \text{Normalize}[\text{eq2}[[2, 4]]]; \psi_{3y} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ \frac{i}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} \\ \frac{1}{4} \\ \frac{i\sqrt{5}}{4} \end{pmatrix}$$

$\psi_{4y} = -i \text{Normalize}[\text{eq2}[[2, 3]]]; \psi_{4y} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ -\frac{i}{4} \\ \frac{1}{2\sqrt{2}} \\ -\frac{i}{2\sqrt{2}} \\ \frac{1}{4} \\ -\frac{i\sqrt{5}}{4} \end{pmatrix}$$

$\psi_{5y} = i \text{Normalize}[\text{eq2}[[2, 2]]]; \psi_{5y} // \text{MatrixForm}$

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ -\frac{3i}{4\sqrt{2}} \\ -\frac{1}{4} \\ -\frac{i}{4} \\ -\frac{3}{4\sqrt{2}} \\ \frac{1}{4} + i\sqrt{\frac{5}{2}} \end{pmatrix}$$

$\psi_6y = -i \text{Normalize}[\text{eq2}[[2, 1]]]; \psi_6y // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ -\frac{1}{4}i\sqrt{\frac{5}{2}} \\ -\frac{\sqrt{5}}{4} \\ \frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ -\frac{i}{4\sqrt{2}} \end{pmatrix}$$

$UyT = \{\psi_1y, \psi_2y, \psi_3y, \psi_4y, \psi_5y, \psi_6y\}; Uy = \text{Transpose}[UyT]; UyH = UyT^*;$

$Uy // \text{MatrixForm}$

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{1}{4\sqrt{2}} \\ \frac{1}{4}i\sqrt{\frac{5}{2}} & \frac{3i}{4\sqrt{2}} & \frac{i}{4} & -\frac{i}{4} & -\frac{3i}{4\sqrt{2}} & -\frac{1}{4}i\sqrt{\frac{5}{2}} \\ -\frac{\sqrt{5}}{4} & -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & -\frac{\sqrt{5}}{4} \\ -\frac{i\sqrt{5}}{4} & \frac{i}{4} & \frac{i}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & -\frac{i}{4} & \frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{i}{4\sqrt{2}} & -\frac{1}{4}i\sqrt{\frac{5}{2}} & \frac{i\sqrt{5}}{4} & -\frac{i\sqrt{5}}{4} & \frac{1}{4}i\sqrt{\frac{5}{2}} & -\frac{i}{4\sqrt{2}} \end{pmatrix}$$

UyH // MatrixForm

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & -\frac{1}{4}i\sqrt{\frac{5}{2}} & -\frac{\sqrt{5}}{4} & \frac{i\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{i}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3i}{4\sqrt{2}} & -\frac{1}{4} & -\frac{i}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4}i\sqrt{\frac{5}{2}} \\ \frac{\sqrt{5}}{4} & -\frac{i}{4} & \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & \frac{1}{4} & -\frac{i\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & \frac{i}{4} & \frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{3i}{4\sqrt{2}} & -\frac{1}{4} & \frac{i}{4} & -\frac{3}{4\sqrt{2}} & -\frac{1}{4}i\sqrt{\frac{5}{2}} \\ \frac{1}{4\sqrt{2}} & \frac{1}{4}i\sqrt{\frac{5}{2}} & -\frac{\sqrt{5}}{4} & -\frac{i\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{i}{4\sqrt{2}} \end{pmatrix}$$

UyH.Uy

$$\{\{1, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 1\}\}$$

11. Angular momentum for $j = 3$

7x7 matrices

$$\hat{J}_x =$$

Jx // MatrixForm

$$\begin{pmatrix} 0 & \sqrt{\frac{3}{2}} \hbar & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} \hbar & 0 & \sqrt{\frac{5}{2}} \hbar & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} \hbar & 0 & \sqrt{3} \hbar & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} \hbar & 0 & \sqrt{3} \hbar & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} \hbar & 0 & \sqrt{\frac{5}{2}} \hbar & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} \hbar & 0 & \sqrt{\frac{3}{2}} \hbar \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} \hbar & 0 \end{pmatrix}$$

$\hat{J}_y =:$

Jy // MatrixForm

$$\begin{pmatrix} 0 & -i\sqrt{\frac{3}{2}} \hbar & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{\frac{3}{2}} \hbar & 0 & -i\sqrt{\frac{5}{2}} \hbar & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{5}{2}} \hbar & 0 & -i\sqrt{3} \hbar & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{3} \hbar & 0 & -i\sqrt{3} \hbar & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{3} \hbar & 0 & -i\sqrt{\frac{5}{2}} \hbar & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{\frac{5}{2}} \hbar & 0 & -i\sqrt{\frac{3}{2}} \hbar \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{3}{2}} \hbar & 0 \end{pmatrix}$$

APPENDIX

Matrix of the angular momentum with $j = 4$ and $9/2$.

$J = 4$

9 x 9 matrix

$$\hat{J}_x = \begin{pmatrix} 0 & \sqrt{2} \hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} \hbar & 0 & \sqrt{\frac{7}{2}} \hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{7}{2}} \hbar & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & \sqrt{5} \hbar & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{5} \hbar & 0 & \sqrt{5} \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} \hbar & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3\hbar}{\sqrt{2}} & 0 & \sqrt{\frac{7}{2}} \hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{2}} \hbar & 0 & \sqrt{2} \hbar \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \hbar & 0 \end{pmatrix}$$

$$\hat{J}_y = \begin{pmatrix} 0 & -i\sqrt{2} \hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{2} \hbar & 0 & -i\sqrt{\frac{7}{2}} \hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{7}{2}} \hbar & 0 & -\frac{3i\hbar}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3i\hbar}{\sqrt{2}} & 0 & -i\sqrt{5} \hbar & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{5} \hbar & 0 & -i\sqrt{5} \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{5} \hbar & 0 & -\frac{3i\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3i\hbar}{\sqrt{2}} & 0 & -i\sqrt{\frac{7}{2}} \hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{7}{2}} \hbar & 0 & -i\sqrt{2} \hbar \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} \hbar & 0 \end{pmatrix}$$

$$\hat{J}_z =$$

$$\begin{pmatrix} 4\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4\hbar \end{pmatrix}$$

$$J = 9/2$$

10 x 10 matrix

$$\hat{J}_x =$$

$$\begin{pmatrix} 0 & \frac{3\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3\hbar}{2} & 0 & 2\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & \frac{\sqrt{21}\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{21}\hbar}{2} & 0 & \sqrt{6}\hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6}\hbar & 0 & \frac{5\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5\hbar}{2} & 0 & \sqrt{6}\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6}\hbar & 0 & \frac{\sqrt{21}\hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{21}\hbar}{2} & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\hbar & 0 & \frac{3\hbar}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\hbar}{2} & 0 \end{pmatrix}$$

$$\hat{J}_y = \begin{pmatrix} 0 & -\frac{3i\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3i\hbar}{2} & 0 & -2i\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2i\hbar & 0 & -\frac{1}{2}i\sqrt{21}\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}i\sqrt{21}\hbar & 0 & -i\sqrt{6}\hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{6}\hbar & 0 & -\frac{5i\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5i\hbar}{2} & 0 & -i\sqrt{6}\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{6}\hbar & 0 & -\frac{1}{2}i\sqrt{21}\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\sqrt{21}\hbar & 0 & -2i\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i\hbar & 0 & -\frac{3i\hbar}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3i\hbar}{2} & 0 \end{pmatrix}$$

$$\hat{J}_z = \begin{pmatrix} \frac{9\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\hbar}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\hbar}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5\hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7\hbar}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9\hbar}{2} \end{pmatrix}$$