Kronecker product

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1. Introduction

In mathematics, the **Kronecker product**, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix. It is a generalization of the outer product (which is denoted by the same symbol) from vectors to matrices, and gives the matrix of the tensor product with respect to a standard choice of basis. The Kronecker product should not be confused with the usual matrix multiplication, which is an entirely different operation. The Kronecker product is named after Leopold Kronecker, even though there is little evidence that he was the first to define and use it. Indeed, in the past the Kronecker product was sometimes called the *Zehfuss matrix,* after Johann Georg Zehfuss.

http://en.wikipedia.org/wiki/Kronecker_product

We consider the tensor product of the two states

$$|\psi_1, \psi_2\rangle .$$

In more formal mathematical notation, we denote as

$$|\psi_1\rangle \otimes |\psi_2\rangle .$$

using a symbol $\otimes$ (Kronecker product).

Here we define the KroneckerProduct.

$$\hat{A} \otimes \hat{B} ,$$

$$|\psi_1\rangle \otimes |\psi_2\rangle ,$$

where $\hat{A}$ and $\hat{B}$ are the operators (matrices) and $|\psi_1\rangle$ and $|\psi_2\rangle$ are the kets.

$$(\hat{A} \otimes \hat{B})(|\psi_1\rangle \otimes |\psi_2\rangle) = \hat{A}|\psi_1\rangle \otimes \hat{B}|\psi_2\rangle .$$

Similarly, we have

$$(\hat{A}_1 \otimes \hat{A}_2 \otimes \hat{A}_3)(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle) = (\hat{A}_1|\psi_1\rangle) \otimes (\hat{A}_2|\psi_2\rangle) \otimes (\hat{A}_3|\psi_3\rangle).$$

Note that the Bell state is defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|+z\rangle \otimes |-z\rangle - |-z\rangle \otimes |+z\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

2. Representation for the Kronecker product in matrix forms

(a)

$$\hat{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix},$$

(b)

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & a_{12}\hat{B} \\ a_{21}\hat{B} & a_{22}\hat{B} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix},$$

(c)

$$\hat{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_1 \hat{B} \\ a_2 \hat{B} \end{pmatrix} = \begin{pmatrix} a_1 b_{11} & a_1 b_{12} \\ a_1 b_{21} & a_1 b_{22} \\ a_2 b_{11} & a_2 b_{12} \\ a_2 b_{21} & a_2 b_{22} \end{pmatrix},$$

(d)

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & a_{12}\hat{B} & a_{13}\hat{B} \\ a_{21}\hat{B} & a_{22}\hat{B} & a_{23}\hat{B} \\ a_{31}\hat{B} & a_{32}\hat{B} & a_{33}\hat{B} \end{pmatrix},$$

or

$$\begin{pmatrix} \text{a11 b11} & \text{a11 b12} & \text{a11 b13} & \text{a12 b11} & \text{a12 b12} & \text{a12 b13} & \text{a13 b11} & \text{a13 b12} & \text{a13 b13} \\ \text{a11 b21} & \text{a11 b22} & \text{a11 b23} & \text{a12 b21} & \text{a12 b22} & \text{a12 b23} & \text{a13 b21} & \text{a13 b22} & \text{a13 b23} \\ \text{a11 b31} & \text{a11 b32} & \text{a11 b3} & \text{a12 b31} & \text{a12 b32} & \text{a12 b3} & \text{a13 b31} & \text{a13 b32} & \text{a13 b3} \\ \text{a21 b11} & \text{a21 b12} & \text{a21 b13} & \text{a22 b11} & \text{a22 b12} & \text{a22 b13} & \text{a23 b11} & \text{a23 b12} & \text{a23 b13} \\ \text{a21 b21} & \text{a21 b22} & \text{a21 b23} & \text{a22 b21} & \text{a22 b22} & \text{a22 b23} & \text{a23 b21} & \text{a23 b22} & \text{a23 b23} \\ \text{a21 b31} & \text{a21 b32} & \text{a21 b3} & \text{a22 b31} & \text{a22 b32} & \text{a22 b3} & \text{a23 b31} & \text{a23 b32} & \text{a23 b3} \\ \text{a31 b11} & \text{a31 b12} & \text{a31 b13} & \text{a32 b11} & \text{a32 b12} & \text{a32 b13} & \text{a33 b11} & \text{a33 b12} & \text{a33 b13} \\ \text{a31 b21} & \text{a31 b22} & \text{a31 b23} & \text{a32 b21} & \text{a32 b22} & \text{a32 b23} & \text{a33 b21} & \text{a33 b22} & \text{a33 b23} \\ \text{a31 b31} & \text{a31 b32} & \text{a31 b3} & \text{a32 b31} & \text{a32 b32} & \text{a32 b3} & \text{a33 b31} & \text{a33 b32} & \text{a33 b3} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] ⧴ Complex[re, -im]};
```

$$A1 = \begin{pmatrix} a1 \\ a2 \end{pmatrix};\ B1 = \begin{pmatrix} b1 \\ b2 \end{pmatrix};$$

```
C1 = KroneckerProduct[A1, B1] // Simplify;
C1 // MatrixForm
```

$$\begin{pmatrix} a1\,b1 \\ a1\,b2 \\ a2\,b1 \\ a2\,b2 \end{pmatrix}$$

$$A2 = \begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix};\ B2 = \begin{pmatrix} b11 & b12 \\ b21 & b22 \end{pmatrix};$$

```
C2 = KroneckerProduct[A2, B2] // Simplify;
C2 // MatrixForm
```

$$\begin{pmatrix} a11\,b11 & a11\,b12 & a12\,b11 & a12\,b12 \\ a11\,b21 & a11\,b22 & a12\,b21 & a12\,b22 \\ a21\,b11 & a21\,b12 & a22\,b11 & a22\,b12 \\ a21\,b21 & a21\,b22 & a22\,b21 & a22\,b22 \end{pmatrix}$$

```
AB12 = KroneckerProduct[A1, B2] // Simplify;
AB12 // MatrixForm
```

$$\begin{pmatrix} a1\,b11 & a1\,b12 \\ a1\,b21 & a1\,b22 \\ a2\,b11 & a2\,b12 \\ a2\,b21 & a2\,b22 \end{pmatrix}$$

$$A3 = \begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix};\ B3 = \begin{pmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \\ b31 & b32 & b3 \end{pmatrix};$$

```
C3 = KroneckerProduct[A3, B3] // Simplify;
C3 // MatrixForm
```

$$\begin{pmatrix} a11\,b11 & a11\,b12 & a11\,b13 & a12\,b11 & a12\,b12 & a12\,b13 & a13\,b11 & a13\,b12 & a13\,b13 \\ a11\,b21 & a11\,b22 & a11\,b23 & a12\,b21 & a12\,b22 & a12\,b23 & a13\,b21 & a13\,b22 & a13\,b23 \\ a11\,b31 & a11\,b32 & a11\,b3 & a12\,b31 & a12\,b32 & a12\,b3 & a13\,b31 & a13\,b32 & a13\,b3 \\ a21\,b11 & a21\,b12 & a21\,b13 & a22\,b11 & a22\,b12 & a22\,b13 & a23\,b11 & a23\,b12 & a23\,b13 \\ a21\,b21 & a21\,b22 & a21\,b23 & a22\,b21 & a22\,b22 & a22\,b23 & a23\,b21 & a23\,b22 & a23\,b23 \\ a21\,b31 & a21\,b32 & a21\,b3 & a22\,b31 & a22\,b32 & a22\,b3 & a23\,b31 & a23\,b32 & a23\,b3 \\ a31\,b11 & a31\,b12 & a31\,b13 & a32\,b11 & a32\,b12 & a32\,b13 & a33\,b11 & a33\,b12 & a33\,b13 \\ a31\,b21 & a31\,b22 & a31\,b23 & a32\,b21 & a32\,b22 & a32\,b23 & a33\,b21 & a33\,b22 & a33\,b23 \\ a31\,b31 & a31\,b32 & a31\,b3 & a32\,b31 & a32\,b32 & a32\,b3 & a33\,b31 & a33\,b32 & a33\,b3 \end{pmatrix}$$

3. Relations to the matrix operations

(a) Bilinearity and associativity: The Kronecker product is a special case of the tensor product, so it is bilinear and associative:

$$\hat{A} \otimes (\hat{B} + \hat{C}) = \hat{A} \otimes \hat{B} + \hat{A} \otimes \hat{C},$$

$$(\hat{A} + \hat{B}) \otimes \hat{C} = \hat{A} \otimes \hat{C} + \hat{B} \otimes \hat{C},$$

$$(k\hat{A}) \otimes \hat{B} = \hat{A} \otimes (k\hat{B}) = k(\hat{A} \otimes \hat{B}),$$

$$(\hat{A}\otimes\hat{B})\otimes\hat{C} = \hat{A}\otimes(\hat{B}\otimes\hat{C}),$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are matrices and k is a scalar.

(b) **Non-commutative:** In general $\hat{A}\otimes\hat{B}$ and $\hat{B}\otimes\hat{A}$ are different matrices.

(c) The mixed-product property and the inverse of a Kronecker product: If $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are matrices of such size that one can form the matrix products $\hat{A}\hat{C}$ and $\hat{B}\hat{D}$, then

$$(\hat{A}\otimes\hat{B})(\hat{C}\otimes\hat{D}) = (\hat{A}\hat{C})\otimes(B\hat{D}).$$

This is called the *mixed-product property,* because it mixes the ordinary matrix product and the Kronecker product. It follows that $\hat{A}\otimes\hat{B}$ is invertible if and only if **A** and **B** are invertible, in which case the inverse is given by

$$\left(\hat{A}\otimes\hat{B}\right)^{-1} = \hat{A}^{-1}\otimes\hat{B}^{-1}.$$

(d) Transpose: The transposition and conjugate transposition are distributive over the Kronecker product:

$$\left(\hat{A}\otimes\hat{B}\right)^{T} = \hat{A}^{T}\otimes\hat{B}^{T}, \quad \text{and} \quad \left(\hat{A}\otimes\hat{B}\right)^{*} = \hat{A}^{*}\otimes\hat{B}^{*}.$$

((Note))
Here we show the very useful formula,

$$(\hat{A}_1\otimes\hat{B}_1)(\hat{A}_2\otimes\hat{B}_2)(\hat{A}_3\otimes\hat{B}_3)...(\hat{A}_n\otimes\hat{B}_n) = (\hat{A}_1\hat{A}_2\hat{A}_3...\hat{A}_n)\otimes(\hat{B}_1\hat{B}_2\hat{B}_3...\hat{B}_n).$$

$$(\hat{A}_1\otimes\hat{B}_1+\hat{A}_2\otimes\hat{B}_2)(\hat{A}_3\otimes\hat{B}_3+\hat{A}_4\otimes\hat{B}_4) = \hat{A}_1\hat{A}_3\otimes\hat{B}_1\hat{B}_3+\hat{A}_1\hat{A}_4\otimes\hat{B}_1\hat{B}_4 + \hat{A}_2\hat{A}_3\otimes\hat{B}_2\hat{B}_3+\hat{A}_2\hat{A}_4\otimes\hat{B}_2\hat{B}_4$$

4. qubits: $|0\rangle\otimes|0\rangle$, $|0\rangle\otimes|1\rangle$, $|1\rangle\otimes|0\rangle$, $|1\rangle\otimes|1\rangle$

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix},$$

$$|0\rangle\otimes|0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |0\rangle\otimes|1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |1\rangle\otimes|0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},$$

$$|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

5. $\sigma_i \otimes \sigma_j$ (*i*, *j* = 1, 2, 3)

The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\sigma_i \otimes \sigma_j$ has the 4x4 matrix. There are 9 combinations.

$$\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\sigma_y \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_y \otimes \sigma_z = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\sigma_z \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \sigma_z \otimes \sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

6. $\sigma_i \otimes \sigma_j \otimes \sigma_k$ (*i*, *j*, *k* = 1, 2, 3)

$\sigma_i \otimes \sigma_j \otimes \sigma_k$ has the 8x8 matrix. There are 27 combinations.

$$\sigma_x \otimes \sigma_x \otimes \sigma_x =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_x \otimes \sigma_y \otimes \sigma_y =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_y \otimes \sigma_z \otimes \sigma_x =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

7. Example (1)

$$(\hat{\sigma}_x \otimes \hat{\sigma}_y)(|0\rangle \otimes |1\rangle) = (\hat{\sigma}_x |0\rangle \otimes (\hat{\sigma}_y)|1\rangle),$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\sigma}_x|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \hat{\sigma}_y|1\rangle = \begin{pmatrix} -i \\ 0 \end{pmatrix},$$

$$(\hat{\sigma}_x|0\rangle \otimes (\hat{\sigma}_y)|1\rangle) = \begin{pmatrix} 0 \\ 0 \\ -i \\ 0 \end{pmatrix}, \qquad (\hat{\sigma}_x \otimes \hat{\sigma}_y)(|0\rangle \otimes |1\rangle) = \begin{pmatrix} 0 \\ 0 \\ -i \\ 0 \end{pmatrix}.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$\psi[0] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\psi[1] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\sigma[1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma[2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$;

$\sigma[3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

```
f1 = (KroneckerProduct[σ[1], σ[2]]); f2 = (KroneckerProduct[ψ[0], ψ[1]]);
f12 = f1.f2; g1 = KroneckerProduct[σ[1].ψ[0], σ[2].ψ[1]];
```

```
f12 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ -i \\ 0 \end{pmatrix}$$

```
g1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ -i \\ 0 \end{pmatrix}$$

```
σ[1].ψ[0] // MatrixForm
```

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

```
σ[2].ψ[1] // MatrixForm
```

$$\begin{pmatrix} -i \\ 0 \end{pmatrix}$$

```
f1 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
f2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

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8. Example (2)

Show that

$$(\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \sigma_z)(|0\rangle \otimes |1\rangle \otimes |0\rangle) = (\hat{\sigma}_x|0\rangle) \otimes (\hat{\sigma}_y|1\rangle \otimes (\hat{\sigma}_z|0\rangle),$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\sigma}_x|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \hat{\sigma}_y|1\rangle = \begin{pmatrix} -i \\ 0 \end{pmatrix}, \qquad \hat{\sigma}_z|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$(\hat{\sigma}_x|0\rangle \otimes (\hat{\sigma}_y|1\rangle) = \begin{pmatrix} 0 \\ 0 \\ -i \\ 0 \end{pmatrix}, \qquad (\hat{\sigma}_x|0\rangle \otimes (\hat{\sigma}_y|1\rangle) \otimes (\hat{\sigma}_z|0\rangle) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -i \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \sigma_z =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|0\rangle \otimes |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \sigma_z)(|0\rangle \otimes |1\rangle \otimes |0\rangle) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -i \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

Then we conclude that

$$(\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \sigma_z)(|0\rangle \otimes |1\rangle \otimes |0\rangle) = (\hat{\sigma}_x|0\rangle) \otimes (\hat{\sigma}_y|1\rangle \otimes (\hat{\sigma}_z|0\rangle) .$$

9. Example (3)

In general,

$$\exp(\hat{A} \otimes \hat{B}) \neq \exp(\hat{A}) \otimes \exp(\hat{B}) .$$

We show one example which supports this.

((**Mathematica**))

We calculate separately $\exp(\hat{\sigma}_x) \otimes \exp(\hat{\sigma}_z)$ and $\exp(\hat{\sigma}_x \otimes \hat{\sigma}_z)$ using the Mathematica. It is shown that

$$\exp(\hat{\sigma}_x) \otimes \exp(\hat{\sigma}_z) \neq \exp(\hat{\sigma}_x \otimes \hat{\sigma}_z) .$$

```
Clear["Global`*"]; σ[1] = (0 1
                          1 0); σ[2] = (0 -ⅈ
                                        ⅈ  0); σ[3] = (1  0
                                                      0 -1);
Σ[i_, j_] := KroneckerProduct[MatrixExp[σ[i]], MatrixExp[σ[j]]] //
  FullSimplify;
χ[i_, j_] := MatrixExp[KroneckerProduct[σ[j], σ[i]]] // FullSimplify;
Σ[1, 3] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\left(1+e^2\right) & 0 & \frac{1}{2}\left(-1+e^2\right) & 0 \\ 0 & \frac{1}{2}\left(1+\frac{1}{e^2}\right) & 0 & \frac{1}{2}-\frac{1}{2e^2} \\ \frac{1}{2}\left(-1+e^2\right) & 0 & \frac{1}{2}\left(1+e^2\right) & 0 \\ 0 & \frac{1}{2}-\frac{1}{2e^2} & 0 & \frac{1}{2}\left(1+\frac{1}{e^2}\right) \end{pmatrix}$$

```
χ[1, 3] // MatrixForm
```

$$\begin{pmatrix} \frac{1+e^2}{2e} & \frac{-1+e^2}{2e} & 0 & 0 \\ \frac{-1+e^2}{2e} & \frac{1+e^2}{2e} & 0 & 0 \\ 0 & 0 & \frac{1+e^2}{2e} & -\frac{-1+e^2}{2e} \\ 0 & 0 & -\frac{-1+e^2}{2e} & \frac{1+e^2}{2e} \end{pmatrix}$$

10. Steeb and Hardy Problem ((2-1))

(i) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus $\{|0\rangle, |1\rangle$ forms a basis. Calculate

$$|0\rangle \otimes |0\rangle, \qquad |0\rangle \otimes |1\rangle, \qquad |1\rangle \otimes |0\rangle, \qquad |1\rangle \otimes |1\rangle.$$

(ii) Consider the Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find

$$\hat{\sigma}_x \otimes \hat{\sigma}_z, \qquad \hat{\sigma}_z \otimes \hat{\sigma}_x.$$

((**Solution**))

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(ii)

$$\sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \sigma_z \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] ↦ Complex[re, -im]};
```

$$\psi 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \psi 2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

```
M11 = KroneckerProduct[ψ1, ψ1] // Simplify; M11 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
M12 = KroneckerProduct[ψ1, ψ2] // Simplify; M12 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

```
M21 = KroneckerProduct[ψ2, ψ1] // Simplify; M21 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

```
M22 = KroneckerProduct[ψ2, ψ2] // Simplify; M22 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
σ1 = (0 1
      1 0); σ2 = (0 -i
                  i  0); σ3 = (1  0
                               0 -1);

A13 = KroneckerProduct[σ1, σ3] // Simplify; A13 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

```
A31 = KroneckerProduct[σ3, σ1] // Simplify; A31 // MatrixForm
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

```
A13 - A31 // Simplify

{{0, -1, 1, 0}, {-1, 0, 0, -1}, {1, 0, 0, 1}, {0, -1, 1, 0}}
```

---

11. Steeb and Hardy Problem (2-4)

The single-bit Walsh-Hadamard transform is the unitary operator $\hat{W}$ given by

$$\hat{W}|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle], \qquad \hat{W}|1\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle],$$

or

$$\hat{W} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The n-bit Walsh-Hadamard transformation $\hat{W}_n$ is defined as

$$\hat{W}_n = \hat{W} \otimes \hat{W} \otimes \hat{W} \otimes ... \otimes \hat{W} \qquad (n \text{ times})$$

Consider n = 2. Find

$$\hat{W}_2|0\rangle \otimes |0\rangle.$$

((**Solution**))

$$\hat{W}_2|0\rangle \otimes |0\rangle = (\hat{W} \otimes \hat{W}) \otimes (|0\rangle \otimes |0\rangle)$$
$$= \hat{W}|0\rangle \otimes \hat{W}|0\rangle$$
$$= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$
$$= \frac{1}{2}[|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle]$$

So $\hat{W}_2|0\rangle \otimes |0\rangle$ generates a linear combination of all states. This also applies to $\hat{W}_n$.

((**Note**))

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$W_2|0\rangle \otimes |0\rangle = W|0\rangle \otimes W|0\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$W_3|0\rangle \otimes |0\rangle \otimes |0\rangle = W|0\rangle \otimes W|0\rangle \otimes W|0\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};\ \psi0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \psi1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

```
χ0 = W.ψ0; χ0 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
Ω1 = KroneckerProduct[χ0, χ0]; Ω1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

```
Ω2 = KroneckerProduct[χ0, χ0, χ0 ]; Ω2 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix}$$

---

12. Steeb and Hardy Problem ((2-12))

Let $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, ..., |n-1\rangle\}$ be an orthonormal basis in the Hilbert space. Is

$$|\psi\rangle = \frac{1}{\sqrt{n}}[\sum_{j=0}^{n-2} |j\rangle \otimes |j+1\rangle + |n-1\rangle \otimes |0\rangle],$$

independent of the chosen orthonormal basis? Prove or disprove.

For $n = 2$,

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle].$$

((**Solution**))
(i)
Using the basis

$$|0\rangle = |+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

(ii)
Using the basis

$$|0\rangle = |+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |1\rangle = |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

(iii)

Using the basis

$$|0\rangle = |+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, \quad |1\rangle = |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix},$$

we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}.$$

In conclusion, $|\psi\rangle$ depends on the chosen basis.

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :→ Complex[re, -im]};
ψzp = (1
       0); ψzn = (0
                  1); ψxp = 1/√2 (1
                                  1); ψxn = 1/√2 (1
                                                  -1); ψyp = 1/√2 (1
                                                                   ⅈ)
ψyn = 1/√2 (1
            -ⅈ); Mz = 1/2 (KroneckerProduct[ψzp, ψzn]
      + KroneckerProduct[ψzn, ψzp]) // Simplify;
Mx = 1/2 (KroneckerProduct[ψxp, ψxn]
      + KroneckerProduct[ψxn, ψxp]) // Simplify;
My = 1/2 (KroneckerProduct[ψyp, ψyn]
      + KroneckerProduct[ψyn, ψyp]) // Simplify;
```

```
Mz // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

```
Mx // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

```
My // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

---

13. Steeb and Hardy Problem ((2-13))

The Bell states are given by

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}[\left|0\right\rangle \otimes \left|0\right\rangle + \left|1\right\rangle \otimes \left|1\right\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}[\left|0\right\rangle \otimes \left|0\right\rangle - \left|1\right\rangle \otimes \left|1\right\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}[\left|0\right\rangle \otimes \left|1\right\rangle + \left|1\right\rangle \otimes \left|0\right\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}[\left|0\right\rangle \otimes \left|1\right\rangle - \left|1\right\rangle \otimes \left|0\right\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and form an orthonormal basis. Here $\{\left|0\right\rangle, \left|1\right\rangle\}$. Let

$$\left|0\right\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}, \qquad \left|1\right\rangle = \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}.$$

(i) Find $\left|\Phi^{+}\right\rangle$, $\left|\Phi^{-}\right\rangle$, $\left|\Psi^{+}\right\rangle$, $\left|\Psi^{-}\right\rangle$ for this basis.

(ii) Consider the case when $\theta = 0$ and $\phi = 0$.

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :→ Complex[re, -im]};

rule1 = {θ → 0, ϕ → 0};
```

$$\psi 1 = \begin{pmatrix} \text{Exp}[i\,\phi]\,\text{Cos}[\theta] \\ \text{Sin}[\theta] \end{pmatrix};\ \psi 2 = \begin{pmatrix} -\text{Exp}[i\,\phi]\,\text{Sin}[\theta] \\ \text{Cos}[\theta] \end{pmatrix};$$

$$\Phi 1 = \frac{1}{\sqrt{2}}\,(\text{KroneckerProduct}[\psi 1, \psi 1] + \text{KroneckerProduct}[\psi 2, \psi 2])\ //\ \text{Simplify};$$

```
Φ1 // MatrixForm
```

$$\begin{pmatrix} \frac{e^{2\,i\,\phi}}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Phi 2 = \frac{1}{\sqrt{2}}\,(\text{KroneckerProduct}[\psi 1, \psi 1] - \text{KroneckerProduct}[\psi 2, \psi 2])\ //\ \text{Simplify};$$

```
Φ2 // MatrixForm
```

$$\begin{pmatrix} \frac{e^{2 i \phi} \text{Cos}[2\theta]}{\sqrt{2}} \\ \sqrt{2}\, e^{i \phi} \text{Cos}[\theta]\, \text{Sin}[\theta] \\ \sqrt{2}\, e^{i \phi} \text{Cos}[\theta]\, \text{Sin}[\theta] \\ -\frac{\text{Cos}[2\theta]}{\sqrt{2}} \end{pmatrix}$$

Φ3 = $\frac{1}{\sqrt{2}}$ (KroneckerProduct[ψ1, ψ2]
+ KroneckerProduct[ψ2, ψ1]) // FullSimplify;
Φ3 // MatrixForm

$$\begin{pmatrix} -\sqrt{2}\, e^{2 i \phi} \text{Cos}[\theta]\, \text{Sin}[\theta] \\ \frac{e^{i \phi} \text{Cos}[2\theta]}{\sqrt{2}} \\ \frac{e^{i \phi} \text{Cos}[2\theta]}{\sqrt{2}} \\ \sqrt{2}\, \text{Cos}[\theta]\, \text{Sin}[\theta] \end{pmatrix}$$

Φ4 = $\frac{1}{\sqrt{2}}$ (KroneckerProduct[ψ1, ψ2]
- KroneckerProduct[ψ2, ψ1]) // FullSimplify;
Φ4 // MatrixForm

$$\begin{pmatrix} 0 \\ \frac{e^{i\phi}}{\sqrt{2}} \\ -\frac{e^{i\phi}}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
Φ1 /. rule1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
Φ2 /. rule1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
Φ3 /. rule1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
Φ4 /. rule1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

---

14. Problem and solution

Let B be the Bell matrix

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find B^{-1} and B^{T}.

(ii) Show that

$$(I_2 \otimes B)(B \otimes I_2)(I_2 \otimes B) = \frac{1}{\sqrt{2}}(I_2 \otimes B^2 + B^2 \otimes I_2).$$

(iii) Solve the eigenvalue problem for the Bell matrix *B*.

((**Mathematica**))
We solve this problem using the Mathematica.

$$\text{Clear["Global`*"]; } B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}; I2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

```
Inverse[B] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
Transpose[B] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
f1 = KroneckerProduct[I2, B]; f2 = KroneckerProduct[B, I2];
f = f1.(f2.f1);
f // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

```
g1 = KroneckerProduct[I2, B.B];
g2 = KroneckerProduct[B.B, I2];
```

$$g = \frac{1}{\sqrt{2}} (g1 + g2);$$

```
g // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

```
Eigensystem[B]
```

$$\left\{\left\{\frac{1+i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}\right\}, \{\{-i, 0, 0, 1\}, \{0, i, 1, 0\}, \{i, 0, 0, 1\}, \{0, -i, 1, 0\}\}\right\}$$

15. Problem and solution

(i) Consider the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{pmatrix}.$$

Show that the inverse T^{-1} of T exists and find the inverse.

(ii) Let

$$F_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Calculate

$$T(F_\alpha \otimes F_\alpha)T^{-1}.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{pmatrix};\ InvT = Inverse[T];$$

$$F\alpha = \begin{pmatrix} Cos[\alpha] & -Sin[\alpha] \\ Sin[\alpha] & Cos[\alpha] \end{pmatrix};\ F\beta = \begin{pmatrix} Cos[\beta] & -Sin[\beta] \\ Sin[\beta] & Cos[\beta] \end{pmatrix};$$

```
A1 = T.KroneckerProduct[Fα, Transpose[Fβ]].
   Inverse[T] // Simplify;
InvT // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

```
A1 // MatrixForm
```

$$\begin{pmatrix} Cos[\alpha+\beta] & 0 & 0 & i\,Sin[\alpha+\beta] \\ 0 & Cos[\alpha-\beta] & -Sin[\alpha-\beta] & 0 \\ 0 & Sin[\alpha-\beta] & Cos[\alpha-\beta] & 0 \\ i\,Sin[\alpha+\beta] & 0 & 0 & Cos[\alpha+\beta] \end{pmatrix}$$

REFERENCES

W.H. Steeb and Y. Hardy, Problems and Solutions in Quantum Computing and Quantum Information (World Scientific 2004).

A. Graham, Kronecker Products and Matrix Calculus; with Applications (Ellis Horwood Limited, 1981).

APPENDIX-I Spin 1/2 systems

We use the following notations for the spin 1/2 system.

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \qquad |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

Then we have

$$|\psi_z\rangle = \frac{1}{2}[|+z\rangle \otimes |-z\rangle + |-z\rangle \otimes |+z\rangle] = \frac{1}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$|\psi_x\rangle = \frac{1}{2}[|+x\rangle \otimes |-x\rangle + |-x\rangle \otimes |+x\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$|\psi_y\rangle = \frac{1}{2}[|+y\rangle \otimes |-y\rangle + |-y\rangle \otimes |+y\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

APPENDIX-II KroneckerProduct (mathematics)

(a) Definition of Kronecker Product

We consider a matrix A and a matrix B which are given by

$$A = \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix} \qquad B = \begin{pmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{pmatrix}$$

Then we get

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

We introduce vectors given by

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Consider two linear transformations

$$x = Az = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_{11}z_1 + a_{12}z_2 \\ a_{21}z_1 + a_{22}z_2 \end{pmatrix},$$

$$y = Bw = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} b_{11}w_1 + b_{12}w_2 \\ b_{21}w_1 + b_{22}w_2 \end{pmatrix};$$

The vectors μ and ν are defined by

$$\begin{aligned} \mu &= x \otimes y \\ &= \begin{pmatrix} x_1 y \\ x_2 y \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix} \\ &= \begin{pmatrix} (a_{11}z_1 + a_{12}z_2)(b_{11}w_1 + b_{12}w_2) \\ (a_{11}z_1 + a_{12}z_2)(b_{21}w_1 + b_{22}w_2) \\ (a_{21}z_1 + a_{22}z_2)(b_{11}w_1 + b_{12}w_2) \\ (a_{21}z_1 + a_{22}z_2)(b_{21}w_1 + b_{22}w_2) \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{11}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}\begin{pmatrix} z_1 w_1 \\ z_1 w_2 \\ z_2 w_1 \\ z_2 w_2 \end{pmatrix} \end{aligned}$$

and

$$\nu = z \otimes w = \begin{pmatrix} z_1 w \\ z_2 w \end{pmatrix} = \begin{pmatrix} z_1 w_1 \\ z_1 w_2 \\ z_2 w_1 \\ z_2 w_2 \end{pmatrix}$$

Then we can calculate

$$\begin{aligned}(A \otimes B)(z \otimes w) &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \begin{pmatrix} z_1 w_1 \\ z_1 w_2 \\ z_2 w_1 \\ z_2 w_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix} = \mu \\ &= x \otimes y \\ &= (Az) \otimes (Bw)\end{aligned}$$

In other words, we have

$$(A \otimes B)(z \otimes w) = (Az) \otimes (Bw)$$

Note that

$$A^{-1} \otimes B^{-1} = (A \otimes B)^{-1}$$

(b) Definition of Kronecker Sum (see the Appendix)

Given a matrix A and a matrix B, their Kronecker Sum denoted by $A \oplus B$ is defined as the expression

$$A \oplus B = A \otimes I_m + I_n \otimes B.$$

where A(n x n), B(m x m), I_m (m x m, identity matrix), and I_n((n x n, identity matrix),

We verify that

$$\exp(A) \otimes I_2 = \exp(A \otimes I_2) .$$

where I_2 is the identity matrix of 2 x 2.

(c) Theorem-1

$$(A\otimes B)(u_i \otimes v_j) = Au_i \otimes Bv_j$$
$$= a_i u_i \otimes b_j v_j$$
$$= a_i b_j (u_i \otimes v_j)$$

where u_i is the eigenvector of A with an eigenvalue a_i and v_j is the eigenvector of B with an eigenvalue b_j and

$$Au_i = a_i u_i, \quad \text{and} \quad Bv_j = b_j v_j.$$

(d) Theorem-2

Theorem:

Note that

$$A\oplus B = A\otimes I_B + I_A \otimes B.$$

where I_A is the identity matrix for the same size of the matrix A and I_B is the identity matrix for the same size of the matrix B. $A\oplus B$ denotes the Kronecker sum, but not a direct sum.

If $\{\lambda_i\}$ and $\{\mu_j\}$ are the eigenvalues of A and B, respectively, then $\{\lambda_i + \mu_j\}$ are the eigenvalues of $A\oplus B$.

$$(A\oplus B)(a_i \otimes b_j) = (A\otimes I)(a_i \otimes b_j) + (I\otimes B)(a_i \otimes b_j)$$
$$= (Aa_i \otimes b_j) + (a_i \otimes Bb_j)$$
$$= (\lambda_i a_i \otimes b_j) + (a_i \otimes \mu_j b_j)$$
$$= (\lambda_i + \mu_j)(a_i \otimes b_j)$$

APPENDIX-III Formula

(a). Formula-1

$$(|a\rangle_1 \otimes |b\rangle_2)({}_1\langle c| \otimes_2 \langle d|) = (|a\rangle_{11}\langle c|) \otimes (|b\rangle_{22}\langle d|).$$

((**Proof**))

```
Clear["Global`*"]; expr_* := expr /. Complex[a_, b_] :> Complex[a, -b];

A1 = {{a1}, {a2}, {a3}}; B1 = {{b1}, {b2}, {b3}}; C1 = {{c1}, {c2}, {c3}}; D1 = {{d1}, {d2}, {d3}};

f1 = KroneckerProduct[A1, B1];

f2 = KroneckerProduct[Transpose[C1], Transpose[D1]];

f12 = f1.f2;

g1 = (A1.Transpose[C1]); g2 = B1.Transpose[D1];

g12 = KroneckerProduct[g1, g2];

f12 - g12 // Simplify

{{0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

(b). Formula-2

$$(\hat{A}\otimes\hat{B})(|\psi_1\rangle\otimes|\psi_2\rangle)=\hat{A}|\psi_1\rangle\otimes\hat{B}|\psi_2\rangle.$$

((**Proof**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
A1 = {{a11, a12, a13}, {a21, a22, a23}, {a31, a32, a33}}; B1 = {{b11, b12, b13}, {b21, b22, b23}, {b31, b32, b33}};
ψ1 = {{α1}, {α2}, {α3}}; ψ2 = {{β1}, {β2}, {β3}}; f1 = KroneckerProduct[A1, B1];
f2 = KroneckerProduct[ψ1, ψ2]; f12 = f1.f2;
g1 = A1.ψ1; g2 = B1.ψ2; g12 = KroneckerProduct[g1, g2];
f12 - g12 // Simplify

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}
```

(c) Formula 3

$(\hat{A}\otimes\hat{B})(\hat{C}\otimes\hat{D}) = (\hat{A}\hat{C})\otimes(B\hat{D})$.

((**Proof**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
A1 = ( a11 a12 a13
       a21 a22 a23
       a31 a32 a33 ); B1 = ( b11 b12 b13
                             b21 b22 b23
                             b31 b32 b33 );
C1 = ( c11 c12 c13
       c21 c22 c23
       c31 c32 c33 );
D1 = ( d11 d12 d13
       d21 d22 d23
       d31 d32 d33 );
f1 = KroneckerProduct[A1, B1];
f2 = KroneckerProduct[C1, D1]; f12 = f1.f2;
g1 = A1.C1; g2 = B1.D1;
g12 = KroneckerProduct[g1, g2];
f12 - g12 // FullSimplify // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

(d). Formula-4

$(\hat{A}\otimes\hat{B})^{-1} = \hat{A}^{-1}\otimes\hat{B}^{-1}$.

((**Proof**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
A1 = ( a11  a12 ); AR1 = Inverse[A1]; B1 = ( b11  b12 );
     ( a21  a22 )                          ( b21  b22 )
BR1 = Inverse[B1];

f1 = KroneckerProduct[A1, B1] // Simplify;
f2 = Inverse[f1] // Simplify;
g1 = KroneckerProduct[AR1, BR1] // Simplify;
f2 - g1 // FullSimplify

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

(e) Formula

$Tr(\hat{A}\otimes\hat{B}) = Tr(\hat{B}\otimes\hat{A}) = Tr(\hat{A})Tr(\hat{B})$.

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
A1 = (a11 a12 a13
      a21 a22 a23
      a31 a32 a33);
B1 = (b11 b12 b13
      b21 b22 b23
      b31 b32 b33);

f1 = KroneckerProduct[A1, B1];
f2 = KroneckerProduct[B1, A1];

g1 = Tr[f1] // Simplify

(a11 + a22 + a33) (b11 + b22 + b33)

g2 = Tr[f2] // Simplify

(a11 + a22 + a33) (b11 + b22 + b33)

g1 - g2

0

g3 = Tr[A1] Tr[B1]

(a11 + a22 + a33) (b11 + b22 + b33)

g1 - g3

0
```

APPENDIX-IV Direct sum

(a) Direct sum of two matrices, $A \oplus B$

Another operation, which is used less often, is the direct sum (denoted by $\oplus$). Note the Kronecker sum is also denoted by $\oplus$; the context should make the usage clear. The direct sum of any pair of matrices $\boldsymbol{A}$ of size $m \times n$ and $\boldsymbol{B}$ of size $p \times q$ is a matrix of size $((m+p)\times(n+q)$ defined as

$$\boldsymbol{A} \oplus \boldsymbol{B} = \begin{pmatrix} \boldsymbol{A} & 0 \\ 0 & \boldsymbol{B} \end{pmatrix}.$$

The direct sum of matrices is a special type of block matrix, in particular the direct sum of square matrices is a block diagonal matrix. In general, the direct sum of *n* matrices is

$$\bigoplus_{i-1}^{n} \boldsymbol{A}_i = diag(\boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_n) = \begin{pmatrix} \boldsymbol{A}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{A}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{A}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{A}_n \end{pmatrix}.$$

where the zeros are actually blocks of zeros, i.e. zero matricies.

We define $A \oplus B$ as follows. Suppose that

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

For example, we have

$$A \oplus B = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} A_{2\times 2} & O_{2\times 3} \\ O_{3\times 2} & B_{3\times 3} \end{pmatrix}.$$

(b) Direct sum of two vectors, $v \oplus w$

$$v \oplus w = \begin{pmatrix} v_1 \\ . \\ v_n \\ w_1 \\ . \\ w_m \end{pmatrix} = \begin{pmatrix} v_{n\times 1} \\ w_{m\times 1} \end{pmatrix}.$$

v is the column matrix (*n* x 1), and *w* is the column matrix (*m* x 1).

(c). $(A\oplus B)(v\oplus w)$

$$(A\oplus B)(v\oplus w) = \begin{pmatrix} A_{2\times 2} & O_{2\times 3} \\ O_{3\times 2} & B_{3\times 3} \end{pmatrix}\begin{pmatrix} v_{2\times 1} \\ w_{3\times 1} \end{pmatrix} = \begin{pmatrix} A_{2\times 2}v_{2\times 1} \\ B_{3\times 3}w_{3\times 1} \end{pmatrix} = (Av)\oplus(Bw).$$

Similarly, we have

$$(A\oplus B)(C\oplus D) = \begin{pmatrix} A_{2\times 2} & O_{2\times 3} \\ O_{3\times 2} & B_{3\times 3} \end{pmatrix}\begin{pmatrix} C_{2\times 2} & O_{2\times 3} \\ O_{3\times 2} & D_{3\times 3} \end{pmatrix} = \begin{pmatrix} A_{2\times 2}C_{2\times 2} & O_{2\times 3} \\ O_{3\times 2} & B_{3\times 3}D_{3\times 3} \end{pmatrix}.$$

(d) Properties

Other useful formula are

$$\det(A\oplus B) = \det A \det B,$$

$$Tr(A\oplus B) = Tr(A) + Tr(B).$$

APPENDIX-V Kronecker sum

The Kronecker sum is different from the direct sum, but is also denoted by $\oplus$. It is defined using the Kronecker product $\otimes$ and normal matrix addition. If ***A*** is $n\times n$, ***B*** is $m\times m$ and I_k denotes the ($k\times k$) identity matrix, then the Kronecker sum is defined by

$$A\oplus B = A\otimes I_m + I_n \otimes B.$$

((**Example**))
When

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

we have the Kronecker sum as

$$\boldsymbol{A} \oplus \boldsymbol{B} = \begin{pmatrix} a_{11}+b_{11} & b_{12} & b_{13} & a_{12} & 0 & 0 \\ b_{21} & a_{11}+b_{22} & b_{23} & 0 & a_{12} & 0 \\ b_{31} & b_{32} & a_{11}+b_{33} & 0 & 0 & a_{12} \\ a_{21} & 0 & 0 & a_{22}+b_{11} & b_{12} & b_{13} \\ 0 & a_{21} & 0 & b_{21} & a_{22}+b_{22} & b_{23} \\ 0 & 0 & a_{21} & b_{31} & b_{32} & a_{22}+b_{33} \end{pmatrix}.$$

We note that the direct sum is given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & b_{21} & b_{22} \end{pmatrix}.$$