

Ammonia Maser
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(Date: November 08, 2020)

Charles Hard Townes (born July 28, 1915) is an American Nobel Prize-winning physicist and educator. Townes is known for his work on the theory and application of the maser, on which he got the fundamental patent, and other work in quantum electronics connected with both maser and laser devices. He shared the Nobel Prize in Physics in 1964 with Nikolay Basov and Alexander Prokhorov.

http://en.wikipedia.org/wiki/Charles_Hard_Townes



http://www.nobelprize.org/nobel_prizes/physics/laureates/1964/townes.jpg

1. Introduction

MASER: **M**icrowave **A**mplification by **S**timulated **E**mission of **R**adiation

“The maser was originally conceived in 1951 by Townes, who noted his thoughts on the back of an envelope during a period of contemplation on a bench in Franklin Park, Washington, DC. The device was realized experimentally by Gordon, Zeiger, and Townes (1954).”

C.H. Townes, *How the Laser Happened Adventures of a Scientist* (Oxford University Press, 1999).

A maser is a device that produces coherent electromagnetic waves through amplification due to stimulated emission. Historically the term came from the acronym "microwave amplification by stimulated emission of radiation", although modern masers emit over a broad portion of the electromagnetic spectrum. This has led some to replace "microwave" with "molecular" in the acronym, as suggested by Townes. When optical coherent oscillators were first developed, they were called optical masers, but it has become more common to refer to these as lasers.

Theoretically, reflecting principles previously discussed by Joseph Weber at the June 1952 conference of the Institute of Radio Engineers, the principle of the maser was described by Nikolay Basov and Alexander Prokhorov from Lebedev Institute of Physics at an All-Union Conference on Radio-Spectroscopy held by USSR Academy of Sciences in May 1952. They subsequently published their results in October 1954. Independently, Charles H. Townes, J. P. Gordon, and H. J. Zeiger built the first maser at Columbia University in 1953. The device used stimulated emission in a stream of energized ammonia molecules to produce amplification of microwaves at a frequency of 23.786 GHz. Townes later worked with Arthur L. Schawlow to describe the principle of the optical maser, or laser, which Theodore H. Maiman first demonstrated in 1960. For their research in this field Townes, Basov and Prokhorov were awarded the Nobel Prize in Physics in 1964.

((Note))

Properties of NH₃ maser

- (i) The microwave frequency:

$$\hbar\omega_0 = 2A$$

$$f_0 = 23.786 \text{ GHz}$$

- (ii) The energy:

$$2A = \hbar(2\pi f_0) = 0.098371 \text{ meV}$$

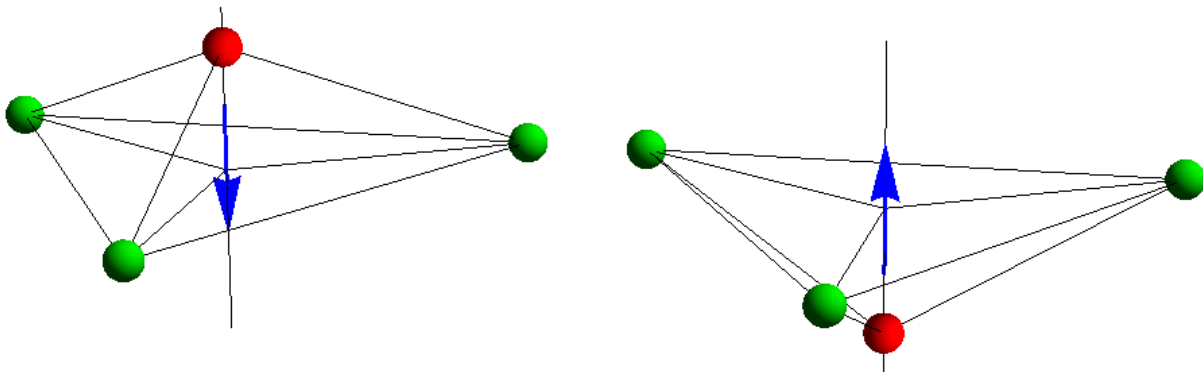
- (iii) The equivalent temperature:

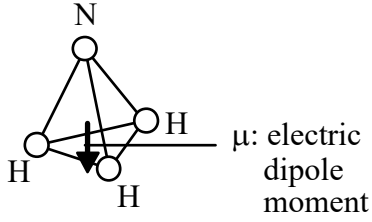
$$T = 1.14155 \text{ K.}$$

- (iv) The wavelength:

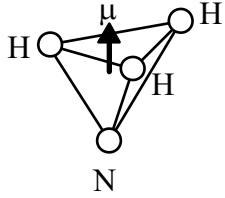
$$\lambda = 1.26037 \text{ cm}$$

2. The state of NH₃ molecule





$|1\rangle$ when the nitrogen is up.



$|2\rangle$ when the nitrogen is down.

We consider the parity operator $\hat{\pi}$, such that

$$\hat{\pi}|1\rangle = |2\rangle \quad \hat{\pi}|2\rangle = |1\rangle.$$

Therefore the kets $|1\rangle$ and $|2\rangle$ are not the eigenkets of $\hat{\pi}$. Since

$$\hat{\pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{\sigma}_x,$$

$\hat{\pi}$ is regarded as the Pauli matrix $\hat{\sigma}_x$. The eigenkets of $\hat{\sigma}_x$ are $|\pm x\rangle$.

$$\hat{\sigma}_x|\pm x\rangle = \pm|\pm x\rangle,$$

with

$$|+x\rangle = |\phi_s\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle); \quad \text{symmetric state.}$$

$$|-x\rangle = |\phi_a\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle); \quad \text{antisymmetric state}$$

These two states are the eigenkets of $\hat{\pi}$. We now consider the Hamiltonian \hat{H} . The symmetry of two physical configuration suggests that

$$\langle 1|\hat{H}|1\rangle = \langle 2|\hat{H}|2\rangle = E_0.$$

What about the off-diagonal elements? The vanishing of $\langle 2|\hat{H}|1\rangle$ would mean that a molecule initially in the state $|1\rangle$ would remain in that state. If $\langle 2|\hat{H}|1\rangle \neq 0$, there is a small amplitude for the system to mix between the two states.

$$\langle 2|\hat{H}|1\rangle = -A,$$

$$\hat{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} = E_0 \hat{1} - A \hat{\sigma}_x.$$

under the basis of $|1\rangle$ and $|2\rangle$. This Hamiltonian commutes with the parity operator: $[\hat{H}, \hat{\pi}] = \hat{0}$ since

$$[\hat{H}, \hat{\pi}] = [E_0 \hat{1} - A \hat{\sigma}_x, \hat{\sigma}_x] = 0.$$

((Eigenvalue problem))

$$\hat{H}|\pm x\rangle = (E_0 \hat{1} - A \hat{\sigma}_x)|\pm x\rangle = (E_0 \mp A)|\pm x\rangle,$$

where

$$\hat{\sigma}_x|\pm x\rangle = \pm|\pm x\rangle.$$

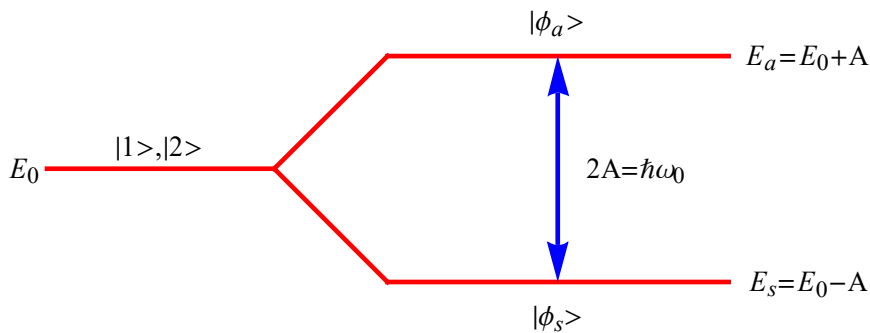
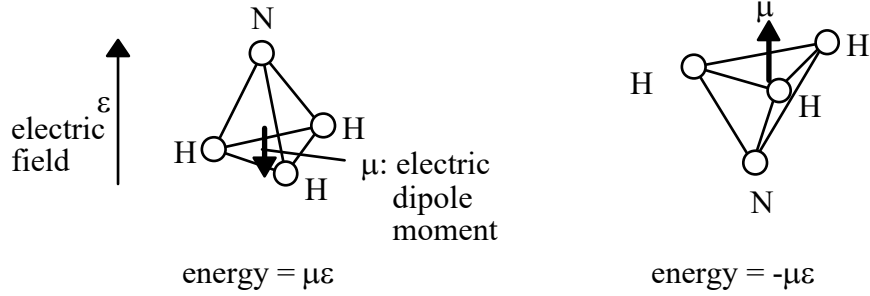


Fig. The splitting of the energy level due to the perturbation A .

3. Application of non-homogeneous static electric field



When the electric field is applied along the x axis (the axis of the electric dipole moment), the Hamiltonian is changed into

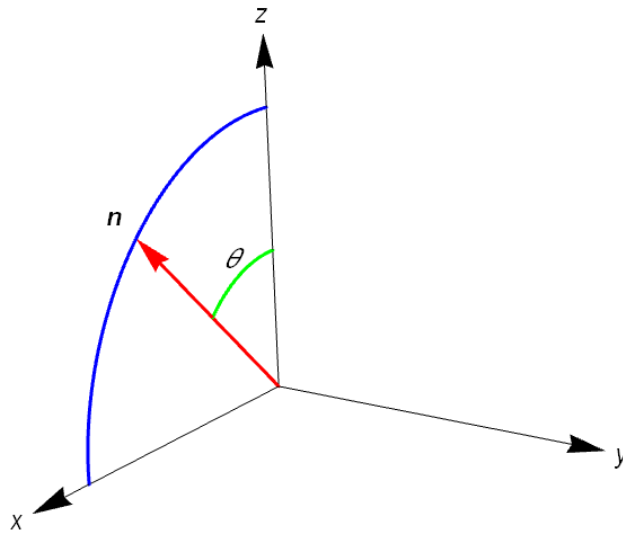
$$\hat{H} = \begin{pmatrix} E_0 + \mu\varepsilon & -A \\ -A & E_0 - \mu\varepsilon \end{pmatrix} = E_0 \hat{1} + \mu\varepsilon \hat{\sigma}_z - A \hat{\sigma}_x.$$

under the basis of $|1\rangle$ and $|2\rangle$. The new Hamiltonian \hat{H} does not commute with the parity operator $\hat{\pi}$ ($=\hat{\sigma}_x$);

$$[\hat{H}, \hat{\pi}] = [\hat{H}, \hat{\sigma}_x] = \mu\varepsilon_0 [\hat{\sigma}_z, \hat{\sigma}_x] = i\mu\varepsilon_0 \hat{\sigma}_y.$$

We have

$$\hat{H} = E_0 \hat{1} + \sqrt{(\mu\varepsilon)^2 + A^2} \left(-\frac{A}{\sqrt{(\mu\varepsilon)^2 + A^2}} \hat{\sigma}_x + \frac{\mu\varepsilon}{\sqrt{(\mu\varepsilon)^2 + A^2}} \hat{\sigma}_z \right).$$



$$\mathbf{n} = \left(-\frac{A}{\sqrt{(\mu\epsilon)^2 + A^2}}, 0, \frac{\mu\epsilon}{\sqrt{(\mu\epsilon)^2 + A^2}} \right),$$

$$\hat{H} = E_0 \hat{1} + \sqrt{(\mu\epsilon)^2 + A^2} \hat{\sigma} \cdot \mathbf{n},$$

$$(\hat{\sigma} \cdot \mathbf{n})|\pm \mathbf{n}\rangle = \pm |\pm \mathbf{n}\rangle,$$

where

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}|2\rangle,$$

and

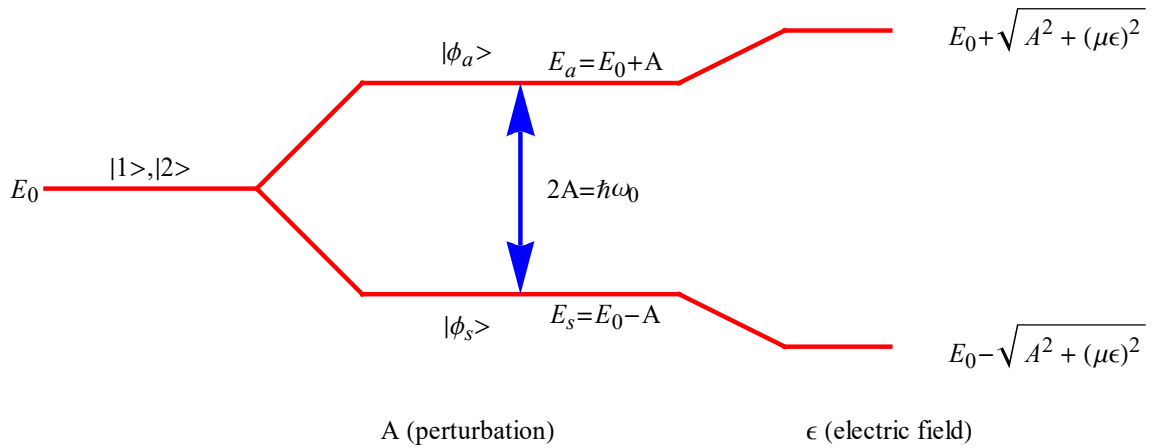
$$|-\mathbf{n}\rangle = -\sin\frac{\theta}{2}|1\rangle + \cos\frac{\theta}{2}|2\rangle,$$

where

$$\sin\theta = -\frac{A}{\sqrt{(\mu\epsilon)^2 + A^2}} \quad \cos\theta = \frac{\mu\epsilon}{\sqrt{(\mu\epsilon)^2 + A^2}}.$$

Thus, we have

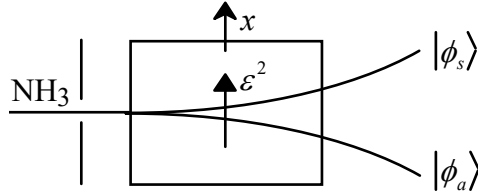
$$\hat{H}|\pm \mathbf{n}\rangle = (E_0 \pm \sqrt{(\mu\epsilon)^2 + A^2})|\pm \mathbf{n}\rangle,$$



In a weak electric field, using the Taylor expansion, we have

$$E_s = E_0 - A\sqrt{1 + \frac{\mu^2 \varepsilon^2}{A^2}} = E_0 - A - \frac{\mu^2 \varepsilon^2}{2A} + \dots$$

$$E_a = E_0 + A\sqrt{1 + \frac{\mu^2 \varepsilon^2}{A^2}} = E_0 + A + \frac{\mu^2 \varepsilon^2}{2A} + \dots$$



Let us consider NH_3 in a region where ε is weak but where ε^2 has a strong gradient in the x -direction (i.e., along the axis of molecules).

$$\frac{d}{dx}(\varepsilon^2) = \lambda.$$

The molecules in the state $|\phi_s\rangle$ are subjected to a force parallel to the x axis:

$$F_s = -\frac{dE_s}{dx} = \frac{1}{2}\lambda\frac{\mu^2}{A}.$$

Similarly, the molecules in the state $|\phi_a\rangle$ are subjected to an opposite force:

$$F_a = -\frac{dE_a}{dx} = -\frac{1}{2}\lambda\frac{\mu^2}{A}.$$

This is the basis of the method which is used in the ammonia maser to sort the molecules and select those in the higher energy state.

We note that just before entering the maser cavity, the states are $\{|\phi_s\rangle, |\phi_a\rangle\}$ since there is no non-homogeneous external electric field.

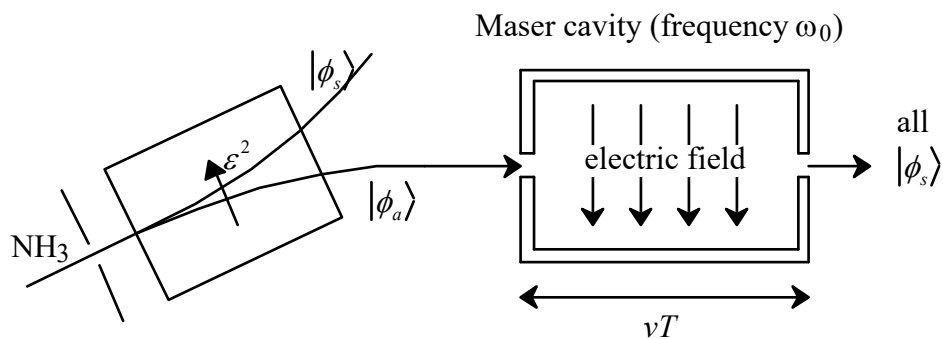


Fig. Diagram of NH_3 maser. Molecules produced by oven are collimated, screened into state $|\phi_a\rangle$ (higher energy state) [focusing field], and passed through maser cavity. They give up their energy to the electric field in accordance with the Rabi flopping formula. No pumping is necessary, unlike lasers.

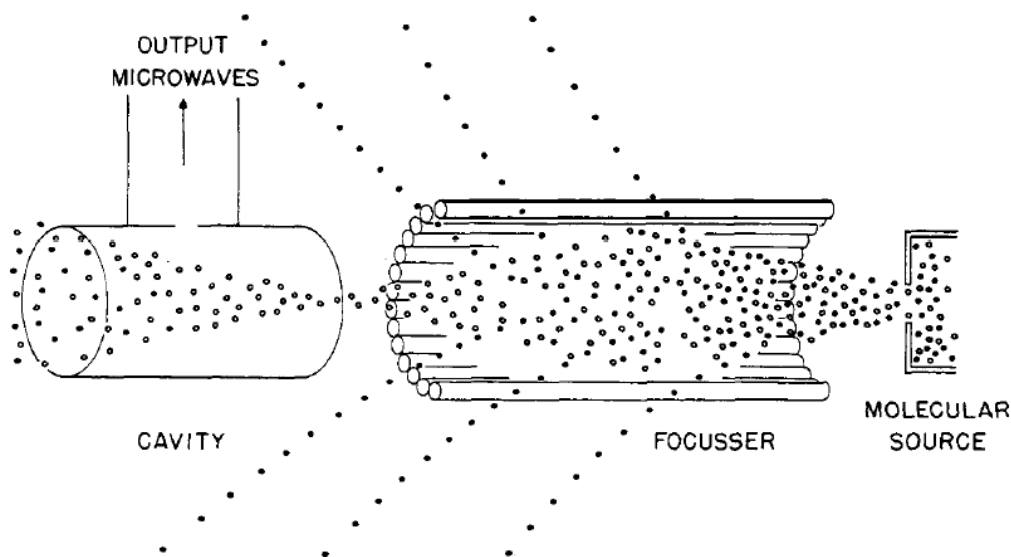


Fig. (C. Townes, Nobel lecture 1964). The ammonia (beam-type) maser. Molecules diffuse from the source into a focuser where the excited molecules (open circles) are focused into a cavity and molecules in the ground state (solid circles) are rejected. A sufficient number of excited molecules will initiate an oscillating electromagnetic field in the cavity, which is emitted as the output microwaves. Because of energy given to the field, some molecules return to the ground state toward the end of their transit through the cavity.

((Charles H. Townes))

Charles H. Townes, *How the Laser Happened: Adventures of a Scientist* (Oxford, 1999), p.57

On that morning in Franklin Park, the goal of boosting energy gave me an incentive to think more deeply about stimulated emission than I had before. How could one get such a nonequilibrium set up? Answers were actually well known; they had been in front of me and the physics community for decades. Rabi, right at Columbia, had been working with molecular and atomic beams (streams of gases) that he manipulated by deflecting atoms in excited states from those of lower energies. The result could be a beam enriched in excited atoms. At Harvard, Ed Purcell and Norman Ramsey had proposed a conceptual name to describe systems with such inverted populations; they had coined the term “negative temperature,” to contrast with the positive temperatures, because these “negative” temperatures inverted the relative excess of lower-level over upper-level states in equilibrated systems. It is perhaps a hackneyed device among dramatists to have a scientist scribble his thinking on the back of an envelope, but that is what I did. I took an envelope from my pocket to try to figure out how many molecules it would take to make an oscillator able to produce and amplify millimeter or submillimeter waves. All the required numbers about my friend, the ammonia molecule, were in my head. Ammonia appeared to be the most favorable medium. I quickly showed that we still needed a resonator, but now we would not have to pump electromagnetic energy into it. We could merely send a stream, or beam, of excited molecules through it, which would do the work! Any resonator has losses, so we would need a certain threshold number of molecules in the flow to keep the wave from dying out. Beyond that threshold, a wave would not only sustain itself bouncing back and forth, but it would gain energy with each pass. The power would be limited only by the rate at which molecules carried energy into the cavity.

3. Change of basis

In the ammonia maser, the beam with molecules in the state $|\phi_a\rangle$ and with the higher energy is sent through a resonant cavity. Here we note that

$$|\phi_s\rangle = |+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \hat{U}|1\rangle, \quad \text{for } (E_0 - A)$$

$$|\phi_a\rangle = |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \hat{U}|2\rangle \quad \text{for } (E_0 + A)$$

where the unitary operator is defined by

$$\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \hat{U}^\dagger = \hat{U}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We now consider the system inside the maser cavity. The Hamiltonian \hat{H} is given by

$$\hat{H}(t) = \begin{pmatrix} E_0 + \mu\mathcal{E}(t) & -A \\ -A & E_0 - \mu\mathcal{E}(t) \end{pmatrix} \quad \text{under the basis of } |1\rangle \text{ and } |2\rangle.$$

in the presence of a time-dependent electric field $\mathcal{E}(t)$ inside the maser cavity. We assume that

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle = c_a(t)|\phi_a\rangle + c_s(t)|\phi_s\rangle$$

where

$$|a_i\rangle = \{|\phi_s\rangle, |\phi_a\rangle\}, \quad |b_i\rangle = \{|1\rangle, |2\rangle\}.$$

$$|\phi_s\rangle = \hat{U}|1\rangle, \quad \text{for } (E_0 - A).$$

$$|\phi_a\rangle = \hat{U}|2\rangle, \quad \text{for } (E_0 + A).$$

(a) The matrix element (I):

$$\begin{aligned} \langle a_i | \psi \rangle &= \sum_j \langle a_i | b_j \rangle \langle b_j | \psi \rangle \\ &= \sum_j \langle b_i | \hat{U}^\dagger | b_j \rangle \langle b_j | \psi \rangle \end{aligned}$$

$$\begin{pmatrix} \langle \phi_s | \psi \rangle \\ \langle \phi_a | \psi \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \langle 1 | \psi \rangle \\ \langle 2 | \psi \rangle \end{pmatrix}$$

or

$$\begin{pmatrix} C_s(t) \\ C_a(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}$$

(b) The matrix element (II):

$$\begin{aligned} \langle b_i | \psi \rangle &= \sum_j \langle b_i | a_j \rangle \langle a_j | \psi \rangle \\ &= \sum_j \langle b_i | \hat{U} | b_j \rangle \langle a_j | \psi \rangle \end{aligned}$$

$$\begin{pmatrix} \langle 1 | \psi \rangle \\ \langle 2 | \psi \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \langle \phi_s | \psi \rangle \\ \langle \phi_a | \psi \rangle \end{pmatrix}$$

or

$$\begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} C_s(t) \\ C_a(t) \end{pmatrix}$$

(c) Schrödinger equation

The Schrödinger equation is given by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle,$$

or

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \langle a_i | \psi(t) \rangle &= \langle a_i | \hat{H} | \psi(t) \rangle \\ &= \sum_j \langle a_i | \hat{H} | a_j \rangle \langle a_j | \psi(t) \rangle \end{aligned}$$

or

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \langle b_i | \psi(t) \rangle &= \langle b_i | \hat{H} | \psi(t) \rangle \\ &= \sum_j \langle b_i | \hat{H} | b_j \rangle \langle b_j | \psi(t) \rangle \end{aligned}$$

Note that

$$\langle a_i | \hat{H} | a_j \rangle = \langle b_i | \hat{U}^\dagger \hat{H} \hat{U} | b_j \rangle$$

and

$$\begin{aligned} \langle b_i | \hat{U}^\dagger \hat{H} \hat{U} | b_j \rangle &\rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E_0 + \mu\varepsilon(t) & -A \\ -A & E_0 - \mu\varepsilon(t) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} E_0 - A & \mu\varepsilon(t) \\ \mu\varepsilon(t) & E_0 + A \end{pmatrix} \end{aligned}$$

Thus we have

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} E_0 + \mu\varepsilon(t) & -A \\ -A & E_0 - \mu\varepsilon(t) \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}. \quad \text{under the basis } |b_i\rangle = \{|1\rangle, |2\rangle\}$$

and

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix} = \begin{pmatrix} E_0 - A & \mu \varepsilon(t) \\ \mu \varepsilon(t) & E_0 + A \end{pmatrix} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix} \quad \text{under the basis } |a_i\rangle = \{|\phi_s\rangle, |\phi_a\rangle\}$$

Thanks to our choice of the basis $|a_i\rangle = \{|\phi_s\rangle, |\phi_a\rangle\}$, when $\varepsilon(t) = 0$,

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix} = \begin{pmatrix} E_0 - A & 0 \\ 0 & E_0 + A \end{pmatrix} \begin{pmatrix} c_s(t) \\ c_a(t) \end{pmatrix}$$

or

$$c_s(t) = \gamma_s e^{-\frac{i(E_0 - A)t}{\hbar}}, \quad c_a(t) = \gamma_a e^{-\frac{i(E_0 + A)t}{\hbar}}.$$

where γ_s and γ_a are constants.

((Mathematica))

```
Clear["Global`*"]; U1 =  $\frac{1}{\sqrt{2}}$   $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ; UR = Inverse[U1];
H1 =  $\begin{pmatrix} E0 + \mu \epsilon & -A \\ -A & E0 - \mu \epsilon \end{pmatrix}$ ; f1 = UR.H1.U1 // Simplify;
f1 // MatrixForm
```

$$\begin{pmatrix} -A + E0 & \epsilon \mu \\ \epsilon \mu & A + E0 \end{pmatrix}$$

When $\varepsilon(t) \neq 0$, first we write

$$c_s(t) = \gamma_s(t) e^{-\frac{i(E_0 - A)t}{\hbar}},$$

$$c_a(t) = \gamma_a(t) e^{-\frac{i(E_0 + A)t}{\hbar}},$$

We obtain a system of coupled differential equations, showing that the electric field (actually polarization vector associated with induced photon) induces transition between the states $\{|\phi_s\rangle, |\phi_a\rangle\}$.

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\varepsilon(t)e^{-i\omega_0 t} \gamma_a(t),$$

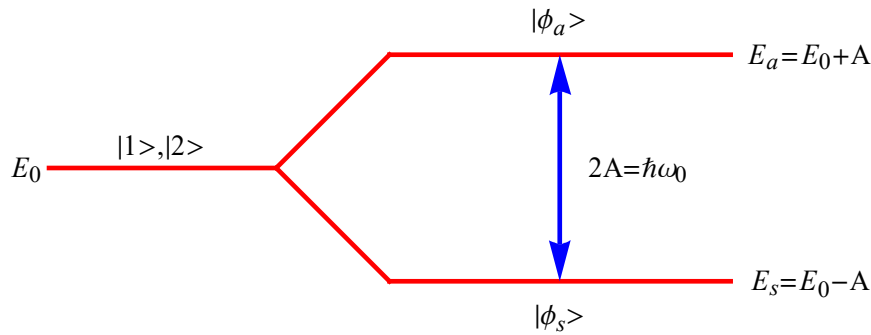
$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\varepsilon(t)e^{i\omega_0 t} \gamma_s(t),$$

where

$$E_a - E_s = \hbar\omega_0 = 2A,$$

with

$$E_a = E_0 + A, \quad E_s = E_0 - A,$$



We consider the case:

$$\varepsilon(t) = 2\varepsilon_0 \cos\omega t = \varepsilon_0 (e^{i\omega t} + e^{-i\omega t}).$$

Then we have

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\varepsilon_0 [e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}] \gamma_a(t),$$

$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\varepsilon_0 [e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}] \gamma_s(t).$$

4. Rotating wave approximation

The terms with $(\omega + \omega_0)$ oscillate very rapidly about an average value of zero and, therefore do not contribute very much on the average to the rate of change of γ .

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\varepsilon_0 e^{i(\omega-\omega_0)t} \gamma_a(t),$$

$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\epsilon_0 e^{-i(\omega-\omega_0)t} \gamma_s(t).$$

5. Rabi angular frequency

5.1 Derivation

The molecules enter the cavity, the cavity field-oscillating at exactly the right frequency-induces transition from the upper to the lower states, and the energy released is fed into the oscillatory field. The molecular energy is converted into the energy of an external electromagnetic field. We define that

$$\Delta = \omega - \omega_0, \quad \Gamma_0 = \frac{\mu\epsilon_0}{\hbar},$$

and

$$\Omega_R = \sqrt{\frac{\Delta^2}{4} + \Gamma_0^2}, \quad (\text{Rabi angular frequency})$$

(we follow the definition of the Rabi angular frequency based on the book of Sakurai and Napolitano), with

$$\hbar\omega_0 = 2A.$$

For convenience, here we use Ω instead of Ω_R

$$\Omega = 2\Omega_R = \sqrt{\Delta^2 + 4\Gamma_0^2}$$

For the following calculations. Then, we get

$$i \frac{d\gamma_s(t)}{dt} = \frac{\mu\epsilon_0}{\hbar} e^{i\Delta t} \gamma_a(t) = \Gamma_0 e^{i\Delta t} \gamma_a(t),$$

$$i \frac{d\gamma_a(t)}{dt} = \frac{\mu\epsilon_0}{\hbar} e^{-i\Delta t} \gamma_s(t) = \Gamma_0 e^{-i\Delta t} \gamma_s(t),$$

with the initial condition

$$\gamma_a(0) = 1, \quad \gamma_s(0) = 0.$$

The solution is as follows, using the Mathematica (which will be shown later)

$$\gamma_s(t) = -\frac{2ie^{-\frac{it\Delta}{2}}\Gamma_0 \sin(\frac{1}{2}\Omega t)}{\Omega} = -\frac{ie^{-\frac{it\Delta}{2}}\Gamma_0 \sin(\Omega_R t)}{\Omega_R},$$

$$\gamma_a(t) = \frac{e^{-\frac{it(\Delta+\Omega)}{2}} [-\Delta + \Omega + e^{it\Omega}(\Delta + \Omega)]}{2\Omega}.$$

The probabilities are given by

$$\begin{aligned} P_s(t) &= \gamma_s(t)^* \gamma_s(t) \\ &= \frac{4\Gamma_0^2 \sin^2(\frac{t}{2}\sqrt{4\Gamma_0^2 + \Delta^2})}{4\Gamma_0^2 + \Delta^2} \\ &= \frac{2\Gamma_0^2 [1 - \cos(t\sqrt{4\Gamma_0^2 + \Delta^2})]}{4\Gamma_0^2 + \Delta^2} \end{aligned}$$

$$P_a(t) = \gamma_a(t)^* \gamma_a(t) = \frac{2\Gamma_0^2 + \Delta^2 + 2\Gamma_0^2 \cos(t\sqrt{4\Gamma_0^2 + \Delta^2})}{4\Gamma_0^2 + \Delta^2}.$$

Note that

$$P_s(t) + P_a(t) = 1.$$

We make a plot of $P_s(t)$ and $P_a(t)$ as a function of t , where $\omega_0 = 10$ and ω is changed as a parameter in the vicinity of $\omega = 10$.

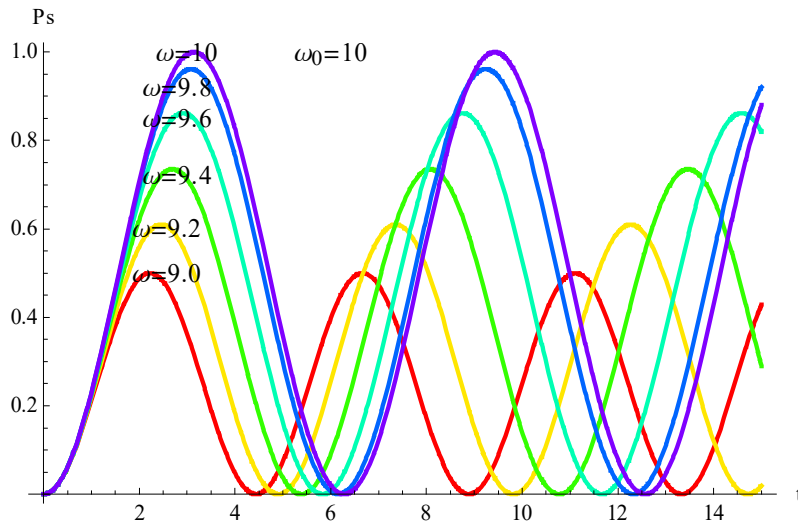


Fig. Probability of finding the NH₃ atoms in the ground state. $\omega_0 = 10$. ω is changed as a parameter between $\omega = 9.0$ and 10.0 . $\Gamma_0 = 0.5$. P_s has a maximum at $\Gamma_0 t = \frac{\pi}{2}$ when $\Delta = 0$.

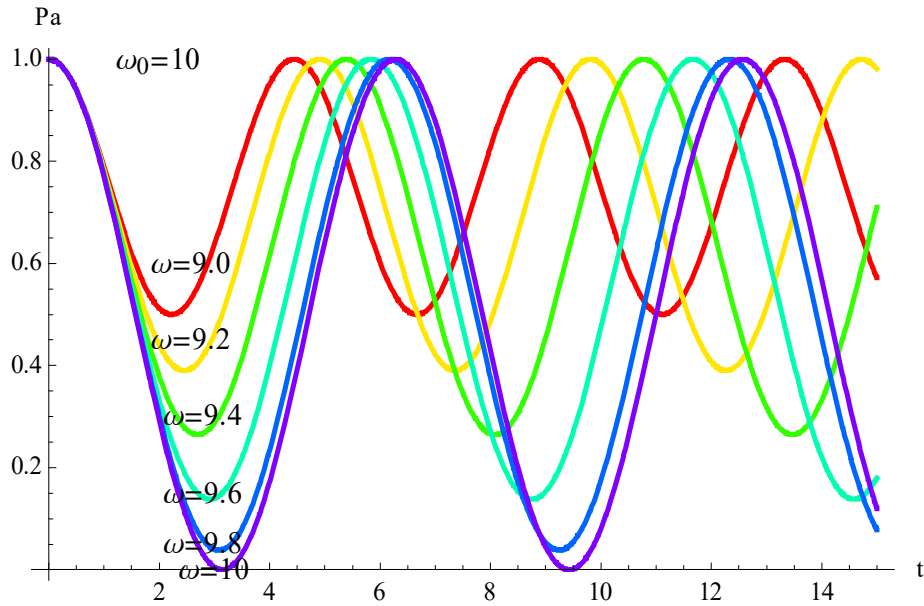


Fig. Probability of finding the NH₃ atoms in the excited state. $\omega_0 = 10$. ω is changed as a parameter between $\omega = 9.0$ and 10.0 . $\Gamma_0 = 0.5$. Note that P_a has a minimum at $\Gamma_0 t = \frac{\pi}{2}$ for $\Delta = 0$ ($\omega = \omega_0$).

5.2. Mathematica

Here we show how to solve the differential equations (time-dependent Schrödinger equations whose Hamiltonian is time dependent) by using the Mathematica.


```
Clear["Global`*"]; SuperStar;
expr_* := expr /. Complex[a_, b_] := Complex[a, -b];
```

$$\Delta = \omega - \omega_0, \quad \sqrt{\frac{4 \epsilon_0^2 \mu^2}{\hbar^2} + \Delta^2} = \Omega, \quad \Gamma_0 = \frac{\mu \epsilon_0}{\hbar}$$

```
eq1 = i D[ya[t], t] == Gamma0 Exp[-i Delta t] ys[t];
```

```
eq2 = i D[ys[t], t] == Gamma0 Exp[i Delta t] ya[t];
```

```
s11 = DSolve[{eq1, eq2, ya[0] == 1, ys[0] == 0}, {ya[t], ys[t]},
t] // Simplify[#, {Delta > 0, Gamma0 > 0}] &;
```

```
s12 = s11 /. {Sqrt[4 Gamma0^2 + Delta^2] -> Omega,
1/Sqrt[4 Gamma0^2 + Delta^2] -> 1/Omega} // Simplify;
```

```
ya[t_] = ya[t] /. s12[[1]] // ComplexExpand // FullSimplify
```

$$\frac{e^{-\frac{1}{2} i t (\Delta + \Omega)} (-\Delta + \Omega + e^{i t \Omega} (\Delta + \Omega))}{2 \Omega}$$

```
ys[t_] = ys[t] /. s12[[1]] // ComplexExpand // FullSimplify
```

$$-\frac{2 i e^{\frac{i t \Delta}{2}} \Gamma_0 \sin\left[\frac{t \Omega}{2}\right]}{\Omega}$$

6. Resonance condition

When $\Delta = 0$ and $\Omega = \Gamma_0 = \frac{\mu \epsilon_0}{\hbar}$,

we have

$$\gamma_a(t) = \cos(\Gamma_0 t), \quad P_a(t) = \cos^2(\Gamma_0 t),$$

and

$$\gamma_s(t) = -i \sin(\Gamma_0 t), \quad P_s(t) = \sin^2(\Gamma_0 t),$$

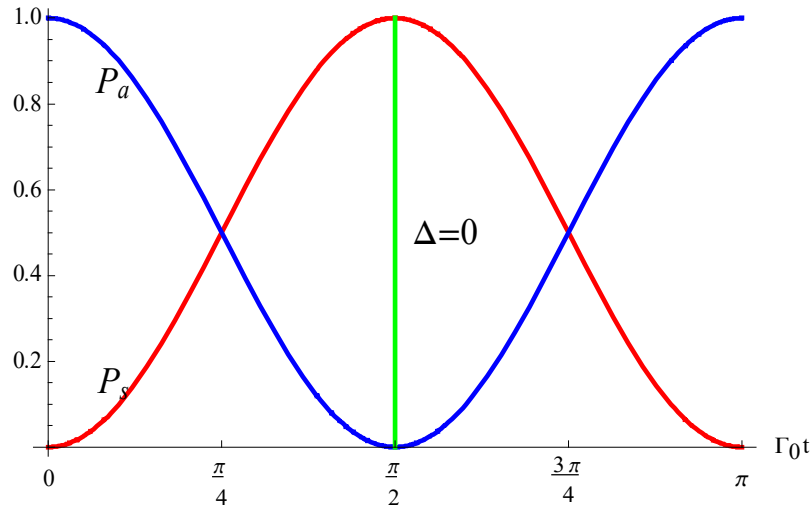


Fig. Probabilities for the two states [$P_a(t)$ and $P_s(t)$] of the ammonia molecules in a sinusoidal electric field. $\Delta = \omega - \omega_0 = 0$. When $\Gamma_0 t = \frac{\pi}{2}$, $P_s = 1$ and $P_a = 0$.

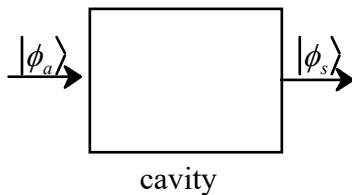
7. Physical meaning

((Feynman, vol.3. 9-12))

Let us suppose that it takes the time $t = T$ to go through the cavity. If we make the cavity just long enough so that

$$\Gamma_0 T = \mu \varepsilon_0 T / \hbar = \pi / 2,$$

then a molecule which enters in the upper state $|\phi_a\rangle$ will certainly leave it in the lower state $|\phi_s\rangle$.



In other words, its energy is decreased, and the loss of energy cannot go anywhere else but into the machinery which generates the field. By energy conservation the molecules deliver energy to the electromagnetic field. This process is called stimulated emission. If the molecules are initially in the state $|\phi_s\rangle$, they will absorb energy from the electromagnetic field in going to the state $|\phi_a\rangle$, a process called stimulated absorption. When the size of the resonant cavity is L , L should be on the same order as $L = vT$, where v is the average velocity of the molecules. Note that the velocity is not the same. They obey the Maxwell distribution.

8. Alternative method for solving the differential equation

8-1 Formulation

Here we use the same technique which is used in the discussion of the magnetic resonance in order to obtain the probability of finding the system in the $|\phi_a\rangle$ and $|\phi_s\rangle$ states. We start with the eqs.,

$$i\hbar \frac{d\gamma_s(t)}{dt} = \mu\varepsilon_0 e^{i(\omega-\omega_0)t} \gamma_a(t),$$

$$i\hbar \frac{d\gamma_a(t)}{dt} = \mu\varepsilon_0 e^{-i(\omega-\omega_0)t} \gamma_s(t).$$

We assume that

$$\Delta = \omega - \omega_0, \quad \Gamma_0 = \frac{\mu\varepsilon_0}{\hbar},$$

$$\Omega_R = \sqrt{\frac{\Delta^2}{4} + \Gamma_0^2}. \quad (\text{Rabi angular frequency})$$

We also use the notation

$$\Omega = 2\Omega_R = \sqrt{\Delta^2 + 4\Gamma_0^2}.$$

For convenience, we put

$$\gamma_s(t) = e^{i\Delta t/2} \chi_s(t),$$

$$\gamma_a(t) = e^{-i\Delta t/2} \chi_a(t)$$

This is related to the relation between the rotating reference frame and the laboratory frame in the case of nuclear magnetic resonance. Substituting these expressions in the above differential equations, we get

$$i\hbar(e^{i\Delta t/2} \dot{\chi}_s + i\frac{\Delta}{2}e^{i\Delta t/2} \chi_s) = \hbar\Gamma_0 e^{i\Delta t} e^{-i\Delta t/2} \chi_a,$$

$$i\hbar(e^{-i\Delta t/2} \dot{\chi}_a - i\frac{\Delta}{2}e^{-i\Delta t/2} \chi_a) = \hbar\Gamma_0 e^{-i\Delta t} e^{i\Delta t/2} \chi_s,$$

or

$$i\hbar(\dot{\chi}_s + i\frac{1}{2}\Delta\chi_s) = \hbar\Gamma_0\chi_a,$$

$$i\hbar(\dot{\chi}_a - i\frac{1}{2}\Delta\chi_a) = \hbar\Gamma_0\chi_s,$$

or

$$i\hbar\dot{\chi}_s = \hbar(\frac{1}{2}\Delta\chi_s + \Gamma_0\chi_a),$$

$$i\hbar\dot{\chi}_a = \hbar(-\frac{1}{2}\Delta\chi_a + \Gamma_0\chi_s).$$

These equations can be rewritten in the form of the Schrödinger equation for $|\psi'(t)\rangle$,

$$i\hbar\frac{\partial}{\partial t}|\psi'(t)\rangle = \hat{H}'|\psi'(t)\rangle,$$

where

$$|\psi'(t)\rangle = \begin{pmatrix} \chi_s(t) \\ \chi_a(t) \end{pmatrix},$$

under the basis of

$$|\phi_a\rangle = |-z\rangle, \quad |\phi_s\rangle = |+z\rangle.$$

(we use this notation for the simplicity, from the analogy from the spin 1/2 case). The new Hamiltonian \hat{H}' is independent of time t and be expressed by

$$\begin{aligned} \hat{H}' &= \hbar \begin{pmatrix} \frac{1}{2}\Delta & \Gamma_0 \\ \Gamma_0 & -\frac{1}{2}\Delta \end{pmatrix} \\ &= \hbar(\frac{1}{2}\Delta\hat{\sigma}_z + \Gamma_0\hat{\sigma}_x) \\ &= \frac{\hbar}{2}\sqrt{4\Gamma_0^2 + \Delta^2} \left(\frac{\Delta}{\sqrt{4\Gamma_0^2 + \Delta^2}}\hat{\sigma}_z + \frac{2\Gamma_0}{\sqrt{4\Gamma_0^2 + \Delta^2}}\hat{\sigma}_x \right) \end{aligned}$$

or

$$\hat{H}' = \frac{\hbar}{2}\sqrt{4\Gamma_0^2 + \Delta^2}(\hat{\sigma} \cdot \mathbf{n}),$$

where \mathbf{n} is the unit vector in the x - z plane. The angle between the z axis and \mathbf{n} is θ .

$$\mathbf{n} = \left(\frac{2\Gamma_0}{\sqrt{4\Gamma_0^2 + \Delta^2}}, 0, \frac{\Delta}{\sqrt{4\Gamma_0^2 + \Delta^2}} \right) = (\sin \theta, 0, \cos \theta).$$

The ket vector $|\pm \mathbf{n}\rangle$ is the eigenvector of $(\hat{\sigma} \cdot \mathbf{n})$ with the eigenvalues ± 1 ,

$$(\hat{\sigma} \cdot \mathbf{n})|\pm \mathbf{n}\rangle = \pm |\pm \mathbf{n}\rangle.$$

Thus $|\pm \mathbf{n}\rangle$ is the eigenket of \hat{H}' with the eigenvalues $\pm E$ with

$$\hat{H}'|\pm \mathbf{n}\rangle = \frac{\hbar}{2}\sqrt{4\Gamma_0^2 + \Delta^2}(\hat{\sigma} \cdot \mathbf{n})|\pm \mathbf{n}\rangle = \pm \frac{\hbar}{2}\sqrt{4\Gamma_0^2 + \Delta^2}|\pm \mathbf{n}\rangle = \pm E|\pm \mathbf{n}\rangle,$$

where

$$E = \frac{\hbar\Omega}{2},$$

with

$$\Omega = \sqrt{4\Gamma_0^2 + \Delta^2}.$$

We note that

$$|+\mathbf{n}\rangle = \cos \frac{\theta}{2}|\phi_s\rangle + \sin \frac{\theta}{2}|\phi_a\rangle = \cos \frac{\theta}{2}|+z\rangle + \sin \frac{\theta}{2}|-z\rangle,$$

and

$$|-\mathbf{n}\rangle = -\sin \frac{\theta}{2}|\phi_s\rangle + \cos \frac{\theta}{2}|\phi_a\rangle = -\sin \frac{\theta}{2}|+z\rangle + \cos \frac{\theta}{2}|-z\rangle,$$

where

$$\cos \theta = \frac{\Delta}{\sqrt{4\Gamma_0^2 + \Delta^2}}, \quad \sin \theta = \frac{2\Gamma_0}{\sqrt{4\Gamma_0^2 + \Delta^2}}.$$

The new state vector $|\psi'(t)\rangle$ is related to the original state vector $|\psi(t)\rangle$ by

$$|\psi(t)\rangle = e^{i\Delta t/2} e^{-(E_0 - A)t/\hbar} \chi_s(t)|+z\rangle + e^{-i\Delta t/2} e^{-(E_0 + A)t/\hbar} \chi_a(t)|-z\rangle,$$

$$|\psi'(t)\rangle = \chi_s(t)|+z\rangle + \chi_a(t)|-z\rangle.$$

8-2. Rabi's formula

We assume that the initial condition is given by

$$|\psi'(t=0)\rangle = |\psi(t=0)\rangle = |\phi_a\rangle = |-z\rangle, \quad (\text{initial condition})$$

which means that

$$\chi_a(t=0) = 1, \quad \chi_s(t=0) = 1.$$

Then we get

$$\begin{aligned} \exp\left(-\frac{i\hat{H}'t}{\hbar}\right) &= \exp\left(-\frac{i\hat{H}'t}{\hbar}\right)(|+\mathbf{n}\rangle\langle+\mathbf{n}| + |-\mathbf{n}\rangle\langle-\mathbf{n}|) \\ &= \exp\left(-\frac{i\hat{H}'t}{\hbar}\right)|+\mathbf{n}\rangle\langle+\mathbf{n}| + \exp\left(-\frac{i\hat{H}'t}{\hbar}\right)|-\mathbf{n}\rangle\langle-\mathbf{n}| \\ &= \exp\left(-\frac{i\Omega t}{2}\right)|+\mathbf{n}\rangle\langle+\mathbf{n}| + \exp\left(\frac{i\Omega t}{2}\right)|-\mathbf{n}\rangle\langle-\mathbf{n}| \end{aligned}$$

$$\begin{aligned} |\psi'(t)\rangle &= \exp\left(-\frac{i\hat{H}'t}{\hbar}\right)|-z\rangle \\ &= \exp\left(-\frac{i\Omega t}{2}\right)|+\mathbf{n}\rangle\langle+\mathbf{n}|-z\rangle + \exp\left(\frac{i\Omega t}{2}\right)|-\mathbf{n}\rangle\langle-\mathbf{n}|-z\rangle \\ &= \exp\left(-i\frac{\Omega}{2}t\right)\sin\left(\frac{\theta}{2}\right)|+\mathbf{n}\rangle + \cos\left(\frac{\theta}{2}\right)\exp\left(i\frac{\Omega}{2}t\right)|-\mathbf{n}\rangle \end{aligned}$$

Using these relations, we have the matrix elements as

$$\begin{aligned} \langle -z|\psi'(t)\rangle &= \exp\left(-i\frac{\Omega}{2}t\right)\sin\left(\frac{\theta}{2}\right)\langle -z|+\mathbf{n}\rangle + \cos\left(\frac{\theta}{2}\right)\exp\left(i\frac{\Omega}{2}t\right)\langle -z|-\mathbf{n}\rangle \\ &= \sin^2\left(\frac{\theta}{2}\right)\exp\left(-i\frac{\Omega}{2}t\right) + \cos^2\left(\frac{\theta}{2}\right)\exp\left(i\frac{\Omega}{2}t\right) \\ &= \frac{1}{2}(1 - \cos\theta)\exp\left(-i\frac{\Omega}{2}t\right) + \frac{1}{2}(1 + \cos\theta)\exp\left(i\frac{\Omega}{2}t\right) \\ &= \cos\left(\frac{\Omega}{2}t\right) + i\cos\theta\sin\left(\frac{\Omega}{2}t\right) \end{aligned}$$

$$\begin{aligned}
\langle +z | \psi'(t) \rangle &= \exp(-i\frac{\Omega}{2}t) \sin(\frac{\theta}{2}) \langle +z | +\mathbf{n} \rangle + \cos(\frac{\theta}{2}) \exp(i\frac{\Omega}{2}t) \langle +z | -\mathbf{n} \rangle \\
&= \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \exp(-i\frac{\Omega}{2}t) - \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \exp(i\frac{\Omega}{2}t) \\
&= -\sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) [\exp(i\frac{\Omega}{2}t) - \exp(-i\frac{\Omega}{2}t)] \\
&= -2i \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \sin(\frac{\Omega}{2}t) \\
&= -i \sin(\theta) \sin(\frac{\Omega}{2}t)
\end{aligned}$$

$P_a(t)$ is the probability for finding the spin in the state $|-z\rangle = |\phi_a\rangle$,

$$\begin{aligned}
P_a(t) &= |\langle -z | \psi(t) \rangle|^2 \\
&= |\langle -z | \psi'(t) \rangle|^2 \\
&= \cos^2(\frac{\Omega}{2}t) + \cos^2 \theta \sin^2(\frac{\Omega}{2}t) \\
&= 1 - \sin^2 \theta \sin^2(\frac{t}{2} \sqrt{(\Delta\omega)^2 + \omega_1^2}) \\
&= 1 - \frac{4\Gamma_0^2}{\Delta^2 + 4\Gamma_0^2} \sin^2(\frac{t}{2} \sqrt{4\Gamma_0^2 + \Delta^2})
\end{aligned}$$

Note that $P_a(t=0) = 1$. $P_s(t)$ is the probability for finding the spin in the state $|+z\rangle = |\phi_s\rangle$.

$$\begin{aligned}
P_s(t) &= |\langle +z | \psi(t) \rangle|^2 \\
&= |\langle +z | \psi'(t) \rangle|^2 \\
&= \sin^2 \theta \sin^2(\frac{\Omega t}{2}) \\
&= \sin^2 \theta \sin^2(\frac{t}{2} \sqrt{(\Delta\omega)^2 + \omega_1^2}) \\
&= \frac{4\Gamma_0^2}{\Delta^2 + 4\Gamma_0^2} \sin^2(\frac{t}{2} \sqrt{4\Gamma_0^2 + \Delta^2})
\end{aligned}$$

((Alternative method))

We use the formula for the spin 1/2.

$$\exp[-\frac{i}{2} \alpha (\hat{\sigma} \cdot \mathbf{n})] = \hat{1} \cos(\frac{\alpha}{2}) - i \sin(\frac{\alpha}{2}) (\hat{\sigma} \cdot \mathbf{n}).$$

The Hamiltonian is given by

$$\hat{H}' = \frac{\hbar}{2} \sqrt{4\Gamma_0^2 + \Delta^2} (\hat{\sigma} \cdot \mathbf{n}) = \frac{\hbar\Omega}{2} (\hat{\sigma} \cdot \mathbf{n}).$$

Then the time evolution is obtained as

$$\exp\left(-\frac{i\hat{H}'t}{\hbar}\right) = \exp\left[-i\frac{\Omega t}{2} (\hat{\sigma} \cdot \mathbf{n})\right] = \hat{1} \cos\left(\frac{\Omega t}{2}\right) - i \sin\left(\frac{\Omega t}{2}\right) (\hat{\sigma} \cdot \mathbf{n}),$$

using the above formula. Then we get

$$\begin{aligned} |\psi'(t)\rangle &= \exp\left(-\frac{i\hat{H}'t}{\hbar}\right) |-z\rangle \\ &= \left[\hat{1} \cos\left(\frac{\Omega t}{2}\right) - i \sin\left(\frac{\Omega t}{2}\right) (\hat{\sigma} \cdot \mathbf{n}) \right] |-z\rangle \\ &= \left[\hat{1} \cos\left(\frac{\Omega t}{2}\right) - i \sin\left(\frac{\Omega t}{2}\right) (\sigma_z \cos\theta + \sigma_x \sin\theta) \right] |-z\rangle \\ &= \begin{pmatrix} \cos\left(\frac{\Omega t}{2}\right) - i \sin\left(\frac{\Omega t}{2}\right) \cos\theta & -i \sin\left(\frac{\Omega t}{2}\right) \sin\theta \\ -i \sin\left(\frac{\Omega t}{2}\right) \sin\theta & \cos\left(\frac{\Omega t}{2}\right) + i \sin\left(\frac{\Omega t}{2}\right) \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -i \sin\left(\frac{\Omega t}{2}\right) \sin\theta \\ \cos\left(\frac{\Omega t}{2}\right) + i \sin\left(\frac{\Omega t}{2}\right) \cos\theta \end{pmatrix} \end{aligned}$$

$$\begin{aligned} P_a(t) &= \left| \langle -z | \psi'(t) \rangle \right|^2 \\ &= \left| \cos\left(\frac{\Omega t}{2}\right) + i \sin\left(\frac{\Omega t}{2}\right) \cos\theta \right|^2 \\ &= \cos^2\left(\frac{\Omega t}{2}\right) + \cos^2\theta \sin^2\left(\frac{\Omega t}{2}\right) \\ &= 1 - \sin^2\theta \sin^2\left(\frac{t}{2} \sqrt{(\Delta\omega)^2 + \omega_1^2}\right) \\ &= 1 - \frac{4\Gamma_0^2}{\Delta^2 + 4\Gamma_0^2} \sin^2\left(\frac{t}{2} \sqrt{4\Gamma_0^2 + \Delta^2}\right) \end{aligned}$$

8-3 The use of Mathematica

We obtain the time dependence of the Schrödinger equation with the time-independent Hamiltonian;

$$\hat{H}' = \hbar \begin{pmatrix} \frac{1}{2}\Delta & \Gamma_0 \\ \Gamma_0 & -\frac{1}{2}\Delta \end{pmatrix}.$$

We calculate the time dependence of the wave function given by

$$\exp\left(-\frac{i}{\hbar}\hat{H}'t\right)|\psi(0)\rangle,$$

using the Mathematica.

((Mathematica))

```
Clear["Global`*"];
exp_ * := exp /. {Complex[re_, im_] := Complex[re, -im]};
H1 =  $\frac{\hbar}{2} \begin{pmatrix} \Delta & 2\Gamma_0 \\ 2\Gamma_0 & -\Delta \end{pmatrix}$ ; M1 = MatrixExp[ $\frac{-i}{\hbar}$  H1 t] // FullSimplify;
rule1 = { $\sqrt{-\Delta^2 - 4\Gamma_0^2} \rightarrow i\Omega$ ,  $\frac{1}{\sqrt{-\Delta^2 - 4\Gamma_0^2}} \rightarrow \frac{-i}{\Omega}$ };
M2 = M1 /. rule1 // Simplify;
```

P11: Probability of finding in the excited state $|\phi_a\rangle$

P21: Probability of finding in the ground state $|\phi_s\rangle$

with

$$\sqrt{\Delta^2 + 4\Gamma_0^2} = \Omega$$

```
P1 = {0, 1}.M2.{0, 1} // Simplify;
```

```
P11 = P1* P1 // Simplify
```

$$\cos\left[\frac{t\Omega}{2}\right]^2 + \frac{\Delta^2 \sin\left[\frac{t\Omega}{2}\right]^2}{\Omega^2}$$

```
P2 = {1, 0}.M2.{0, 1} // Simplify;
```

```
P21 = P2* P2 // Simplify
```

$$\frac{4\Gamma_0^2 \sin\left[\frac{t\Omega}{2}\right]^2}{\Omega^2}$$

```
Clear["Global`*"]; SuperStar;
expr_* := expr /. Complex[a_, b_] :=> Complex[a, -b];
```

$$\Delta = \omega - \omega_0, \quad \sqrt{\frac{4 \epsilon_0^2 \mu^2}{\hbar^2} + \Delta^2} = \Omega, \quad \Gamma_0 = \frac{\mu \epsilon_0}{\hbar}$$

```
eq1 = i D[\gammaa[t], t] == \Gamma_0 Exp[-i \Delta t] \gamma_s[t];
```

```
eq2 = i D[\gamma_s[t], t] == \Gamma_0 Exp[i \Delta t] \gammaa[t];
```

```
s11 = DSolve[{eq1, eq2, \gammaa[0] == 1, \gamma_s[0] == 0}, {\gammaa[t], \gamma_s[t]}, t] //
Simplify[#, {\Delta > 0, \Gamma_0 > 0}] &;
```

```
s12 = s11 /. {Sqrt[4 \Gamma_0^2 + \Delta^2] -> \Omega, 1/Sqrt[4 \Gamma_0^2 + \Delta^2] -> 1/\Omega} // Simplify;
```

```
\gammaa[t_] = \gammaa[t] /. s12[[1]] // ComplexExpand // FullSimplify
```

$$\frac{e^{-\frac{1}{2} i t (\Delta + \Omega)} (-\Delta + \Omega + e^{i t \Omega} (\Delta + \Omega))}{2 \Omega}$$

```
\gamma_s[t_] = \gamma_s[t] /. s12[[1]] // ComplexExpand // FullSimplify
```

$$-\frac{2 i e^{\frac{i t \Delta}{2}} \Gamma_0 \sin\left[\frac{t \Omega}{2}\right]}{\Omega}$$

10. Transition of resonance ((Feynman))

We consider the case that the electric field is small, and, also the period of time T is small. So that

$$\Gamma_0 T = \frac{\mu \epsilon_0}{\hbar} T,$$

is much less than 1. The probability $P_s(t)$ is still small at $t = T$. Note that $P_s(T)$ is the probability which is related to the absorption of energy from the high energy state of NH_3 molecules into the system (cavity) at time $t = T$.

$$P_s(T) = \frac{4 \Gamma_0^2 \sin^2\left(\frac{T}{2} \sqrt{4 \Gamma_0^2 + \Delta^2}\right)}{4 \Gamma_0^2 + \Delta^2} \approx \Gamma_0^2 T^2 \frac{\sin^2\left(\frac{T \Delta}{2}\right)}{\left(\frac{T \Delta}{2}\right)^2},$$

or

$$y = \frac{P_s(T)}{\Gamma_0^2 T^2} = \frac{\sin^2\left(\frac{T\Delta}{2}\right)}{\left(\frac{T\Delta}{2}\right)^2} = \frac{\sin^2(x)}{x^2},$$

We make a plot of y as a function of $x = \frac{T\Delta}{2} = \frac{T}{2}(\omega - \omega_0)$.

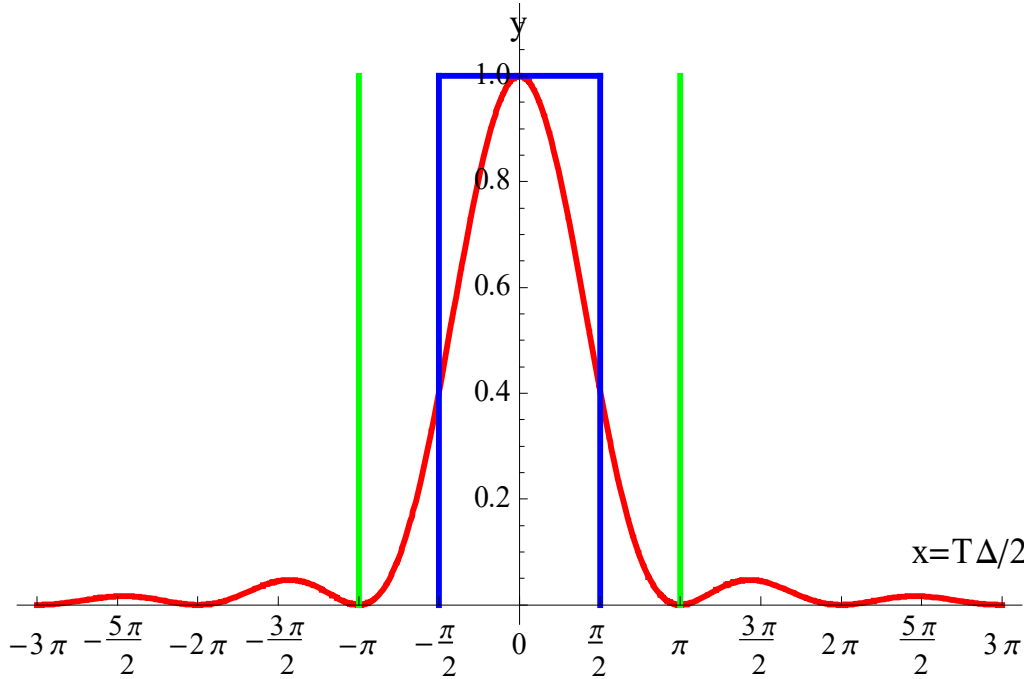


Fig. Plot of $y = \frac{P_s(T)}{\Gamma_0^2 T^2}$ as a function of $x = \frac{T\Delta}{2}$. Note that T is a fixed time, not a temperature.

The curve falls rather abruptly to zero for $x = (T\Delta)/2 = \pm\pi$ and never regains significant size for larger frequency derivations. In fact, by far the greatest part of the area under the curve lies within the range $\pm\pi/2$. We note that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

Thus the area under the curve is π and is equal to the area of the rectangle enclosed by the blue lines ($|x| < \frac{\pi}{2}$ and $0 < y < 1$). In other words, y is described by a Dirac delta function which has a peak at $\omega = \omega_0$. The value of y is equal to 1 for

$$|\omega - \omega_0| < \frac{\pi}{T},$$

and zero for

$$|\omega - \omega_0| > \frac{\pi}{T}.$$

((Example))

In the NH₃ maser,

$$\left| \frac{\omega - \omega_0}{\omega_0} \right| = \left| \frac{f - f_0}{f_0} \right| < \frac{\pi}{T\omega_0} = \frac{1}{2f_0T}.$$

Suppose that we assume $T = 1$ ms. Since $f_0 = 23.786$ GHz, we have

$$\left| \frac{f - f_0}{f_0} \right| < \frac{1}{2f_0T} = 2.1 \times 10^{-8}.$$

leading to the quality factor $Q = \frac{f_0}{\Delta f} < 4.8 \times 10^7$ (we will discuss later). The frequency must be very close to f_0 , in order to get a significant transition probability.

11. Stimulated emission of light

We have discussed the behavior of a NH₃ molecule under the influence of an electric field, whether the electric field is confined in a cavity or not. So, we could be simply shining a beam of light at the microwave frequencies at the molecule. Here we calculate the transition probability of the stimulate emission, where the light shining on such a system is not exactly monochromatic. The light has the intensity $I(\omega)d\omega$ between ω and $\omega + d\omega$, covering a broad range including $\omega = \omega_0$. To this end, we start with the probability given by

$$P_s(T) = \left(\frac{\mu\epsilon_0}{\hbar} \right)^2 T^2 \frac{\sin^2\left(\frac{T\Delta}{2}\right)}{\left(\frac{T\Delta}{2}\right)^2},$$

where

$$\Gamma_0 T = \frac{\mu\epsilon_0}{\hbar} T, \quad \Delta = \omega - \omega_0.$$

From the APPENDIX described below, we have the expression for the time-average of the energy density $\langle u \rangle$ and the magnitude of the Poynting vector $\langle S \rangle$,

$$\begin{cases} \langle u \rangle = \frac{1}{8\pi} \varepsilon_0^2 \text{ (erg/cm}^3\text{)} \\ \langle S \rangle = c \langle u \rangle \left(\text{erg} \frac{\text{s}}{\text{cm}^3} \right) \rightarrow I(\omega) d\omega \end{cases}$$

where

$$\varepsilon_0 = |\tilde{\mathbf{E}}_0| \quad (\text{the maximum of the electric field}),$$

and

$$I(\omega) = \left[\frac{\text{erg} \cdot \text{s}}{\text{cm}^3} \cdot \frac{\text{cm}}{\text{s}} \right] = \left[\text{erg} \frac{1}{\text{cm}^2} \right].$$

Then we have

$$\frac{1}{8\pi} \varepsilon_0^2 \rightarrow \frac{1}{c} I(\omega) d\omega.$$

The probability transition now becomes

$$P_s(T) = 8\pi \frac{\mu^2 T^2}{\hbar^2 c} \int_0^\infty I(\omega) d\omega \frac{\sin^2 \left[\frac{T(\omega - \omega_0)}{2} \right]}{\left[\frac{T(\omega - \omega_0)}{2} \right]^2}.$$

Here we note that $I(\omega)$ will vary much more slowly with ω than the sharp resonance term (which forms such a Dirac delta function). In such a case, we can replace $I(\omega)$ by its value $I(\omega_0)$ at the center of the sharp resonance curve and take it outside of the integral,

$$P_s(T) = 8\pi \frac{\mu^2 T^2}{\hbar^2 c} I(\omega_0) \int_0^\infty d\omega \frac{\sin^2 \left[\frac{T(\omega - \omega_0)}{2} \right]}{\left[\frac{T(\omega - \omega_0)}{2} \right]^2}.$$

Noting that

$$\int_0^{\infty} d\omega \frac{\sin^2 \left[\frac{T(\omega - \omega_0)}{2} \right]}{\left[\frac{T(\omega - \omega_0)}{2} \right]^2} = \frac{2}{T} \int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \frac{2\pi}{T},$$

we get the result that

$$P_s(T) = 16\pi^2 \frac{\mu^2}{\hbar^2 c} I(\omega_0) T.$$

This is the expression of the **stimulated emission (or absorption) of light** by molecular or atomic system. First, this probability is proportional to T . In other words, there is a constant probability per unit time that transitions will occur. Second, it is proportional the intensity of light on the system. Third, it is proportional to μ^2 .

((Note)) The spontaneous emission and the stimulated emission in maser

In maser, the rate of the spontaneous emission A_{21} is much smaller than the rate of the stimulated emission B_{21} ;

$$\frac{A_{21}}{B_{21}} = \frac{8\pi\hbar f_0^3}{c^3} \propto \omega_0^3 \ll 1$$

since $f_0 = 23.786$ GHz is much lower than frequencies of visible light (600 THz for $\lambda = 500$ nm). So that, the effect of the spontaneous emission can be neglected.

12. Polarization vector

Here we note the property of electric field in the cavity. There is a photon inside the cavity. The electric field (the polarization vector) is expressed by the plane wave with the angular frequency $\omega = \frac{2A}{\hbar}$. The electric dipole moment has a perturbation energy which is given by

$$\hat{H}' = -\boldsymbol{\mu} \cdot \boldsymbol{\varepsilon}$$

where

$$\boldsymbol{\varepsilon} = 2\varepsilon_0 \cos \omega t = \varepsilon_0 (e^{i\omega t} + e^{-i\omega t})$$

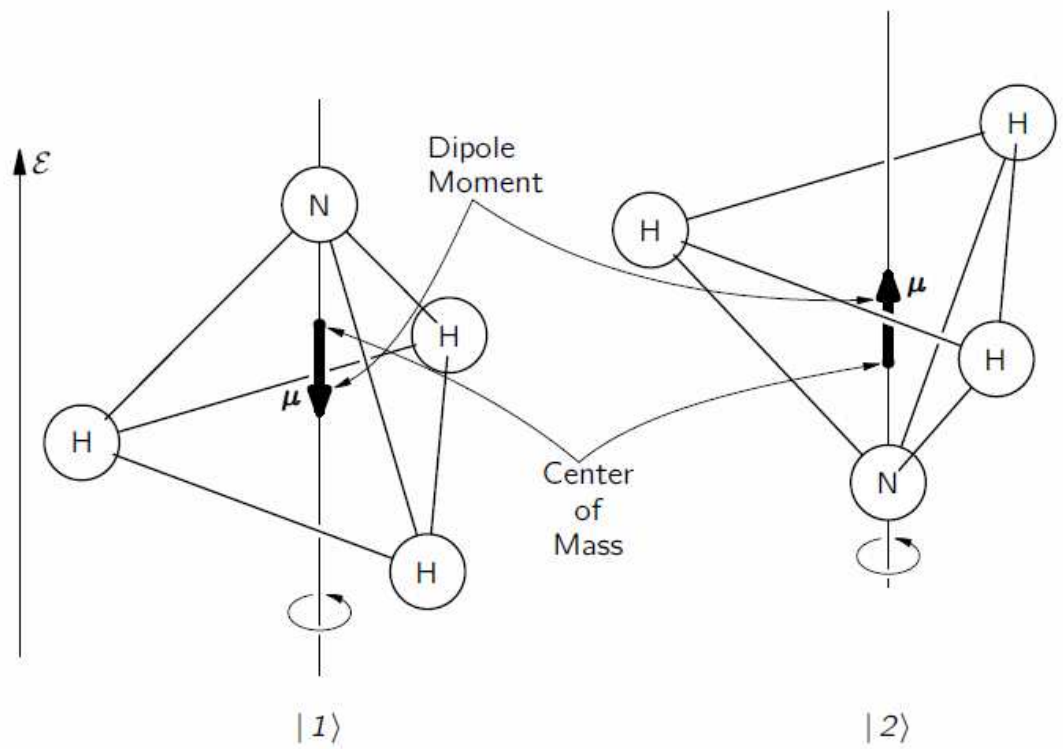


Fig. A physical model of two base states for the ammonia molecule. These states have the electric dipole moments μ . (R.P. Feynman, Lectures on Physics III).

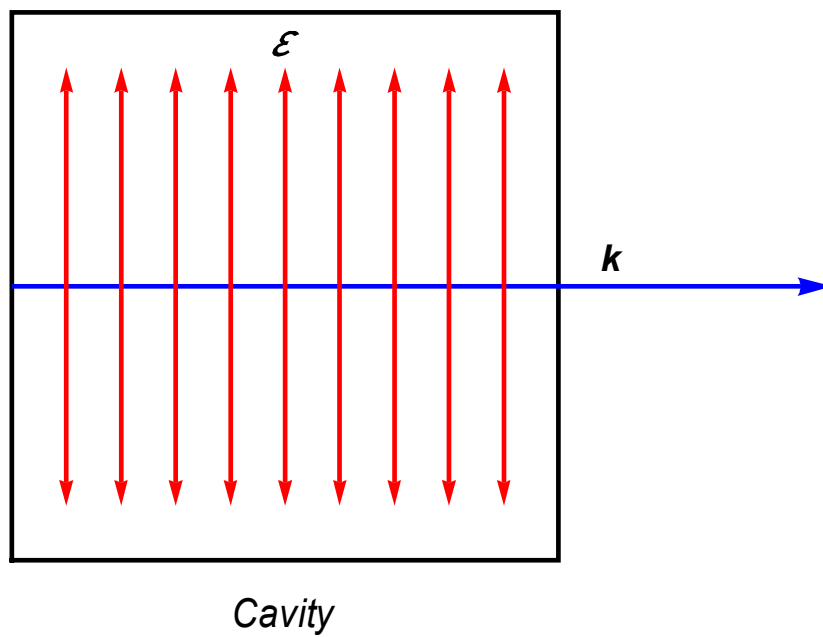
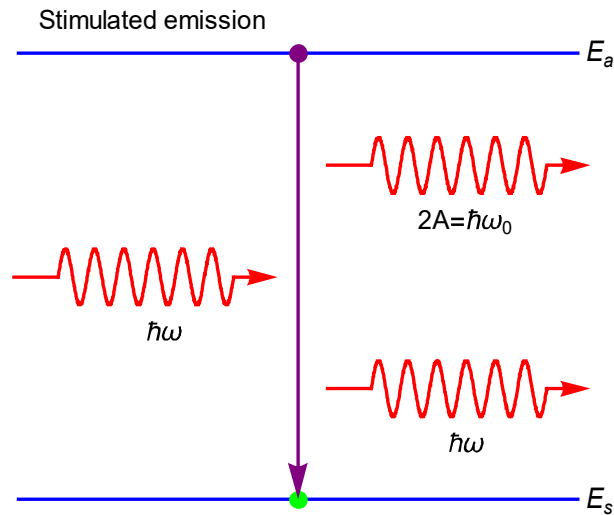


Fig. The cavity. The polarization vector (the direction of the electric field) is along the z axis. The direction of the wave vector for the plane wave is perpendicular to the polarization vector. A possible electric field (polarization vector) of one photon due to the spontaneous emission may give rise to a subsequent chain reactions of stimulated emission of many photons in the cavity.

Using the above figure, we have

$$\hat{H}'|1\rangle = \mu\varepsilon|1\rangle, \quad \hat{H}'|2\rangle = -\mu\varepsilon|2\rangle$$

Inside the cavity consisting of two parallel mirrors, when photons with $\omega = \frac{2A}{\hbar}$ interacts with the ammonia atoms with excited state (inversion population), leading to the stimulated emission. One at a time there are two photons, one is the incident photon and the other is from the stimulated emission. When these events are repeated many times, the photon becomes perfectly coherent wave with the angular frequency with $\omega = \frac{2A}{\hbar}$.



13. Feynman's discussion



Excellent discussion on the ammonia maser is obtained by Feynman (The Feynman Lectures on Physics, Feynman, Leighton, Sands, volume III 9-1).

Summary A two state system with constant H has two energy levels given by solving

$$\text{Det} \begin{pmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} = 0 = (H_{11} - E)(H_{22} - E) - H_{12}H_{21}$$

for symmetrical system, $H_{11} = H_{22} = E_0$ say; $|H_{12}|^2 = A^2$ say. $E_{I,II} = E_0 \pm A$.

$^{14}\text{NH}_3$ $|1\rangle$ $|2\rangle$; Lowest state $E_{II} = E_0 - A$; Highest $E_I = E_0 + A$. $|II\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$; $|I\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

MASER In electric field \mathcal{E} :  can be separated by inhomogeneous electric field \mathcal{E} .  **MASER CAVITY**

If all in upper state I go into cavity (of resonant frequency $2A/\hbar$) where electric field is oscillating at frequency $\frac{E_I - E_{II}}{\hbar}$ transitions $I \rightarrow II$ are induced; the energy goes into maintaining the osc. in the cavity.

Details can be got by solving eqn. $i\hbar \dot{C}_1 = H_{11}C_1 + H_{12}C_2$ with H varying with time (due to electric field)
 $i\hbar \dot{C}_2 = H_{21}C_1 + H_{22}C_2$

The molecule enters the cavity, the cavity field – oscillating at exactly the right frequency – induces transitions from the upper to the lower state, and that the energy released is fed into the oscillating field. In an operating maser the molecules deliver enough energy to maintain the cavity oscillations – not only providing enough power to make up for the cavity losses but even providing small amounts of excess power that can be drawn from the cavity. Thus, the molecular energy is converted into the energy of an external electromagnetic field.

14. Evaluation for the Q factor and number density n

(a) Velocity of NH_3 molecules inside the cavity

Here we discuss the velocity of NH_3 running inside the cavity. Suppose that the gas of NH_3 molecules in the oven obey the Maxwell-Boltzmann distribution in the velocity. The temperature oven is T_s (K). The most probable velocity of NH_3 molecules (leading to the peak value of the probability), is

$$v_{mp} = \sqrt{\frac{2k_B T_s}{m_{\text{NH}_3}}}$$

Where k_B is the Boltzmann constant. Using the mass of NH_3 as 17.031 g/mol, the most probable velocity is

$$v_{mp} = 31.3473 \sqrt{T_s} \quad (\text{m/s}).$$

When $T_s = 300$ K, $v_{mp} = 0.5421$ km/s, and for $T_s = 500$ K, $v_{mp} = 0.69871$ km/s. When the length of the cavity is 1m, the time t_0 for the NH_3 molecules to stay in the cavity is

$$t_0 = \frac{L}{v_{mp}} = 1.42 \text{ ms},$$

at $T_s = 500 \text{ K}$.

(b) The electric dipole moment of NH₃ molecule

The electric dipole moment of NH₃ molecule is 1.47 D, where D is the Debye unit and is

$$1\text{D} = 1 \times 10^{-18} \text{ esu-cm}, \quad 1\text{D} = 3.34 \times 10^{-30} \text{ Cm}$$

In cgs units, $\mu = 1.47 \times 10^{-18} \text{ esu-cm}$. In S.I. units, $\mu = 4.9098 \times 10^{-30} \text{ Cm}$.

(c) Evaluation of electric field

We use a rough estimation for the maximum electric field as follows.

$$\Gamma t_0 = \frac{\mu \epsilon_0 t_0}{\hbar} = \frac{\pi}{2}$$

or

$$t_0 = \frac{\pi \hbar}{2 \mu \epsilon_0} = \frac{L}{v}$$

Using $v = v_{mp}$, we have

$$\epsilon_0 = \frac{\pi \hbar v_{mp}}{2 \mu L} = 7.87 \times 10^{-7} \text{ statvolt/cm (in c.g.s. units).}$$

or

$$\epsilon_0 = \frac{\pi \hbar v_{mp}}{2 \mu L} = 2.36 \times 10^{-2} \text{ V/m (in SI units).}$$

(d) Quality factor ((Sargent, Scully, and Lamb))

Reference: M. Sargent III, M.O. Scully, and W.E. Lamb, Jr. Laser Physics (Addison-Wesley, 1974).

Here we discuss the quality factor Q (c.g.s. units). Note that the related discussion based on the SI units is made in the above Reference. Only difference is the use of the averaged electromagnetic energy.

The condition for oscillation is that the energy lost in the cavity per time (loss) is equal to the energy gained per unit time (saturated gain) from the NH_3 molecules, that is,

$$\text{Loss} = \text{saturated gain.}$$

We write the loss in terms of an important ratio Q called the cavity factor and defined by

$$Q = 2\pi \frac{\text{energy stored in field}}{\text{energy dissipated per cycle}} = \omega_0 \frac{\text{energy stored in field}}{\text{energy lost/second}}.$$

The energy stored in the field is given by

$$\frac{V}{8\pi} \langle E^2 + B^2 \rangle_{time} = \frac{V}{4\pi} \langle E^2 \rangle_{time} = \frac{V}{8\pi} \varepsilon_0^2 \quad (\text{in c.g.s. units})$$

where V is the volume of the maser cavity and ε_0 is the maximum of the electric field. Thus, we have the energy lost per second:

$$\text{Loss} = \frac{1}{8\pi} \frac{\omega_0}{Q} \varepsilon_0^2 V.$$

The energy gained per second is

$$\text{Saturated gain} = N \frac{\nu}{L} P_s(t) \hbar \omega_0.$$

Here N is the number of molecules in the cavity. $N \frac{\nu}{L}$ is the number of molecules leaving the cavity per sec. The frequency ν of the electric field is close to the cavity frequency (for central tuning), although a quantitative relationship cannot be determined from our present, simple analysis. Since

$$\frac{1}{8\pi} \frac{\omega_0}{Q} \varepsilon_0^2 V = \frac{N\nu}{L} P_s(t) \hbar \omega_0$$

with

$$P_s(t) = \frac{4\Gamma_0^2}{\Delta^2 + 4\Gamma_0^2} \sin^2 \left(\frac{t_0}{2} \sqrt{\Delta^2 + 4\Gamma_0^2} \right) \quad \text{with } t_0 = \frac{L}{\nu},$$

where

$$\Delta = \omega - \omega_0, \quad \Gamma_0 = \frac{\mu \varepsilon_0}{\hbar}.$$

Then the quality factor Q can be obtained as

$$\frac{1}{8\pi} \frac{\omega_0}{Q} \varepsilon_0^2 V = \hbar \omega_0 \frac{Nv}{L} \frac{4\Gamma_0^2}{\Delta^2 + 4\Gamma_0^2} \sin^2\left(\frac{t_0}{2} \sqrt{\Delta^2 + 4\Gamma_0^2}\right),$$

or

$$\frac{1}{Q} = \frac{32\pi\mu^2}{\hbar} \frac{nv}{L} \frac{1}{\Delta^2 + 4\Gamma_0^2} \sin^2\left(\frac{L}{2v} \sqrt{\Delta^2 + 4\Gamma_0^2}\right),$$

where $n = \frac{N}{V}$ (density) and $\varepsilon_0 = \frac{\hbar\Gamma_0}{\mu}$. When $\Delta = 0$ (in resonance), we get

$$\frac{1}{Q} = \frac{8\pi\mu^2}{\hbar} \frac{nL}{v} \frac{\sin^2\left(\frac{\Gamma_0 L}{v}\right)}{\left(\frac{\Gamma_0 L}{v}\right)^2}.$$

In the limit of $\frac{\Gamma_0 L}{v} \rightarrow 0$,

$$\frac{1}{Q} = \frac{8\pi\mu^2}{\hbar} \frac{nL}{v}.$$

We use $\mu = 1.47 \times 10^{-18}$ esu-cm., $L = 100$ cm, $v = v_{mp} = 0.69871$ km/s. Thus, we get

$$Q = \frac{1.357 \times 10^{10}}{n} \quad \text{or} \quad n = \frac{1.357 \times 10^{10}}{Q}.$$

If Q is required to be 36000 (for example), the number density should be on the order of $n = 3.77 \times 10^5 \text{ cm}^{-3}$. From the expression for Q given by $Q = f_0 / \Delta f$, the uncertainty in frequency Δf is 660 kHz. The number density of photon is surprisingly small, possibly because of the use of gaseous state of NH_3 molecules.

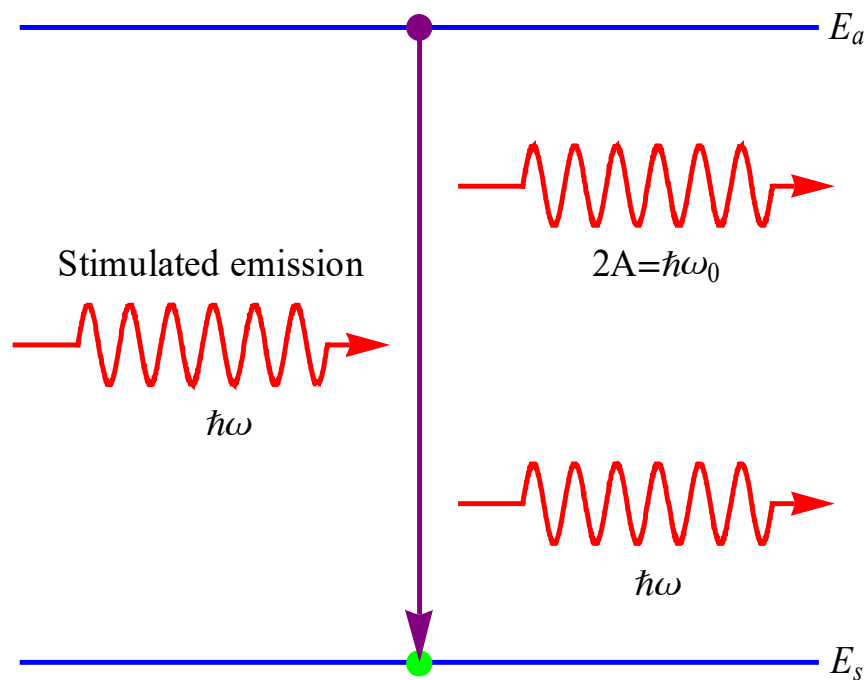
If we use $n = 1.5651 \times 10^6 \text{ cm}^{-3}$, which is evaluated in the APPENDIX C, the quality factor Q can be evaluated as

$$Q = 8.67 \times 10^3 .$$

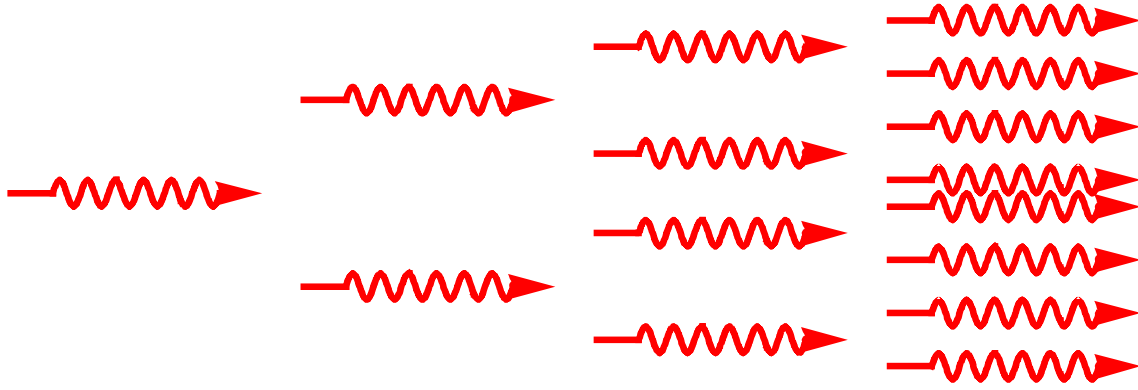
15. Summary

In the ammonium (NH_3) masers, the transition on which laser action occurs is related to a two-level system. In these devices a molecular beam is prepared in the excited state by a Stern–Gerlach type method, allowing selection of the excited molecules only. This beam of totally inverted molecules then traverses a resonant cavity which performs a role analogous to that of the optical cavity of a laser. A significant fraction of the molecules is then transferred to the ground state by stimulated emission. The molecules then leave the cavity. The ground state molecules are in this way physically removed from the cavity, thus ensuring that the population inversion is maintained.

((Stimulated emission))



((Light amplification))



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APPENDIX-A

A.1 Plane wave representation of the energy density and the Poynting vector

We use the Maxwell's equation in the units of c.g.s. which is described by the plane wave representation. Here we suppose that

$$\mathbf{E} = \text{Re}[\tilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] = \text{Re}[\tilde{\mathbf{E}}_0 e^{i\theta}] = \frac{\tilde{\mathbf{E}}_0 e^{i\theta} + \tilde{\mathbf{E}}_0^* e^{-i\theta}}{2},$$

$$\mathbf{B} = \text{Re}[\tilde{\mathbf{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] = \text{Re}[\tilde{\mathbf{B}}_0 e^{i\theta}] = \frac{\tilde{\mathbf{B}}_0 e^{i\theta} + \tilde{\mathbf{B}}_0^* e^{-i\theta}}{2},$$

where

$$\theta = \mathbf{k} \cdot \mathbf{r} - \omega t.$$

From

$$\nabla \cdot \mathbf{E} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0.$$

we have

$$\mathbf{k} \cdot \tilde{\mathbf{E}}_0 = 0, \quad \text{and} \quad \mathbf{k} \cdot \tilde{\mathbf{B}}_0 = 0.$$

From

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}.$$

we have

$$i(\mathbf{k} \times \tilde{\mathbf{E}}_0) = \frac{i\omega}{c} \tilde{\mathbf{B}}_0,$$

or

$$(\mathbf{k} \times \tilde{\mathbf{E}}_0) = \tilde{\mathbf{B}}_0,$$

or

$$\tilde{\mathbf{B}}_0 = \hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0.$$

where $\hat{\mathbf{k}}$ is the unit vector along the wave vector \mathbf{k} .

((Note)) Dispersion relation

From

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E},$$

we have a dispersion relation

$$\omega = ck = c|\mathbf{k}|.$$

The energy density is defined by

$$u = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2).$$

The Poynting vector is defined by

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}).$$

Calculation of \mathbf{E}^2

$$\mathbf{E}^2 = \left(\frac{\tilde{\mathbf{E}}_0 e^{i\theta} + \tilde{\mathbf{E}}_0^* e^{-i\theta}}{2} \right)^2 = \frac{1}{4} (\tilde{\mathbf{E}}_0^2 e^{2i\theta} + \tilde{\mathbf{E}}_0^{*2} e^{-2i\theta} + 2\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^*),$$

$$\langle \mathbf{E}^2 \rangle = \frac{1}{T} \int_0^T \mathbf{E}^2 dt = \frac{1}{2} \tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^* = \frac{1}{2} |\tilde{\mathbf{E}}_0|^2.$$

where $T = \frac{2\pi}{\omega}$ and $\frac{1}{T} \int_0^T e^{\pm 2i\theta} dt = 0$. Similarly, we have

$$\langle \mathbf{B}^2 \rangle = \frac{1}{T} \int_0^T \mathbf{B}^2 dt = \frac{1}{2} \tilde{\mathbf{B}}_0 \cdot \tilde{\mathbf{B}}_0^* = \frac{1}{2} |\tilde{\mathbf{B}}_0|^2.$$

The time-average of the energy density is given by

$$\langle u \rangle = \frac{1}{8\pi} (\langle \mathbf{E}^2 \rangle + \langle \mathbf{B}^2 \rangle) = \frac{1}{16\pi} (|\tilde{\mathbf{E}}_0|^2 + |\tilde{\mathbf{B}}_0|^2).$$

Here we note

$$|\tilde{\mathbf{B}}_0|^2 = (\hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0) \cdot (\hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0^*) = [\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^* - (\hat{\mathbf{k}} \cdot \tilde{\mathbf{E}}_0)(\hat{\mathbf{k}} \cdot \tilde{\mathbf{E}}_0^*)] = |\tilde{\mathbf{E}}_0|^2.$$

Then we have

$$\langle u \rangle = \frac{1}{8\pi} |\tilde{\mathbf{E}}_0|^2 = \frac{1}{8\pi} |\tilde{\mathbf{B}}_0|^2.$$

Calculation of Poynting vector \mathbf{S}

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) = \frac{c}{4\pi} \left(\frac{\tilde{\mathbf{E}}_0 e^{i\theta} + \tilde{\mathbf{E}}_0^* e^{-i\theta}}{2} \right) \times \left(\frac{\tilde{\mathbf{B}}_0 e^{i\theta} + \tilde{\mathbf{B}}_0^* e^{-i\theta}}{2} \right),$$

$$\langle \mathbf{S} \rangle = \frac{c}{16\pi} (\tilde{\mathbf{E}}_0 \times \tilde{\mathbf{B}}_0^* + \tilde{\mathbf{E}}_0^* \times \tilde{\mathbf{B}}_0) = \frac{c}{8\pi} \text{Re}[\tilde{\mathbf{E}}_0 \times \tilde{\mathbf{B}}_0^*].$$

Note that

$$\tilde{\mathbf{E}}_0 \times \tilde{\mathbf{B}}_0^* = \tilde{\mathbf{E}}_0 \times (\hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0^*) = \hat{\mathbf{k}} |\tilde{\mathbf{E}}_0|^2.$$

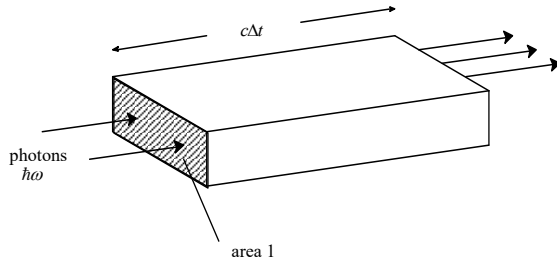
Then we have

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{c}{8\pi} \text{Re}[\tilde{\mathbf{E}}_0 \times \tilde{\mathbf{B}}_0^*] \\ &= \frac{c}{8\pi} \text{Re}[\hat{\mathbf{k}} |\tilde{\mathbf{E}}_0|^2] \\ &= \frac{c}{8\pi} \hat{\mathbf{k}} |\tilde{\mathbf{E}}_0|^2 \end{aligned}$$

or

$$\langle \mathbf{S} \rangle = c\hat{\mathbf{k}}\langle u \rangle.$$

A.2 Physical meaning of the Poynting vector



$\langle \mathbf{S} \rangle$ is the energy flux (energy per unit area per unit time). We define the intensity I given by

$$I = \langle S \rangle = c\langle u \rangle = \frac{c}{8\pi} |\tilde{\mathbf{E}}_0|^2.$$

We now consider the photon (the velocity is c) flows. During the time Δt , the total energy passing through the area A is

$$E = c\Delta t A \langle u \rangle = A\Delta t \langle S \rangle.$$

or

$$\frac{E}{A\Delta t} = \langle S \rangle.$$

where the volume is $c\Delta t A$ and the energy density is $\langle u \rangle$.

APPENDIX B.

Key words in maser physics

Maser (Microwave amplification by stimulated emission of radiation)

Stimulated emission

Negative temperature

Population inversion

Resonant cavity

Rabi frequency

Rabi formula

Rotating wave approximation

Coherent radiation

((Hydrogen maser))

A **hydrogen maser**, also known as hydrogen frequency standard, is a specific type of maser that uses the intrinsic properties of the hydrogen atom to serve as a precision frequency reference.

Both the proton and electron of a hydrogen atom have spins. The atom has a higher energy if both are spinning in the same direction, and a lower energy if they spin in opposite directions. The amount of energy needed to reverse the spin of the electron is equivalent to a photon at the frequency of 1.420,405,751.786 GHz, which corresponds to the 21 cm line in hydrogen spectrum. See LN hydrogen 21 cm line (hyperfine structure).

APPENDIX-C

Rough evaluation of number density of photon

We start with the expression

$$N\hbar\omega_0 = \frac{V}{8\pi} E_{\max}^2$$

where V is the volume of the system (cavity), E_{\max} is the maximum value of the electric field, and N is the number of photon with ω_0 . Thus, we have

$$E_{\max} = \sqrt{8\pi n\hbar\omega_0}$$

Where n is the number density of photons having the same ω_0 . Suppose that

$$t_0 = \frac{\pi}{2\Gamma_0} = \frac{\pi\hbar}{2\mu E_{\max}} \approx \frac{L}{v}$$

in order to avoid the absorption process. Using this condition, we can evaluate

$$n = \frac{\pi \hbar v^2}{32 \mu^2 \omega_0 L^2} = 1.565 \times 10^6 / \text{cm}^3.$$

where we use

$$\begin{aligned} v &= 0.69871 \text{ km/s}, \quad L = 1 \text{ m} = 100 \text{ cm}, \\ \mu &= 1.47D = 1.47 \times 10^{-18} \text{ statC.m}, \quad \omega_0 = 2\pi f_0, \text{ with } f_0 = 23.786 \text{ GHz} \end{aligned}$$