Eigenstates of three particles with *S* = 1/2 and *S* = 1

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We present a method to obtain the eigenstates of three particles having either S = 1/2 or L = 1 using the Mathematica program (Kronecker Product and Eigensystem). For simplicity, We use the unit of $\hbar$ = 1.

1. Eigenstates of three particles with spin 1/2

$$\begin{aligned} D_{1/2}\times D_{1/2}\times D_{1/2} &= (D_1 + D_0)\times D_{1/2} \\ &= D_{3/2} + 2D_{1/2} \end{aligned}$$

There are 8 states ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, ξ_7, ξ_8, defined by

$$\xi_1 = \alpha\otimes\alpha\otimes\alpha = \alpha\alpha\alpha = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \qquad \xi_2 = \alpha\alpha\beta = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix},$$

$$\xi_3 = \alpha\beta\alpha = \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \qquad \xi_4 = \alpha\beta\beta = \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix},$$

$$\xi_5 = \beta\alpha\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_6 = \beta\alpha\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\xi_7 = \beta\beta\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_8 = \beta\beta\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where

$$\alpha = |+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These state are not the eigenstates of $\hat{\boldsymbol{M}}^2$ (the square of the total angular momentum) and $\hat{M}_z$ (the *z* component of the total angular momentum). We need to find the eigenstates and eigenvalues of $\hat{\boldsymbol{M}}^2$ and $\hat{M}_z$, where

$$[\hat{\boldsymbol{M}}^2, \hat{M}_z] = 0.$$

The eigenstates are the superposition of the eight states, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, ξ_7, ξ_8. The matrices of $\hat{\boldsymbol{M}}^2$ and $\hat{M}_z$ (the size 8 x 8) are obtained as

$$
\begin{aligned}
\hat{\boldsymbol{M}}^2 = 2(&\hat{S}_x \otimes \hat{S}_x \otimes \hat{I}_2 + \hat{S}_y \otimes \hat{S}_y \otimes \hat{I}_2 + \hat{S}_z \otimes \hat{S}_z \otimes \hat{I}_2 \\
&+ \hat{S}_x \otimes \hat{I}_2 \otimes \hat{S}_x + \hat{S}_y \otimes \hat{I}_2 \otimes \hat{S}_y + \hat{S}_z \otimes \hat{I}_2 \otimes \hat{S}_z \\
&+ \hat{I}_2 \otimes \hat{S}_x \otimes \hat{S}_x + \hat{I}_2 \otimes \hat{S}_y \otimes \hat{S}_y + \hat{I}_2 \otimes \hat{S}_z \otimes \hat{S}_z) \\
&+ 3S(S+1)\hat{I}_2 \otimes \hat{I}_2 \otimes \hat{I}_2 \\
&= \begin{pmatrix}
15/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7/4 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 7/4 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 7/4 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 7/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 7/4 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 7/4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 15/4
\end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\hat{M}_z &= \hat{S}_z \otimes \hat{I}_2 \otimes \hat{I}_2 + \hat{I}_2 \otimes \hat{S}_z \otimes \hat{I}_2 + \hat{I}_2 \otimes \hat{I}_2 \otimes \hat{S}_z \\
&= \begin{pmatrix}
3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/2
\end{pmatrix},
\end{aligned}
$$

where $S = \frac{1}{2}$,

$$
\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
$$

$$
\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(identity matrix)}
$$

We use the Mathematica programs (KroneckerProduct and Eigensystem) to solve the eigenvalue problems for $\hat{\boldsymbol{M}}^2$ and $\hat{M}_z$. The results are as follows.

(a) j = 3/2 (symmetric)

$$|\,j=3/2, m=3/2\rangle = U\xi_1 = \alpha\alpha\alpha\,,$$

$$|\,j=3/2, m=1/2\rangle = U\xi_2 = \frac{1}{\sqrt{3}}(\alpha\alpha\beta+\alpha\beta\alpha+\beta\alpha\alpha)\,,$$

$$|\,j=3/2, m=-1/2\rangle = \hat{U}\xi_3 = \frac{1}{\sqrt{3}}(\alpha\beta\beta+\beta\alpha\beta+\beta\beta\alpha)\,,$$

$$|\,j=3/2, m=-3/2\rangle = U\xi_4 = \beta\beta\beta\,.$$

(b) j = 1/2

$$|\,j=1/2, m=1/2\rangle = \hat{U}\xi_5 = \frac{1}{\sqrt{2}}(\alpha\alpha\beta-\beta\alpha\alpha)\,,$$

$$|\,j=1/2, m=-1/2\rangle = \hat{U}\xi_6 = \frac{1}{\sqrt{2}}(\alpha\beta\beta-\beta\beta\alpha)\,,$$

(c) j = 1/2

$$|\,j=1/2, m=1/2\rangle = \hat{U}\xi_7 = \frac{1}{\sqrt{6}}(\alpha\alpha\beta-2\alpha\beta\alpha+\beta\alpha\alpha)\,,$$

$$|\,j=1/2, m=-1/2\rangle = \hat{U}\xi_8 = \frac{1}{\sqrt{6}}(\alpha\beta\beta-2\beta\alpha\beta+\beta\beta\alpha)\,.$$

The corresponding unitary operator is given by

$\hat{U}$ =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\hat{U}^{+}\hat{\boldsymbol{M}}^{2}\hat{U} =$

$$\begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{15}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

$\hat{U}^{+}\hat{M}_{z}\hat{U} =$

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

These matrices ($\hat{U}^{+}\hat{\boldsymbol{M}}^{2}\hat{U}$ and $\hat{U}^{+}\hat{M}_{z}\hat{U}$) can be written as block-diagonal form. It is formed of (4x4) matrix denoted by the irreducible representation $D_{3/2}$ (for j = 3/2) and two (2x2) matrices denoted by the irreducible representation $D_{1/2}$ (for j = 1/2).

2. Eigenstates of three particles with the spin $S = L = 1$.

We now consider the state of three particles with L = 1 (we use L = 1 instead of S = 1).

$$D_1 \times D_1 \times D_1 = (D_2 + D_1 + D_0) \times D_1 = D_3 + 2D_2 + 3D_1 + D_0 .$$

The wave function of three particles with S = 1. We consider the 27 states ξ_1, ξ_2, ξ_3, ξ_4, ..., ξ_{26}, ξ_{27}, which are defined by

$$\xi_1 = \alpha \otimes \alpha \otimes \alpha = \alpha\alpha\alpha ,$$
$$\xi_2 = \alpha \otimes \alpha \otimes \beta = \alpha\alpha\beta ,$$
$$\xi_3 = \alpha \otimes \alpha \otimes \gamma = \alpha\alpha\gamma ,$$
$$\xi_4 = \alpha \otimes \beta \otimes \alpha = \alpha\beta\alpha ,$$
$$\xi_5 = \alpha \otimes \beta \otimes \beta = \alpha\beta\beta ,$$
$$\xi_6 = \alpha \otimes \beta \otimes \gamma = \alpha\beta\gamma ,$$
$$\xi_7 = \alpha \otimes \gamma \otimes \alpha = \alpha\gamma\alpha ,$$
$$\xi_8 = \alpha \otimes \gamma \otimes \beta = \alpha\gamma\beta ,$$
$$\xi_9 = \alpha \otimes \gamma \otimes \gamma = \alpha\gamma\gamma ,$$
$$\xi_{10} = \beta \otimes \alpha \otimes \alpha = \beta\alpha\alpha ,$$
$$\xi_{11} = \beta \otimes \alpha \otimes \beta = \beta\alpha\beta ,$$

$$\xi_{12} = \beta \otimes \alpha \otimes \gamma = \beta\alpha\gamma \ ,$$
$$\xi_{13} = \beta \otimes \beta \otimes \alpha = \beta\beta\alpha \ ,$$
$$\xi_{14} = \beta \otimes \beta \otimes \beta = \beta\beta\beta \ ,$$
$$\xi_{15} = \beta \otimes \beta \otimes \gamma = \beta\beta\gamma \ ,$$
$$\xi_{16} = \beta \otimes \gamma \otimes \alpha = \alpha\gamma\alpha \, ,$$
$$\xi_{17} = \beta \otimes \gamma \otimes \beta = \alpha\gamma\beta \, ,$$
$$\xi_{18} = \beta \otimes \gamma \otimes \gamma = \alpha\gamma\gamma \, ,$$
$$\xi_{19} = \gamma \otimes \alpha \otimes \alpha = \gamma\alpha\alpha \ ,$$
$$\xi_{20} = \gamma \otimes \alpha \otimes \beta = \gamma\alpha\beta \ ,$$
$$\xi_{21} = \gamma \otimes \alpha \otimes \gamma = \gamma\alpha\gamma \ ,$$
$$\xi_{22} = \gamma \otimes \beta \otimes \alpha = \gamma\beta\alpha \ ,$$
$$\xi_{23} = \gamma \otimes \beta \otimes \beta = \gamma\beta\beta \, ,$$
$$\xi_{24} = \gamma \otimes \beta \otimes \gamma = \gamma\beta\gamma \ ,$$
$$\xi_{25} = \gamma \otimes \gamma \otimes \alpha = \gamma\gamma\alpha \, ,$$
$$\xi_{26} = \gamma \otimes \gamma \otimes \beta = \gamma\gamma\beta \ ,$$
$$\xi_{18} = \gamma \otimes \gamma \otimes \gamma = \gamma\gamma\gamma \ ,$$

where

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left| l = 1, m = 1 \right\rangle ,$$

$$\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \left| l = 1, m = 0 \right\rangle ,$$

$$\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left| l = 1, m = -1 \right\rangle .$$

These state are not the eigenstates of $\hat{\boldsymbol{M}}^2$ and $\hat{M}_z$. We need to find the eigenstates and eigenvalues of $\hat{\boldsymbol{M}}^2$ and $\hat{M}_z$, where

$$[\hat{\boldsymbol{M}}^2, \hat{M}_z] = 0 .$$

We have the matrices of $\hat{\boldsymbol{M}}^2$ and $\hat{M}_z$ as follows. Note that these matrices has a size of 27x27.

$$\begin{aligned}\hat{\boldsymbol{M}}^2 &= 2(\hat{L}_x \otimes \hat{L}_x \otimes \hat{I}_3 + \hat{L}_y \otimes \hat{L}_y \otimes \hat{I}_3 + \hat{L}_z \otimes \hat{L}_z \otimes \hat{I}_3 + \hat{L}_x \otimes \hat{I}_3 \otimes \hat{L}_x \\ &\quad + \hat{L}_y \otimes \hat{I}_3 \otimes \hat{L}_y + \hat{L}_z \otimes \hat{I}_3 \otimes \hat{L}_z + \hat{I}_3 \otimes \hat{L}_x \otimes \hat{L}_x + \hat{I}_3 \otimes \hat{L}_y \otimes \hat{L}_y + \hat{I}_3 \otimes \hat{L}_z \otimes \hat{L}_z) \\ &\quad + 3L(L+1)\hat{I}_3 \otimes \hat{I}_3 \otimes \hat{I}_3 \\ &=\end{aligned}$$

$$\begin{pmatrix}
12&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&8&0&2&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&4&0&2&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&8&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&6&0&2&0&0&0&2&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&4&0&2&0&0&0&2&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&4&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&4&0&0&0&0&0&2&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&2&0&2&0&0&0&0&0&0&0&0&0&0\\
0&2&0&2&0&0&0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&2&0&0&0&0&0&6&0&2&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&4&0&2&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&2&0&2&0&0&0&2&0&6&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&2&0&0&0&2&0&6&0&2&0&0&0&2&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&6&0&2&0&0&0&2&0&2&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&4&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&6&0&0&0&0&0&2&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&8&0&0&0&0&0&2&0&2&0\\
0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&0&0&4&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&4&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&2&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&2&0&6&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&8&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&4&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&8&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&12
\end{pmatrix}$$

$$\begin{aligned}\hat{M}_z &= \hat{L}_z \otimes \hat{I}_3 \otimes \hat{I}_3 + \hat{I}_3 \otimes \hat{L}_z \otimes \hat{I}_3 + \hat{I}_3 \otimes \hat{I}_3 \otimes \hat{L}_z \\ &=\end{aligned}$$

$$\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{pmatrix}$$

where $\hat{L}_i$ ($i = x, y, z$) is the operator of orbital angular momentum with $L = 1$.

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{L}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\hat{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \text{(identity matrix, 3 x 3)}$$

We use the Mathematica programs with KroneckerProduct and Eigensystem to solve the eigenvalue problem. The results are as follows.

$\hat{U}^{+}\hat{\boldsymbol{M}}^{2}\hat{U} =$

$$
\begin{pmatrix}
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$\hat{U}^{+}\hat{M}_{z}\hat{U} =$

$$\begin{pmatrix}
3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Eigenstates:

$$\left| j=3, m=3 \right\rangle = U\xi_1 = \alpha\alpha\alpha\,,$$

$$\left| j=3, m=2 \right\rangle = U\xi_2 = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)\,,$$

$$\left| j=3, m=1 \right\rangle = U\xi_3 = \frac{1}{\sqrt{15}}(\alpha\alpha\gamma + 2\alpha\beta\beta + \alpha\gamma\alpha + 2\beta\alpha\beta + 2\beta\beta\alpha + \gamma\alpha\alpha)\,,$$

$$\left| j=3, m=0 \right\rangle = U\xi_4 = \frac{1}{\sqrt{10}}(\alpha\beta\gamma + \alpha\gamma\beta + \beta\alpha\gamma + 2\beta\beta\beta + \beta\gamma\alpha + \gamma\alpha\beta + \gamma\beta\alpha)\,,$$

$$\left| j=3, m=-1 \right\rangle = U\xi_5 = \frac{1}{\sqrt{15}}(\alpha\gamma\gamma + 2\beta\beta\gamma + 2\beta\gamma\beta + \gamma\alpha\gamma + 2\gamma\beta\beta + \gamma\gamma\alpha)\,,$$

$$\left| j=3, m=-2 \right\rangle = U\xi_6 = \frac{1}{\sqrt{3}}(\beta\gamma\gamma + \gamma\beta\gamma + \gamma\gamma\beta),$$

$$\left| j=3, m=-3 \right\rangle = U\xi_7 = \gamma\gamma\gamma .$$

$$\left| j=2, m=2 \right\rangle = U\xi_8 = \frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha),$$

$$\left| j=2, m=2 \right\rangle = U\xi_9 = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha),$$

$$\left| j=2, m=1 \right\rangle = U\xi_{10} = \frac{1}{2\sqrt{3}}(2\alpha\alpha\gamma + \alpha\beta\beta - \alpha\gamma\alpha + \beta\alpha\beta - 2\beta\beta\alpha - \gamma\alpha\alpha),$$

$$\left| j=2, m=1 \right\rangle = U\xi_{11} = \frac{1}{2}(\alpha\beta\beta + \alpha\gamma\alpha - \beta\alpha\beta - \gamma\alpha\alpha),$$

$$\left| j=2, m=0 \right\rangle = U\xi_{12} = \frac{1}{2\sqrt{2}3}(\alpha\beta\gamma + 2\alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha - 2\gamma\alpha\beta - \gamma\beta\alpha),$$

$$\left| j=2, m=0 \right\rangle = U\xi_{13} = \frac{1}{2}(\alpha\beta\gamma + \beta\alpha\gamma - \beta\gamma\alpha - \gamma\beta\alpha),$$

$$\left| j=2, m=-1 \right\rangle = U\xi_{14} = \frac{1}{2\sqrt{3}}(2\alpha\gamma\gamma + \beta\beta\gamma + \beta\gamma\beta - \gamma\alpha\gamma - 2\gamma\beta\beta - \gamma\gamma\alpha),$$

$$\left| j=2, m=-1 \right\rangle = U\xi_{15} = \frac{1}{2}(\beta\beta\gamma - \beta\gamma\beta + \gamma\alpha\gamma - \gamma\gamma\alpha),$$

$$\left| j=2, m=-2 \right\rangle = U\xi_{16} = \frac{1}{\sqrt{6}}(\beta\gamma\gamma - 2\gamma\beta\gamma + \gamma\gamma\beta),$$

$$\left| j=2, m=-2 \right\rangle = U\xi_{17} = \frac{1}{\sqrt{2}}(\beta\gamma\gamma - \gamma\gamma\beta).$$

$$\left| j=1, m=1 \right\rangle = U\xi_{18} = \frac{1}{2\sqrt{15}}(\alpha\alpha\gamma - 3\alpha\beta\beta + 6\alpha\gamma\alpha + 2\beta\alpha\beta - 3\beta\beta\alpha + \gamma\alpha\alpha),$$

$$\left| j=1,m=1 \right\rangle = U\xi_{19} = \frac{1}{2}(\alpha\alpha\gamma - \alpha\beta\beta + \beta\beta\alpha - \gamma\alpha\alpha \,,$$

$$\left| j=1,m=1 \right\rangle = U\xi_{20} = \frac{1}{\sqrt{3}}(\alpha\alpha\gamma - \beta\alpha\beta + \gamma\alpha\alpha) \,,$$

$$\left| j=1,m=0 \right\rangle = U\xi_{21} = \frac{1}{2\sqrt{10}}(\alpha\beta\gamma + \alpha\gamma\beta - 4\beta\alpha\gamma + 2\beta\beta\beta - 4\beta\gamma\alpha + \gamma\alpha\beta + \gamma\beta\alpha) \,,$$

$$\left| j=1,m=0 \right\rangle = U\xi_{22} = \frac{1}{2\sqrt{6}}(\alpha\beta\gamma - 3\alpha\gamma\beta + 2\beta\beta\beta - 3\gamma\alpha\beta + \gamma\beta\alpha) \,,$$

$$\left| j=1,m=0 \right\rangle = U\xi_{23} = \frac{1}{\sqrt{3}}(\alpha\beta\gamma - \beta\beta\beta + \gamma\beta\alpha) \,,$$

$$\left| j=1,m=-1 \right\rangle = U\xi_{24} = \frac{1}{2\sqrt{15}}(\alpha\gamma\gamma - 3\beta\beta\gamma + 2\beta\gamma\beta + 6\gamma\alpha\gamma - 3\gamma\beta\beta + \gamma\gamma\alpha) \,,$$

$$\left| j=1,m=-1 \right\rangle = U\xi_{25} = \frac{1}{2}(\alpha\gamma\gamma - \beta\beta\gamma + \gamma\beta\beta - \gamma\gamma\alpha) \,,$$

$$\left| j=1,m=-1 \right\rangle = U\xi_{26} = \frac{1}{\sqrt{3}}(\alpha\gamma\gamma - \beta\gamma\beta + \gamma\gamma\alpha) \,.$$

$$\left| j=0,m=0 \right\rangle = U\xi_{27} = \frac{1}{\sqrt{6}}(\alpha\beta\gamma - \alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha + \gamma\alpha\beta - \gamma\beta\alpha) \,.$$

Note that these matrices ($\hat{U}^{+}\hat{\boldsymbol{M}}^{2}\hat{U}$ and $\hat{U}^{+}\hat{M}_{z}\hat{U}$) can be written as block-diagonal form. It is formed of (7x7) matrix denoted by the irreducible representation D_3 (for j = 3), two (5x5) matrices denoted by the irreducible representation D_2 (for j = 2), three (3x3) matrices denoted by the irreducible representation D_2 (for j = 1), and one (1x1) matrices denoted by the irreducible representation D_2 (for j = 0),