Dirac Delta function

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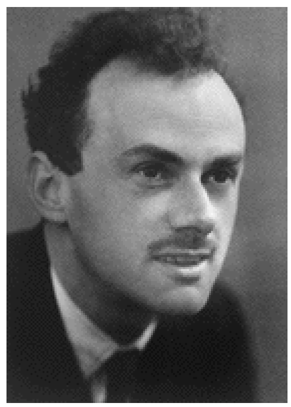
(Date: October 30, 2011)

Paul Adrien Maurice Dirac (8 August 1902 – 20 October 1984) was a British theoretical physicist. Dirac made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. He held the Lucasian Chair of Mathematics at the University of Cambridge and spent the last fourteen years of his life at Florida State University. Among other discoveries, he formulated the Dirac equation, which describes the behavior of fermions. This led to a prediction of the existence of antimatter. Dirac shared the Nobel Prize in physics for 1933 with Erwin Schrödinger, "for the discovery of new productive forms of atomic theory."



http://en.wikipedia.org/wiki/Paul_Dirac

The Dirac delta function $\delta(x)$ is a useful function which was proposed by in 1930 by Paul Dirac in his mathematical formalism of quantum mechanics. The Dirac delta function is not a mathematical function according to the usual definition because it does not have a definite value when x is zero. Nevertheless, it has many applications in physics.

1. Dirac delta function

When $f(x)$ is a well-defined function at $x = x_0$,

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0) \int_{-\infty}^{\infty} \delta(x - x_0) dx = f(x_0) .$$

The Dirac delta function $\delta(x - x_0)$ has a sharp peak at $x = x_0$.

$$\delta(x - x_0) = 0 \text{ if } x \neq x_0 \text{ and } +\infty \text{ if } x = x_0 ,$$

and

$$\int_{-\infty}^{\infty} \delta(x-x_0)dx = 1 .$$

$\delta(x-x_0)$ is a generalization of the Kronecker delta function.

2. Principle properties of the Dirac delta function

1. $\delta(x) = \delta(-x)$.

2. $\delta(ax) = \dfrac{1}{|a|}\delta(x) \qquad (a \neq 0)$.

3. $\delta[g(x)] = \sum_n \dfrac{1}{|g'(x_n)|}\delta(x-x_n)$.

where $g(x_n) = 0$ and $g'(x_n) \neq 0$. Note that $g(x) = g(x_n) + g'(x_n)(x-x_n) = g'(x_n)(x-x_n)$ around $x = x_n$.

$$\delta[g(x)] = \delta[\sum_n g'(x_n)(x-x_n)] = \sum_n \delta[g'(x_n)(x-x_n)] = \sum_n \frac{\delta[(x-x_n)]}{|g'(x_n)|} .$$

4. $x\delta(x) = 0$.

5. $f(x)\delta(x-a) = f(a)\delta(x-a)$.

6. $\int \delta(x-y)\delta(y-a)dy = \delta(x-a)$.

7. $\delta(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{ikx}dk$. (Fourier transform)

((Note)) The formula (7) is extensively used in physics.

3. Derivation of (1) $\delta(x) = \delta(-x)$

$$I_1 = \int_{-\infty}^{\infty} \delta(x) f(x)dx = f(0) .$$

By putting that $y = -x$, $dy = -dx$,

$$I_2 = \int_{-\infty}^{\infty} \delta(-x) f(x) dx = \int_{\infty}^{-\infty} \delta(y) f(-y)(-dy) = \int_{-\infty}^{\infty} \delta(y) f(-y) dy = f(0) = I_1 .$$

Thus we have

$$\delta(-x) = \delta(x) .$$

4. Derivation of (2) $\delta(ax) = \frac{1}{|a|}\delta(x)$

(i) For a>0, we put $y = ax$.

$$I_1 = \int_{-\infty}^{\infty} \delta(ax) f(x) dx = \int_{-\infty}^{\infty} \delta(y) f(\frac{y}{a}) \frac{dy}{a}$$

$$= \frac{1}{a} f(0) = \frac{1}{a} \int_{-\infty}^{\infty} \delta(x) f(x) dx$$

Thus we have $\delta(ax) = \frac{1}{a}\delta(x)$.

(ii) For a<0, we put $y = ax$.

$$I_2 = \int_{-\infty}^{\infty} \delta(ax) f(x) dx = \int_{\infty}^{-\infty} \delta(y) f(\frac{y}{a}) \frac{dy}{a}$$

$$= -\int_{-\infty}^{\infty} \delta(y) f(\frac{y}{a}) \frac{dy}{a} = -\frac{1}{a} f(0) = -\frac{1}{a} \int_{-\infty}^{\infty} \delta(x) f(x) dx$$

Thus we have $\delta(ax) = -\frac{1}{a}\delta(x)$. Combining (i) and (ii), we get the final result:

$$\delta(ax) = \frac{1}{|a|}\delta(x) .$$

5. Derivation of (4). $x\delta(x) = 0$.

$$I = \int_{-\infty}^{\infty} x\delta(x) f(x) dx = 0 f(0) = 0 .$$

6. Example of $\delta[g(x)] = \sum_n \frac{1}{|g'(x_n)|}\delta(x - x_n)$

(i) $\delta[(x-a)(x-b)] = \frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)]$.

where $a-b \neq 0$.

((Proof))

$$g(x) = (x-a)(x-b),$$

$$g'(x) = 2x-a-b,$$

then

$$\delta[(x-a)(x-b)] = \frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)].$$

Another method to derive this equation is as follows.

$$\begin{aligned}\delta[(x-a)(x-b)] &= [\delta[(a-b)(x-a)]+\delta[(x-b)(b-a)]] \\ &= \frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)]\end{aligned}.$$

(ii) $\delta(\sqrt{x}-\sqrt{a}) = 2\sqrt{a}\delta(x-a)$.

((**Proof**))

$$\delta(\sqrt{x}-\sqrt{a}) = \delta\left(\frac{x-a}{\sqrt{x}+\sqrt{a}}\right) = \delta\left(\frac{x-a}{2\sqrt{a}}\right) = 2\sqrt{a}\delta(x-a).$$

(iii) $\delta(\sqrt{x-a}) = 0$.

((**Proof**))

$$g(x) = \sqrt{x-a}, \qquad g'(x) = \frac{1}{2\sqrt{x-a}},$$

$$\delta(g(x)) = \frac{1}{|g'(a)|}\delta(x-a) = 2\sqrt{x-a}\,|_{x\to a}\,\delta(x-a) = 0.$$

(iv) $\delta(\sqrt{x^2-a^2}) = 0$.

((**Proof**))

$$g(x)=\sqrt{x^2-a^2}\,,\qquad g'(x)=\frac{x}{\sqrt{x^2-a^2}}\,,$$

$$\delta(\sqrt{x^2-a^2})=\frac{1}{|g'(a)|}\delta(x-a)+\frac{1}{|g'(-a)|}\delta(x+a)$$
$$=\frac{\sqrt{x^2-a^2}}{a}|_{x\to a}\ \delta(x-a)+\frac{\sqrt{x^2-a^2}}{a}|_{x\to -a}\ \delta(x+a)=0$$
.

7. Derivative of the Dirac function

1. $\int_{-\infty}^{\infty}\delta^{(m)}(x)f(x)dx=(-1)^m f^{(m)}(0)$.

2. $\delta^{(m)}(x)=(-1)^m\delta^{(m)}(-x)$.

3. $\int_{-\infty}^{\infty}\delta^{(m)}(x-y)\delta^{(m)}(y-a)dy=\delta^{(m+n)}(x-a)$.

4. $x^{m+1}\delta^{(m)}(x)=0$.

5. $\int_{-\infty}^{\infty}\delta'(x)f(x)dx=-\int_{-\infty}^{\infty}\delta(x)f'(x)dx=-f'(0)$.

6. $\delta'(x)=-\delta'(-x)$.

7. $\int_{-\infty}^{\infty}\delta'(x-y)\delta(y-a)dy=\delta'(x-a)$.

8. $x\delta'(x)=-\delta(x)$.

9. $x^2\delta'(x)=0$.

10. $\delta'(x)=\frac{i}{2\pi}\int_{-\infty}^{\infty}ke^{ikx}dk$.

8. Derivation of (1) $\int_{-\infty}^{\infty}\delta^{(m)}(x)f(x)dx=(-1)^m f^{(m)}(0)$

$$\int_{-\infty}^{\infty}\delta^{(m)}(x)f(x)dx = (-1)^m f^{(m)}(0)\,.$$

((**Proof**))

$$\int_{-\infty}^{\infty}\delta^{(m)}(x)f(x)dx = (-1)^m\int_{-\infty}^{\infty}\delta(x)f^{(m)}(x)dx = (-1)^m f^{(m)}(0)\,.$$

((Note)): in quantum mechanics, we use in general,

$$\int_{-\infty}^{\infty} f^{(m)}(x)g(x)dx = (-1)^m\int_{-\infty}^{\infty} f(x)g^{(m)}(x)dx\,,$$

since $f^{(n)}(-\infty) = f^{(n)}(\infty) = g^{(n)}(-\infty) = g^{(n)}(\infty) = 0$ for n = 0, 1, 2, …..

9. Derivation of (8) $x\delta'(x) = -\delta(x)$

$$x\delta'(x) = -\delta(x)\,.$$

((Proof))

$$\int_{-\infty}^{\infty} x\delta'(x)f(x)dx = \left[xf(x)\delta(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(x)\frac{d}{dx}[xf(x)]dx$$

$$= -\int_{-\infty}^{\infty}\delta(x)[f(x) + x\frac{d}{dx}f(x)]dx$$

$$= -\int_{-\infty}^{\infty}\delta(x)f(x)dx$$

10. Properties of Dirac delta function (Mathematica)

```
Clear["Global`*"]
```

$$\int_{-\infty}^{\infty} \text{DiracDelta}[x]\, dx$$

```
1
```

$$\int_{-\infty}^{\infty} f[x]\, \text{DiracDelta}[x - a]\, dx \text{ // Simplify}[\#, a > 0]\ \&$$

```
f[a]
```

$$\int_{-\infty}^{\infty} f[x]\, \text{DiracDelta}[-x]\, dx$$

```
f[0]
```

$$\int_{-\infty}^{\infty} f[x]\, \text{DiracDelta}\left[\frac{x}{a}\right] dx \text{ // Simplify}[\#, a > 0]\ \&$$

```
a f[0]
```

$$\int_{-\infty}^{\infty} f[x]\, \text{DiracDelta}\left[\frac{x}{a}\right] dx \text{ // Simplify}[\#, a < 0]\ \&$$

```
Abs[a] f[0]
```

$$\int_{-\infty}^{\infty} x\, \text{DiracDelta}'[x]\, dx$$

```
-1
```

$$\int_{-\infty}^{\infty} f[x]\, \text{DiracDelta}'[x]\, dx$$

```
-f'[0]
```

$$\int_{-\infty}^{\infty} f[x]\, x\, \text{DiracDelta}'[x]\, dx$$

```
-f[0]
```

$$\int_{-\infty}^{\infty} f[x]\ \text{DiracDelta}''[x]\, dx$$

$f''[0]$

$$\text{Table}\left[\left\{n, \int_{-\infty}^{\infty} f[x]\ D[\text{DiracDelta}[x], \{x, n\}]\, dx\right\}, \{n, 1, 10\}\right] \text{ // TableForm}$$

1	$-f'[0]$
2	$f''[0]$
3	$-f^{(3)}[0]$
4	$f^{(4)}[0]$
5	$-f^{(5)}[0]$
6	$f^{(6)}[0]$
7	$-f^{(7)}[0]$
8	$f^{(8)}[0]$
9	$-f^{(9)}[0]$
10	$f^{(10)}[0]$

$$\int_{-\infty}^{\infty} f[x]\ \text{DiracDelta}[(x - a)\ (x - b)]\, dx$$

$$\frac{f[a] + f[b]}{\text{Abs}[a - b]}$$

$$\int_{-\infty}^{\infty} f[x]\ \text{DiracDelta}\left[\left(x^3 - 1\right)\right] dx$$

$$\frac{f[1]}{3}$$

$$\int_{-\infty}^{\infty} f[x]\ \text{DiracDelta}\left[\left(x^2 - 4\, x + 3\right)\right] dx$$

$$\frac{1}{2}\ (f[1] + f[3])$$

$$\int_{-\infty}^{\infty} f[x]\ \text{DiracDelta}\left[\sqrt{x} - \sqrt{a}\right] dx \text{ // Simplify}[\#, a > 0]\ \&$$

$2\sqrt{a}\ f[a]$

11. Representative of Delta function

Representative as the limit of a kernel of an integral operator

1. $\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \qquad (n \to \infty)$. (Gaussian)

2. $\delta(x) = \lim_{\eta \to \infty} \frac{1}{\pi} \frac{\sin(\eta x)}{x}$. (Sinc sequence)

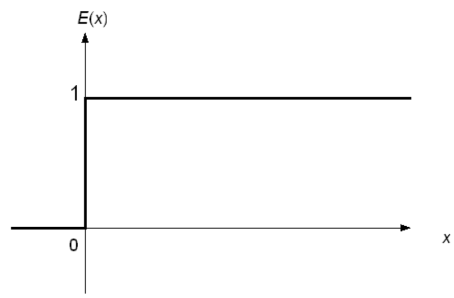
3. $\delta(x) = \lim_{\eta \to \infty} \frac{1}{\pi} \frac{1 - \cos(\eta x)}{\eta x^2}$. (Sinc squared)

4. $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \left(\frac{\varepsilon}{x^2 + \varepsilon^2} \right)$. (Lorentzian, or resonance)

5. $\delta(x) = \lim_{\varepsilon \to 0} \frac{E(x + \varepsilon) - E(x)}{\varepsilon}$.

where $E(x)$ is the unit step function

$E(x) = 1$ if $x>0$ and 0 if $x<0$.



6. $2\pi\delta(x) = 1 + \sum_{n=1}^{\infty} 2\cos(nx)$.

7. $2\pi\delta(x) = \sum_{n=-\infty}^{\infty} e^{inx}$.

8. $2\pi\delta(x) = \int_{-\infty}^{\infty} e^{ikx} dk$.

9. $\pi\delta(x) = \int_{-\infty}^{\infty} \cos(kx) dk$.

12. Gaussian

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}, \qquad (n \to \infty)$$

((**Mathematica**))

```
Clear["Global`*"]

f[x_, n_] := n/Sqrt[π] Exp[-n^2 x^2]

Integrate[f[x, n], {x, -∞, ∞}, Assumptions → {n > 0}]

1

f1 = Plot[Evaluate[Table[f[x, n], {n, 200, 1000, 200}]],
   {x, -0.005, 0.005},
   PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 5}],
   PlotPoints → 100,
   PlotRange → {{-0.005, 0.005}, {0, 700}},
   AxesLabel → {"x", "δn(x)"}];

f2 =
  Graphics[{Text[Style["n = 200", Red, 10], {-0.0001, 110}],
    Text[Style[" 400", Yellow, 10], {0, 220}],
    Text[Style[" 600", Green, 10], {0, 330}],
    Text[Style[" 800", Blue, 10], {0, 450}],
    Text[Style[" 1000", Purple, 10], {0, 560}]}];

Show[f1, f2, PlotRange → All]
```

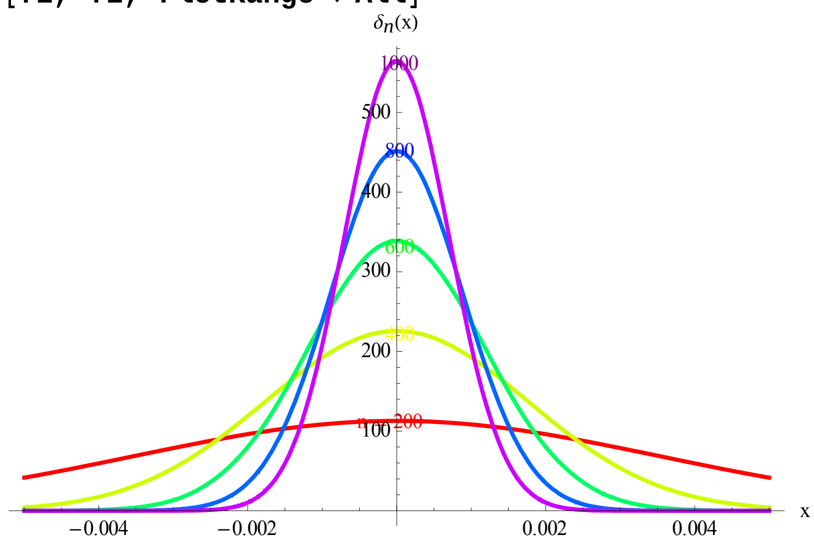


Fig. n is changed as a parameter; $n = 200$, 400, 600, 800, and 1000.

13. Sine sequence function

$$\delta(x) = \lim_{\eta \to \infty} \frac{1}{\pi} \frac{\sin(\eta x)}{x}.$$

The sine sequence function at an finite value of η, has a peak (η/π) at $x = 0$, and becomes zero at $x = \pi/\eta$.

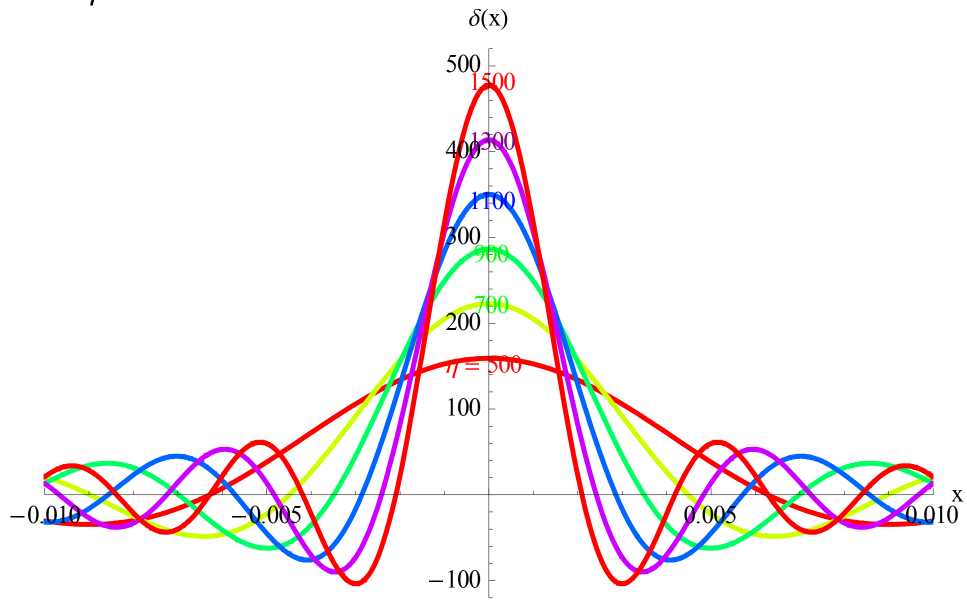


Fig. η is changed as a parameter; η = 500, 700, 900, 1100, 1300, and 1500.

14. Lorentzian

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{1 + n^2 x^2}, \qquad (n \to \infty)$$

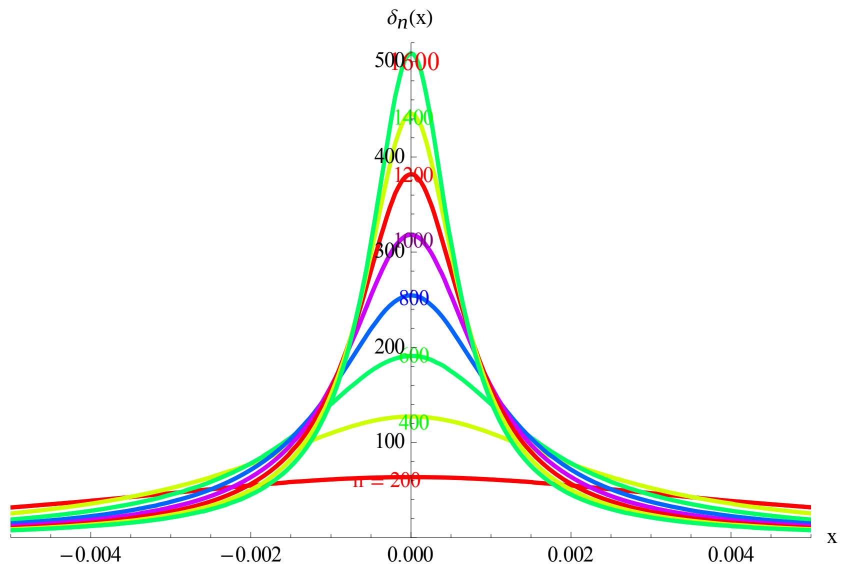
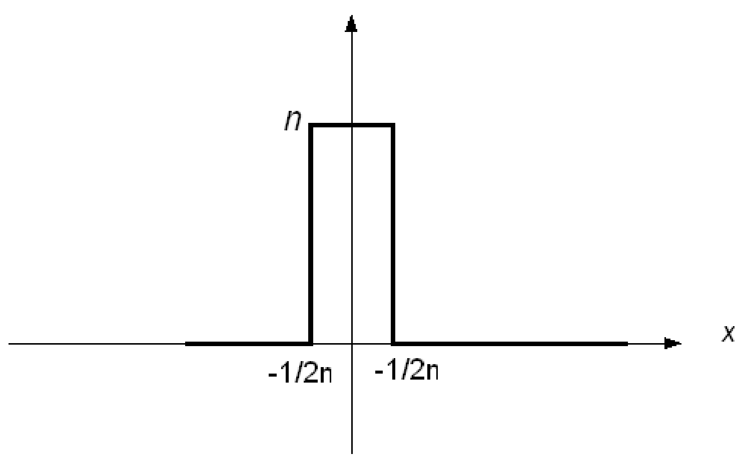


Fig. *n* is changed as a parameter; *n* = 200, 400, 600, 800, 1000, 1200, 1400, and 1600.

15. δ-sequence function

A step function with the height *n* and the width (1/2*n*) centered at *x* = 0.



16. Proof

Here we show that $\delta(x) = \lim_{\eta\to\infty}\frac{1}{\pi}\frac{\sin(\eta x)}{x}$.

using the knowledge of the Fourier transformation.

((**Proof**))

Using the formula derived from the Fourier transform, we have

$$2\pi\delta(x) = \int_{-\infty}^{\infty} e^{ikx} dk = \lim_{\eta\to\infty} \int_{-\eta}^{\eta} e^{ikx} dk = \lim_{\eta\to\infty} 2\int_{0}^{\eta} \cos(kx) dk = \lim_{\eta\to\infty} \frac{2}{x} \sin(\eta x),$$

or

$$\delta(x) = \lim_{\eta\to\infty} \frac{\sin(\eta x)}{\pi x}.$$

((**Note**))

$$\lim_{\eta\to\infty} \frac{\sin(\eta x)}{x} = \pi\delta(x).$$

This implies that

$$I = \int_{-\infty}^{\infty} \frac{\sin(\eta x)}{x} dx = \int_{-\infty}^{\infty} \pi\delta(x) dx = \pi. \qquad (\eta \to \infty)$$

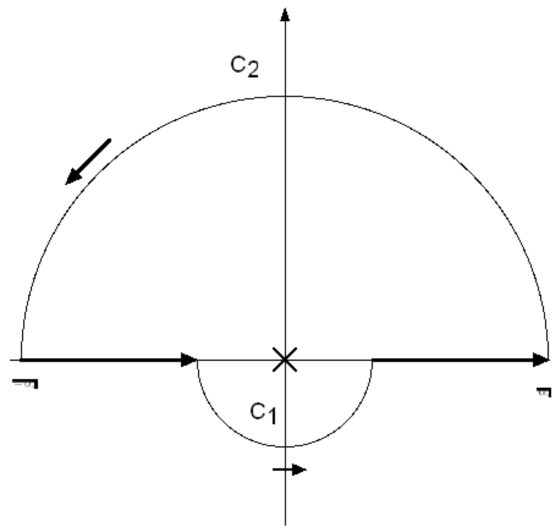
We put $y = \eta x$. Then we have

$$I = \int_{-\infty}^{\infty} dx \frac{\sin(\eta x)}{x} = \int_{-\infty}^{\infty} \frac{dy}{\eta} \frac{\sin(y)}{\frac{y}{\eta}} = \int_{-\infty}^{\infty} dy \frac{\sin y}{y} = \pi,$$

which is independent of η and is equal to π. Note that

$$\int_{-\infty}^{\infty} dy \frac{\sin y}{y} = \int_{-\infty}^{\infty} dy \left(\frac{\cos y + i \sin y}{iy} \right) = \frac{1}{i} \int_{-\infty}^{\infty} dy \frac{e^{iy}}{y}.$$

Using the Cauchy theorem, we calculate $\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x}$,



Consider the path integral

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx+\int_{C_1}\frac{e^{iz}}{z}dz+\int_{C_2}\frac{e^{iz}}{z}dz=2\pi i\,\mathrm{Re}\,s(z=0)=2\pi i\,.$$

The contour integral around the path C_2: $(z=\mathrm{Re}^{i\theta})$

$$\int_{C_2}\frac{e^{iz}}{z}dz=\int_0^{\pi}\frac{\mathrm{Re}^{i\theta}}{\mathrm{Re}^{i\theta}}id\theta e^{i(\mathrm{Re}^{i\theta})}=\int_0^{\pi}id\theta e^{i(R\cos\theta+iR\sin\theta)}=i\int_0^{\pi}d\theta e^{iR\cos\theta}e^{-R\sin\theta}\to 0\,,$$

The contour integral around the path C_1: $(z=\varepsilon e^{i\theta})$,

$$\int_{C_1}\frac{e^{iz}}{z}dz=\int_{-\pi}^{0}\frac{\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}}id\theta e^{i(\varepsilon e^{i\theta})}=-\int_{-\pi}^{0}id\theta e^{i(\varepsilon\cos\theta+i\varepsilon\sin\theta)}=i\int_{-\pi}^{0}d\theta=i\pi\,,$$

or

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx=\pi i=i\int_{-\infty}^{\infty}dx\frac{\sin x}{x}\,,$$

or

$$\int_{-\infty}^{\infty}dx\frac{\sin x}{x}=\pi\,.$$

17. Fourier transformation-I

Fourier transformation and inverse Fourier transformation (definition)

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx\,,$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk\,.$$

From these definitions, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}dk\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{ik\xi}d\xi = f(x)\,,$$

or

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi f(\xi)\int_{-\infty}^{\infty} dke^{ik(\xi-x)}=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi f(\xi)2\pi\delta(\xi-x)\,,$$

or

$$\delta(\xi-x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(\xi-x)}dk\,.$$

18. Fourier transformation II

We consider the Fourier transform of $\delta(x-\xi)$.

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-\xi)e^{ikx}dx=\frac{1}{\sqrt{2\pi}}e^{ik\xi}\,.$$

The inverse Fourier transform is

$$\delta(x-\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{ik\xi}e^{-ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)}dk\,.$$

19. Fourier transform III

We consider the Fourier transformation of $f(x) = 1$.

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}dx=\frac{1}{\sqrt{2\pi}}2\pi\delta(k)=\sqrt{2\pi}\delta(k)\,,$$

or

$$\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\delta(k),$$

or

$$\int_{-\infty}^{\infty} e^{ikx} dk = 2\pi\delta(x),$$

20. Definition of the Dirac delta function

((**Mathematica**))

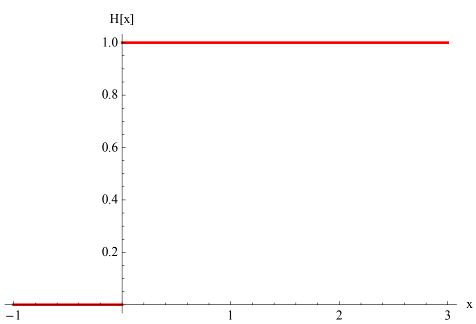
```
Clear["Global`*"]

FourierTransform[1, k, x] // Simplify

√(2 π) DiracDelta[x]
```

21. Shape of the step function

The derivative of the step function $H(x)$ with respect to x, yields the Dirac delta function.

$$H'(x) = \delta(x),$$

where $H(x) = 1$ for $x>0$ and 0 for $x<0$.



In other words,

$$H(x) = \int_{-\infty}^{x} \delta(t) dt .$$

We choose the Gaussian delta function.

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} . \qquad (n\to\infty)$$

Then we have

$$H_n(x) = \frac{1}{2}[1 + erf(nx)]$$

where erf(x) is the error function.

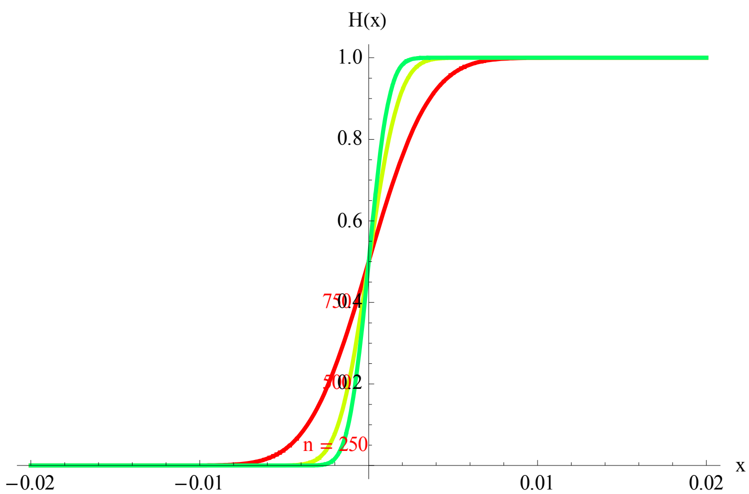


Fig. n is changed as a parameter; n = 250, 500, and 750.

22. Shape of $\delta(x)$

We consider the shape of $\delta(x)$ using the Gaissian delta function

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} . \qquad (n\to\infty)$$

The derivative of $\delta_n(x)$ with respect to x is given by

$$\delta_n'(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} (-2n^2 x) . \qquad (n\to\infty)$$

This function has a odd function with respect to x and has a local maximum (= $n^2\sqrt{\frac{2}{e\pi}}$) at $x = -\frac{1}{\sqrt{2}n}$.

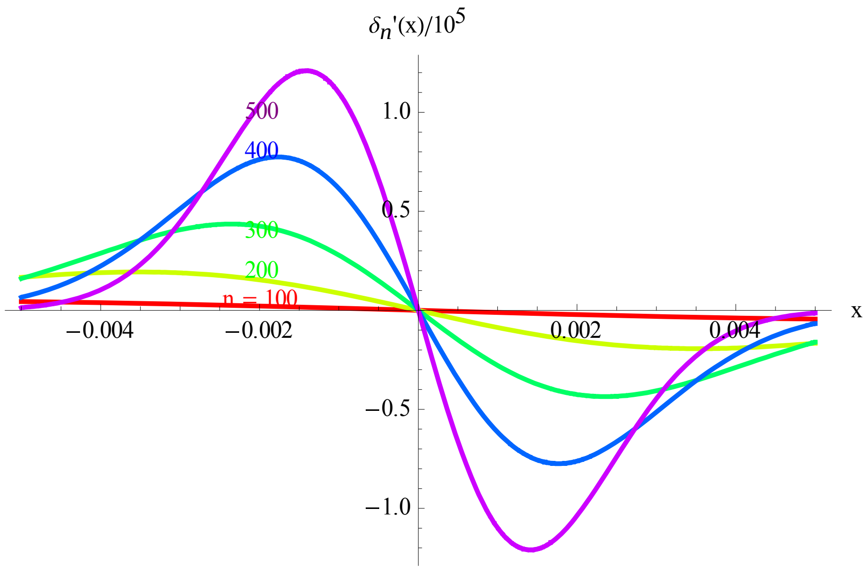


Fig. *n* is changed as a parameter; *n* = 100, 200, 300, 400, and 500.

APPENDIX Mathematrica

Properties of the Dirac function
from Michael Trott's book

```
rule1 =
 { f_[x_] Derivative[ν_][DiracDelta][x_] :> (-1)^ν Derivative[ν] [f][x] DiracDelta[x],
  DiracDelta[x_ - c_] f_[x_] :> f[c] DiracDelta[x - c], x^n_. DiracDelta[-c_ + x_] :> c^n,
  x_^n_. Derivative[ν_][DiracDelta][x_] :>
       (-1)^n ν! / (ν - n)! Derivative[ν - n][DiracDelta][x]
       }

{f_[x_] DiracDelta^(ν_)[x_] :> (-1)^ν f^(ν)[x] DiracDelta[x],
 DiracDelta[-c_ + x_] f_[x_] :> f[c] DiracDelta[x - c], x^n_. DiracDelta[-c_ + x_] :> c^n,
 x_^n_. DiracDelta^(ν_)[x_] :> ((-1)^n ν! DiracDelta^(ν-n)[x]) / (ν - n)!}

eq1 = f[x] DiracDelta'[x] /. rule1

-DiracDelta[x] f′[x]

eq2 = f[x] DiracDelta''[x] /. rule1

DiracDelta[x] f″[x]

eq3 = f[x] DiracDelta'''[x] /. rule1

-DiracDelta[x] f^(3)[x]
```

Technique of pure function

```
F = Function[{x}, Cos[x]]
```

```
Function[{x}, Cos[x]]
```

```
F[x]
```

```
Cos[x]
```

```
G = Function[{x}, x^4 - x^2 + 1]
```

$$\text{Function}\left[\{x\},\ x^4 - x^2 + 1\right]$$

```
G[x]
```

$$1 - x^2 + x^4$$

```
G'[x]
```

$$-2\,x + 4\,x^3$$

```
G''[x]
```

$$-2 + 12\,x^2$$

```
eq11 = eq1 /. f → G
```

$$-\left(-2\,x + 4\,x^3\right)\ \text{DiracDelta}[x]$$

```
eq12 = eq1 /. f → F
```

```
DiracDelta[x] Sin[x]
```

```
eq31 = eq3 /. f → G
```

```
-24 x DiracDelta[x]
```

```
x DiracDelta'[x] /. rule1
```

```
-DiracDelta[x]
```

```
f[x] DiracDelta[x - b] /. rule1
```

```
DiracDelta[-b + x] f[b]
```

```
h[x] DiracDelta'[x] /. rule1
```

```
-DiracDelta[x] h′[x]
```

x^4 `D[DiracDelta[x], {x, 4}] /. rule1`

```
24 DiracDelta[x]
```

x^2 `D[DiracDelta[x], {x, 2}] /. rule1`

```
2 DiracDelta[x]
```

`h = Function[{x},` $x^4 - 3x^3 + 2x$`]`

`Function[{x},` $x^4 - 3x^3 + 2x$`]`

```
f[x] D[DiracDelta[x], {x, 3}] /. rule1 /. f → h
```

```
-(-18 + 24 x) DiracDelta[x]
```

Comment

$(x^4 - 3x^3 + 2x)$ `D[DiracDelta[x], {x, 3}] /. rule1`

$(2x - 3x^3 + x^4)$ `DiracDelta`$^{(3)}$`[x]`

x^3 `D[DiracDelta[x], {x, 3}] /. rule1`

```
-6 DiracDelta[x]
```

23. Delta function in the spherical and cylindrical coordiunates

How is the delta function represented in curvilinear coordinates? First we refer to the basic integration property

$$\int_V \delta(\mathbf{r} - \mathbf{r}')d^3\mathbf{r} = 1$$

when r' is in the volume *V*. We also recall that

$$d^3\mathbf{r} = \frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} dq_1 dq_2 dq_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3 = h(q) dq_1 dq_2 dq_3$$

where Jacobian determinant is defined as;

$$h(q) = \frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix}$$

Accordingly, we must have

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{h(q)} \delta(q_1 - q_1')\delta(q_2 - q_2')\delta(q_3 - q_3')$$

(a) Spherical coordinate

$$\begin{aligned} \delta(\mathbf{r} - \mathbf{r}') &= \frac{1}{r^2 \sin\theta} \delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi') \\ &= \frac{1}{r^2} \delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \\ &= \frac{1}{r^2} \delta(r - r')\delta(\mu - \mu')\delta(\phi - \phi') \end{aligned}$$

where $\mu = \cos\theta$, since

$$\begin{aligned} \delta(\cos\theta - \cos\theta') &= \delta[(\theta - \theta')(-\sin\theta')] \\ &= \frac{1}{\sin\theta'} \delta(\theta - \theta') \\ &= \frac{1}{\sin\theta} \delta(\theta - \theta') \end{aligned}$$

(b) Cylindrical co-ordinate

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{\rho} \delta(\rho - \rho')\delta(\phi - \phi')\delta(\mathrm{z} - \mathrm{z}')$$

REFERENCE:

J. Schwinger et al, Classical Electrodynamics (Perseus Books, Reading, MA, 1998).

APPENDIX

Derivation of Green's function

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r}),$$

where

$$\mathbf{r} = (x, y, z), \qquad r = \sqrt{x^2 + y^2 + z^2}.$$

We consider a sphere with radius ε ($\varepsilon \to 0$)

$$\int d\mathbf{r} \nabla \cdot \nabla \frac{1}{r} = \int d\mathbf{r} \Delta \frac{1}{r} = \int d\mathbf{a} \cdot \nabla \frac{1}{r} = \int da (\mathbf{n} \cdot \nabla \frac{1}{r})$$

where

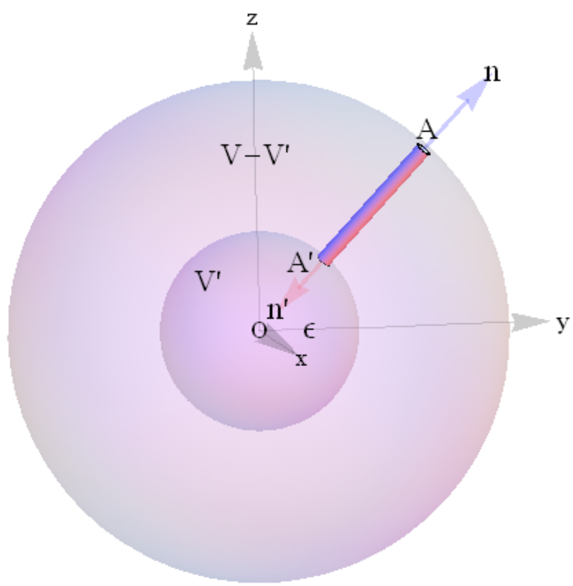
$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \mathbf{n} = \frac{\mathbf{r}}{r} = \mathbf{e}_r = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r}), \qquad d\mathbf{a} = \mathbf{n} da$$

and

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}, \qquad \mathbf{n} \cdot \nabla \frac{1}{r} = \hat{r} \cdot (-\frac{\mathbf{r}}{r^3}) = -\frac{1}{r^2}$$

$\nabla \cdot \nabla (\frac{1}{r}) = 0$ except at the origin.

We now consider the volume integral over the whole volume (V - V') between the surface A and the surface of sphere A' (volume V', radius $\varepsilon \to 0$). We note that the outer surface and the inner surface are connected to an appropriate cylinder.



Since $\nabla \cdot \nabla(\frac{1}{r}) = 0$ over the whole volume *V* - *V*' we have

Using the Gauss's law, we get

$$\int_{V-V'} d\mathbf{r} \nabla \cdot \nabla \frac{1}{r} = \int_{V-V'} d\mathbf{r} \nabla^2 \frac{1}{r}$$

$$= \int_A da(\mathbf{n} \cdot \nabla \frac{1}{r}) + \int_{A'} da'(\mathbf{n}' \cdot \nabla \frac{1}{r}) = 0$$

or

$$\int_A da(\mathbf{n} \cdot \nabla \frac{1}{r}) = -\int_{A'} da'(\mathbf{n}' \cdot \nabla \frac{1}{r}) = \int_{A'} da'(\mathbf{n} \cdot \nabla \frac{1}{r})$$

where $\mathbf{n}' = -\mathbf{n} = -\hat{r}$ and d***r*** is over the volume integral. Then we have

$$\int_A da(\mathbf{n} \cdot \nabla \frac{1}{r}) = \int da(-\frac{1}{r^2}) = -4\pi\varepsilon^2 \frac{1}{\varepsilon^2} = -4\pi = -4\pi \int d\mathbf{r} \delta(\mathbf{r})$$

Using the Gauss's law, we have

$$\int_A da(\mathbf{n}\cdot\nabla\frac{1}{r}) = \int_V d\mathbf{r}(\nabla\cdot\nabla\frac{1}{r}) = -4\pi\int_V d\mathbf{r}\delta(\mathbf{r})$$

or

$$\Delta\frac{1}{r} = -4\pi\delta(\mathbf{r}).$$

or

$$\Delta(\frac{1}{4\pi r}) = -\delta(\mathbf{r}).$$

((**Mathematica**))

```
Clear["Gobal`"];

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]]

Cartesian[x, y, z]

r1 = {x, y, z}; r = √r1.r1

√(x^2 + y^2 + z^2)

Grad[1/r] // Simplify

{-x/(x^2 + y^2 + z^2)^(3/2), -y/(x^2 + y^2 + z^2)^(3/2), -z/(x^2 + y^2 + z^2)^(3/2)}

Laplacian[1/r] // Simplify

0
```