Laplace-Runge-Lenz Triangles in Feynman Hodograph Diagram: the Kepler's Model and Sommerfeld's Model

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(Date: November 11, 2022)

The Kepler's problem (inverse-square law of force) admits a conserved vector that lies in the plane of motion. This vector has been associated with the names of Laplace, Runge, and Lenz (LRL) among the conservation of the energy and the angular momentum. The property and geometry of the LRL vectors are discussed in terms of the Feynman hodograph diagram, which is extensively revised and improved here. It occurs with our finding of a construction line in the diagram. In this revised diagram, a various size of so-called LRL triangles (typically similar four triangles) clearly can be seen. In order to obtain the precise diagram, we calculate the coordinates of the intersections of circles and straight lines by using the Mathematica.

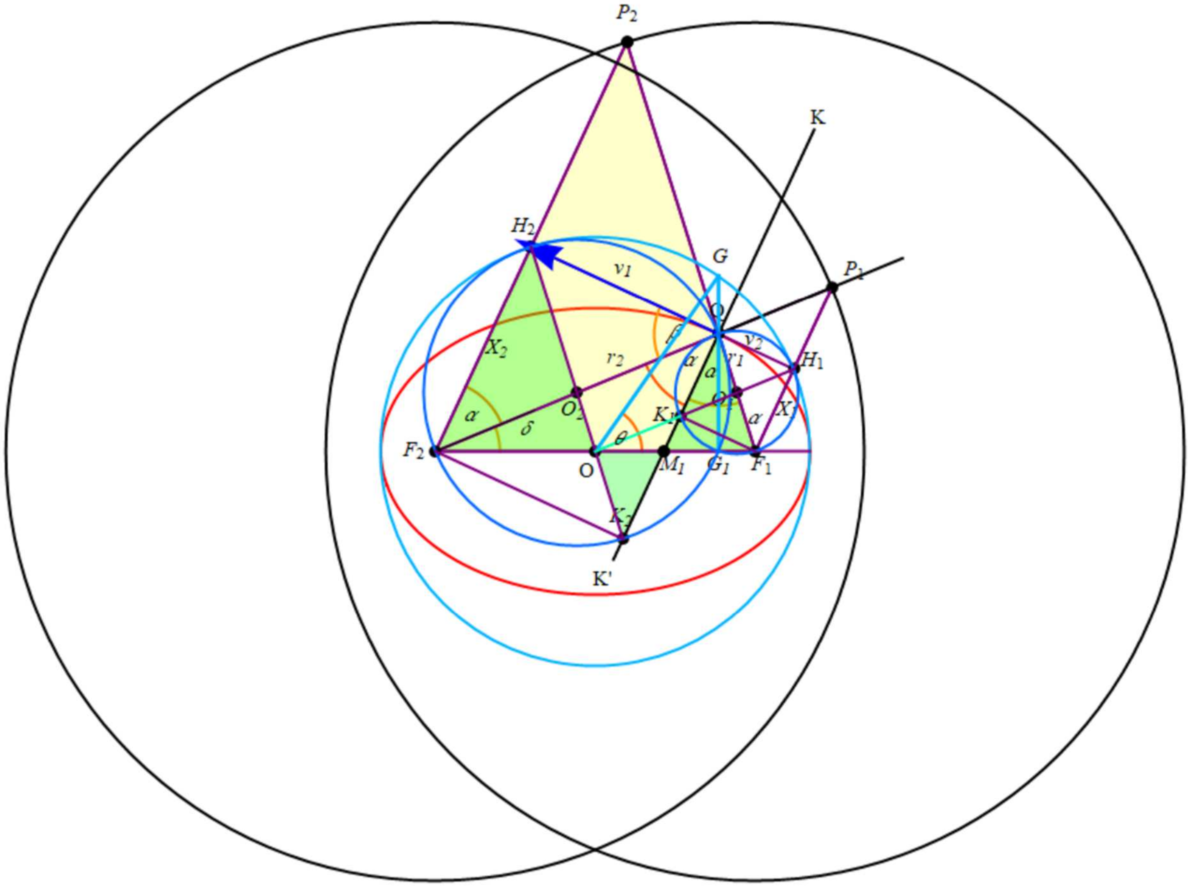

Fig.1

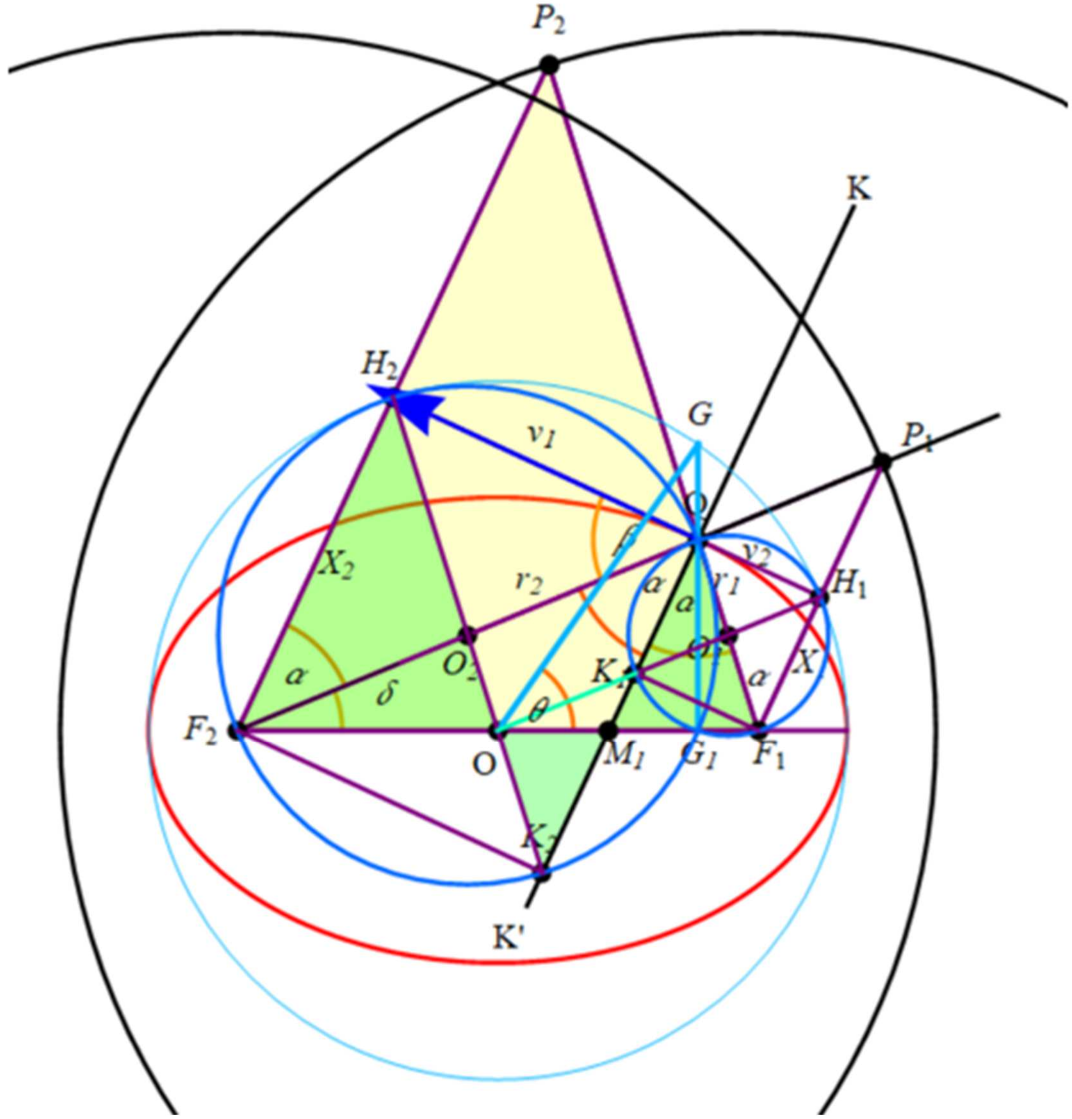

Fig.2

Figs.1 and 2 Feynman hodograph diagram, where the Laplace-Runge-Lenz triangles are clearly seen. The construction line QG_1 plays an important role to the revised version of Feynman hodograph diagram. The particle is located at the point Q on the elliptic orbit.

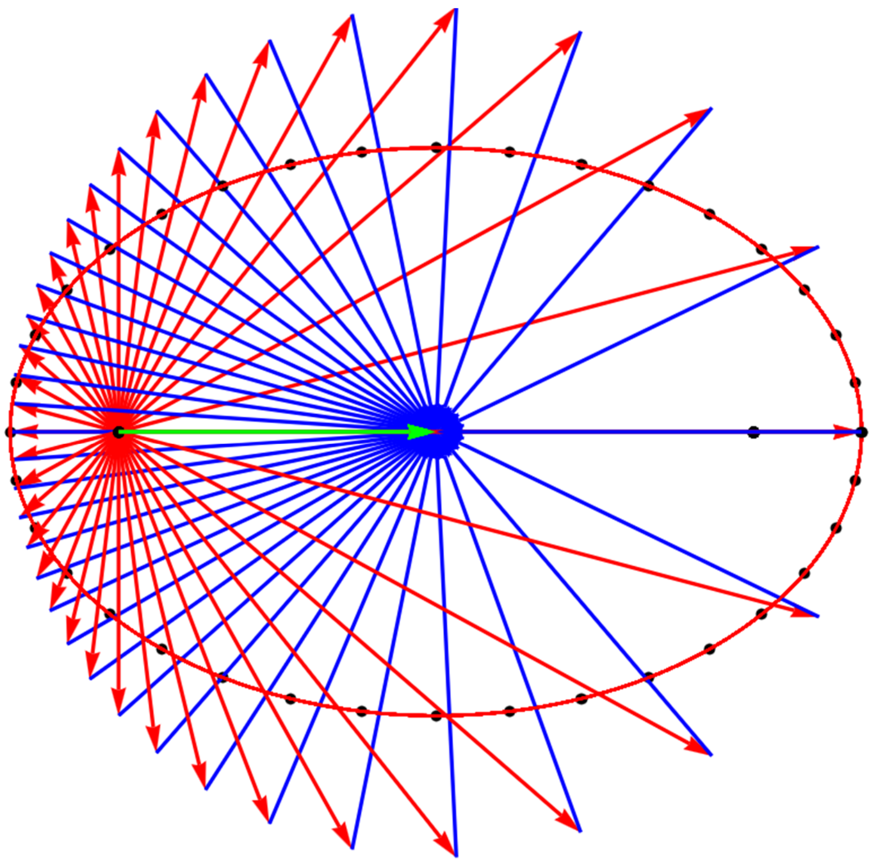

Fig.3 Laplace-Runge-Lenz (LRL) triangles in the Feynman hodograph diagram. The LRL vector connecting between two focus points, is denoted by the green line

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1. Overview

It is well known that a planet (such Earth) undergoes an elliptic orbit around the Sun (as one of focus in the ellipse). The eccentricity of the ellipse is positive and smaller than unity. There is a gravitational attractive interaction between the planet and the sun, forming the central force problem (Kepler model). The magnitude of the force is inversely proportional to the square of the distance between the planet and Sun. The following three laws are well-known as the Kepler's law's three laws.

1. Planets move around the sun in ellipse, with the Sun at one focus.
2. The line connecting the sun to a plane sweeps equal areas in equal times (the angular momentum conservation).
3. The square of the orbital period of a plane is proportional to the cube (3rd power) of the semi-major axis (distance) of the ellipse.

In 1913, Niels Bohr proposed a Bohr model for electron in hydrogen atom, where an electron (negative charge) undergoes a circular orbit around a proton (positive charge) in the nucleus. There is an attractive Coulomb interaction between electron and proton. The magnitude of the force is inversely proportional to the square of the distance between electron and proton. The energy of the electron is quantized. The Bohr model is the first success to the construction of quantum mechanics.

In 1916, the Bohr's model was revised by Sommerfeld. Sommerfeld showed that actually electron shows an elliptic orbit around the protons. The two quantum conditions are used in the form of

$$\oint p_\phi d\phi = n_\phi \hbar \text{ and } \oint p_r dr = n_r \hbar ,$$

where n_ϕ is the azimuthal quantum number and n_r is the radial quantum number. When the principal quantum number n is defined by

$$n = n_r + n_\phi ,$$

the energy of electron is obtained by the same form derived by Bohr. The significance of this extension may make it possible to account for the fine structure of the energy levels and spectrum lines in hydrogen and hydrogen-like atoms. We note that Wilson, Sommerfeld, and Ishiwara postulated, about the same time and quite independently, that each degree of freedom must be quantized separately. In other words, each degree of freedom should be fixed by its own separate quantum number.

Sommerfeld's analysis was quickly overtaken by the development of quantum mechanics by Heisenberg, Schrödinger, Dirac, and others. Application of the Schrödinger equation led to the quantized energy levels which is the same as those derived by Bohr and Sommerfeld. Once the Schrödinger equation appeared, no special attention on the Kepler-model and Sommerfeld-model were seriously paid. Nevertheless, we think that it is necessary to discuss again the Kepler-model and Sommerfeld model in terms of the Feynman hodograph model and the LRL vector.. The analysis described in this article may be useful to undergraduate students who want to know how the transition occurs from the simple Bohr model to the full solution of the Schrödinger equation in electron of hydrogen atom. Here we discuss the significant role of the Laplace-Runge-Lenz (LRL) vector in the Feynman hodograph diagram (the Kepler-model and the Sommerfeld model without the quantized condition and relativistic correction). The LRL vector is always parallel to the semi-major axis, and the magnitude is universally constant. In previous our article, we have discussed the detail of the Feynman hodograph diagram in the Kepler problem. Here, the Feynman hodograph diagram will be greatly extended and improved. The LRL triangles are clearly seen in the revised Feynman hodograph diagram. In order to get the revised Feynman hodograph diagram exactly, all the coordinates of the intersections of the straight lines and circles are calculated by using Mathematica, in spite of the usefulness of the geometry.

Thanks to Bohr, Sommerfeld, and the others, we realize that the electron undergoes an elliptic orbit around the protons in the nucleus. In general, the number of protons inside the nucleus is given by the atomic number *Z*; $Z = 1$ for hydrogen atom. The Sommerfeld model is the first success in visualizing the shape of elliptic orbits in atoms, depending on the discrete energy and orbital angular momentum. Unfortunately, experimental results on atomic spectra data made at that times, could not be explained by the Sommerfeld model, partly because the existence of spins is not taken into account.

One of our motivations to write this article is that we feel it necessary to understand the essential points of the Sommerfeld model. We tried to understand the experimental result of the energy separation of double (D)-lines of alkali metal atoms by using the spin-orbit interaction. We derived the effective proton charge Z_{eff} for *n*P double energy separation. Unexpectedly, we get $Z_{eff} > 1$ even for 3*P* double line energy separation. The *nP* orbits ($n \geq 3$) penetrate inside the 2*P* orbits near the perihelion. As the eccentricity approaches unity, the velocity drastically increases. Relativistic correction should be necessary. The result of $Z_{eff} > 1$ may be related to the Slater's rule and Sommerfeld's puzzle (these will be discussed in detail in other articles).

2. Definition of Laplace-Runge-Lenz vector in the Kepler's model and Sommerfeld's model

Laplace-Runge-Lenz (LRL) vector ((**Wikipedia**))

We find the definition of the LRL vector in Wikipedia. It is given as follows. In classical mechanics, the **Laplace–Runge–Lenz (LRL) vector** is a vector used chiefly to describe the shape and orientation of the orbit of one astronomical body around another, such as a binary star or a planet revolving around a star. For two bodies interacting by Newtonian gravity, the LRL vector is a constant of motion, meaning that it is the same no matter where it is calculated on the orbit; equivalently, the LRL vector is said to be conserved. More generally, the LRL vector is conserved in all problems in which two bodies interact by a central force that varies as the inverse square of the distance between them; such problems are called Kepler problems. The hydrogen atom is a Kepler problem, since it comprises two charged particles interacting by Coulomb's law of electrostatics, another inverse-square central force. The LRL vector was essential in the first quantum mechanical derivation of the spectrum of the hydrogen atom,[7][8] before the development of the Schrödinger equation. However, this approach is rarely used today.

https://en.wikipedia.org/wiki/Laplace%E2%80%93Runge%E2%80%93Lenz_vector

We have presented an article titled "The Hodographic solution to the Kepler problem in 2015 in the **Bingweb**, where planet undergoes an elliptic orbit around the Sun as one of the focus. There is an attractive gravitational interaction between the planet and the Sun (the Kepler model)

$$U_G = -\frac{GMm}{r},$$

where G is the gravitational constant, M is the mass of Sun, and m is the mass of the planet.

Recently we had an opportunity to study on the spin-orbit interaction in alkali metal atoms. The experimental results (NIST Atomic Spectra Data) on the energy separations of *np* orbits ($n\ {}^2P_{3/2}$ and $n\ {}^2P_{1/2}$) are compared with the theoretical calculations such as Sommerfeld model (relativistic) and Dirac relativistic electron theory. In 1916, Sommerfeld realized that an electron (a charge -e, e>0) undergoes an elliptic orbit around the nucleus with protons with positive charges Ze), where Z is the atomic number. There is a Coulomb interaction between An electron In a hydrogen-like atom, an electron In order to understand the physics, we need that the elliptic orbit. The we are interested in the Sommerfeld model where an electron (a charge -e, e>0) is rotated around the nucleus (charges Ze) with atomic number Z. There is a attractive Coulomb interaction between the electron and protons (the Sommerfeld model). The potential energy is

$$U_e = -\frac{Ze^2}{r}.$$

Since the potential energy for both models is inversely proportional to the distance r, classically the property of the elliptic orbits is equivalent. Here we use the notations comment to both models.

$$|E| = \frac{k_0 m}{2a} = \frac{k}{2a},$$

$$k_0 = \begin{cases} \dfrac{Ze^2}{m} & \text{(Coulomb)} \\ \\ GM & \text{(Kepler)} \end{cases}, \qquad \text{(in units of cm}^3/\text{s}^2\text{)}$$

or

$$k = k_0 m = \begin{cases} Ze^2 & \text{(Coulomb)} \\ \\ GMm & \text{(Kepler)} \end{cases}$$

Note that we also use the notation of eccentricity ε, but not *e*, since we use -*e* for the charge of electron.

Here we discuss on the Feynman hodograph diagram in the Kepler problem and Sommerfeld model. The Kepler model is a purely classical model, In thee Sommerfeld model is that the energy and the angular momentum are quantized in the Sommerfeld model. In the classical Kepler model, the energy and the angular momentum are continuous, but nit quantized. The hodographic solution to the Kepler problem was discussed in our web site (**Bingweb**), Here we continue to discuss the hodographic solution to both models such model which may be more useful to our understanding of physics of these models (classically). In another article, we will discuss the spin-orbit interactions of alkali metal atoms with the relativistic theory of Sommerfeld model and Dirac relativistic electron, by comparison with the corresponding classical versions. To this end, we use Mathematica for the schematic hodographic diagram. We use algebraic calculations, with the help of our knowledge on the elementary geometry.

3. Eccentricity anomaly and polar coordinate

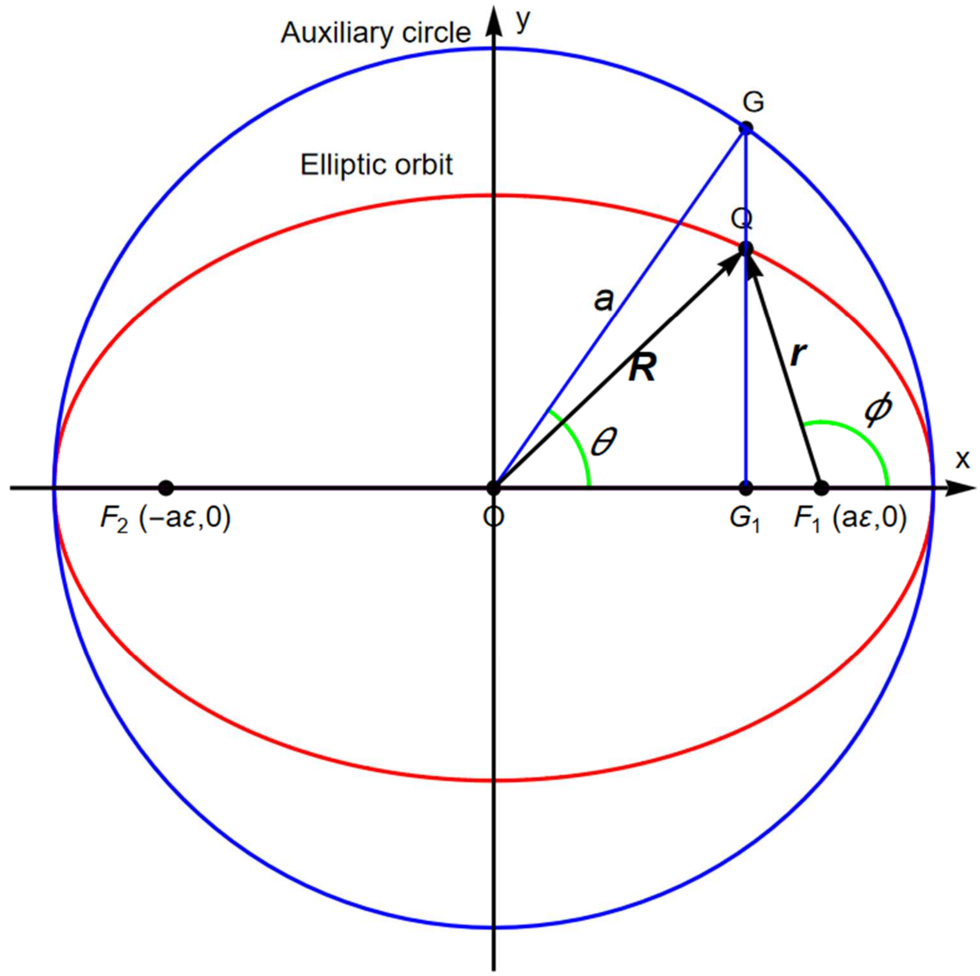


Fig.4 Elliptic orbit of particle (focus at the point F_1). Auxiliary orbit with radius (semi-major axis a). θ is called the eccentricity anomaly. F_1 and F_2 are the focal points. The particle is located at the point Q. $\{r, \phi\}$ are the two-dimensional polar coordinates. ϕ is the azimuthal angle. The equation of elliptic orbit is given by

$$\frac{(x+a\varepsilon)^2}{a^2}+\frac{y^2}{b^2}=1 .$$

$\mathbf{G}_1=(a\varepsilon\cos\theta,0)$. $\mathbf{F}_1=(a\varepsilon,0)$. $\mathbf{Q}=(a\cos\theta,b\sin\theta)=(r\cos\phi,r\sin\phi)$

((**Position vector**))

We circumscribe the ellipse with an auxiliary circle of radius a centered at the origin O, and project the point Q (defined by $\boldsymbol{R}$) on to the circle at the point G. We consider a particle at the point Q on the elliptic orbit. The position vector at the point Q is described by the Cartesian coordinates as

$$\overrightarrow{OQ} = \mathbf{R} = (x, y) = (a\cos\theta, b\sin\theta) = a(\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta),$$

where a is the semi-major axis, b is the semi-minor axis and ε is the eccentricity.
We also use the polar coordinates $\mathbf{r}$ with the point F_1 as a

$$\mathbf{R} = \overrightarrow{OF_1} + \overrightarrow{F_1Q} = \mathbf{F}_1 + \mathbf{r},$$

with

$$\overrightarrow{OF_1} = \mathbf{F}_1 = (a\varepsilon, 0),$$

and

$$\overrightarrow{F_1Q} = \mathbf{r} = (r\cos\phi, r\sin\phi),$$

where r is the distance between the points Q and F_1 and ϕ. From these relations we have

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{r} = (a\varepsilon, 0) + (r\cos\phi, r\sin\phi),$$

or

$$x = a\cos\theta = a\varepsilon + r\sin\phi,$$

$$y = b\sin\theta = r\sin\phi.$$

From the definition, the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ can be obtained as

$$\begin{aligned}\dot{\mathbf{R}} &= \dot{\mathbf{r}}\\ &= \mathbf{v}\\ &= (-a\sin\theta, b\cos\theta)\dot{\theta}\\ &= a(-\sin\theta, \sqrt{1-\varepsilon^2}\cos\theta)\dot{\theta}\end{aligned}$$

since $\dot{\mathbf{F}}_1 = 0$. The acceleration vector is simply defined by

$$\ddot{\mathbf{R}} = \ddot{\mathbf{r}} = \mathbf{a},$$

since $\ddot{\mathbf{F}}_1 = 0$.

We consider the expression of the angular momentum with respect to the origin ($\mathbf{L}_R$) and the angular momentum with respect the point F$_1$ ($\mathbf{L}$),

$$\mathbf{R} = \mathbf{r} + \mathbf{F}_1.$$

((**Angular momentum**))

$\mathbf{L}_R$: The angular momentum around the origin O

$$\mathbf{L}_R = \mathbf{R} \times \mathbf{P}_R.$$

where $\mathbf{P}_R = m\dot{\mathbf{R}}$.

$\mathbf{L}$: The angular momentum around the F$_1$ (focus)

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

with the linear momentum

$$\mathbf{p}_R = m\mathbf{v}_R = m\mathbf{v} = \mathbf{p}, \qquad \mathbf{v}_R = \mathbf{v}.$$

We have

$$\begin{aligned} \mathbf{L}_R &= \mathbf{R} \times m\mathbf{v}_R \\ &= (\mathbf{r} + \mathbf{F}_1) \times m\mathbf{v} \\ &= \mathbf{L} + \mathbf{F}_1 \times m\mathbf{v}_1 \end{aligned}$$

or

$\mathbf{L} = \mathbf{L}_R - \mathbf{F}_1 \times m\mathbf{v}_1$.

4. Central force problem (2D polar coordinates)

We consider a system consisting of two bodies. A central force between two bodies is defined as a force which is directed along the line connecting of the two bodies.

The angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ around the focus (point F_1. The torque is related to $\mathbf{L}$ as

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}.$$

For the central force with $\mathbf{F}$ parallel to $\mathbf{r}$, $\boldsymbol{\tau} = 0$ and $\boldsymbol{L}$ is a constant of motion and therefore is conserved under the action of a central force. Since $\mathbf{L} \cdot \mathbf{r} = 0$, the motion of the system under a central force is confined to a 2D plane. The motion can be described in plane polar coordinates whose variables are r and ϕ. The angular momentum vector is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{v}$.Since $\mathbf{L}$ is constant, the plane is invariant.

The motion is confined to a plane perpendicular to $\mathbf{L}$. The central force is along $\mathbf{r}$ and can exerts no torque on the mass. Thus, the angular momentum $\mathbf{L}$ is a constant of motion, both in direction and in magnitude. $\mathbf{r}$ is always perpendicular to $\mathbf{L}$ by the properties of cross product. Because $\mathbf{L}$ is fixed in direction, the plane of the motion is also fixed, and $\boldsymbol{r}$ can only move in a plane perpendicular to $\mathbf{L}$.

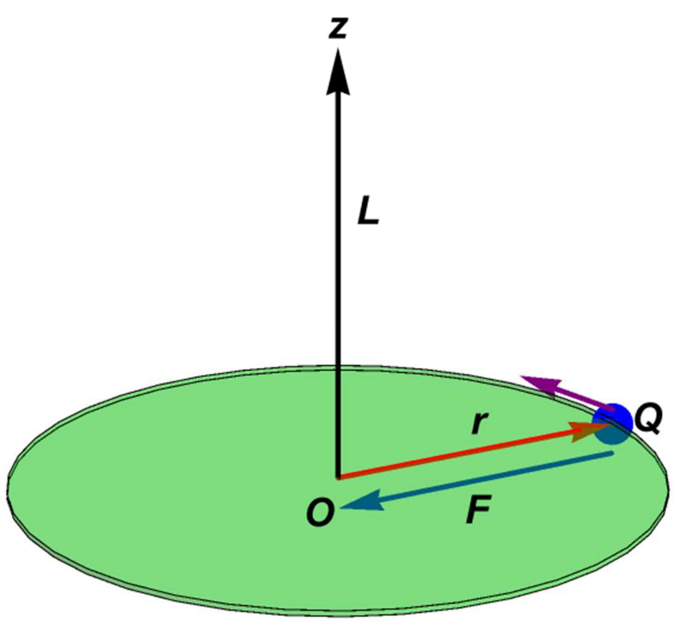


Fig.5 Central force problem. $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ around the focus (point F_1). The torque is related to $\mathbf{L}$ as $\boldsymbol{\tau} = \dfrac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}$. For the central force with $\mathbf{F}$ parallel to $\mathbf{r}$, $\boldsymbol{\tau} = 0$ and $\boldsymbol{L}$ is a constant of motion and therefore is conserved under the action of a central force. Since $\mathbf{L} \cdot \mathbf{r} = 0$, the motion of the system under a central force is confined to a 2D plane.

5. Lagrange equation of motion: equation of $r(\phi)$ in the polar coordinates ((Goldstein))

We consider the central force problem where the force is directed along the radial component of the position vector ***r***.

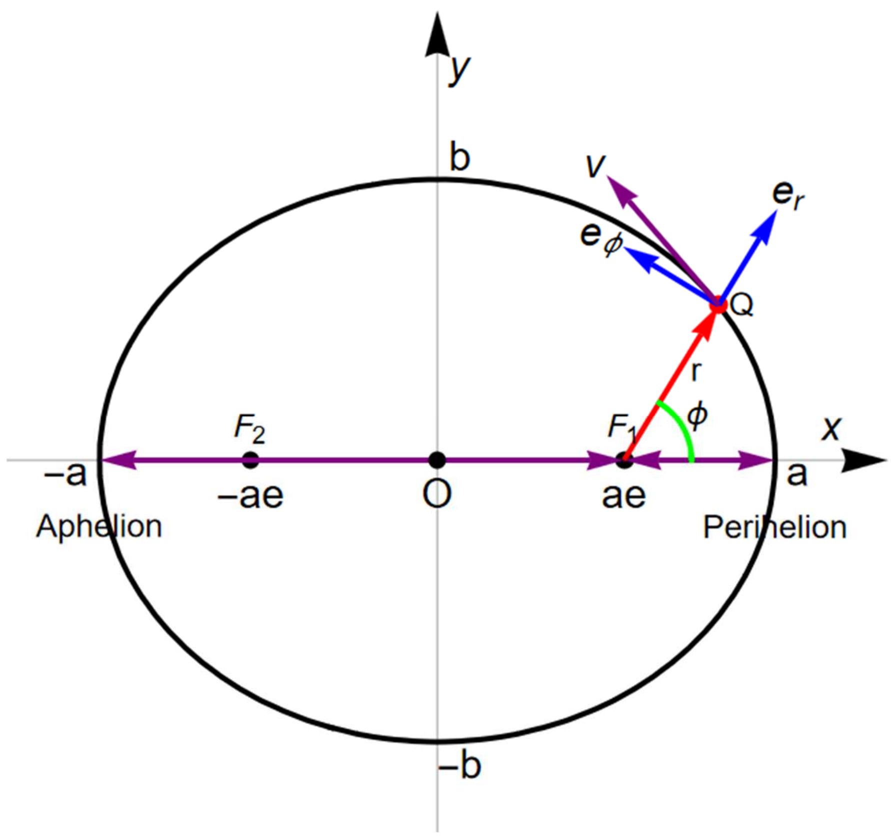


Fig.6 Elliptic orbit (2D polar coordinates with r and ϕ). $v_r = \dot{r}$ and $v_\phi = r\dot{\phi}$. $r_a = a(1+\varepsilon)$ (aphelion). $r_p = a(1-\varepsilon)$ (perihelion). ε is the eccentricity. $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are unit vectors with $\mathbf{e}_r \cdot \mathbf{e}_\phi = 0$.

Here, we start with a Lagrangian of the system given by

$$
\begin{aligned}
L(r,\phi;\dot{r},\dot{\phi}) &= T - V \\
&= \frac{1}{2}m(v_r{}^2 + v_\phi{}^2) - U(r) \\
&= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)
\end{aligned}
$$

where T is the kinetic energy and $U(r)$ is the potential energy

$$U(r) = -\frac{k}{r}.$$

where

$$k = k_0 m = \begin{cases} GmM & \text{(Kepler)} \\ \\ Ze^2 & \text{(Coulomb)} \end{cases}$$

The velocity of the particle is

$$v_r = \dot{r}, \qquad v_\phi = r\dot{\phi}$$

The canonical momentum is

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \qquad \text{(angular momentum)}$$

The Lagrange equation is

$$\dot{p}_r = \frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) = m\ddot{r} = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + f(r)$$

and

$$\dot{p}_\phi = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\phi}}) = \frac{\partial L}{\partial \phi} = 0,$$

or

$$p_\phi = mr^2\dot{\phi}. \qquad \text{(constant of motion, Noether's theorem)}$$

Note that the angular momentum around the focus F_1 is defined as

$$\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p} \\
&= (mr\mathbf{e}_r) \times (v_r\mathbf{e}_r + v_\phi \mathbf{e}_\phi) \\
&= mrv_\phi (\mathbf{e}_r \times \mathbf{e}_\phi) \\
&= mr^2\dot{\phi}\ \mathbf{e}_z
\end{aligned}$$

So that, we have

$$L_z = mr^2\dot{\phi} = \text{constant} \qquad \text{(angular momentum conservation)}$$

The change of area swept by the orbit motion during a short time dt;

$$dA = \frac{1}{2}r^2 d\phi.$$

The areal velocity is related to the z component of the angular momentum as

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{1}{2m}(mr^2\dot{\phi}) = \frac{1}{2m}L_z,$$

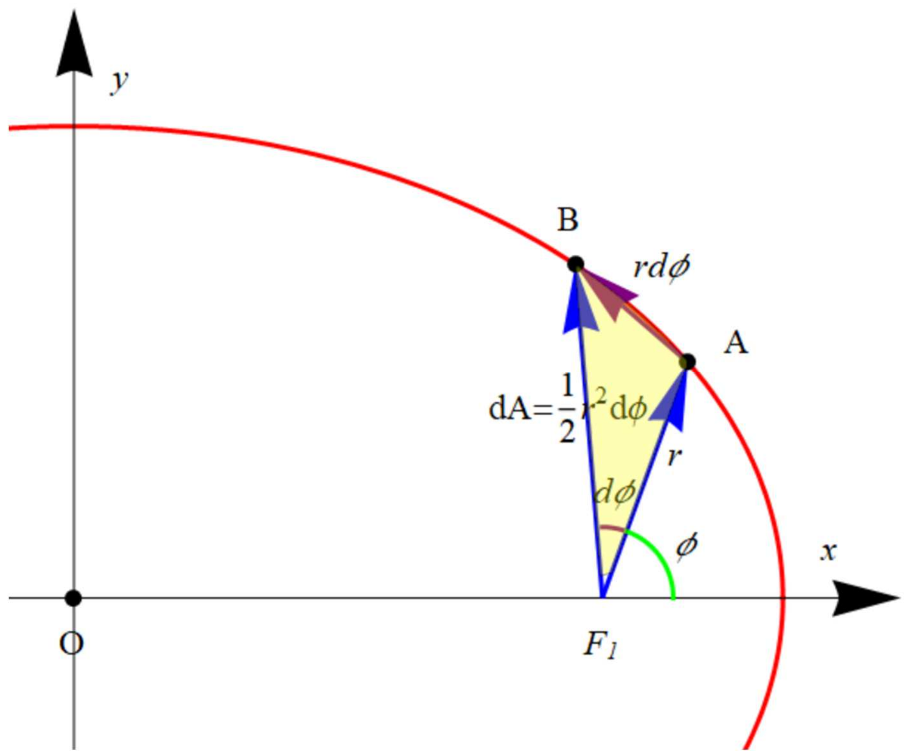


Fig.7 Polar coordinates for the Kepler problem with sun (or proton) at the focus F_1, Areal velocity in a central field. Definition of areal velocity. $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{1}{2} r^2 \dot{\phi}$.

which is independent of time t (Kepler's 2nd law). This law will be discussed later in more detail. The Hamiltonian can be derived from the Lagrangian as

$$
\begin{aligned}
H &= p_r \dot{r} + p_\phi \dot{\phi} - L \\
&= \frac{1}{m} p_r^{\ 2} + \frac{L_z^{\ 2}}{mr^2} - \frac{1}{2m}(p^2 + \frac{L_z^{\ 2}}{2mr^2}) + U(r) \\
&= \frac{1}{2m}(p_r^{\ 2} + \frac{L_z^{\ 2}}{r^2}) + U(r)
\end{aligned}
$$

Equation of motion (Lagrange equation):

$$
m(\ddot{r} - r\dot{\phi}^2) = f(r) = -\frac{d}{dr} U(r) .
$$

or

$$\begin{aligned}
m\ddot{r} &= mr\dot{\phi}^2 - \frac{d}{dr}U(r) \\
&= mr(\frac{L_z}{mr^2})^2 - \frac{d}{dr}U(r) \\
&= \frac{L_z^{\ 2}}{mr^3} - \frac{d}{dr}U(r) \\
&= -\frac{d}{dr}[\frac{L_z^{\ 2}}{2mr^2} + U(r)]
\end{aligned}$$

The effective potential is defined by

$$U_{eff}(r) = \frac{L_z^{\ 2}}{2mr^2} + U(r)\,.$$

The energy conservation law:
Multiplying $\dot{r}$ on both sides, we have

$$m\ddot{r}\dot{r} = -\dot{r}\frac{d}{dr}[\frac{L_z^{\ 2}}{2mr^2} + U(r)]\,,$$

or

$$\frac{d}{dt}(\frac{m}{2}\dot{r}^2) = -\frac{d}{dt}[\frac{L_z^{\ 2}}{2mr^2} + U(r)]\,,$$

or

$$\frac{1}{2}m\dot{r}^2 + \frac{L_z^{\ 2}}{2mr^2} + U(r) = E\,. \qquad \text{(energy eigenvalue; } E < 0 \text{ for the bound state)}$$

For

$$U(r) = -\frac{k}{r}\,,$$

we have

$$\frac{1}{2}m\dot{r}^2 + \frac{L_z^{\ 2}}{2mr^2} - \frac{k}{r} = E\,,$$

where

$$U_{eff} = -\frac{k}{r} + \frac{L_z^{\ 2}}{2mr^2}\,,$$

is the effective potential. Note that $E = -|E| < 0$ (elliptic orbit).

$$\frac{U_{eff}}{U_0} = \frac{1-2x}{2x^2}\,,$$

where $x = r/r_0$, $U_0 = \dfrac{mk^2}{L_z^{\ 2}}$, and $r_0 = \dfrac{L_z^{\ 2}}{mk}$.

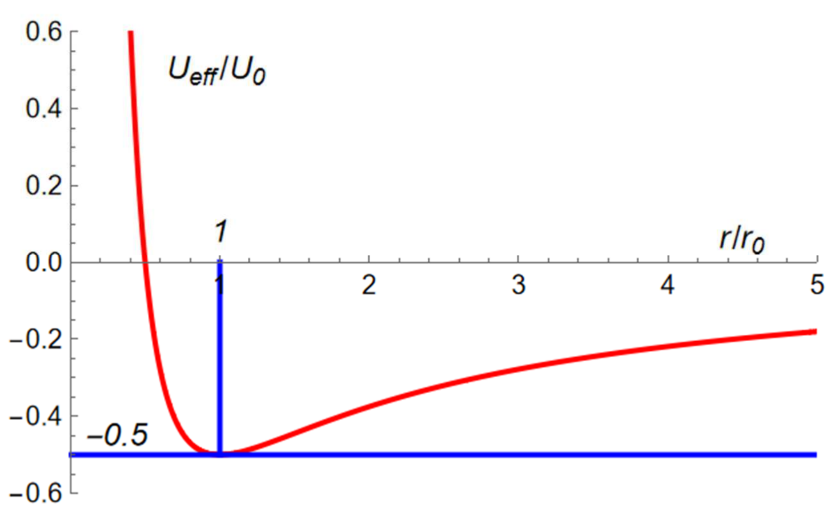


Fig.8 Normalized effective potential in the central force problem. . $y = U_{eff}/U_0$ vs $x = r/r_0$. $U_0 = \dfrac{mk^2}{L_z^{\ 2}}$. $r_0 = \dfrac{L_z^{\ 2}}{mk}$. $E < 0$ for the bound state.

We now solve

$$m\ddot{r} - \frac{L_z^{\ 2}}{mr^3} = -\frac{d}{dr}U(r) = f(r)\,. \qquad \text{(central-force problem)}$$

r depends only on ϕ.

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{L_z}{mr^2}\frac{d}{d\phi}\,,$$

$$\frac{d}{dt}(\frac{d}{dt}) = \frac{L_z}{mr^2}\frac{d}{d\phi}\frac{L_z}{mr^2}\frac{d}{d\phi}\,,$$

$$\frac{L_z}{r^2}\frac{d}{d\phi}[\frac{L_z}{mr^2}\frac{dr}{d\phi}] - \frac{L_z^{\ 2}}{mr^3} = -\frac{d}{dr}U(r) = f(r)\,.$$

We use

$$u = \frac{1}{r}\,.$$

$$\frac{1}{r^2}\frac{\partial r}{\partial\phi} = u^2\frac{\partial}{\partial\phi}\frac{1}{u} = -\frac{\partial u}{\partial\phi}\,.$$

$$\frac{L_z^{\ 2}}{mr^2}\frac{d}{d\phi}[\frac{1}{r^2}\frac{dr}{d\phi}] - \frac{L_z^{\ 2}}{mr^3} = -\frac{d}{dr}U(r) = f(r)\,.$$

$$\frac{L_z^{\ 2}u^2}{m}(\frac{d^2u}{d\phi^2} + u) = -f(\frac{1}{u})\,.$$

Note that

$$f(r) = -\frac{d}{dr}U(r) = \frac{d}{dr}\frac{k}{r} = -\frac{k}{r^2} = -ku^2\,.$$

Thus, we get

$$\frac{L_z^2 u^2}{m}\left(\frac{d^2u}{d\phi^2}+u\right)=ku^2,$$

or

$$\boxed{\frac{d^2u}{d\phi^2}+u=\frac{mk}{L_z^2}}.$$

The solution of this equation (harmonic-oscillator type) is given by

$$u(\phi)=\frac{1+\varepsilon\cos\phi}{a(1-\varepsilon^2)}=\frac{1}{r(\phi)},$$

such that $u(-\theta)=u(\theta)$.

$$\boxed{r(\phi)=\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\phi}=\frac{\Lambda}{1+\varepsilon\cos\phi}}.$$

where ε is the eccentricity of ellipse ($0<\varepsilon<1$) and Λ is the semi-latus rectum and $r=\Lambda$ at $\phi=\frac{\pi}{2}$. Λ is called the semi-latus rectum

$$\Lambda=a(1-\varepsilon^2)=\frac{b^2}{a}=\frac{L_z^2}{mk}.$$

We use the notation ε instead of e, since e is used as a charge. We now consider the energy conservation when $\dot{r}=0$. The perihelion r_{p} (the nearest distance) and aphelion r_{ap} (the farthest distance) are given by

$$r_p=a(1-\varepsilon) \qquad \text{and} \qquad r_{ap}=a(1+\varepsilon).$$

We use semi-major axis a and semi-minor axis b,and $E=-|E|<0$ (the bound state).

$$\frac{L_z^2}{2mr^2}-\frac{k}{r}=-|E|,$$

or

$$|E|r^2 - kr + \frac{L_z^2}{2m} = 0.$$

r_p and r_{ap} $(> r_{ap})$ are the roots of this quadratic equation. Physically, r_p (the perihelion) and r_{ap} (the aphelion) are the distances of the particle from the center (the origin O), when it is at each end of the major axis of the elliptic orbit.

$$r_p + r_{ap} = \frac{k}{|E|} = a(1-\varepsilon) + a(1+\varepsilon) = 2a,$$

$$r_p r_{ap} = \frac{L_z^2}{2m|E|} = a(1-\varepsilon)a(1+\varepsilon) = a^2(1-\varepsilon^2) = a\Lambda,$$

where

$$r_p = a(1-\varepsilon), \quad \text{and} \qquad r_{ap} = a(1+\varepsilon).$$

So that, we have

$$a = \frac{k}{2|E|},$$

and

$$b^2 = a^2(1-\varepsilon^2) = \frac{L_z^2}{2m|E|},$$

or

$$1-\varepsilon^2 = \frac{b^2}{a^2} = \frac{L_z^2}{2m|E|a^2}.$$

6. Feynman hodograph diagram in the Kepler's model (revisited)

We use the same definitions which are discussed in our article. Since the problem is spherically symmetric, the total angular momentum vector, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ around the focus at the point F_1 is conserved. It therefore follows that $\mathbf{r}$ is always perpendicular to the fixed direction of $\mathbf{L}$ in space. This can be true only if $\mathbf{r}$ always lies in a plane whose normal is parallel to $\mathbf{L}$.

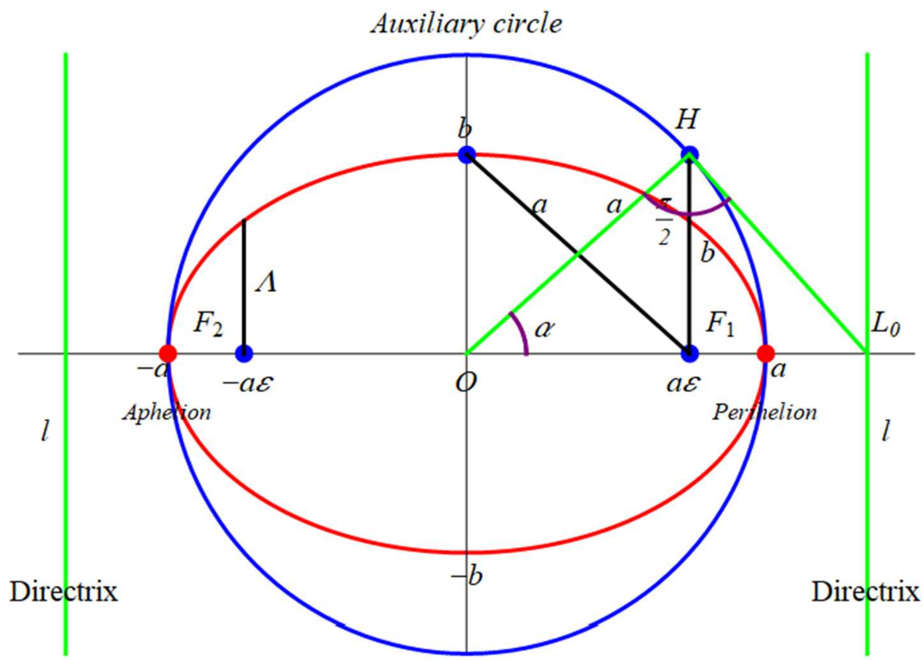


Fig.9 Kepler ellipse with semi-major axis and semi-minor axis; perihelion, aphelion, eccentricity.

Definition of elliptic orbit. The focus is at the point F_1.

O: center.

a: semi-major axis.

b: semi-minor axis.

ε: eccentricity.

l: directrix.

Λ : semi-latus rectum ($\Lambda = \frac{b^2}{a}$).

α: eccentric anomaly

Auxiliary circle (denoted by blue line) of radius a centered at the origin O.

$$a^2 = b^2 + a^2\varepsilon^2 \qquad b^2 = a^2(1-\varepsilon^2)$$

In **Fig.9**, we have the relations

$$\cos\alpha = \frac{a\varepsilon}{a} = \varepsilon\,,$$

$$\overline{OL_0} = \frac{a}{\cos\alpha} = \frac{a}{\varepsilon}\,,$$

$$\begin{aligned}\overline{F_1L_0} &= \frac{a}{\varepsilon} - a\varepsilon \\ &= \frac{a(1-\varepsilon^2)}{\varepsilon} \\ &= \frac{b^2}{a\varepsilon}\end{aligned}$$

We consider triangles $\Delta\mathrm{OHF_1}$ and $\Delta\mathrm{OL_0H}$ which are similar. Their corresponding angles are congruent and corresponding sides are in equal portion.

$$\frac{\overline{\mathrm{OH}}}{\overline{\overline{\mathrm{OL_0}}}} = \frac{\overline{\mathrm{HF_1}}}{\overline{\overline{\mathrm{L_0}H}}} = \frac{\overline{\mathrm{F_1O}}}{\overline{\overline{\mathrm{HO}}}}\,,$$

or

$$\frac{\mathrm{a}}{\overline{\overline{\mathrm{OL_0}}}} = \frac{b}{\overline{\overline{\mathrm{L_0}H}}} = \frac{a\varepsilon}{\mathrm{a}} = \varepsilon\,,$$

yielding the sides of $\Delta\mathrm{OL_0H}$ as

$$\overline{\mathrm{OL_0}} = \frac{\mathrm{a}}{\varepsilon}, \qquad \overline{\mathrm{L_0}H} = \frac{b}{\varepsilon}.$$

Parametric equation of the ellipse:

$$x = a\cos\theta\,, \qquad\qquad y = b\sin\theta\,.$$

The angle θ is called the eccentricity anomaly The circle of radius a centered at the origin is called the **auxiliary circle**.

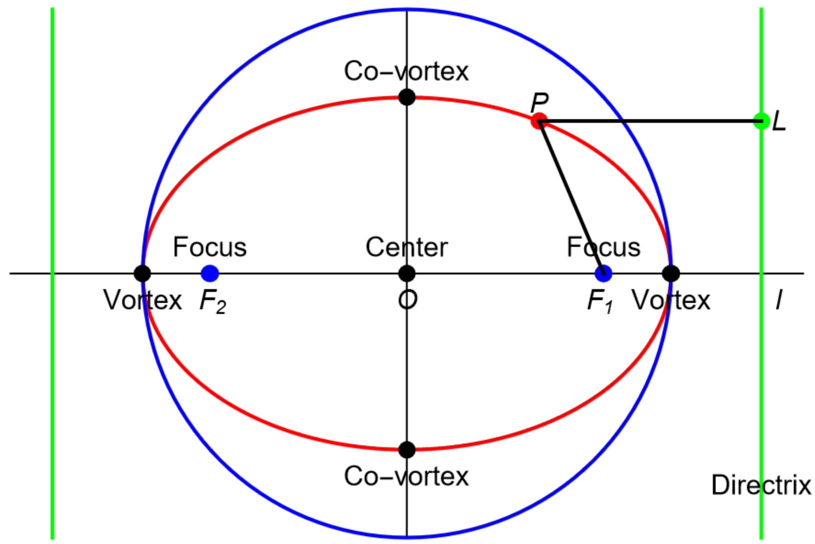


Fig.10 Definition of directrix. Elliptic orbit with $\dfrac{\overline{PF_1}}{\overline{PL}} = \varepsilon$ (definition for the ellipse)

$$\overline{PF_1} = \sqrt{(x-a\varepsilon)^2 + y^2}\ , \qquad \overline{PL} = \left|\frac{a}{\varepsilon} - x\right|.$$

where the particle is located at the position (x, y).

$$\varepsilon\left|\frac{a}{\varepsilon} - x\right| = \sqrt{(x-a\varepsilon)^2 + y^2}\ ,$$

$$\varepsilon^2\left(\frac{a}{\varepsilon} - x\right)^2 = (x-a\varepsilon)^2 + y^2\ ,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{(ellipse)}$$

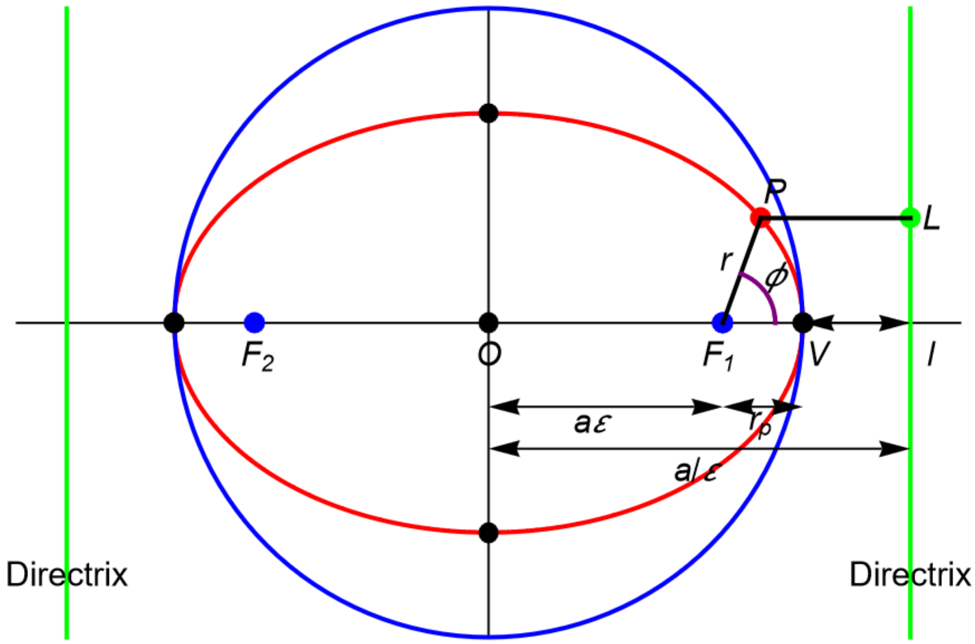


Fig.11 Definition of the directrix in the ellipse.

$$\overline{PL} = \frac{a}{\varepsilon} - (a\varepsilon + r\cos\phi),$$

$$\overline{PF_1} = r,$$

$$\frac{\overline{PF_1}}{\overline{PL}} = \varepsilon,$$

$$\frac{r}{\frac{a}{\varepsilon} - (a\varepsilon + r\cos\phi)} = \varepsilon.$$

The radial distance as s function of azimuthal angle ϕ,

$$r = \frac{b^2(1-\varepsilon^2)}{1+\varepsilon\cos\phi)} = \frac{\Lambda}{1+\varepsilon\cos\phi)}.$$

7. Velocity and angular momentum conservation (J.C. Maxwell): Hodographic solution

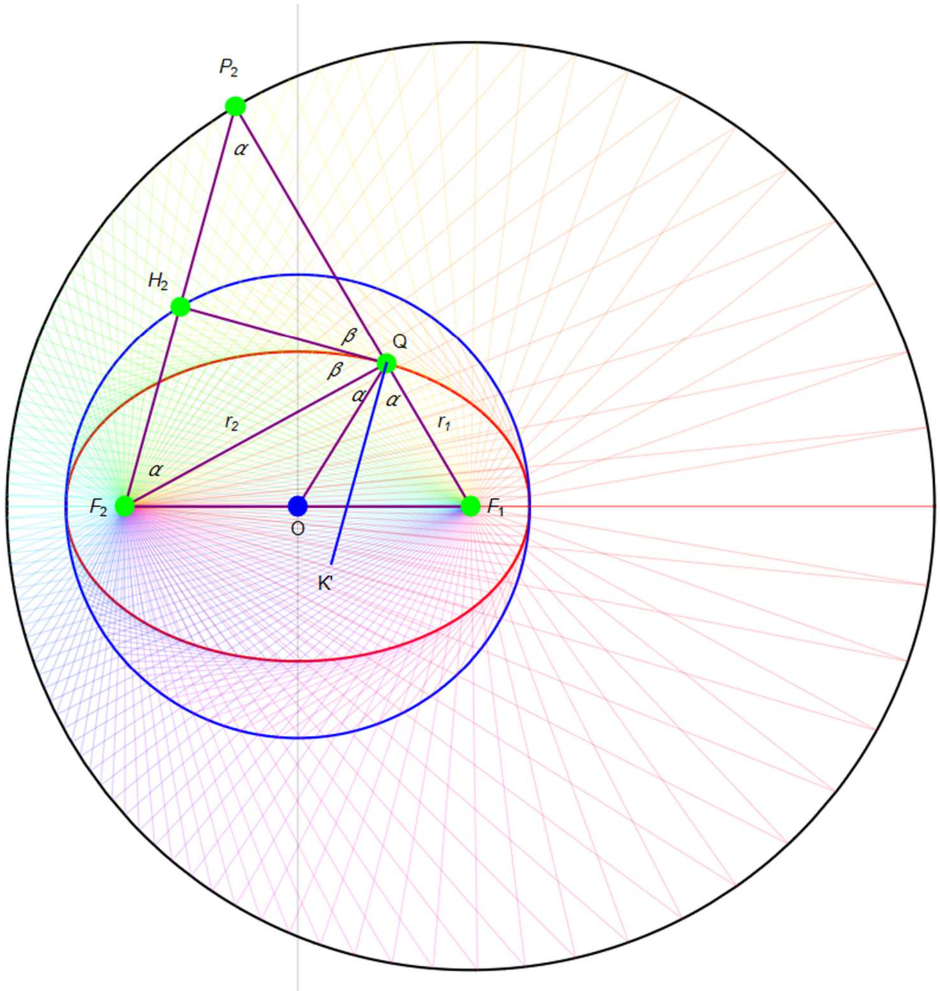


Fig.12 Original Feynman hodograph diagram. Hodographic solution to the Kepler problem (inverse-square law of force). Elliptic orbit. F_1 and F_2 are foci. The point H_2 is on the circle of radius a centered at the origin O. $r_1 + r_2 = r_p + r_{ap} = 2a$ (M.S. Suzuki and Itsuko S. Suzuki). We will discuss much more detail below.

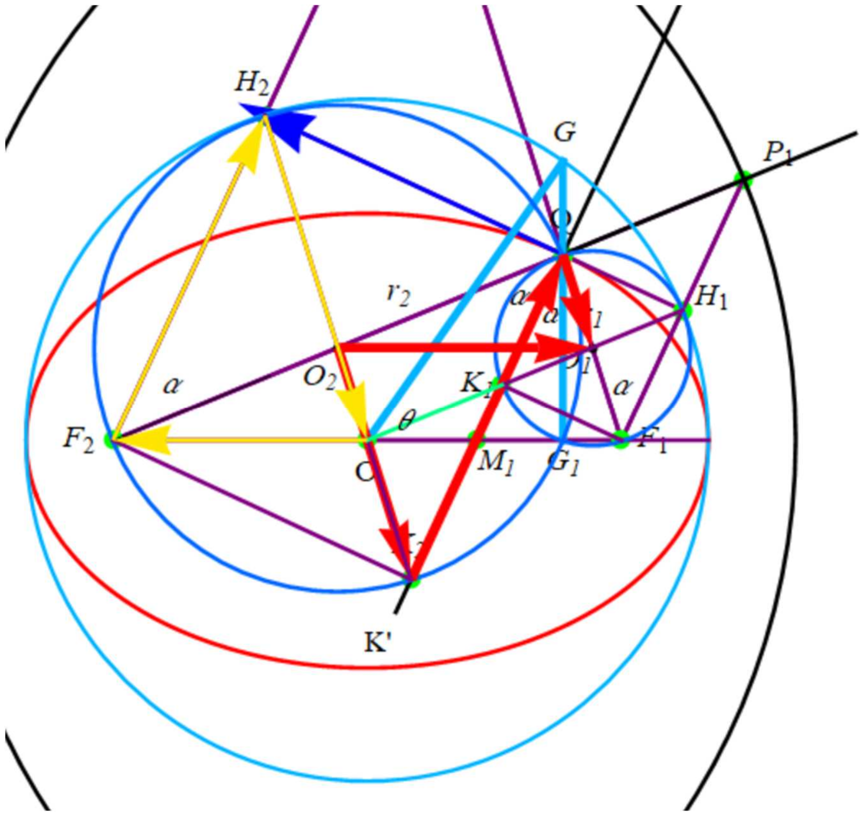


Fig.13 LRL vectors in the Feynman hodograph diagram. The line QG1 is a construction line which plays a significant role in constructing a revised Feynman hodograph diagram. The line $\overline{O_1O_2}$ is parallel to the x axis; $\overline{O_1O_2} = a\varepsilon$. This is independent of the eccentricity anomaly θ.

Here we show that the magnitude of the velocity at the point Q on the elliptic orbit is given by

$$v = \frac{|E|}{L_z}\overline{F_2P_2} \qquad \text{(James C. Maxwell)}$$

where

$$|E| = \frac{k}{2a}, \qquad \frac{L_z^{\ 2}}{2m|E|} = b^2 = a^2(1-\varepsilon^2),$$

or

$$\frac{L_z}{|E|} = \sqrt{\frac{4m}{k} ab^2} = \sqrt{\frac{4}{k_0} ab^2} \,,$$

with $b = a(1-\varepsilon^2)^{1/2}$. Since $\dfrac{|E|}{L_z}$ is conserved, the velocity v is proportional to the distance $\overline{FP}$. The proof is given below. From the property of elliptic orbit, we have

$$r_1 + r_2 = r_p + r_{ap} = a(1-\varepsilon) + a(1+\varepsilon) = 2a\,, \qquad (1)$$

and

$$\angle \mathrm{F_1QK'} = \angle \mathrm{F_2QK'} = \alpha$$

Note that the direction of the velocity at the point Q is tangential to the orbit and is along the direction of the vector $\overrightarrow{\mathrm{QH_2}}$. For the triangle F_2QF_1, we apply the cosine law as

$$(2a\varepsilon)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(2\alpha)\,, \qquad (2)$$

where $\overline{\mathrm{F_1F_2}} = 2a\varepsilon$, $\overline{\mathrm{F_2Q}} = r_2$, $\overline{\mathrm{F_1Q}} = r_1$, and $\angle \mathrm{F_1QF_2} = 2\alpha$. From Eqs.(1) and (2), we get

$$\begin{aligned}4a^2\varepsilon^2 &= (r_1+r_2)^2 - 2r_1r_2 - 2r_1r_2\cos(2\alpha)\\ &= 4a^2 - 2r_1r_2[1+\cos(2\alpha)]\\ &= 4a^2 - 4r_1r_2\cos^2\alpha\end{aligned}$$

or

$$r_1r_2\cos^2\alpha = a^2(1-\varepsilon^2) = b^2\,.$$

We now consider the angular momentum, which is directed along the z direction (perpendicular to the orbital plane). The angular momentum is conserved since the torque is equal to zero

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = 0 .$$

Note that the position vector ***r*** is always parallel to the force ***F*** (the inverse-square law of force). The elliptic orbit lies in the (x, y) plane perpendicular to the angular momentum ***L*** (along the z axis for simplicity). Thus, we have

$$L_z = p_\phi = mvr_1 \sin\beta = mvr_1 \sin(\frac{\pi}{2} - \alpha) = mvr_1 \cos\alpha$$

(angular momentum conserved)

Thus, we get the velocity as

$$\begin{aligned} v &= \frac{L_z}{mr_1 \cos\alpha} \\ &= \frac{L_z}{mr_1 r_2 \cos^2\alpha} r_2 \cos\alpha \\ &= \frac{L_z}{mb^2} r_2 \cos\alpha \\ &= \frac{L_z}{ma^2(1-\varepsilon^2)} r_2 \cos\alpha \end{aligned}$$

Note that that the energy and the angular momentum are conserved,

$$|E| = \frac{k}{2a} = \frac{k_0 m}{2a}, \qquad (E < 0 \text{ for the elliptic orbit})$$

and

$$\begin{aligned} L_z^2 &= 2m|E|b^2 \\ &= 2m|E|a^2(1-\varepsilon^2) \\ &= m^2 k_0 a(1-\varepsilon^2) \quad , \\ &= m^2 k_0 \frac{b^2}{a} \end{aligned}$$

or

$$L_z = m\sqrt{\frac{k_0 b^2}{a}} \; ,$$

where the unit of k is [erg cm], and the unit of k_0 is [cm^3/s^2].
Then we get

$$\begin{aligned} v &= \frac{{L_z}^2}{ma^2(1-\varepsilon^2)L_z} r_2 \cos\alpha \\ &= \frac{2m|E|a^2(1-\varepsilon^2)}{ma^2(1-\varepsilon^2)L_z} r_2 \cos\alpha \\ &= \frac{|E|}{L_z} 2r_2 \cos\alpha \\ &= \frac{|E|}{L_z} \overline{F_2P_2} \end{aligned}$$

or

$$v = \frac{|E|}{L_z}\overline{F_2P_2}, \qquad \text{(J.C. Maxwell)}$$

where

$$\overline{F_2P_2} = 2r_2 \cos\alpha \, .$$

$$\frac{|E|}{L_z} = \sqrt{\frac{k_0}{4\,ab^2}} \; ,$$

$$\Lambda = \frac{b^2}{a}, \qquad (\Lambda\text{: semi-latus rectum})$$

$$|E| = \frac{k}{2a} = \frac{k_0 m}{2a} \, ,$$

$$k_0 = \begin{cases} \dfrac{Ze^2}{m} & \text{(Coulomb)} \\ \\ GM & \text{(Kepler)} \end{cases}$$

The velocity at the perihelion can be evaluated as

$$\overline{F_2P_2} = 2\overline{F_2H_2} = 2a(1+\varepsilon),$$

$$\begin{aligned} v_p &= \frac{|E|}{L_z} 2\overline{F_2P_2} \\ &= \sqrt{\frac{k_0}{4\,ab^2}} 2a(1+\varepsilon) \\ &= \sqrt{\frac{k_0}{a}} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \end{aligned}$$

In the limit of $\varepsilon \to 1$, the velocity v_p may be close to the velocity of light. The velocity at the aphelion can be evaluated as

$$\overline{F_2P_2} = 2\overline{F_2H_2} = 2a(1-\varepsilon),$$

$$\begin{aligned} v_{ap} &= \frac{|E|}{L_z} 2\overline{F_2H_2} \\ &= \sqrt{\frac{k_0}{4\,ab^2}} 2a(1-\varepsilon) \\ &= \sqrt{\frac{k_0}{a}} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \end{aligned}$$

We note that

$$\overrightarrow{F_2H_2} = \mathbf{H}_2 - \mathbf{F}_2 = \left(\frac{a(1-\varepsilon^2)\cos\theta}{1-\varepsilon\cos\theta}, \frac{a\sqrt{1-\varepsilon^2}\sin\theta}{1-\varepsilon\cos\theta}\right).$$

Noting that the direction of the velocity vector is perpendicular to $\overrightarrow{F_2H_2}$, the velocity vector can be expressed by

$$\begin{aligned}\mathbf{v} &= \sqrt{\frac{k_0}{ab^2}}\left(-\frac{a\sqrt{1-\varepsilon^2}\sin\theta}{1-\varepsilon\cos\theta}, \frac{a(1-\varepsilon^2)\cos\theta}{1-\varepsilon\cos\theta}\right) \\ &= \sqrt{\frac{k_0}{a}}\left(-\frac{\sin\theta}{1-\varepsilon\cos\theta}, \frac{\sqrt{1-\varepsilon^2}\cos\theta}{1-\varepsilon\cos\theta}\right)\end{aligned}$$

The magnitude of the velocity is

$$|\mathbf{v}| = \sqrt{\frac{k_0}{a}}\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}}.$$

For the perihelion,

$$|\mathbf{v}| = \sqrt{\frac{k_0}{a}}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}.$$

For the aphelion,

$$|\mathbf{v}| = \sqrt{\frac{k_0}{a}}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}.$$

Note that the velocity at the perihelion is much larger than that at the aphelion when the eccentricity ε tends to unity.

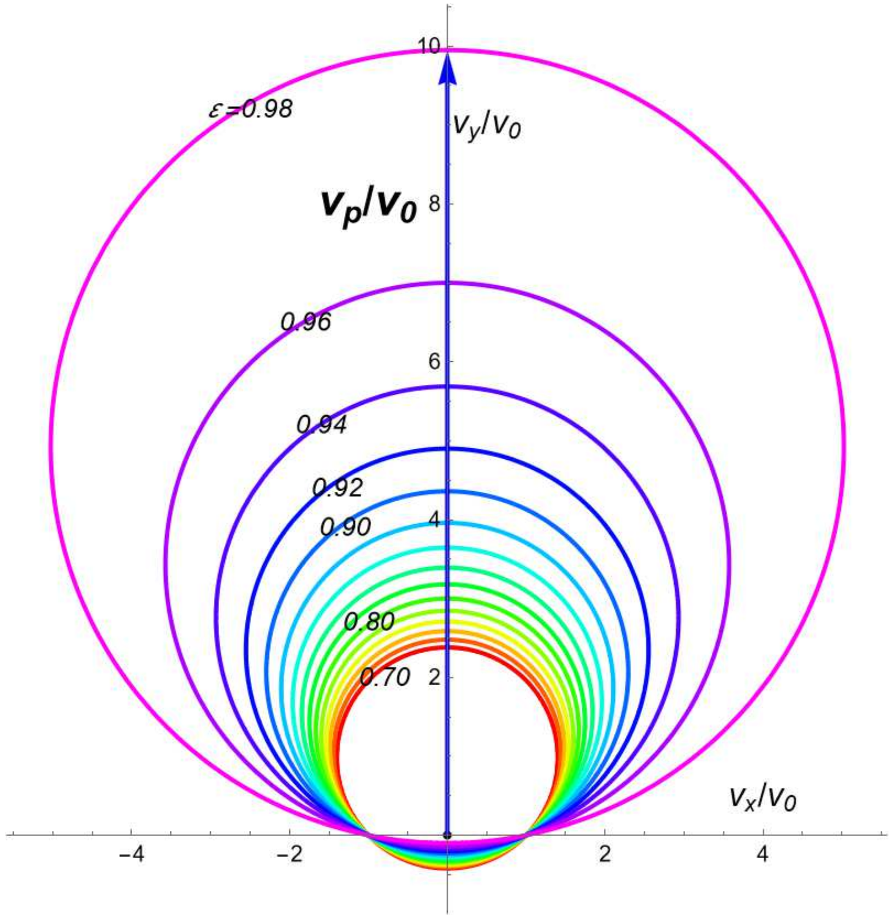


Fig.14 Velocity vector $\mathbf{v}$ (in the units of velocity $v_0 = \sqrt{k_0 / a}$) in the *x*-*y* plane, with the eccentric angle θ being changed as parameter. The eccentricity ε is changed as a parameter. $\varepsilon = 0.98 - 0.70$ with $\Delta\varepsilon = 0.02$.

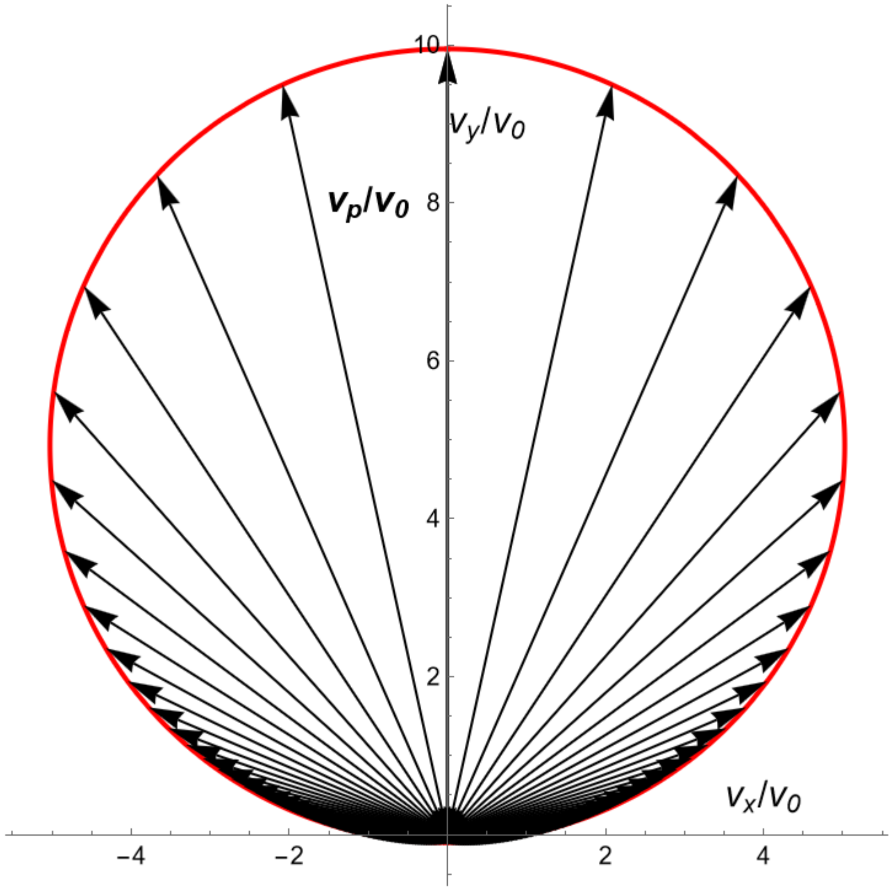


Fig.15 Velocity vector $\mathbf{v}$ (in the units of velocity $v_0 = \sqrt{k_0 / a}$) in the *x*-*y* plane, with the eccentricity anomaly θ being changed as parameter. The origin is the center of the elliptic orbit. The eccentricity is fixed as $\varepsilon = 0.98$. The velocity increases in the vicinity of the perihelion. $\theta = 0° - 355^{\circ}$ with $\Delta\theta = 5^{\circ}$. The velocity direction at $\theta = 0^{\circ}$ coincides with the v_y axis. The center of the velocity circle is $(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})$, and the radius is $\frac{1}{\sqrt{1-\varepsilon^2}}$. in the $(v_x / v_0,\ v_y / v_0)$. The definition of the eccentricity anomaly θ is the same as shown in **Fig.17.**

8. Centripetal acceleration in the central force problem

We start with the expression of the position vector

$$\mathbf{R} = (a\cos\theta, b\sin\theta) = a(\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta).$$

Here we note that

$$\mathbf{R} = (a\varepsilon, 0) + \mathbf{r} \,.$$

From the definition, the velocity $\mathbf{v}_1$ can be obtained as

$$\begin{aligned}
\mathbf{v} &= \dot{\mathbf{r}} \\
&= \dot{\mathbf{R}} \\
&= (-a\sin\theta, b\cos\theta)\dot{\theta} \\
&= a(-\sin\theta, \sqrt{1-\varepsilon^2}\cos\theta)\dot{\theta}
\end{aligned}$$

We use the expression of the velocity obtained above.

$$\begin{aligned}
\mathbf{v} &= \sqrt{\frac{k_0}{a}}\left(-\frac{\sin\theta}{1-\varepsilon\cos\theta}, \frac{\sqrt{1-\varepsilon^2}\cos\theta}{1-\varepsilon\cos\theta}\right) \\
&= \sqrt{\frac{k_0}{a}}(-\sin\theta, \sqrt{1-\varepsilon^2}\cos\theta)\frac{1}{1-\varepsilon\cos\theta}
\end{aligned}$$

Since $\theta = 0$ and $\theta = \pi$, $v_x = 0$, the center of velocity circle is located on the v_y axis, and is calculated as

$$\begin{aligned}
{v_y}^c &= \frac{1}{2}[v_y(\theta = 0) + v_y(\theta = \pi)] \\
&= \frac{1}{2}\sqrt{\frac{k_0}{a}}\sqrt{1-\varepsilon^2}\left(\frac{1}{1-\varepsilon} - \frac{1}{1+\varepsilon}\right) \\
&= \sqrt{\frac{k_0}{a}}\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \\
&= \sqrt{\frac{k_0 a}{b^2}}\varepsilon
\end{aligned}$$

The radius of the velocity circle is

$$v_R = \frac{1}{2}\sqrt{\frac{k_0}{a}}\sqrt{1-\varepsilon^2}(\frac{1}{1-\varepsilon}+\frac{1}{1+\varepsilon})$$

$$= \frac{1}{2}\sqrt{\frac{k_0}{a}}\sqrt{1-\varepsilon^2}(\frac{2}{1-\varepsilon^2})$$

$$= \sqrt{\frac{k_0}{a}}\frac{1}{\sqrt{1-\varepsilon^2}}$$

$$= \sqrt{\frac{k_0 a}{b^2}}$$

9. Derivation of the Kepler's third law from the angular velocity of eccentricity anomaly

((**Kepler's 3rd law**))

The squares of the orbital periods of the planets are directly proportional to the cubes of the semi-major axes of their orbits.

From two equations for ***v*** above, we get the **Kepler's 3rd law (**$T^2 / a^3 = \text{constnt}$**)**

$$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{k_0}{a^3}}\frac{1}{1-\varepsilon\cos\theta},$$

or

$$\sqrt{\frac{k_0}{a^3}}dt = \int_0^\theta (1-\varepsilon\cos\theta)d\theta,$$

or

$$\omega_0 t = \theta - \varepsilon\sin\theta. \qquad \text{(Kepler's equation)}$$

In this equation, θ is called the eccentricity anomaly and $M = \omega_0 t$ is called the mean anomaly.

The period $T = \dfrac{2\pi}{\omega_0}$ is obtained as

$$
\begin{aligned}
T &= \int_0^T dt \\
&= \sqrt{\frac{a^3}{k_0}} \int_0^{2\pi} (1 - \varepsilon \cos\theta) d\theta \\
&= 2\pi \sqrt{\frac{a^3}{k_0}} \\
&= \frac{2\pi}{\omega_0}
\end{aligned}
$$

with

$$
\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{k_0}{a^3}}
$$

Note that we do not use the area of the ellipse ($A = \pi ab$).

((**Note**))

$$
\frac{dA}{dt} = \frac{L_z}{2m} = \frac{1}{2}\sqrt{k_0 \Lambda} ,
$$

$$
\frac{\pi ab}{T} = \frac{1}{2}\sqrt{k_0 \Lambda} ,
$$

$$
T = \frac{2\pi ab}{\sqrt{k_0 \Lambda}} = \frac{2\pi ab}{\sqrt{k_0 \frac{b^2}{a}}} = 2\pi \sqrt{\frac{a^3}{k_0}} ,
$$

$$
\begin{aligned}
\frac{\pi ab}{T} t &= \int dA \\
&= \frac{1}{2} \int_0^{\phi} r^2 d\phi \\
&= \frac{1}{2} \int_0^{\phi} \frac{\Lambda^2}{(1 + \varepsilon \cos\phi)^2} d\phi
\end{aligned}
$$

We use the Mathematica to calculate the integral.

$$\int_0^{\phi}\frac{1}{(1+\varepsilon\cos\phi)^2}d\phi=\frac{1}{(1-\varepsilon^2)^{3/2}}[2\arctan(\frac{(1-\varepsilon)\tan\frac{\phi}{2}}{\sqrt{1-\varepsilon^2}})-\frac{\varepsilon\sqrt{1-\varepsilon^2}\sin\phi}{1+\varepsilon\cos\phi}]$$

which is the same as that given in the book of Marion.

$$\frac{2\pi ab}{T}t=\frac{\Lambda^2}{(1-\varepsilon^2)^{3/2}}[2\arctan(\frac{(1-\varepsilon)\tan\frac{\phi}{2}}{\sqrt{1-\varepsilon^2}})-\frac{\varepsilon\sqrt{1-\varepsilon^2}\sin\phi}{1+\varepsilon\cos\phi}]$$

or

$$\frac{2\pi t}{T}=\omega_0 t=2\arctan(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\phi}{2})-\frac{\varepsilon\sqrt{1-\varepsilon^2}\sin\phi}{1+\varepsilon\cos\phi}\qquad\text{(Kepler's equation)}$$

where

$$\frac{\Lambda^2}{(1-\varepsilon^2)^{3/2}}=ab$$

$$\frac{\theta}{2}=\arctan(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\phi}{2})$$

$$\tan\frac{\theta}{2}=\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\phi}{2}$$

From the definition, we have

$$
\begin{aligned}
\sin\theta &= \frac{r}{b}\sin\phi \\
&= \frac{1}{b}\frac{\Lambda\sin\phi}{1+\varepsilon\cos\phi} \\
&= \frac{b}{a}\frac{\sin\phi}{1+\varepsilon\cos\phi} \\
&= \frac{\sqrt{1-\varepsilon^2}\sin\phi}{1+\varepsilon\cos\phi}
\end{aligned}
$$

10. Derivation of the Kepler's third law from the areal velocity

This law can be derived from the relation between the areal velocity and the angular momentum along the z axis

$$\frac{dA}{dt} = \frac{1}{2m}L_z = \frac{1}{2}\sqrt{k_0 a(1-\varepsilon^2)} = \frac{1}{2}\sqrt{k_0\Lambda}\,.$$

Noting that the total area of the elliptic orbit is $A = \pi ab$, the period T is obtained as

$$
\begin{aligned}
T &= \int_0^T dt \\
&= \frac{2A}{\sqrt{k_0 a(1-\varepsilon^2)}} \\
&= \frac{2\pi ab}{\sqrt{k_0 a(1-\varepsilon^2)}} \\
&= \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{\sqrt{k_0 a(1-\varepsilon^2)}} \\
&= 2\pi\sqrt{\frac{a^3}{k_0}}
\end{aligned}
$$

showing the Kepler's third law.

11. Centripetal acceleration vector $\mathbf{a}_c$

We now calculate the acceleration vector $\mathbf{a}_c$, which should be parallel to $\boldsymbol{r}$ because of the Newton's 2nd law. We use the Mathematica.

$$
\begin{aligned}
\mathbf{a}_c &= \frac{d\mathbf{v}}{dt} \\
&= \frac{d\mathbf{v}_1}{dt} \\
&= \sqrt{\frac{k_0}{a}}\dot{\theta}\left(\frac{\varepsilon - \cos\theta}{(1-\varepsilon\cos\theta)^2}, \frac{-\sqrt{1-\varepsilon^2}\sin\theta}{(1-\varepsilon\cos\theta)^2}\right) \\
&= \frac{k_0}{a^2}\left(\frac{\varepsilon - \cos\theta}{(1-\varepsilon\cos\theta)^3}, \frac{-\sqrt{1-\varepsilon^2}\sin\theta}{(1-\varepsilon\cos\theta)^3}\right)
\end{aligned}
$$

where

$$
\dot{\theta} = \sqrt{\frac{k_0}{a^3}}\frac{1}{1-\varepsilon\cos\theta}.
$$

and

$$
\begin{aligned}
\ddot{\theta} &= \sqrt{\frac{k_0}{a^3}}\frac{\varepsilon\sin\theta}{(1-\varepsilon\cos\theta)^2}\dot{\theta} \\
&= \frac{k_0}{a^3}\frac{\varepsilon\sin\theta}{(1-\varepsilon\cos\theta)^3}
\end{aligned}
$$

We note the position vector is expressed by

$$
\begin{aligned}
\overrightarrow{F_1Q} &= \mathbf{r} \\
&= (a\cos\theta, b\sin\theta) - (a\varepsilon, 0) \\
&= (-a\varepsilon + a\cos\theta, b\sin\theta) \\
&= a(-\varepsilon + \cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)
\end{aligned}
$$

with

$$
\left|\overrightarrow{F_1Q}\right| = r_1 = a(1-\varepsilon\cos\theta)
$$

Thus, we have the relation between **a** and $\overrightarrow{F_1Q}$ as

$$
\begin{aligned}
\mathbf{a}_c &= -\frac{k_0}{a^2}\frac{1}{(1-\varepsilon\cos\theta)^3}(-\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta) \\
&= -\frac{k_0}{a^3}\frac{1}{(1-\varepsilon\cos\theta)^3}\overrightarrow{F_1Q} \\
&= -k_0\frac{\overrightarrow{F_1Q}}{\left|\overrightarrow{F_1Q}\right|^3}
\end{aligned}
$$

So that, $\mathbf{a}_c$ is the centripetal acceleration vector (the unit of a_c is cm/s^2}. We make a ParametricPlot of the acceleration vector a divided by k_0 / a^3

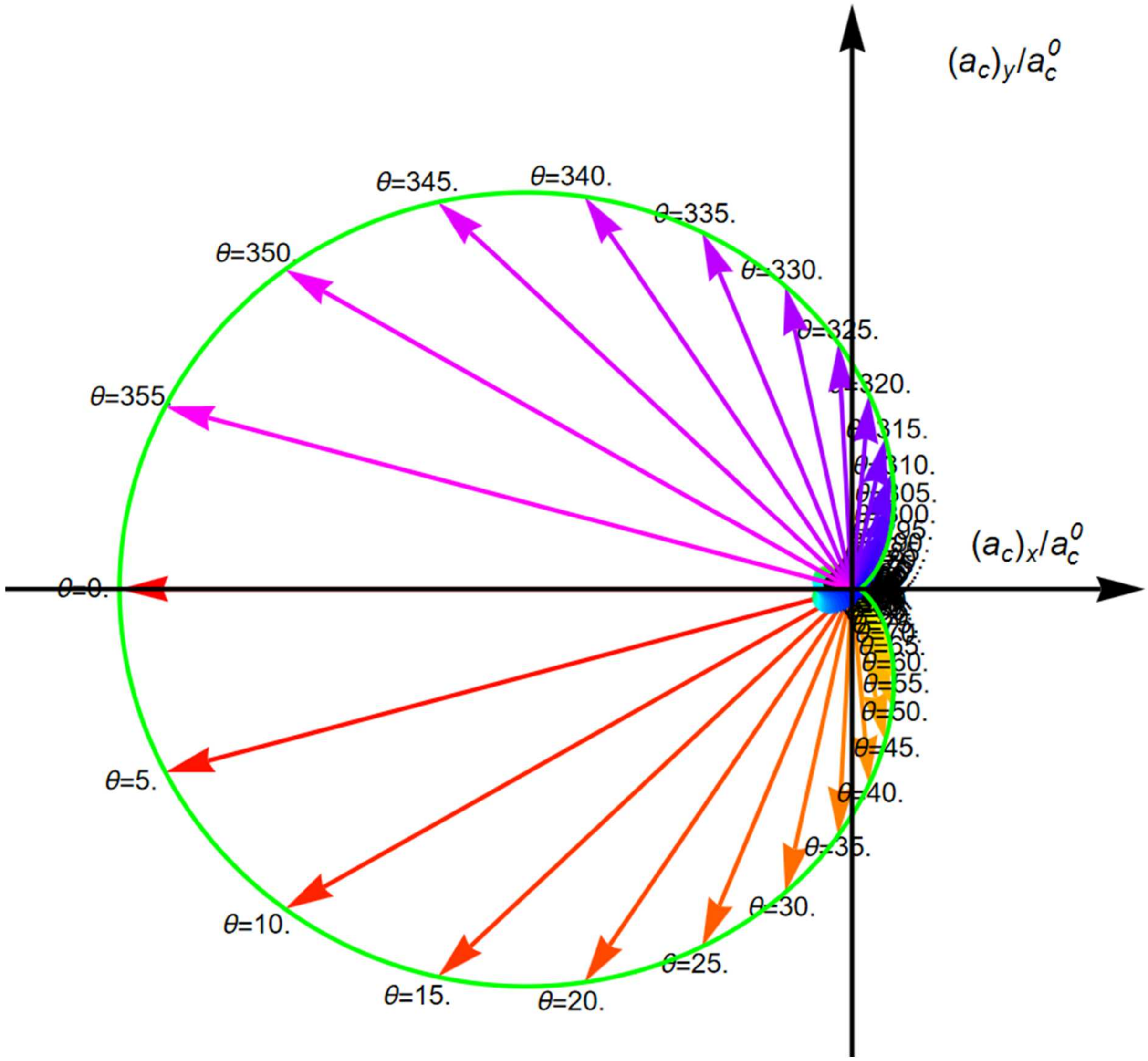
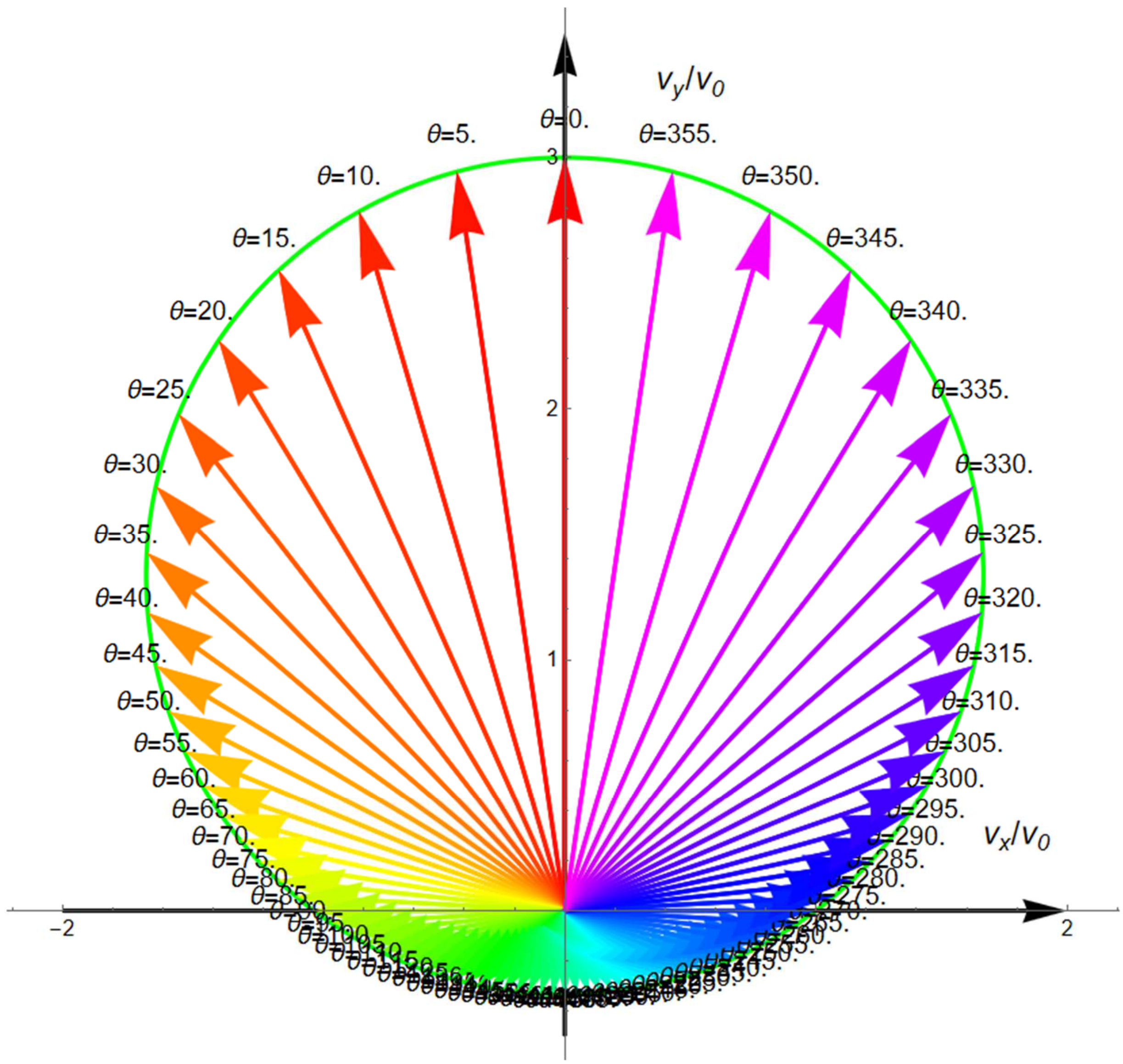


Fig.16 ParametricPlot of $[(\mathbf{a}_c)_x / a_c{}^0, (\mathbf{a}_c)_y / a_c{}^0]$. $\varepsilon = 0.8$. Centripetal acceleration vector where the eccentric angle is changed as a parameter. Centripetal acceleration vector $\mathbf{a}_c$ (in the units of acceleration $a_c{}^0 = \dfrac{k_0}{a^2}$; cm/s^2) in the *x*-*y* plane, with the eccentric angle θ (units of degree) being changed as parameter. $\theta = 0° - 355^{\circ}$ with $\Delta\theta = 5^{\circ}$.



Fig,17 Centripetal acceleration vector $\mathbf{a}_c$ (in the units of acceleration $a_c^{\ 0} = \frac{k_0}{a^2}$) in the x-y plane, with the eccentricity anomaly θ (units of degree) being changed as parameter. The origin is the center of the elliptic orbit. $\varepsilon = 0.8$. $\theta = 0° - 355^{\circ}$ with $\Delta\theta = 5^{\circ}$.

((**Note**))

Unit of acceleration: $a_c = \frac{k_0}{a^2}$ [cm/s^2]

Unit of velocity	$v_0 = \sqrt{k_0 / a}$	(cm/s)
Unit of k_0	k_0	(cm^3/s^2)
Unit of k	$k = k_0 m$	(erg cm)

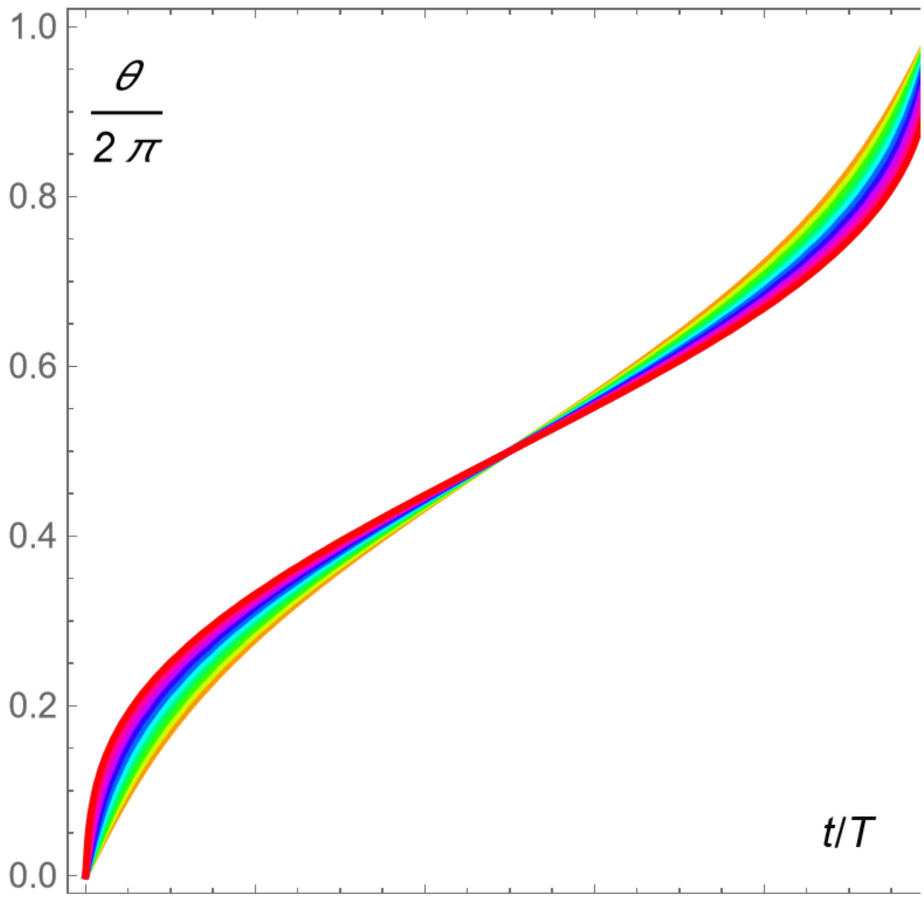


Fig.18 Time dependence of the eccentricity anomaly θ , where ε is changed as a parameter. $\varepsilon = 0.5 - 0.95$ with $\Delta\varepsilon = 0.05$.

12. Kepler's 3rd law (revisited)

$$\frac{d\theta}{dt} = \sqrt{\frac{k_0}{a^3}} \frac{1}{1-\varepsilon\cos\theta},$$

$$t = \sqrt{\frac{a^3}{k_0}} \int_0^\theta (1-\varepsilon\cos\theta) d\theta = \sqrt{\frac{a^3}{k_0}} (\theta - \varepsilon\cos\theta),$$

where

$$T = 2\pi\sqrt{\frac{a^3}{k_0}},$$

$$\frac{t}{T} = \frac{\theta}{2\pi} - \frac{\varepsilon}{2\pi}\sin[2\pi(\frac{\theta}{2\pi})].$$

13. Kepler's equation

((**Mathematica**)) Trajectory of elliptic orbit as s function of time using the Kepler's equation

Plot of the elliptic orbit as a function of t0=t/T, using the Kepler's equation

```
Clear["Global`*"];
a = 3; b = 2.0;
e1 = (√(a^2 - b^2)) / a;
Θ[e11_, t0_] := Module[{s1, e12, eq1, θ1}, e12 = e11;
  s1 = θ - e12 Sin[θ] - 2 π t0;
  eq1 = NSolve[s1 == 0, θ, Reals];
  θ1 = θ /. eq1[[1]]];
X[t0_] := a Cos[Θ[e1, t0]];
Y[t0_] := b Sin[Θ[e1, t0]];
g1[t0_] :=
 Graphics[{PointSize[0.02], Hue[t0], Point[{X[t0], Y[t0]}],
   Arrowheads[0.03], Arrow[{{a e1, 0}, {X[t0], Y[t0]}}]}];
g2 = ParametricPlot[{a Cos[θ], b Sin[θ]}, {θ, 0, 2 π},
  PlotStyle → {Red, Thick}];
Show[g2, Table[g1[t0], {t0, 0, 1, 0.01}]]
```

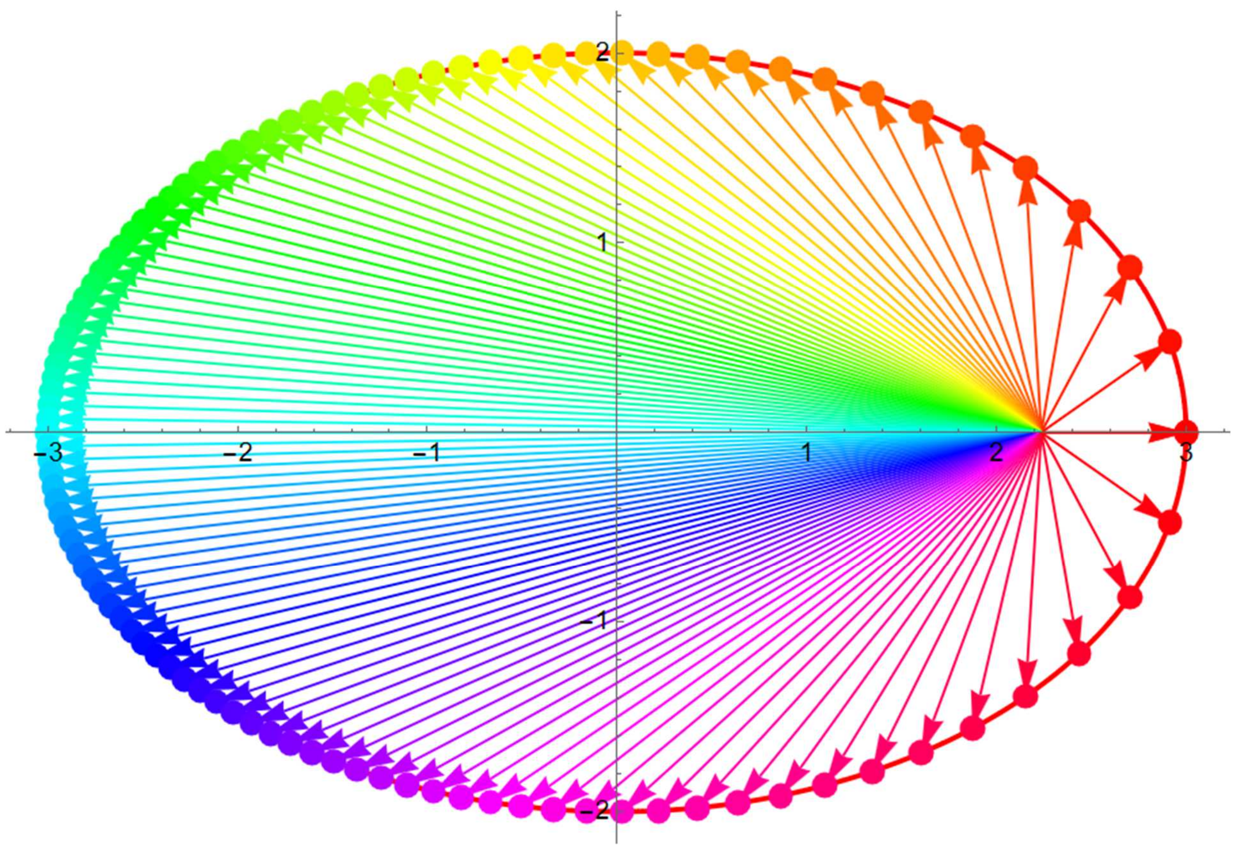


Fig.19 Elliptic orbit with a = 3 and b = 2. $\varepsilon = \sqrt{5}/3 = 0.74436$, where the time $t_0 = t/T$ being changed as a parameter. $t_0 = 0-1$. $\Delta t_0 = 0.01$.

Since

$$\mathbf{R} = (a\cos\theta, b\sin\theta) = a(\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)$$

$$\mathbf{v}_R = \mathbf{v} = \sqrt{\frac{k_0}{a}}(-\sin\theta, \sqrt{1-\varepsilon^2}\cos\theta)\frac{1}{1-\varepsilon\cos\theta}$$

$$\begin{aligned}\mathbf{L}_R &= m(\mathbf{R}\times\mathbf{v}_R) \\ &= m(\mathbf{R}\times\mathbf{v}) \\ &= m\sqrt{\frac{k_0}{a}}\frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}\mathbf{e}_z\end{aligned}$$

and

$$\mathbf{F}_1 \times m\mathbf{v}_1 = m\sqrt{\frac{k_0}{a}}\frac{1}{1-\varepsilon\cos\theta}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a\varepsilon & 0 & 0 \\ -\sin\theta & \sqrt{1-\varepsilon^2}\cos\theta & 0 \end{vmatrix}$$

$$= m\sqrt{\frac{k_0}{a}}\frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}\varepsilon\cos\theta\mathbf{e}_z$$

we have

$$\begin{aligned}
\mathbf{L} &= \mathbf{L}_R - \mathbf{F}_1 \times m\mathbf{v} \\
&= m\sqrt{\frac{k_0}{a}}\frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}\mathbf{e}_z - m\sqrt{\frac{k_0}{a}}\frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}\varepsilon\cos\theta\mathbf{e}_z \\
&= m\sqrt{\frac{k_0}{a}}\frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}\mathbf{e}_z(1-\varepsilon\cos\theta) \\
&= m\sqrt{k_0 a}\sqrt{1-\varepsilon^2}\mathbf{e}_z
\end{aligned}$$

or

$$L_z = m\sqrt{k_0 a(1-\varepsilon^2)} = m\sqrt{k_0\frac{b^2}{a}},$$ (angular momentum is conserved

which is conserved.

13. Laplace-Runge-Lenz vector A

We use the vector $\mathbf{A}$ as

$$\mathbf{A} = \frac{1}{m}(\mathbf{p}\times\mathbf{L}) - \frac{k}{r}\mathbf{r} = \mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r},$$

$$\mathbf{v} = \sqrt{\frac{k_0}{a}}(-\sin\theta, \sqrt{1-\varepsilon^2}\cos\theta)\frac{1}{1-\varepsilon\cos\theta},$$

$$\mathbf{L} = m\sqrt{k_0 a}\sqrt{1-\varepsilon^2}\mathbf{e}_z = m\sqrt{k_0 \frac{b^2}{a}}\mathbf{e}_z = L_z \mathbf{e}_z,$$

So that, we have

$$\begin{aligned}
\mathbf{v}\times\mathbf{L} &= L_z(v_y, -v_x, 0) \\
&= m\sqrt{k_0 \frac{b^2}{a}}\sqrt{\frac{k_0}{a}}\frac{1}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta) \\
&= k_0 m \frac{b}{a}\frac{1}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta) \\
&= k_0 m \frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta)
\end{aligned}$$

$$\mathbf{r} = \mathbf{R} - \mathbf{F}_1 = a(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)$$

$$\begin{aligned}
\frac{k\mathbf{r}}{r} &= \frac{k_0 m a}{a(1-\varepsilon\cos\theta)}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta) \\
&= \frac{k_0 m}{1-\varepsilon\cos\theta}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)
\end{aligned}$$

where

$$L_z = m\sqrt{k_0 \frac{b^2}{a}}_z .$$

Finally, we have

$$
\begin{aligned}
\mathbf{A} &= \mathbf{v}\times\mathbf{L}-\frac{k}{r}\mathbf{r} \\
&= k_0 m\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta,\sin\theta)-\frac{k_0 m}{1-\varepsilon\cos\theta}(\cos\theta-\varepsilon,\sqrt{1-\varepsilon^2}\sin\theta) \\
&= k_0 m\varepsilon\mathbf{e}_x \\
&= k\varepsilon\mathbf{e}_x \\
&= \frac{k}{a}(a\varepsilon\mathbf{e}_x) \\
&= 2\left|E\right|(a\varepsilon\mathbf{e}_x)
\end{aligned}
$$

Thus, the vector A is independent of t. and θ.

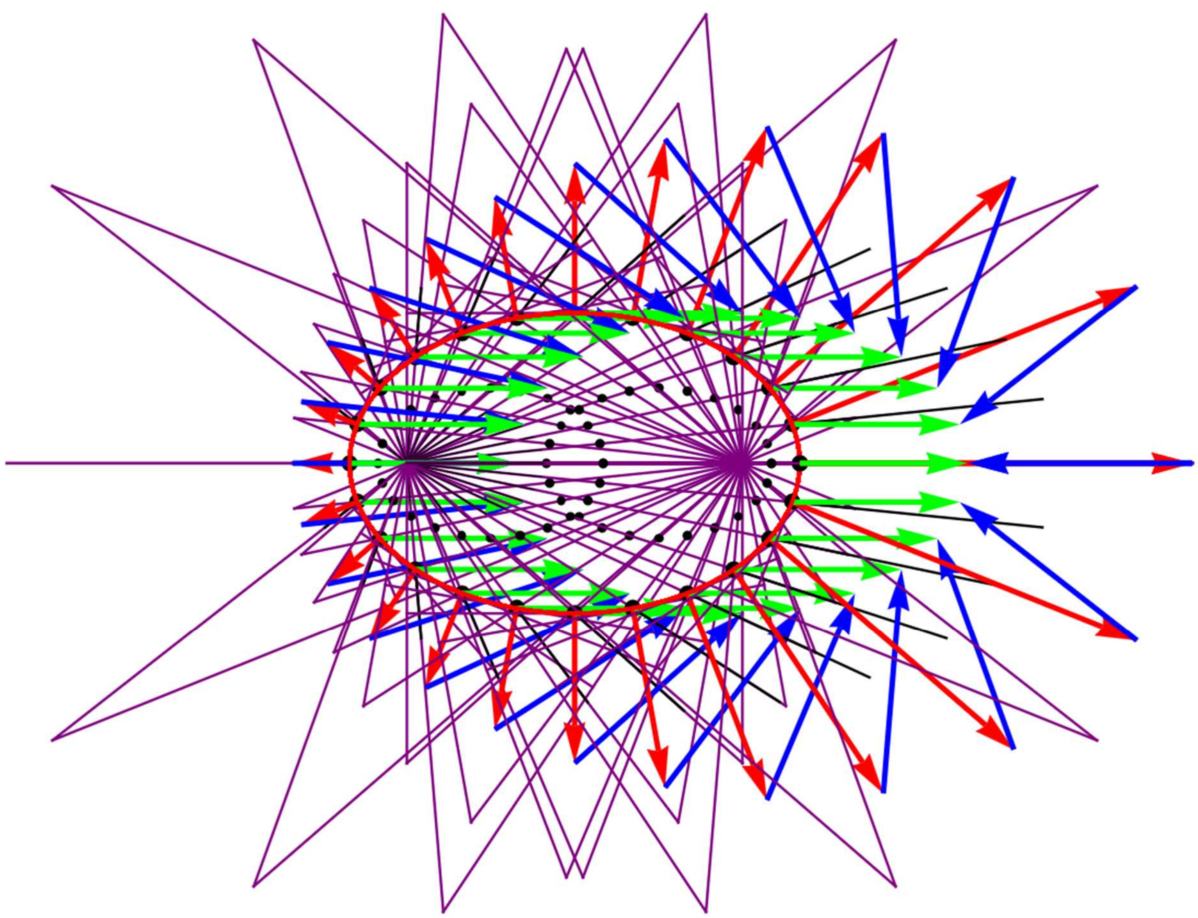


Fig.20 LRL triangles in the Feynman hodograph diagram, where the point F2 of the LRL triangle (ΔH_2F_2O) shifts to the position (Q) of the particle on the elliptic orbit. In the LRL vector, ($\overrightarrow{F_2O}$) is denoted by the green line. ($\overrightarrow{F_2H_2}$) is denoted by the red line. ($\overrightarrow{H_2O}$) is denoted by the blue line. $\left|\overrightarrow{F_2O}\right| = a\varepsilon$.

$$\mathbf{A} = \mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r} = \frac{1}{m}(\mathbf{p}\times\mathbf{L} - \frac{mk}{r}\mathbf{r}),$$

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{A}\cdot &= \mathbf{r}\cdot(\mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r}) \\
&= \mathbf{r}\cdot(\mathbf{v}\times\mathbf{L}) - \frac{k}{r}\mathbf{r}\cdot\mathbf{r} \\
&= \mathbf{r}\cdot(\mathbf{v}\times\mathbf{L}) - kr \\
&= \mathbf{L}\cdot(\mathbf{r}\times\mathbf{v}) - kr \\
&= \frac{1}{m}\mathbf{L}^2 - kr \\
&= \frac{1}{m}L_z^{\ 2} - kr
\end{aligned}$$

Using the relation $\mathbf{A} = k_0 m\varepsilon\mathbf{e}_x = k\varepsilon\mathbf{e}_x$

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{A}\cdot &= k\varepsilon(\mathbf{r}\cdot\mathbf{e}_x) \\
&= \frac{1}{m}\mathbf{L}^2 - kr \\
&= \frac{1}{m}L_z^{\ 2} - kr
\end{aligned}$$

or

$$k\varepsilon r\cos\phi = \frac{1}{m}L_z^{\ 2} - kr,$$

or

$$r = \frac{\frac{L_z^{\ 2}}{mk}}{1+\varepsilon\cos\phi} = \frac{\Lambda}{1+\varepsilon\cos\phi},$$

with

$$\Lambda = \frac{L_z^{\ 2}}{mk} = \frac{1}{mk}\frac{mkb^2}{a} = \frac{b^2}{a}.$$

Note that

$$r_p = \frac{\Lambda}{1+\varepsilon}, \qquad \text{(at } \phi = 0\text{)},$$

$$r_{ap} = \frac{\Lambda}{1-\varepsilon}. \qquad \text{(at } \phi = \pi\text{)}.$$

Using the property of ellipse, we have

$$r_p + r_{ap} = \frac{\Lambda}{1+\varepsilon} + \frac{\Lambda}{1-\varepsilon} = 2a,$$

leading to the relation

$$\Lambda = a(1-\varepsilon^2) = \frac{b^2}{a}.$$

((**Note**))

$$\mathbf{A}\cdot\mathbf{L} = (\mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r})\cdot\mathbf{L} = 0.$$

So that, $\mathbf{A}$ is perpendicular to $\mathbf{L}$ at all points of the motion. It lies in the plane of motion.

14. Laplace-Runge-Lenz law for Kepler's law

((Goldstein))

We start with

$$mr^2\dot{\phi} = L_z\ ,$$

$$d\phi = \frac{L_z}{mr^2}dt\ ,$$

$$\begin{aligned}
\frac{1}{2}m\dot{r}^2 &= -|E| + \frac{k}{r} - \frac{L_z^{\ 2}}{2mr^2} \\
&= -\frac{2mr^2\,|E| - 2mkr + L_z^{\ 2}}{2mr^2} \\
&= \frac{-2m|E|(r-r_p)(r-r_{ap})}{2mr^2} \\
&= \frac{-|E|(r-r_p)(r-r_{ap})}{r^2} \\
&= \frac{k}{2a}\frac{1}{r^2}(r-r_p)(r_{ap}-r)
\end{aligned}$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{k}{ma}}\frac{1}{r}\sqrt{(r-r_p)(r_{ap}-r)}\ ,$$

$$dt = \sqrt{\frac{ma}{k}}\frac{rdr}{\sqrt{(r-r_p)(r_{ap}-r)}}\ ,$$

$$\begin{aligned}
d\phi &= \frac{L_z}{mr^2}dt \\
&= L_z\sqrt{\frac{a}{mk}}\frac{dr}{r\sqrt{(r-r_p)(r_{ap}-r)}}
\end{aligned}$$

$$\phi = L_z\sqrt{\frac{a}{mk}}\int_{r_p}^{r}\frac{dr}{r\sqrt{(r-r_p)(r_{ap}-r)}}\ ,$$

we use $u = \frac{1}{r}$. Since $du = -\frac{1}{r^2}dr = -u^2 dr$, using Mathematica we get

$$\phi = -L_z \sqrt{\frac{a}{mkr_p r_{ap}}} \int_{1/r_p}^{u} \frac{du}{\sqrt{(\frac{1}{r_p} - u)(u - \frac{1}{r_{ap}})}}$$

$$= -\int_{1/r_p}^{u} \frac{du}{\sqrt{(\frac{1}{r_p} - u)(u - \frac{1}{r_{ap}})}}$$

$$= 2\arctan\left(\frac{\sqrt{\frac{1}{r_p} - u}}{\sqrt{u - \frac{1}{r_{ap}}}} \right)$$

where

$$L_z \sqrt{\frac{a}{mkr_p r_{ap}}} = m\sqrt{\frac{k_0 b^2}{a}} \sqrt{\frac{a}{m^2 k_0 r_p r_{ap}}}$$

$$= \sqrt{\frac{b^2}{a}} \sqrt{\frac{a}{a^2(1-\varepsilon^2)}}$$

$$= 1$$

$$L_z = m\sqrt{\frac{k_0 b^2}{a}},$$

$$k = mk_0,$$

$$\tan^2(\frac{\phi}{2}) = \frac{\frac{1}{r_p} - u}{u - \frac{1}{r_{ap}}},$$

or

$$\tan^2(\frac{\phi}{2})+1=\frac{\frac{1}{r_p}-u}{u-\frac{1}{r_{ap}}}+1$$

$$=\frac{\frac{1}{r_p}-u+u-\frac{1}{r_{ap}}}{u-\frac{1}{r_{ap}}}$$

$$=\frac{\frac{r_{ap}-r_p}{r_p r_{ap}}}{u-\frac{1}{r_{ap}}}$$

leading to the relation

$$\frac{1}{\cos^2\frac{\phi}{2}}=\frac{\frac{2\varepsilon}{a(1-\varepsilon^2)}}{u-\frac{1}{r_{ap}}},$$

or

$$u=\frac{1}{r_{ap}}+\frac{2\varepsilon}{a(1-\varepsilon^2)}\cos^2\frac{\phi}{2}$$

$$=\frac{1}{a(1+\varepsilon)}+\frac{\varepsilon}{a(1-\varepsilon^2)}(1+\cos\phi)$$

$$=\frac{1}{a(1-\varepsilon^2)}(1+\varepsilon\cos\phi)$$

or

$$r=\frac{\Lambda}{1+\varepsilon\cos\phi},$$

with

$$\Lambda = a(1-\varepsilon^2) = \frac{b^2}{a}.$$

15. Relation between the azimuthal angle ϕ and the eccentric angle θ

$$r = a(1-\varepsilon\cos\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\phi}.$$

$$\begin{aligned}\cos\theta &= 2\cos^2\frac{\theta}{2} - 1\\ &= \frac{2}{1+\tan^2\frac{\theta}{2}} - 1\\ &= \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\end{aligned}$$

$$1-\varepsilon\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) = \frac{1-\varepsilon^2}{1+\varepsilon\frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}},$$

or

$$\frac{1+\tan^2\frac{\theta}{2} - \varepsilon(1-\tan^2\frac{\theta}{2})}{1+\tan^2\frac{\theta}{2}} = \frac{(1-\varepsilon^2)(1+\tan^2\frac{\phi}{2})}{1+\tan^2\frac{\phi}{2} + \varepsilon(1-\tan^2\frac{\phi}{2})}.$$

Using Mathematica, we get the relation

$$\tan\frac{\phi}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\theta}{2}.$$

Thus, we have a unique correspondence between θ and ϕ.

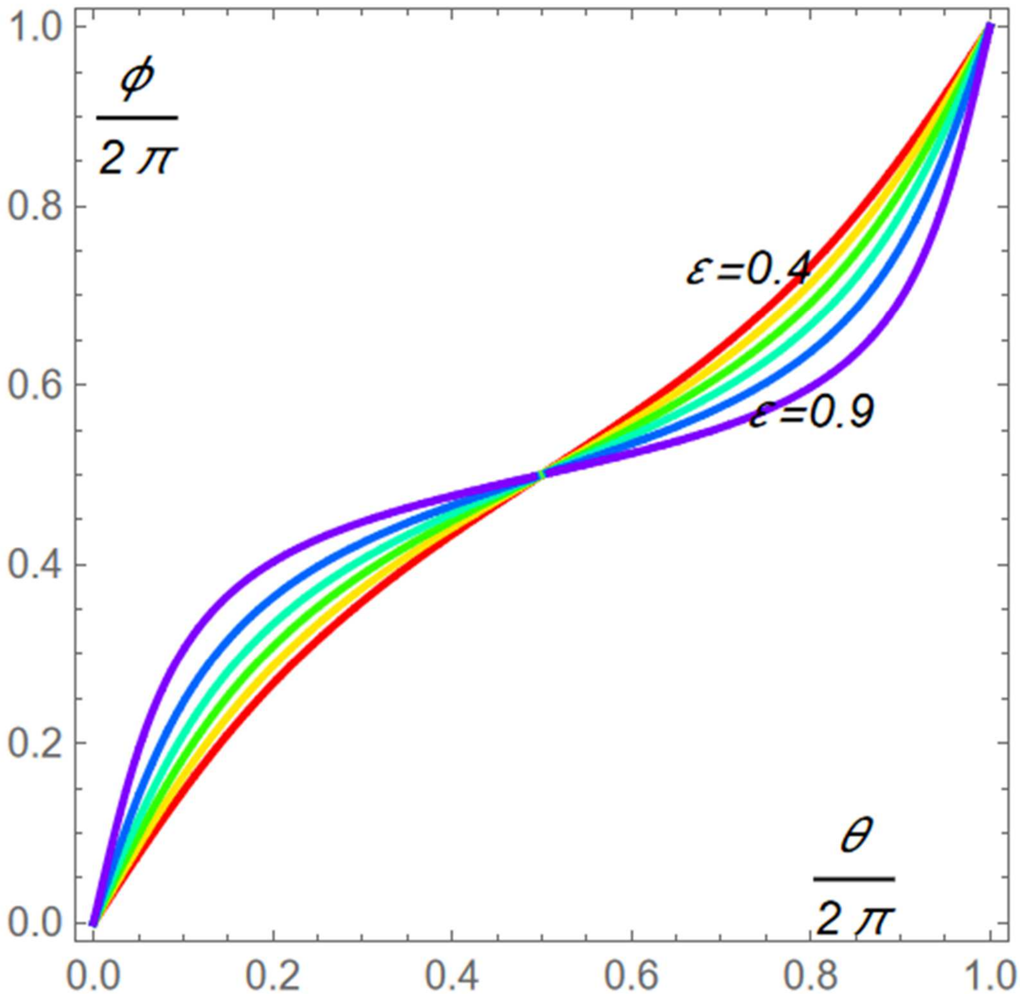


Fig. 21 Relation between the azimuthal angle ϕ and eccentric anomaly θ. The eccentricity ε is changed as a parameter. $\varepsilon = 0.4 - 0.9$ with $\Delta\varepsilon = 0.1$.

16. The Laplace-Runge-Lenz vector; $\frac{d\mathbf{A}}{dt} = 0$

((**Goldstein**))

The Laplace Runge-Lentz vector provides still another way of deriving the orbit equation for the Kepler problem. The vector ***A*** is in the direction of the radius vector to the perihelion point on the orbit, and has a magnitude $k_0 m\varepsilon$. For the Kepler problem we have identified two vector constants of the motion ***L*** and ***A***, and a scalar *E*.

We take a derivative of $\mathbf{A}$ with respect to *t*,

$$\mathbf{A} = \mathbf{v} \times \mathbf{L} - \frac{k}{r}\mathbf{r}\,.$$

The derivative $\dfrac{d\mathbf{A}}{dt}$;

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \frac{d\mathbf{v}}{dt} \times \mathbf{L} + \mathbf{v} \times \frac{d\mathbf{L}}{dt} - \frac{k}{r}\frac{d\mathbf{r}}{dt} + \frac{k}{r^2}\frac{dr}{dt}\mathbf{r} \\
&= \frac{1}{m}\mathbf{F} \times \mathbf{L} + \mathbf{v} \times \boldsymbol{\tau} - \frac{k}{r}\frac{d\mathbf{r}}{dt} + \frac{k}{r^2}(\mathbf{v} \cdot \frac{\mathbf{r}}{r})\mathbf{r}
\end{aligned}$$

where the torque is given by

$$\begin{aligned}
\boldsymbol{\tau} &= \frac{d\mathbf{L}}{dt} \\
&= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\
&= m(\mathbf{v} \times \mathbf{v}) + (\mathbf{r} \times \mathbf{F}) \\
&= \mathbf{r} \times \mathbf{F}
\end{aligned}$$

with

$$\begin{aligned}
\frac{1}{m}\mathbf{F} \times \mathbf{L} &= \mathbf{F} \times (\mathbf{r} \times \mathbf{v}) \\
&= (\mathbf{F} \cdot \mathbf{v})\mathbf{r} - (\mathbf{F} \cdot \mathbf{r})\mathbf{v}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} \times \boldsymbol{\tau} &= \mathbf{v} \times (\mathbf{r} \times \mathbf{F}) \\
&= (\mathbf{v} \cdot \mathbf{F})\mathbf{r} - (\mathbf{v} \cdot \mathbf{r})\mathbf{F}
\end{aligned}$$

$$\mathbf{F} = -\frac{k}{r^3}\mathbf{r}\,. \qquad \text{(central force)}$$

So that, we have

$$
\begin{aligned}
\frac{d\mathbf{A}}{dt} &= (\mathbf{F}\cdot\mathbf{v})\mathbf{r} - (\mathbf{F}\cdot\mathbf{r})\mathbf{v} + (\mathbf{v}\cdot\mathbf{F})\mathbf{r} - (\mathbf{v}\cdot\mathbf{r})\mathbf{F} - \frac{k}{r}\frac{d\mathbf{r}}{dt} + \frac{k}{r^2}(\mathbf{v}\cdot\frac{\mathbf{r}}{r})\mathbf{r} \\
&= -(\mathbf{F}\cdot\mathbf{r})\mathbf{v} + (\mathbf{v}\cdot\mathbf{F})\mathbf{r} - \frac{k}{r}\mathbf{v} + \frac{k}{r^3}(\mathbf{v}\cdot\mathbf{r})\mathbf{r} \\
&= \frac{k}{r}\mathbf{v} - \frac{k}{r^3}(\mathbf{v}\cdot\mathbf{r})\mathbf{r} - \frac{k}{r}\mathbf{v} + \frac{k}{r^3}(\mathbf{v}\cdot\mathbf{r})\mathbf{r} \\
&= 0
\end{aligned}
$$

which means that $\boldsymbol{A}$ is independent of time t.

17. Equation of velocity circle

We start with

$$\frac{k}{r}\mathbf{r} = \mathbf{v}\times\mathbf{L} - \mathbf{A}\,.$$

Thus, we get

$$\frac{k}{r}\mathbf{r}\cdot\frac{k}{r}\mathbf{r} = (\mathbf{v}\times\mathbf{L} - \mathbf{A})\cdot(\mathbf{v}\times\mathbf{L} - \mathbf{A})\,,$$

or

$$m^2 k_0^{\ 2} = (L_z v_y - m k_0 \varepsilon)^2 + L_{z^2} v_x^{\ 2}\,,$$

or

$$(v_y - \frac{m k_0 \varepsilon}{L_z})^2 + v_x^{\ 2} = (\frac{m k_0}{L_z})^2\,, \qquad \text{(velocity circle)}$$

with

$$\frac{m k_0}{L_z} = \sqrt{\frac{k_0 a}{b^2}}\,.$$

This is in agreement with the equation of velocity circle which is derived in different approach (described above).

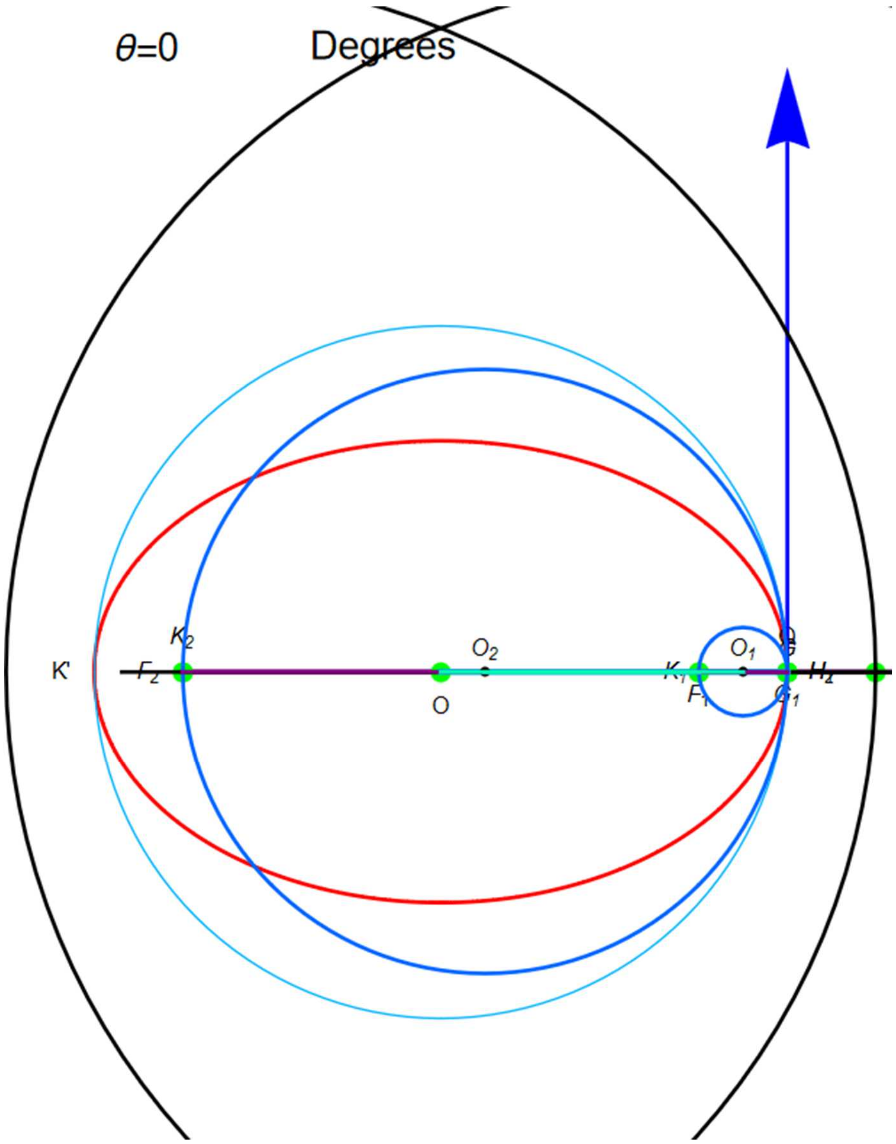


Fig.22 Feynman hodograph diagram with $\theta = 0$. The direction of the velocity at the perihelion. The magnitude of the velocity becomes the largest one.

At $\theta = 0$, we have

$$\mathbf{v} \times \mathbf{L} = k_0 m(1+\varepsilon, 0),$$

$$\frac{k}{r}\mathbf{r} = mk_0(1,0).$$

So that, we have

$$\begin{aligned}\mathbf{A} &= \mathbf{v}\times\mathbf{L} - \frac{k\mathbf{r}}{r} \\ &= mk_0(1+\varepsilon,0) - mk_0(1,0) \\ &= (mk_0\varepsilon,0)\end{aligned}$$

We note that

$$\mathbf{O}_1 = \frac{1}{2}a(\cos\theta+\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta),$$

and

$$\mathbf{O}_2 = \frac{1}{2}a(\cos\theta-\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta).$$

Thus, we have

$$\mathbf{O}_1 - \mathbf{O}_2 = (a\varepsilon,0),$$

$$\mathbf{O}_1 + \mathbf{O}_2 = \mathbf{Q} = (a\cos\theta, b\sin\theta).$$

We may conclude that the vector $\mathbf{O}_1 - \mathbf{O}_2 = (a\varepsilon,0)$ is independent of the eccentric anomaly θ and is related as

$$\mathbf{A} = \frac{mk_0}{a}(a\varepsilon,0) = \frac{mk_0}{a}(\mathbf{O}_1 - \mathbf{O}_2).$$

Note that the position vector $\mathbf{O}_1 - \mathbf{O}_2$ is parallel to the x axis, independent of the eccentricity anomaly θ.

((Determination of the energy from the Laplace-Runge-Lenz equation))

$$\mathbf{A}^2 = (\mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r}).(\mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r})$$

$$= (\mathbf{v}\times\mathbf{L})\cdot(\mathbf{v}\times\mathbf{L}) - 2\frac{k}{r}\mathbf{r}.(\mathbf{v}\times\mathbf{L}) + k^2$$

$$= \mathbf{v}^2\mathbf{L}^2 - (\mathbf{v}.\mathbf{L})^2 - 2\frac{k}{r}\mathbf{L}.(\mathbf{r}\times\mathbf{v}) + k^2$$

$$= (\mathbf{v}^2 - 2\frac{k}{mr})\mathbf{L}^2 + k^2$$

$$= \frac{2E}{m}\mathbf{L}^2 + k^2$$

where $\mathbf{v}.\mathbf{L} = 0$, and

$$\mathbf{v}^2 - \frac{2k}{mr} = 2\frac{E}{m}$$

So that, we have

$$E = \frac{m}{2L_z^{\ 2}}(\mathbf{A}^2 - k^2)$$

$$= -\frac{mk^2}{2L_z^{\ 2}}(1-\varepsilon^2)$$

$$= -\frac{mk^2}{2mk\Lambda}(1-\varepsilon^2)$$

$$= -\frac{k}{2\Lambda}(1-\varepsilon^2)$$

where

$$L_z^{\ 2} = mk\Lambda .$$

$$E = -\frac{ka}{2b^2}(1-\varepsilon^2) = -\frac{k}{2a} .$$

We also note that

$$
\begin{aligned}
\mathbf{A}.\mathbf{L} &= (\mathbf{v}\times\mathbf{L})\cdot\mathbf{L} - \frac{k}{r}\mathbf{r}\cdot\mathbf{L} \\
&= -- \frac{k}{r}\mathbf{r}\cdot(\mathbf{r}\times\mathbf{p}) \\
&= 0
\end{aligned}
$$

K_1Q_1

$$
\begin{aligned}
\mathbf{K}_1 &= \mathbf{F}_1 + (\mathbf{Q}-\mathbf{H}_1) \\
&= \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{K_1Q} &= \mathbf{Q}-\mathbf{K}_1 \\
&= -\mathbf{F}_1 + \mathbf{H}_1 \\
&= (a\cos\theta, a\sqrt{1-\varepsilon^2}\sin\theta) - \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta) + \\
&= \frac{a\sqrt{1-\varepsilon^2}}{1+\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{K_2Q} &= \mathbf{Q}-\mathbf{K}_2 \\
&= \frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}\times\mathbf{L} &= k_0 m\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta) \\
&= \frac{k_0 m}{a}\overrightarrow{K_2Q}
\end{aligned}
$$

$$
\mathbf{O}_2 = \frac{1}{2}a(\cos\theta-\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta),
$$

$$
\begin{aligned}
\frac{k}{r}\mathbf{r} &= \frac{k_0 m}{a(1-\varepsilon\cos\theta)}(-a\varepsilon + a\cos\theta, b\sin\theta \\
&= \frac{k_0 m}{1-\varepsilon\cos\theta}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta) \\
&= \frac{k_0 m}{a}(\overrightarrow{\mathrm{O_1Q}} + \overrightarrow{\mathrm{K_2O_2}})
\end{aligned}
$$

where

$$
\begin{aligned}
\overrightarrow{\mathrm{O_1Q}} + \overrightarrow{\mathrm{K_2O_2}} &= (\mathbf{Q}-\mathbf{O}_1) + (\mathbf{O}_2 - \mathbf{K}_2) \\
&= \frac{a}{1-\varepsilon\cos\theta}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A} &= \mathbf{v}\times\mathbf{L} - \frac{k}{r}\mathbf{r} \\
&= \frac{k_0 m}{a}(\mathbf{Q}-\mathbf{K}_2) - \frac{k_0 m}{a}(\mathbf{Q}-\mathbf{O}_1 + \mathbf{O}_2 - \mathbf{K}_2) \\
&= \frac{k_0 m}{a}(\mathbf{O}_1 - \mathbf{O}_2) \\
&= \frac{k_0 m}{a} a\varepsilon\mathbf{e}_x
\end{aligned}
$$

$$
\mathbf{K}_2 = \frac{a\varepsilon\cos\theta}{1-\varepsilon\cos\theta}(\varepsilon - \cos\theta, -\sqrt{1-\varepsilon^2}\sin\theta),
$$

$$
\begin{aligned}
\mathbf{K}_1 - \mathbf{K}_2 &= \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon + \cos\theta, \sqrt{1-\varepsilon^2}\sin\theta) \\
&\quad - \frac{a\varepsilon\cos\theta}{1-\varepsilon\cos\theta}(\varepsilon - \cos\theta, -\sqrt{1-\varepsilon^2}\sin\theta) \\
&= \frac{2a\varepsilon\sqrt{1-\varepsilon^2}\cos\theta}{1-\varepsilon^2\cos^2\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta)
\end{aligned}
$$

$$
\mathbf{Q} - \mathbf{K}_1 = \frac{a\sqrt{1-\varepsilon^2}}{1+\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta),
$$

$$\mathbf{Q}-\mathbf{K}_2=\frac{a\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta,\sin\theta)\,,$$

$$\mathbf{v}\times\mathbf{L}=\frac{k_0 m}{a}\overrightarrow{K_2Q}\,,$$

$$\overrightarrow{K_2H_2}=\mathbf{H}_2-\mathbf{K}_2=a\left(\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}\right)(\cos\theta-\varepsilon,\sqrt{1-\varepsilon^2}\sin\theta)\,,$$

$$\begin{aligned}\frac{k}{r}\mathbf{r}&=\frac{k_0 m}{1-\varepsilon\cos\theta}(-\varepsilon+\cos\theta,\sqrt{1-\varepsilon^2}\sin\theta)\\&=\frac{k_0 m}{a(1+\varepsilon\cos\theta)}\overrightarrow{K_2H_2}\end{aligned}$$

$$\begin{aligned}\overrightarrow{K_2O_2}&=\mathbf{O}_2-\mathbf{K}_2\\&=\frac{a}{2}\left(\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}\right)(\cos\theta-\varepsilon,\sqrt{1-\varepsilon^2}\sin\theta)\end{aligned}$$

$$\begin{aligned}\frac{k}{r}\mathbf{r}&=\frac{k_0 m}{1-\varepsilon\cos\theta}(-\varepsilon+\cos\theta,\sqrt{1-\varepsilon^2}\sin\theta)\\&=\frac{2k_0 m}{a(1+\varepsilon\cos\theta)}\overrightarrow{K_2O_2}\end{aligned}$$

$$\mathbf{v}\times\mathbf{L}=\frac{k_0 m}{1-\varepsilon\cos\theta}((1-\varepsilon^2)\cos\theta,\sqrt{1-\varepsilon^2}\sin\theta)\,,$$

$$\begin{aligned}\mathbf{A}&=\mathbf{v}\times\mathbf{L}-\frac{k}{r}\mathbf{r}\\&=\frac{k_0 m}{1-\varepsilon\cos\theta}((1-\varepsilon^2)\cos\theta,\sqrt{1-\varepsilon^2}\sin\theta)-\frac{k_0 m}{1-\varepsilon\cos\theta}(-\varepsilon+\cos\theta,\sqrt{1-\varepsilon^2}\sin\theta)\\&=\frac{k_0 m}{1-\varepsilon\cos\theta}(\varepsilon(1-\varepsilon\cos\theta),0)\\&=k_0 m\varepsilon(1,0)\end{aligned}$$

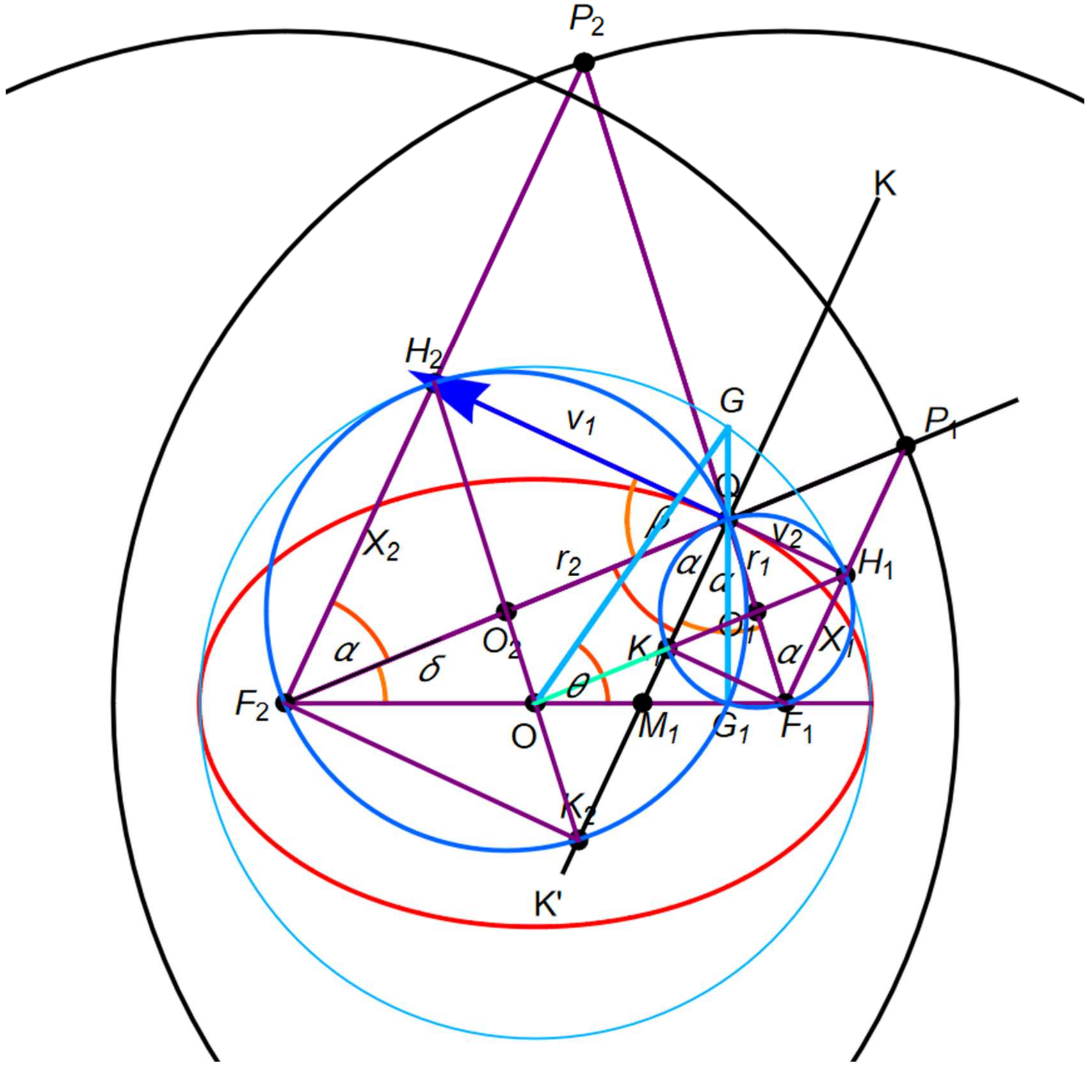


Fig.23 Detail of the Feynman hodograph diagram.

18. Geometry of Feynman hodograph diasgram

$$\mathbf{O}_1 = \frac{1}{2}(\mathbf{Q} + \mathbf{F}_1). \qquad \mathbf{O}_2 = \frac{1}{2}(\mathbf{Q} + \mathbf{F}_2)$$

$$\mathbf{H}_2 = \frac{1}{2}(\mathbf{P}_2 + \mathbf{F}_2).$$

$$\mathbf{P}_2 - \mathbf{F}_1 = 2\mathbf{H}_2 - (\mathbf{F}_1 + \mathbf{F}_2) = 2\mathbf{H}_2.$$

We also note that

$$\begin{aligned}\mathbf{O}_1-\mathbf{O}_2&=\frac{1}{2}(\mathbf{Q}+\mathbf{F}_1)-\frac{1}{2}(\mathbf{Q}+\mathbf{F}_2)\\&=\frac{1}{2}(\mathbf{F}_1-\mathbf{F}_2)\\&=\mathbf{F}_1\end{aligned}$$

Using the two vectors

$$\begin{aligned}\mathbf{K}'-\mathbf{O}_2&=\mathbf{O}_2-\mathbf{H}_2\\&=\frac{1}{2}(\mathbf{F}_2+\mathbf{Q})-\mathbf{H}_2\end{aligned}$$

$$\begin{aligned}\mathbf{O}_1-\mathbf{Q}&=\frac{1}{2}(\mathbf{F}_1+\mathbf{Q})-\mathbf{Q}\\&=\frac{1}{2}(\mathbf{F}_1-\mathbf{Q})\end{aligned}$$

$$\begin{aligned}&\mathbf{K}'-\mathbf{O}_2+\mathbf{O}_1-\mathbf{Q}\\&=\frac{1}{2}(\mathbf{F}_2+\mathbf{Q})-\mathbf{H}_2+\frac{1}{2}(\mathbf{F}_1-\mathbf{Q})\\&=\frac{1}{2}(\mathbf{F}_2+\mathbf{F}_1)-\mathbf{H}_2\\&=-\mathbf{H}_2\end{aligned}$$

If a line is drawn in a triangle so that it is parallel to one of the sides and it intersects the other two sides then the segments are of proportional lengths:

19. The geometrical analysis (Mathematica)

The hodographic solution to the Kepler problem which was discussed previously by us, is now discussed in more detail, using the Mathematica.

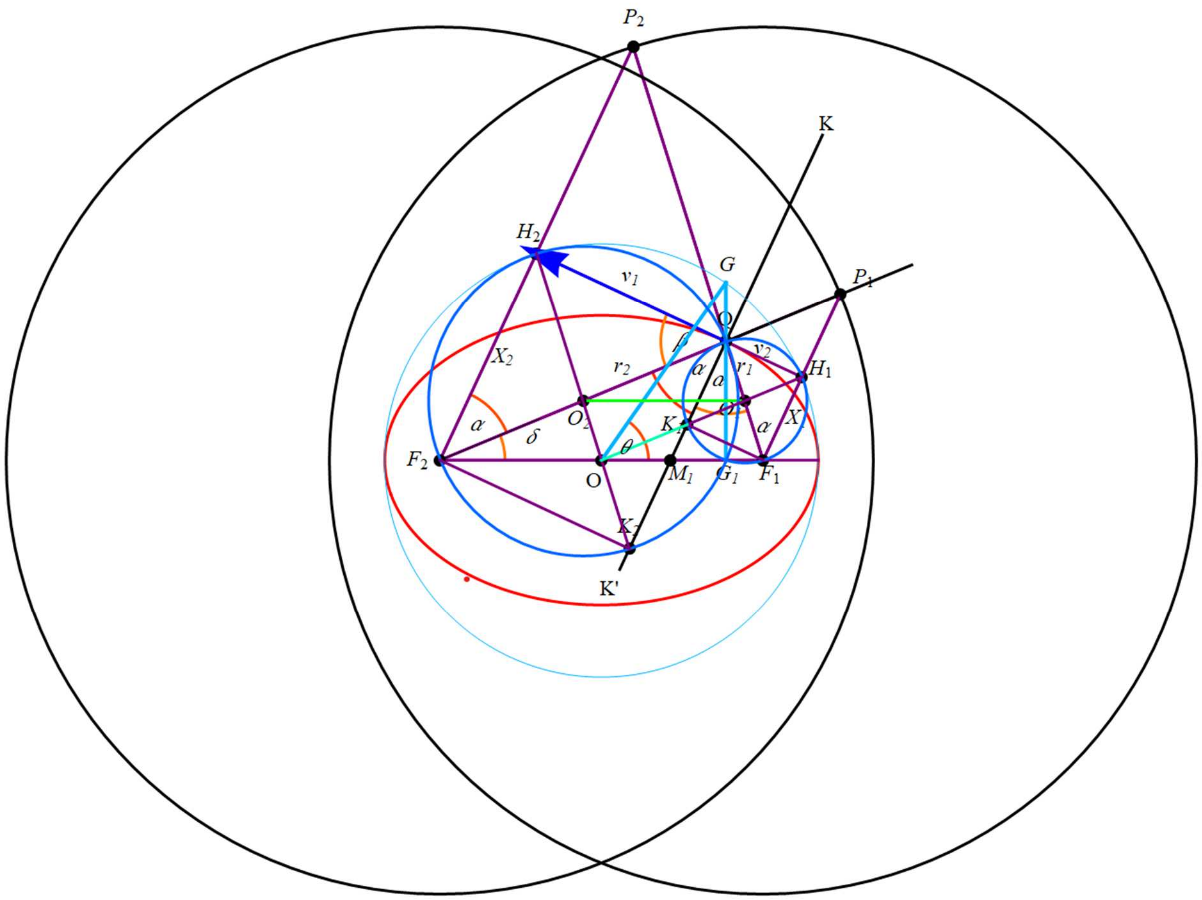


Fig.24 The point Q is on the elliptic orbit. F_1 and F_2 are foci of the elliptic orbit. (i) $\overline{QF_1} = r_1 . \overline{QF_2} = r_2$, $r_1 + r_2 = 2a$. (ii) $\angle F_1QK' = \angle F_2QK' = \alpha$. (iii) The points at H_1 and H_2 are located on a circle of radius a (semi major axis) centered at the origin. (iv) The length F_2P_2 is proportional to the magnitude of the velocity of charge at the point Q. (v) The product of the lengths F_1H_1 and F_2H_2 is constant and is equal to b^2, where b is the semi minor axis. $r_1 r_2 \cos^2\alpha = b^2$. (vi) The points k_1, k_2, and L_2 are on the same circle of the radius $a\varepsilon\cos\theta$ centered at the origin O. When $\theta = \pi/2$, these points coincide with the origin.

O:	$\mathbf{O} = (0,0)$,	(the origin)
F_1:	$\mathbf{F}_1 = (ae, 0)$	(the focus)
F_2;	$\mathbf{F}_2 = (-ae, 0)$	(the focus)
Q:	$\mathbf{Q} = (a\cos\theta, b\sin\theta)$	the point on the elliptic orbit

Tangential line at the point Q on the elliptic orbit

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta),$$

where θ is an eccentric angle. Note that the tangential line crosses the x axis at $x = \dfrac{a}{\cos\theta}$. The auxiliary circle is the circle of radius a centered at the origin

$$x^2 + y^2 = a^2.$$

The intersections (H_1 and H_2) of the tangential line and this circle of radius a;

H_2:

$$\mathbf{H}_2 = \frac{a}{1 - \varepsilon\cos\theta}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta).$$

H_1:

$$\mathbf{H}_1 = \frac{a}{1 + \varepsilon\cos\theta}(\varepsilon + \cos\theta, \sqrt{1-\varepsilon^2}\sin\theta).$$

P_2:

$$\mathbf{P}_2 = (-a\varepsilon + \frac{2a(1-\varepsilon^2)\cos\theta}{1-\varepsilon\cos\theta}, \frac{2a\sqrt{1-\varepsilon^2}\sin\theta}{1-\varepsilon\cos\theta}).$$

P_1:

$$\mathbf{P}_1 = (-a\varepsilon + \frac{2a(\varepsilon + \cos\theta)}{1+\varepsilon\cos\theta}, \frac{2a\sqrt{1-\varepsilon^2}\sin\theta}{1+\varepsilon\cos\theta}).$$

O_1:

$$\mathbf{O}_1 = \frac{a}{2}(\cos\theta + \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)\,.$$

Relation between $\boldsymbol{H}_1$ and $\boldsymbol{O}_1$;

$$\mathbf{H}_1 = \frac{2}{1+\varepsilon\cos\theta}\mathbf{O}_1\,.$$

Since the angle $\angle QG_1F_1 = \angle QH_1F_1 = \dfrac{\pi}{2}$, it is found that the points Q, G_1, F_1, and H_1 are on the same circle of the radius $r_1/2$. The center of the circle is at the point O_1,

$$\begin{aligned}\mathbf{O}_1 &= \frac{1}{2}(\mathbf{Q}+\mathbf{F}_1)\\ &= \frac{1}{2}a(\cos\theta + \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)\end{aligned}$$

K_1:

$$\begin{aligned}\mathbf{K}_1 &= \mathbf{F}_1 + (\mathbf{Q}-\mathbf{H}_1)\\ &= (a\varepsilon + a\cos\theta - x_1, a\sqrt{1-\varepsilon^2}\sin\theta - y_1)\\ &= (\frac{a\varepsilon\cos\theta(\varepsilon+\cos\theta)}{1+\varepsilon\cos\theta}, \frac{a\varepsilon\sqrt{1-\varepsilon^2}\sin\theta\cos\theta}{1+\varepsilon\cos\theta})\\ &= \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)\end{aligned}$$

Relation between $\boldsymbol{K}_1$ and $\boldsymbol{H}_1$;

$$\begin{aligned}\mathbf{K}_1 &= \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)\\ &= \varepsilon\cos\theta\ \mathbf{H}_1\end{aligned}$$

We also note that

$$\begin{aligned}\mathbf{K}_1 - \mathbf{O}_1 &= \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta) - \frac{1}{2}a(\cos\theta+\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta) \\ &= \frac{a}{2}\left(\frac{\varepsilon\cos\theta-1}{1+\varepsilon\cos\theta}\right)(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)\end{aligned}$$

with

$$\left|\mathbf{K}_1 - \mathbf{O}_1\right| = \frac{a}{2}(1-\varepsilon\cos\theta) = \frac{1}{2}r_1,$$

and

$$\left|\mathbf{K}_1\right| = a\varepsilon\cos\theta,$$

where

$$\begin{aligned}\mathbf{K}_1 - \mathbf{F}_1 &= \frac{a\varepsilon\cos\theta}{1+\varepsilon\cos\theta}(\varepsilon+\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta) - (a\varepsilon, 0) \\ &= \frac{a\varepsilon\sin\theta}{1+\varepsilon\cos\theta}(-\varepsilon\sin\theta, \sqrt{1-\varepsilon^2}\cos\theta)\end{aligned}$$

The line $\overline{O_1O_2}$ is parallel to the line $\overline{F_1F_2}$,

$$\mathbf{O}_1 - \mathbf{O}_2 = (a\varepsilon, 0).$$

From these, it is concluded that the point K_1 is on the line $\overline{KK'}$ and is on the circle of radius $r_1/2$ centered at the point O_1. The line $\overline{KK'}$ is an angle bisector divides in the angle into two angles with equal measures. $H_1QK_1F_1$ forms a square where the points H_1, Q, K_1, and F_1 are on the same circle.

$$\begin{aligned}\overline{F_2H_2}^2 &= \left|\mathbf{H}_2 - \mathbf{F}_2\right|^2 \\ &= \frac{a^2(1-\varepsilon^2)(1+\varepsilon\cos\theta)}{1-\varepsilon\cos\theta}\end{aligned}$$

$$\overline{F_1H_1}^2 = |\mathbf{H}_1 - \mathbf{F}_1|^2$$
$$= \frac{a^2(1-\varepsilon^2)(1-\varepsilon\cos\theta)}{1+\varepsilon\cos\theta}$$

The product of $\overline{F_1H_1}$ and $\overline{F_2H_2}$ is constant, independent of the angle θ.

$$\overline{F_1H_1}\ \overline{F_2H_2} = a^2(1-\varepsilon^2) = b^2.$$

Since the angle $\angle QG_1F_2 = \angle QH_2F_2 = \frac{\pi}{2}$, it is found that the points Q, G_1, F_2, and H_2 are on the same circle of the radius $r_2/2$. The center of the circle is at the point O_2.

O_2: $$\mathbf{O}_2 = \frac{1}{2}(\mathbf{Q}+\mathbf{F}_2) = \frac{1}{2}a(-(\varepsilon-\cos\theta), \sqrt{1-\varepsilon^2}\sin\theta).$$

Relation between $\boldsymbol{H}_2$ and $\boldsymbol{O}_2$;

$$\mathbf{H}_2 = \frac{2}{1-\varepsilon\cos\theta}\mathbf{O}_2.$$

The intersection (K_2) of extension line of $\overline{H_2O_2}$ with the circle of radius $r_2/2$ centered at the point O_2.

K_2:

$$\mathbf{K}_2 = \left(\frac{a\varepsilon(\varepsilon-\cos\theta)\cos\theta}{1-\varepsilon\cos\theta}, \frac{a\varepsilon\sqrt{1-\varepsilon^2}\sin\theta\cos\theta}{1-\varepsilon\cos\theta}\right)$$
$$= \frac{a\varepsilon\cos\theta}{1-\varepsilon\cos\theta}(\varepsilon-\cos\theta, \sqrt{1-\varepsilon^2}\sin\theta)$$

and

$$|\mathbf{K}_2| = a\varepsilon\cos\theta.$$

The point L_2 on the x axis is defined as

$$\mathbf{L}_2 = (a\varepsilon\cos\theta, 0)\,.$$

We note that the points K_2, L_2, and K_1 are on the same circle of the radius $a\varepsilon\cos\theta$, centered at the origin O. We also note that G_1 is the intersection of the extension $\overline{G_1Q}$ and the circle of radius a centered at the origin O.

G: $\mathbf{G} = (a\cos\theta, a\sin\theta)\,.$

G_1: $\mathbf{G}_1 = (a\cos\theta, 0)\,.$

$$\overline{F_1H_1}\ \overline{F_2H_2} = a^2(1-\varepsilon^2) = b^2$$

$$\overline{F_2H_2} = b\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}}\,, \qquad \overline{F_1H_1} = b\sqrt{\frac{1-\varepsilon\cos\theta}{1+\varepsilon\cos\theta}}$$

M_1: $\mathbf{M}_1 = (a\varepsilon^2\cos\theta, 0)$

$$\overline{QM_1} = b\sqrt{(1+\varepsilon\cos\theta)(1-\varepsilon\cos\theta)}$$

20. Summary: the property of elliptic orbit

For the elliptic orbit, we have

$$r_1 + r_2 = 2a\,.$$

The line $\overline{KK'}$ is perpendicular to the tangential line at the point Q on the ellipse orbit. This line bisects the angle $\angle F_1QF_2 = 2\alpha$;

$$\angle F_1QK' = \angle F_2QK' = \alpha\,.$$

The distance between Q and F_1 is r_1 and is calculated as

$$
\begin{aligned}
\overline{QF_1}^2 &= |\mathbf{F}_1 - \mathbf{Q}|^2 \\
&= r_1^2 \\
&= (a\cos\theta - a\varepsilon)^2 + (b\sin\theta)^2 \\
&= a^2[(\cos\theta - \varepsilon)^2 + (1-\varepsilon^2)\sin^2\theta] \\
&= a^2(1-\varepsilon\cos\theta)^2
\end{aligned}
$$

Similarly, the distance between ***Q*** and ***F***2 is r_2 and is calculated as

$$
\begin{aligned}
\overline{QF_2}^2 &= |\mathbf{F}_2 - \mathbf{Q}|^2 \\
&= r_2^2 \\
&= (a\cos\theta + a\varepsilon)^2 + (b\sin\theta)^2 \\
&= a^2[(\cos\theta + \varepsilon)^2 + (1-\varepsilon^2)\sin^2\theta] \\
&= a^2(1+\varepsilon\cos\theta)^2
\end{aligned}
$$

Thus, we have

$$r_1 = a(1-\varepsilon\cos\theta), \qquad r_2 = a(1+\varepsilon\cos\theta).$$

These relations can be understood as follows.

We note that $\overline{OK_1} = a\varepsilon\cos\theta$, $\overline{K_1H_1} = r_1$, and $\overline{OH_1} = a$

$$\overline{OH_1} = \overline{OK_1} + \overline{K_1H_1}, \qquad \text{(on the straight line)}$$

or

$$r_1 = a(1-\varepsilon\cos\theta).$$

We also note that $\overline{OK_2} = a\varepsilon\cos\theta$, $\overline{K_2H_2} = r_2$, and $\overline{OH_2} = a$

$$\overline{K_2H_2} = \overline{K_2O} + \overline{OH_2}, \qquad \text{(on the straight line)}$$

or

$$r_2 = a(1+\varepsilon\cos\theta).$$

We apply the cosine law for the triangle QF_1F_2.

$$r_1^2 + r_2^2 - 2r_1r_2\cos(2\alpha) = 4a^2\varepsilon^2. \tag{1}$$

Using the relation ($r_1 + r_2 = 2a$)

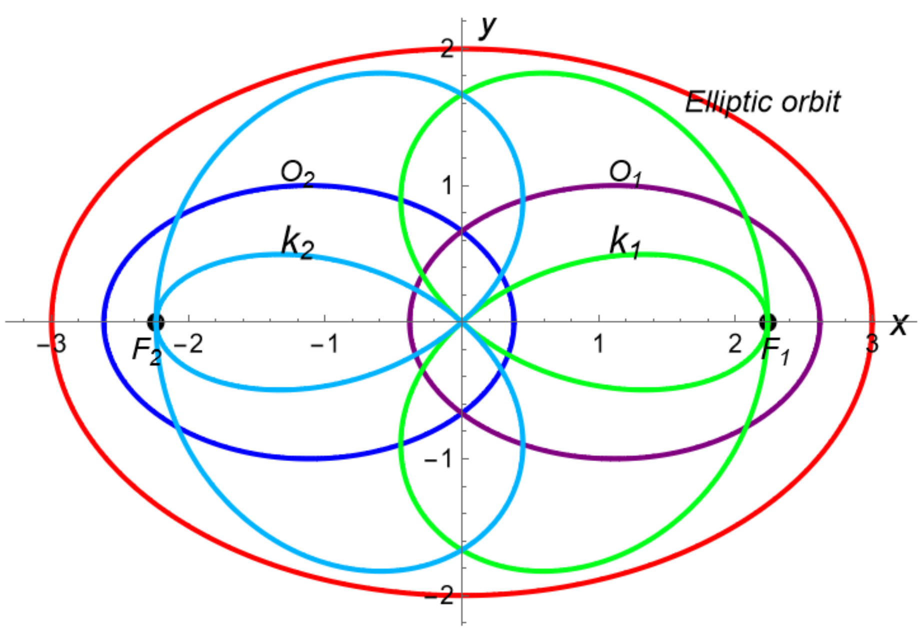
$$r_1^2 + r_2^2 + 2r_1r_2 = 4a^2. \tag{2}$$

From Eqs.(1) and (2), we get

$$r_1r_2\cos^2\alpha = b^2 = a^2(1-\varepsilon^2),$$

or

$$\overline{F_1H_1}\ \overline{F_2H_2} = b^2 = a^2(1-\varepsilon^2).$$



Fig,25 Elliptic orbit (red) with $a = 3$ and $b = 2$. The trajectories of the points K_1. K_2, O_1, and O_2, where the eccentricity anomaly θ is changed as a parameter ($0 \le \theta \le 2\pi$).

((The relationship between the angles α and θ))

$$\cos^2 \alpha = \frac{1-\varepsilon^2}{1-\varepsilon^2 \cos^2 \theta}.$$

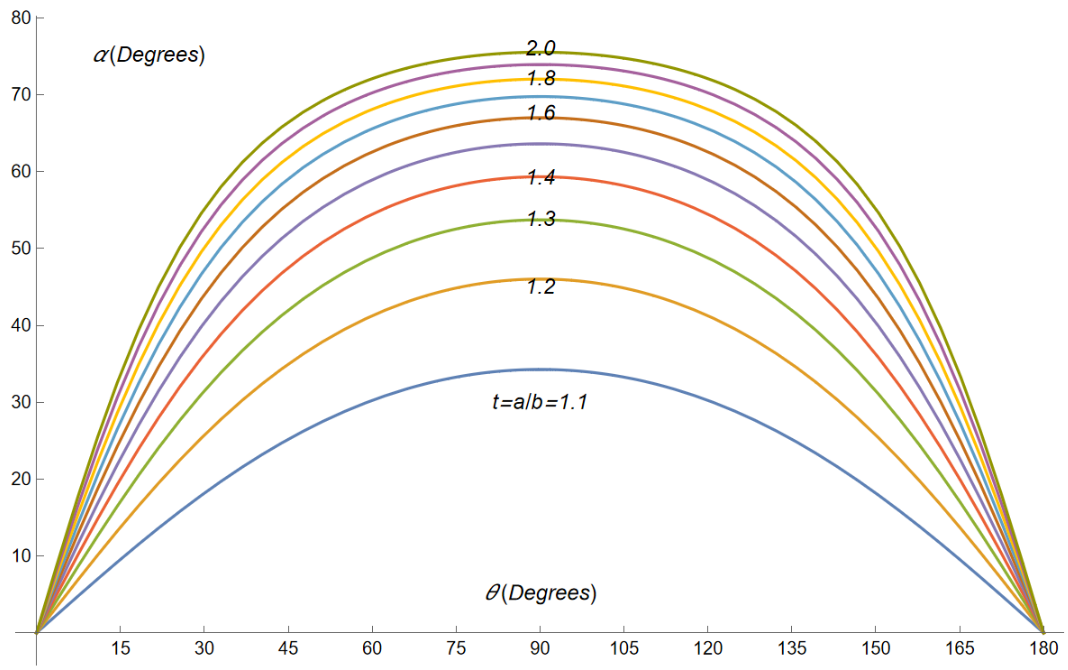


Fig.26 The plot of the angle a as a function of the eccentricity anomaly θ

21. The relationship between the angles ϕ and the eccentric angle θ;

Using the relations

$$r_1 \cos\phi + a\varepsilon = a\cos\theta,$$

and

$$r_1 = a(1-\varepsilon\cos\phi) = \frac{a(1-e^2)}{1+\varepsilon\cos\phi},$$

we get

$$\frac{(1-\varepsilon^2)}{1+\varepsilon\cos\phi}\cos\phi + \varepsilon = \cos\theta,$$

or

$$\frac{\varepsilon+\cos\phi}{1+\varepsilon\cos\phi} = \cos\theta.$$

22. Examples: Feynman hodograph diagram in the Kepler's model and Sommerfeld's model

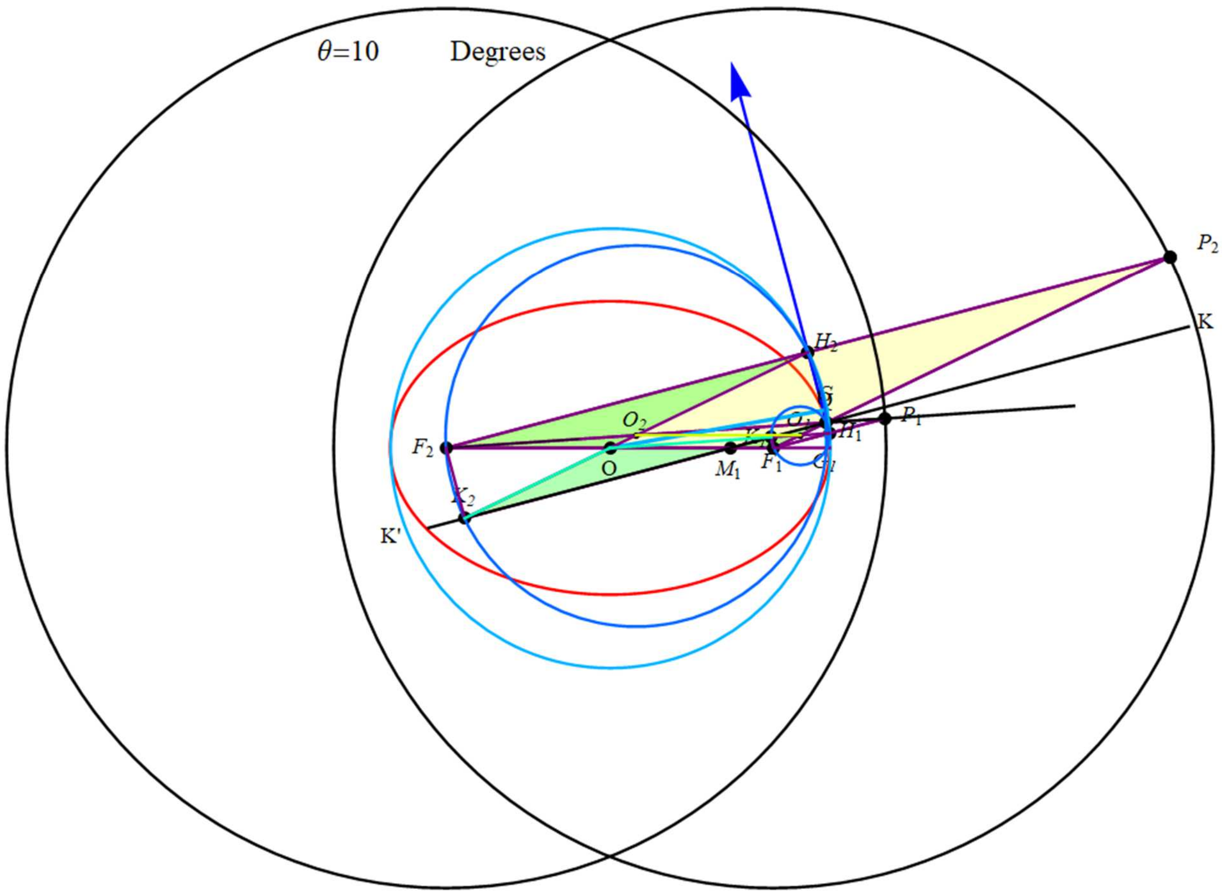


Fig.27 (a) $\theta = 10°$.

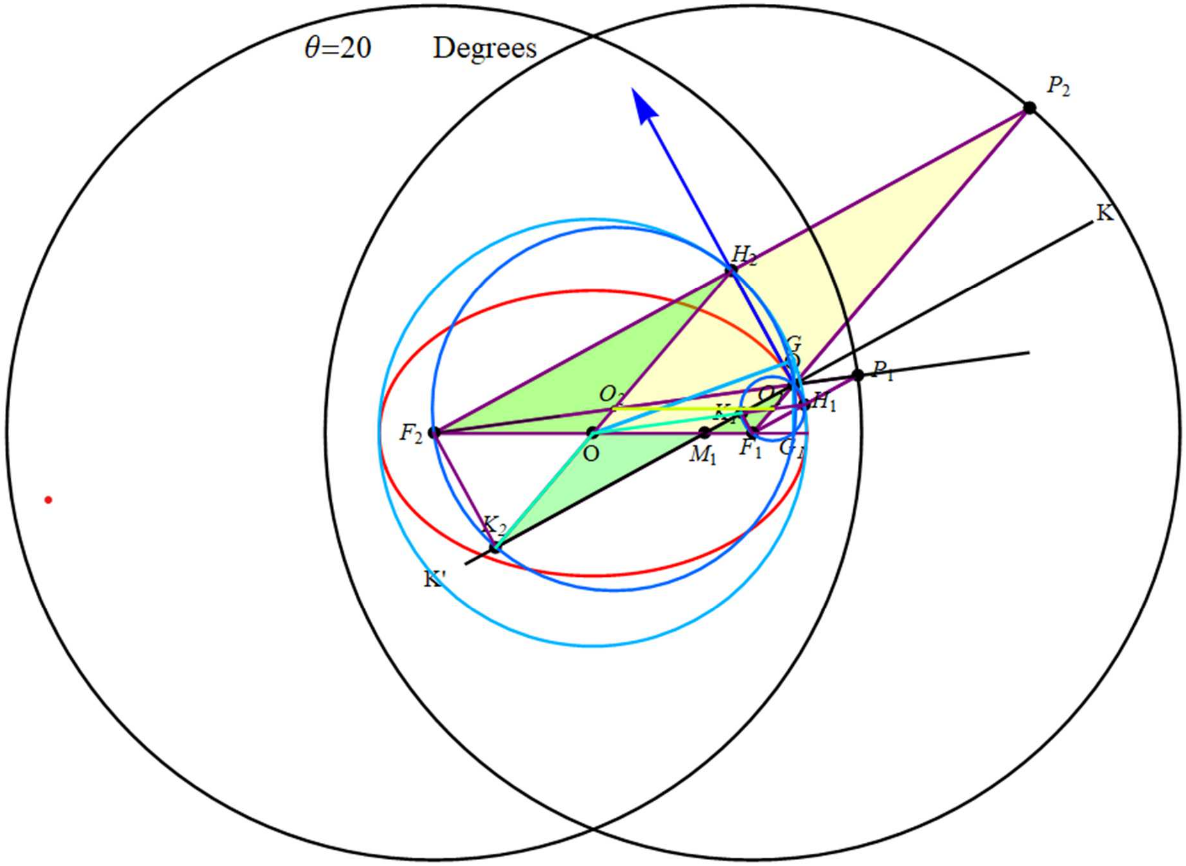


Fig.27 (b) $\theta = 20°$.

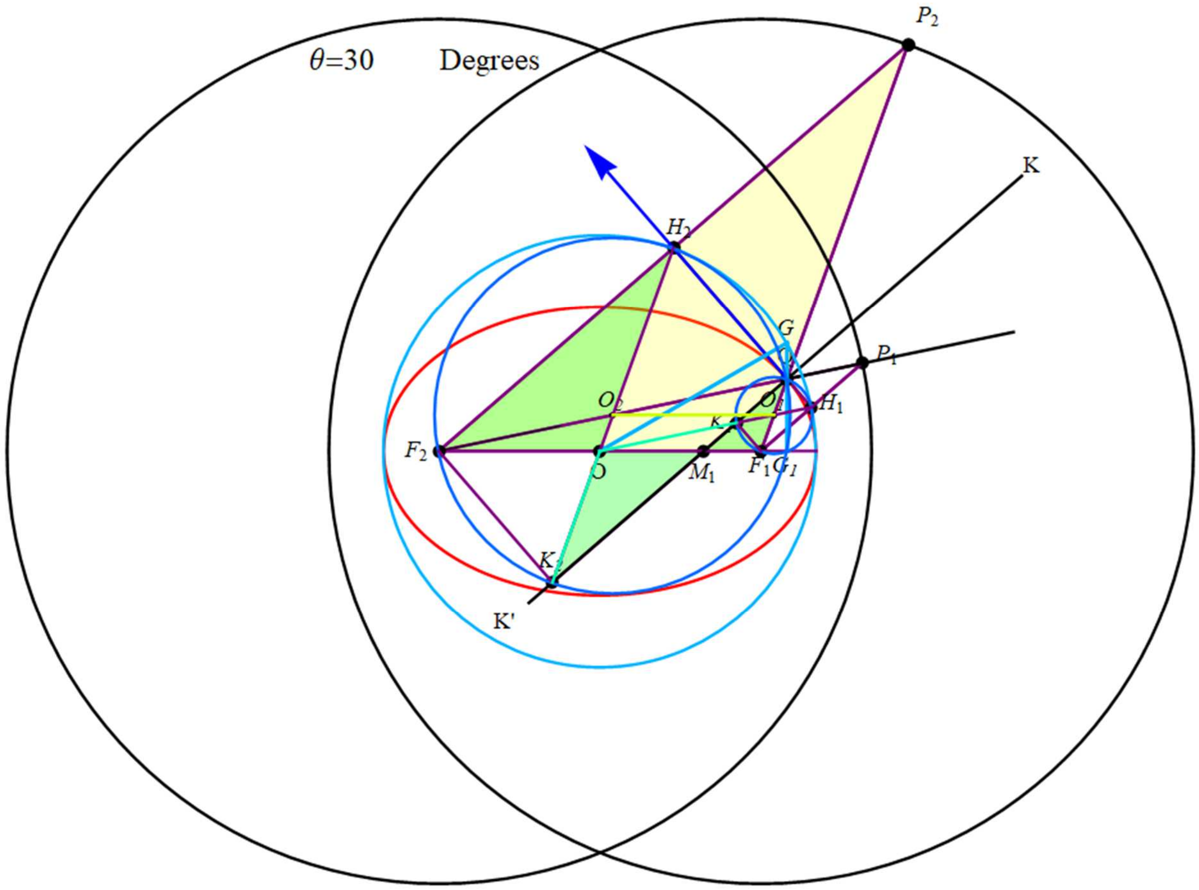


Fig.27 (c) $\theta = 30°$.

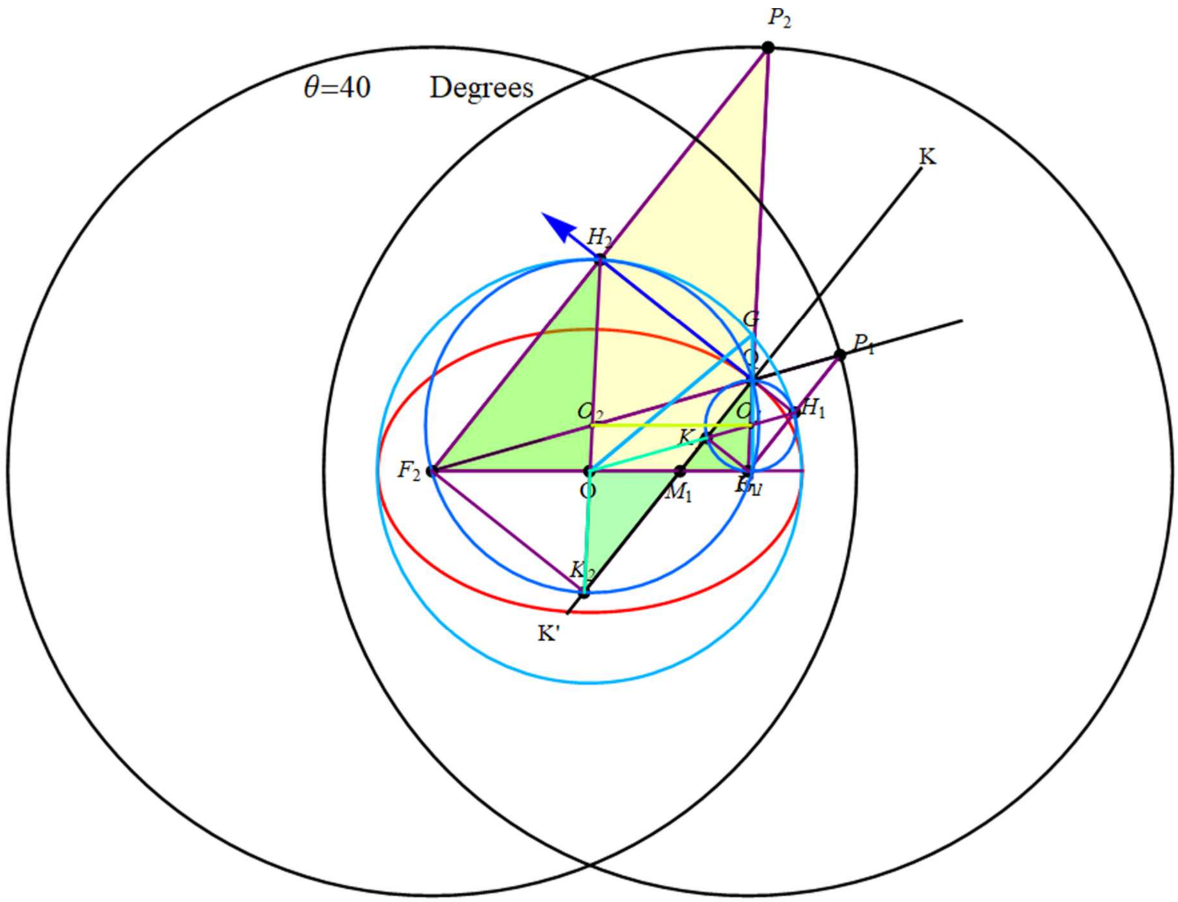


Fig.27 (d) $\theta = 40°$.

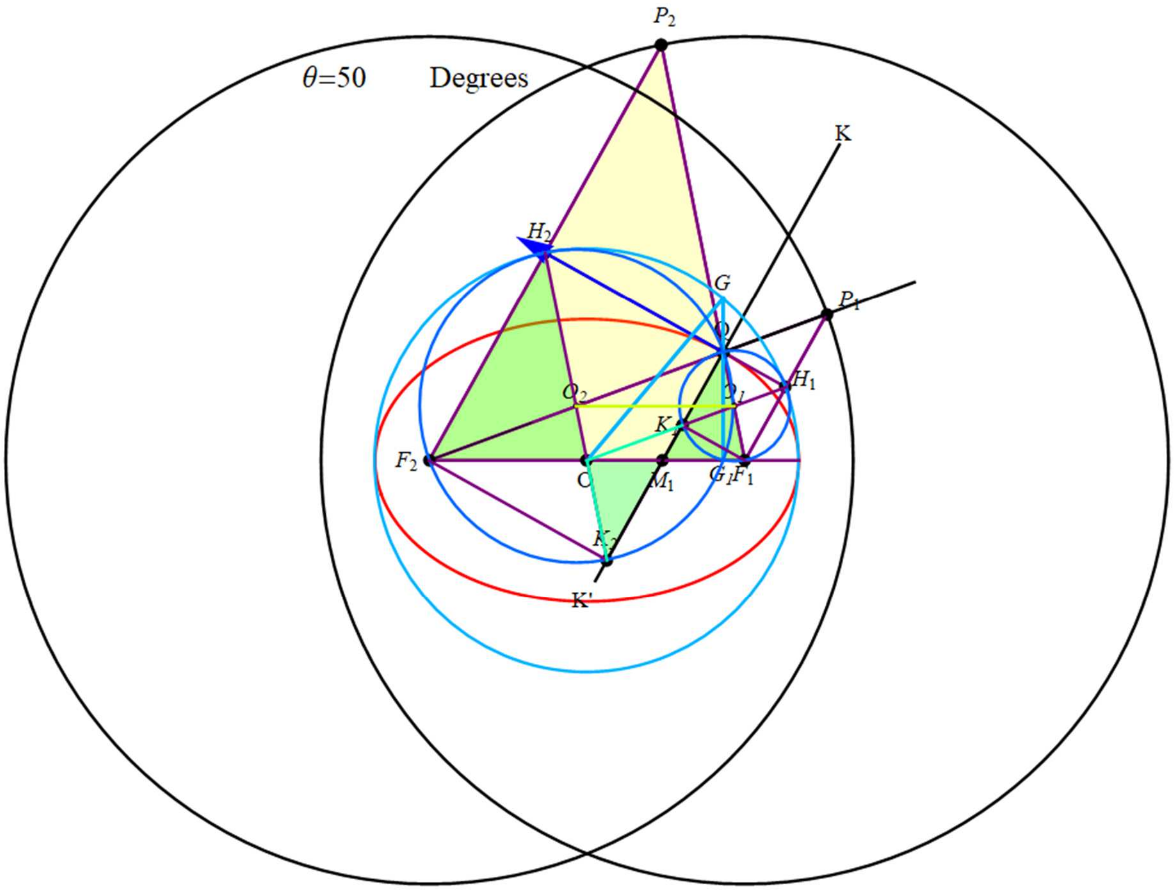


Fig.27 (e) $\theta = 50°$.

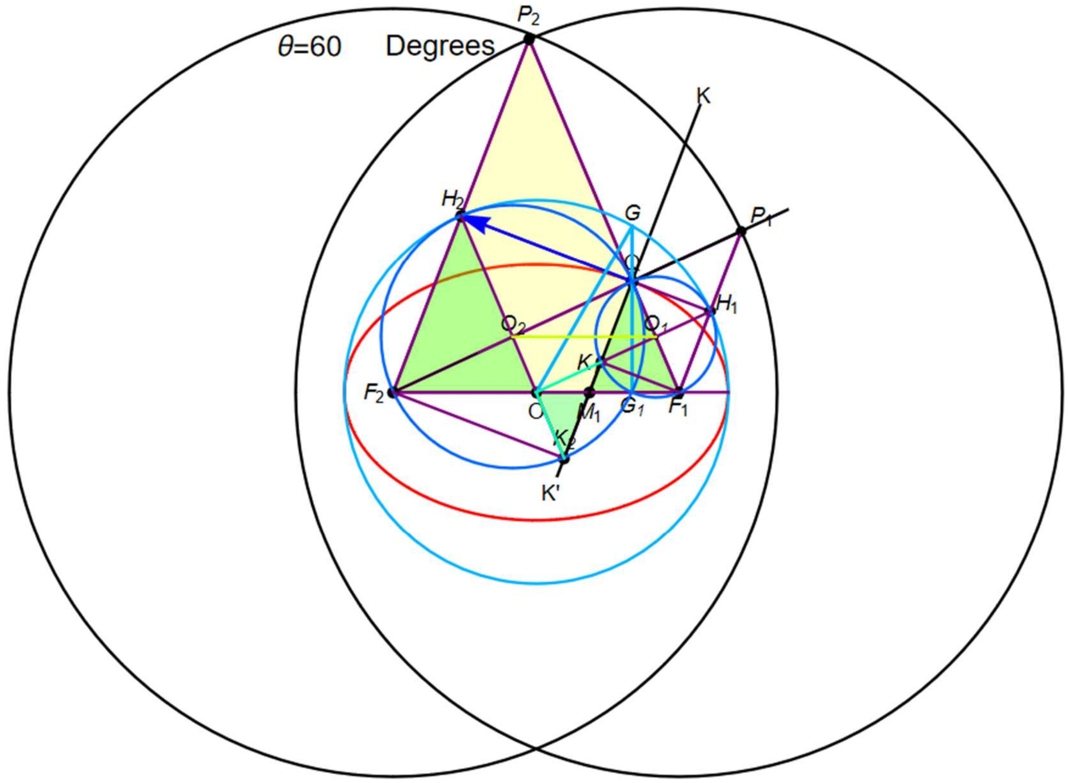


Fig.27 (f) $\theta = 60^\circ$.

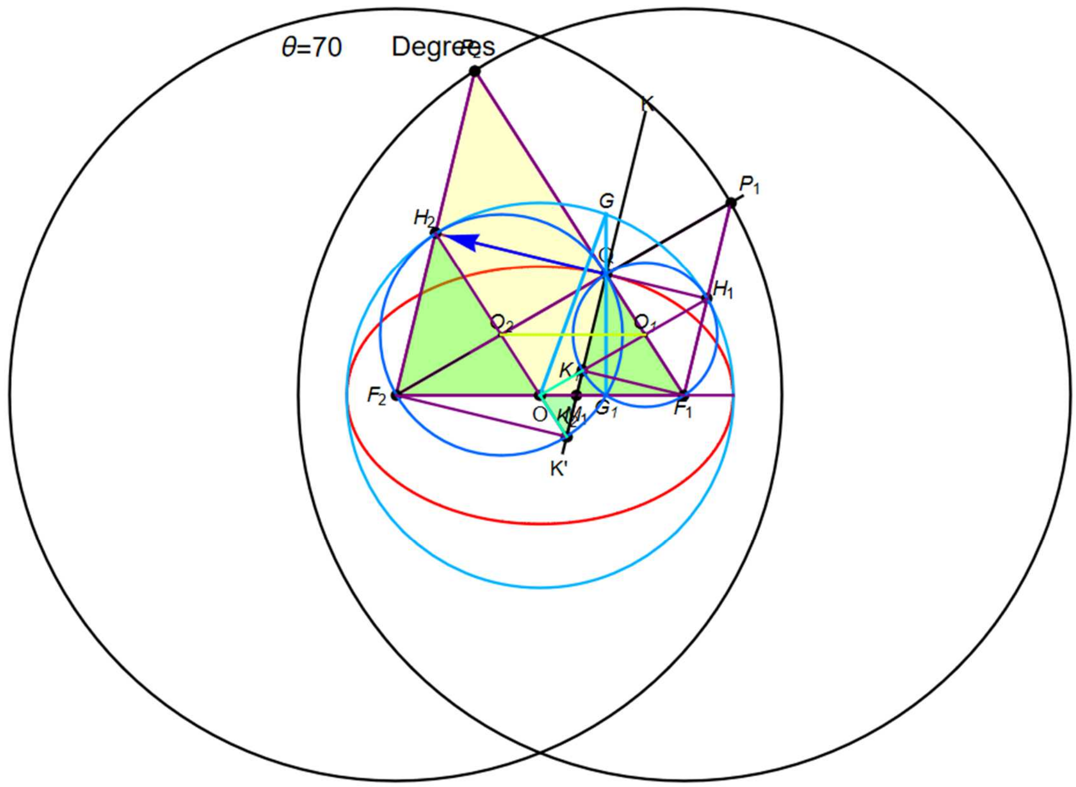


Fig.27 (g) $\theta = 70°$.

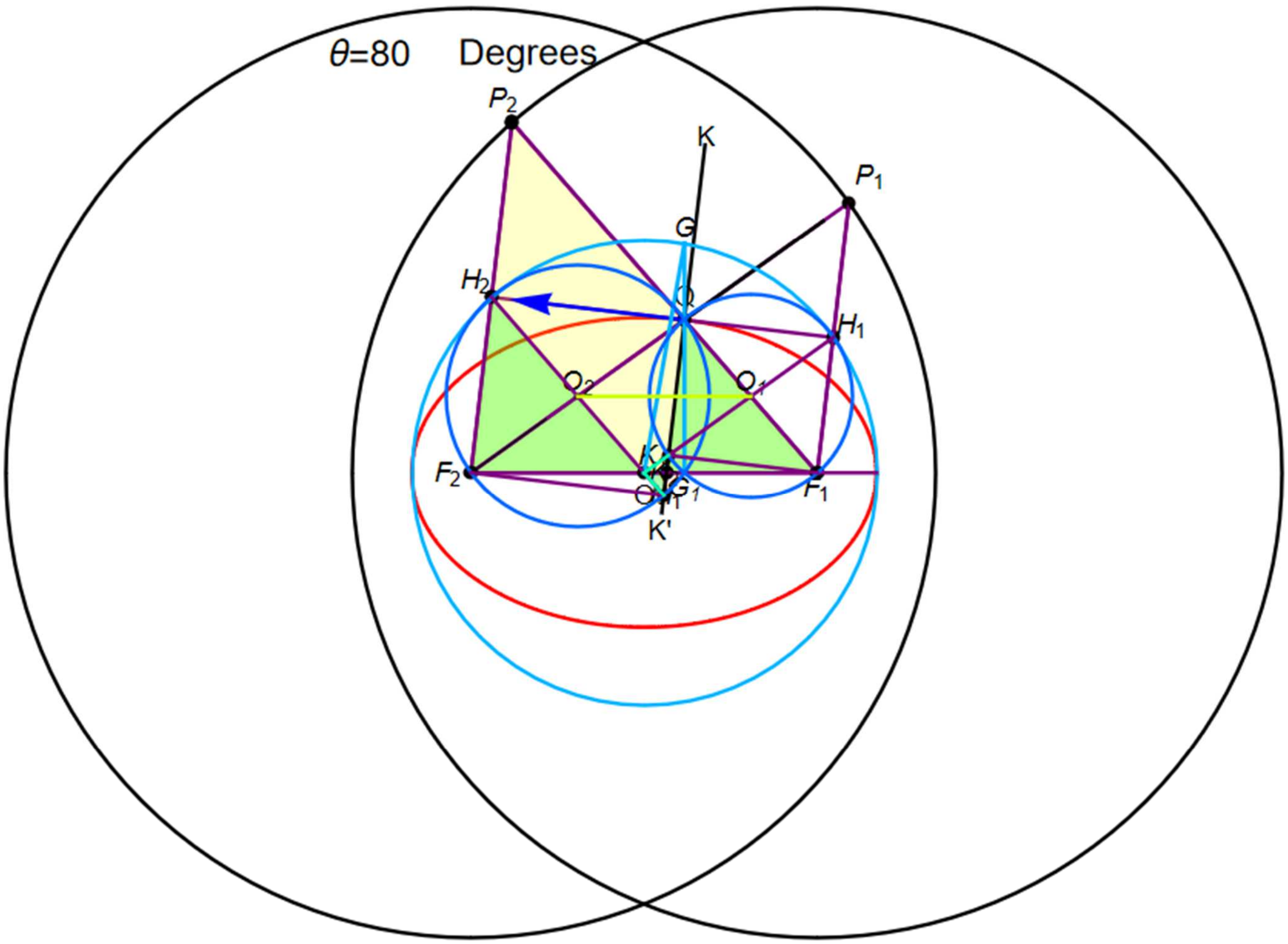


Fig.27 (h) $\theta = 80^\circ$ **.**

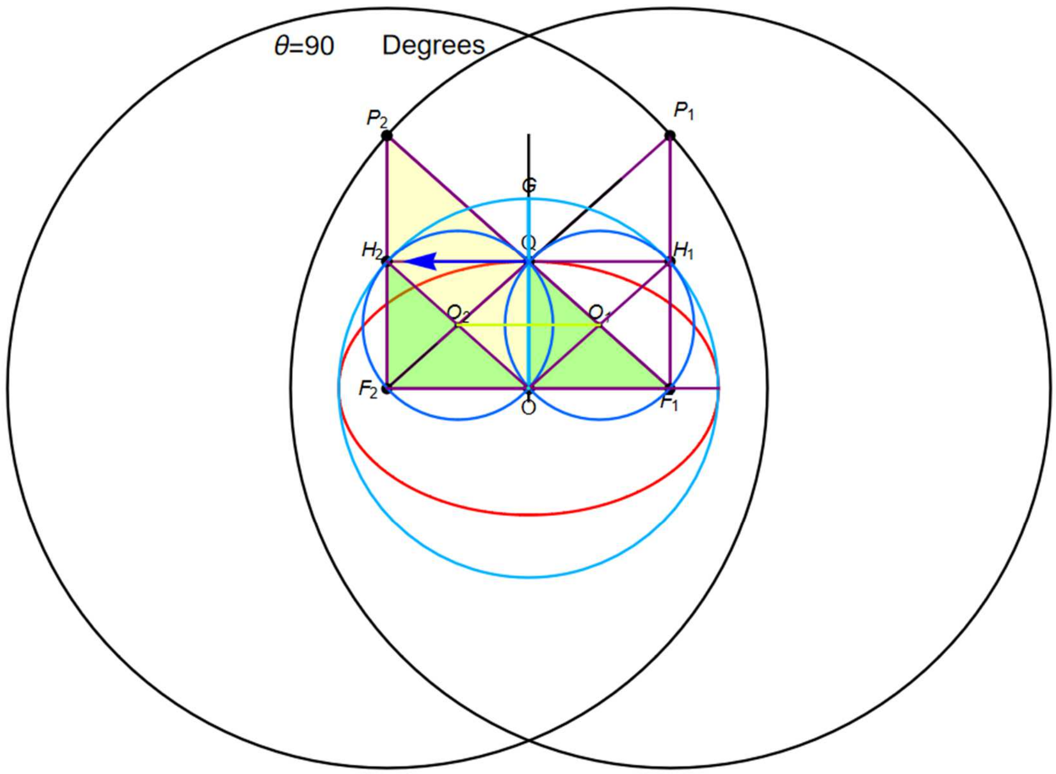


Fig.27 (i) $\theta = 90°$ **.**

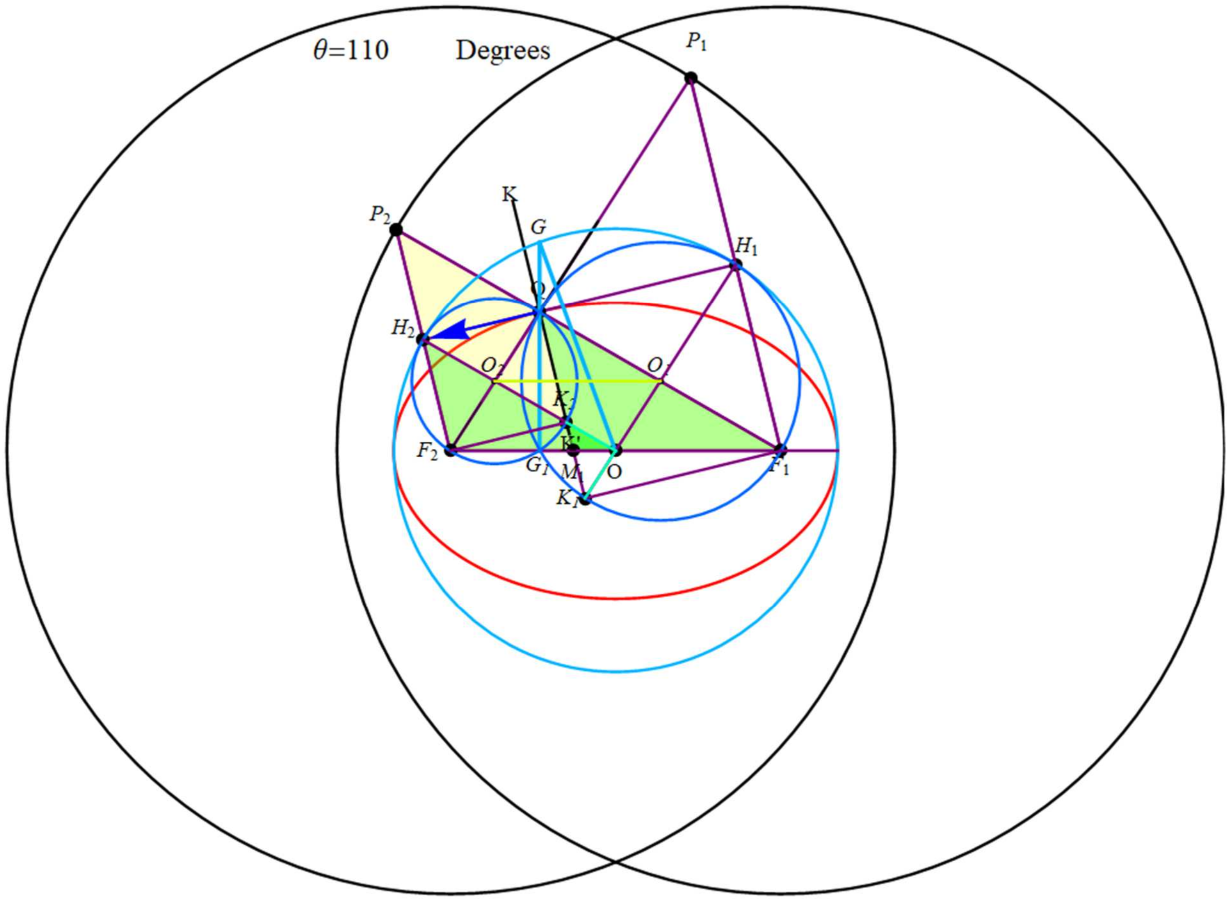


Fig.27 (j) $\theta = 110°$.

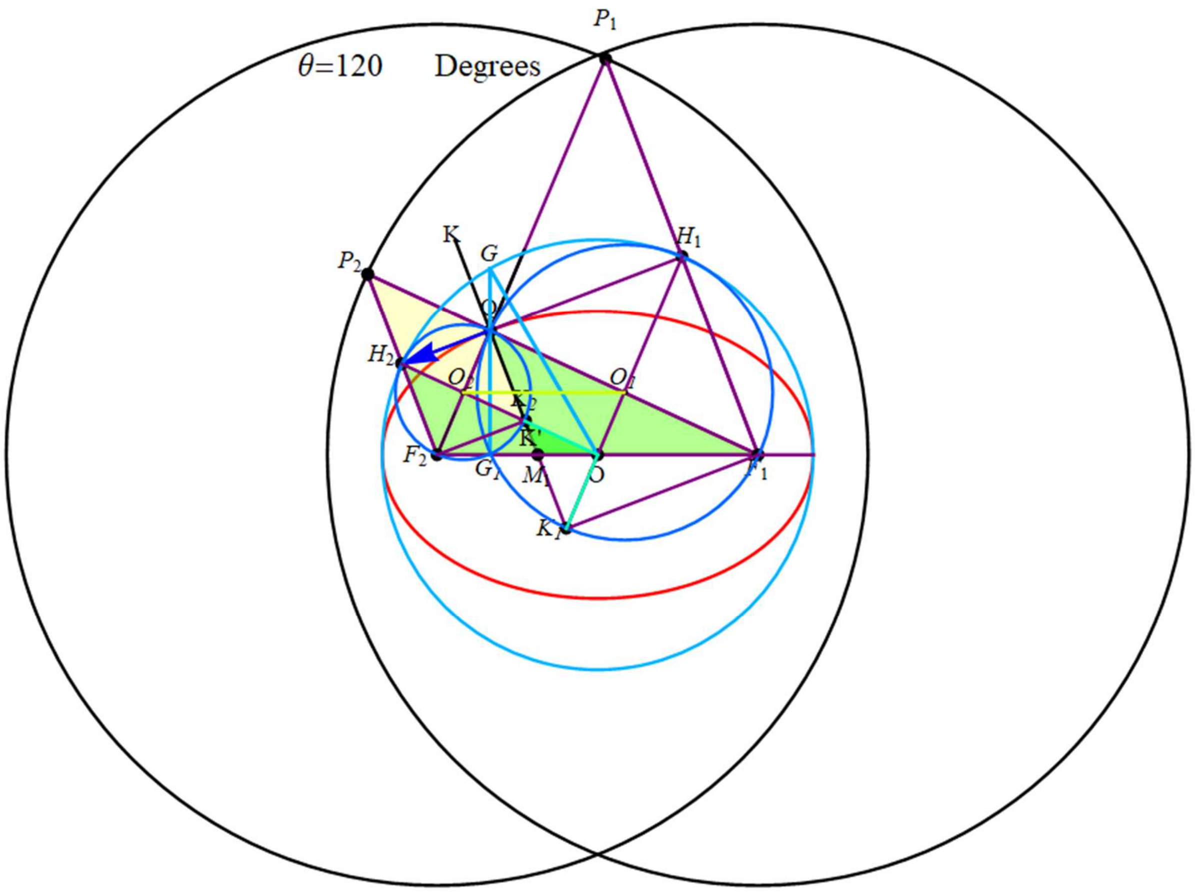


Fig.27 (k) $\theta = 120°$.

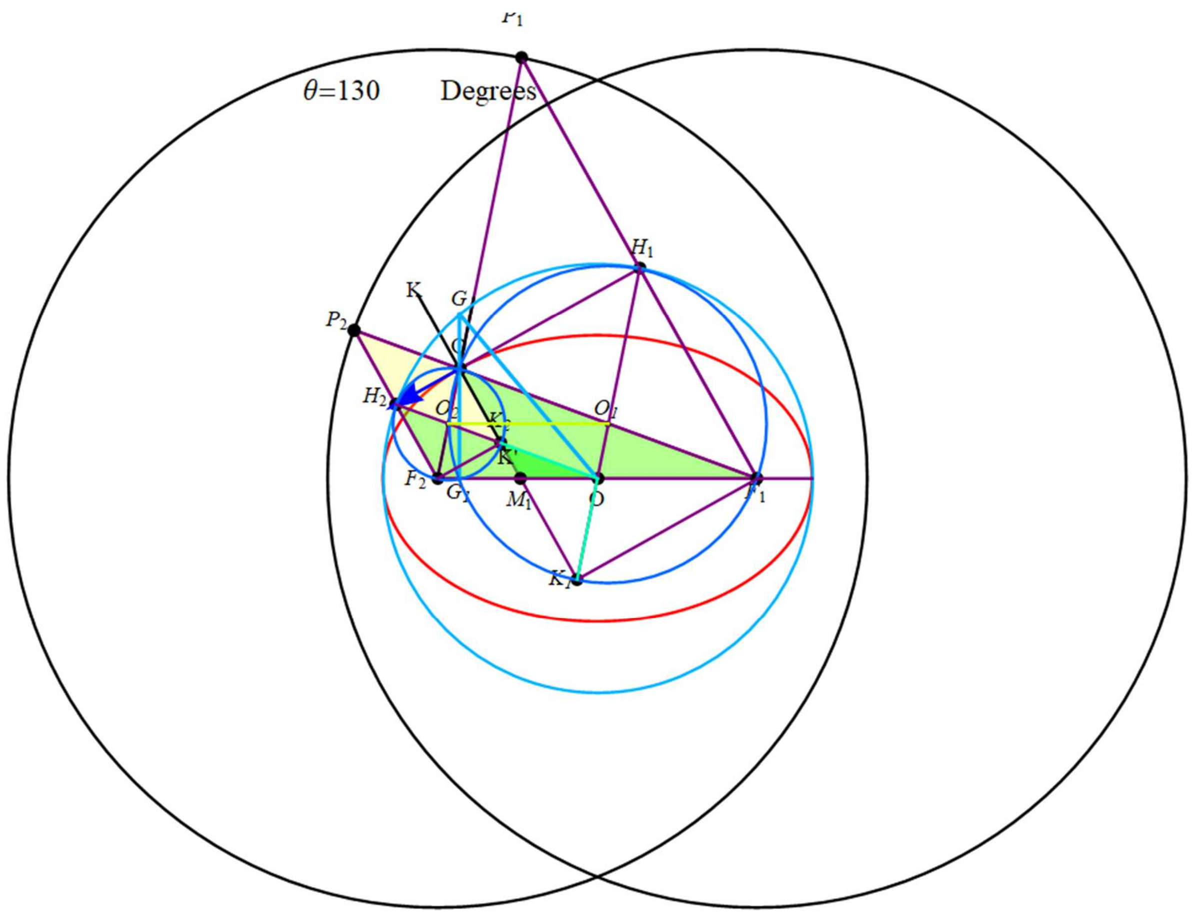


Fig.27 (j) $\theta = 130°$.

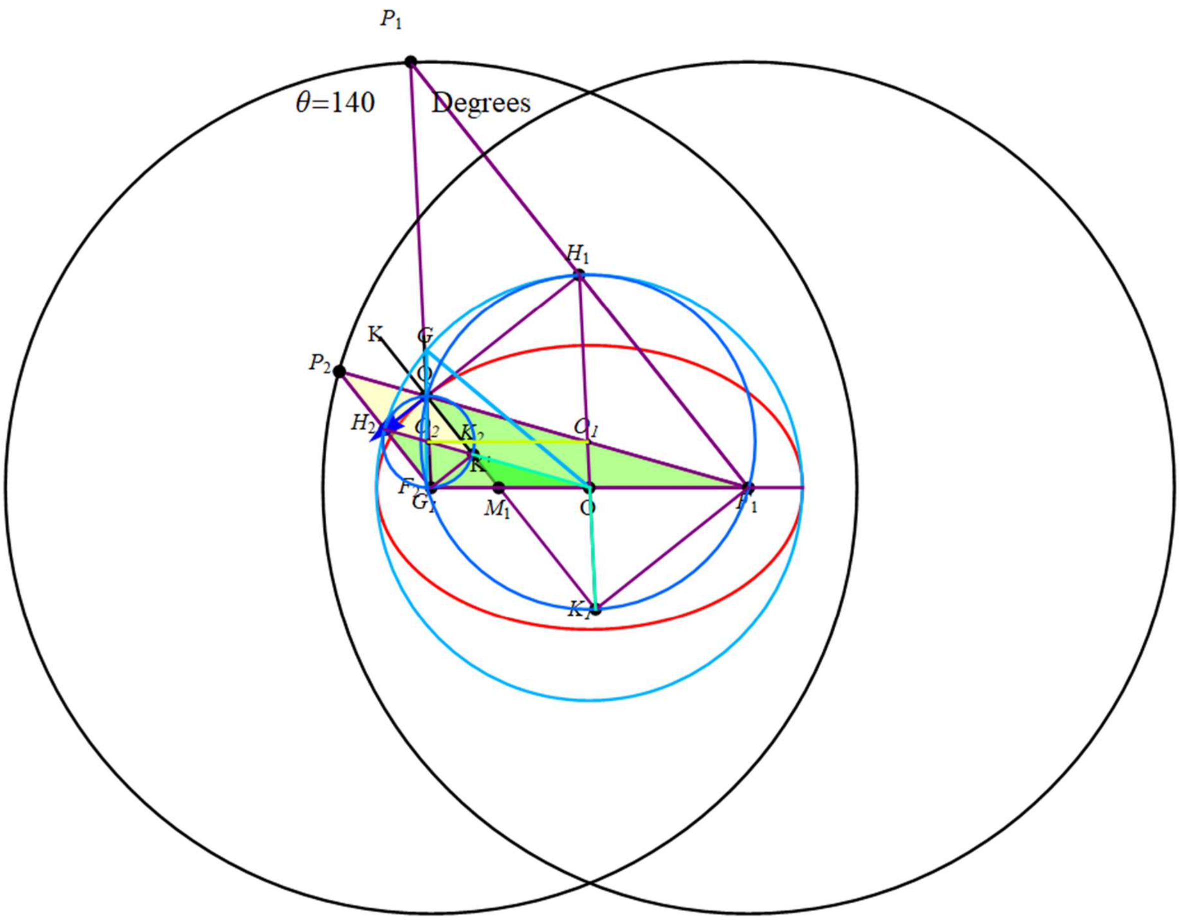


Fig.27 (k) $\theta = 140°$.

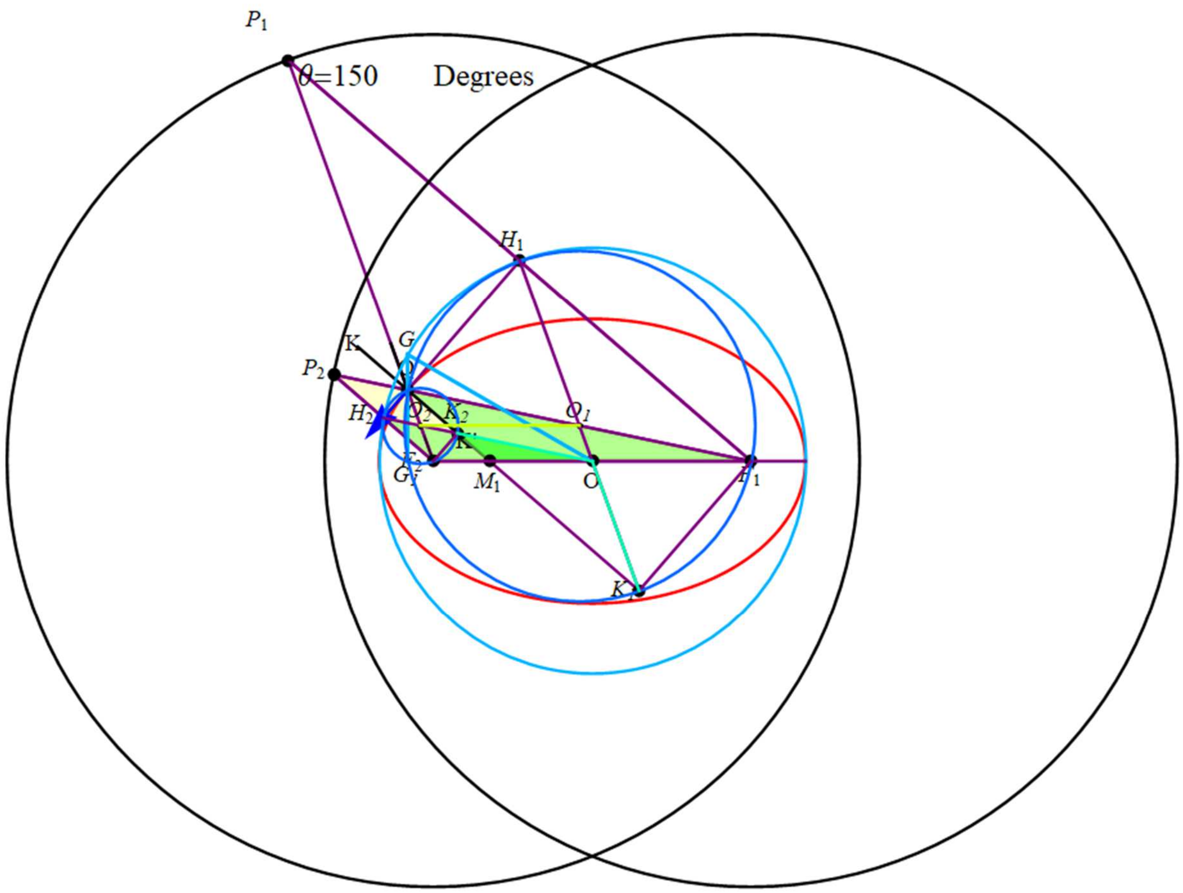


Fig.27(l) $\theta = 150°$.

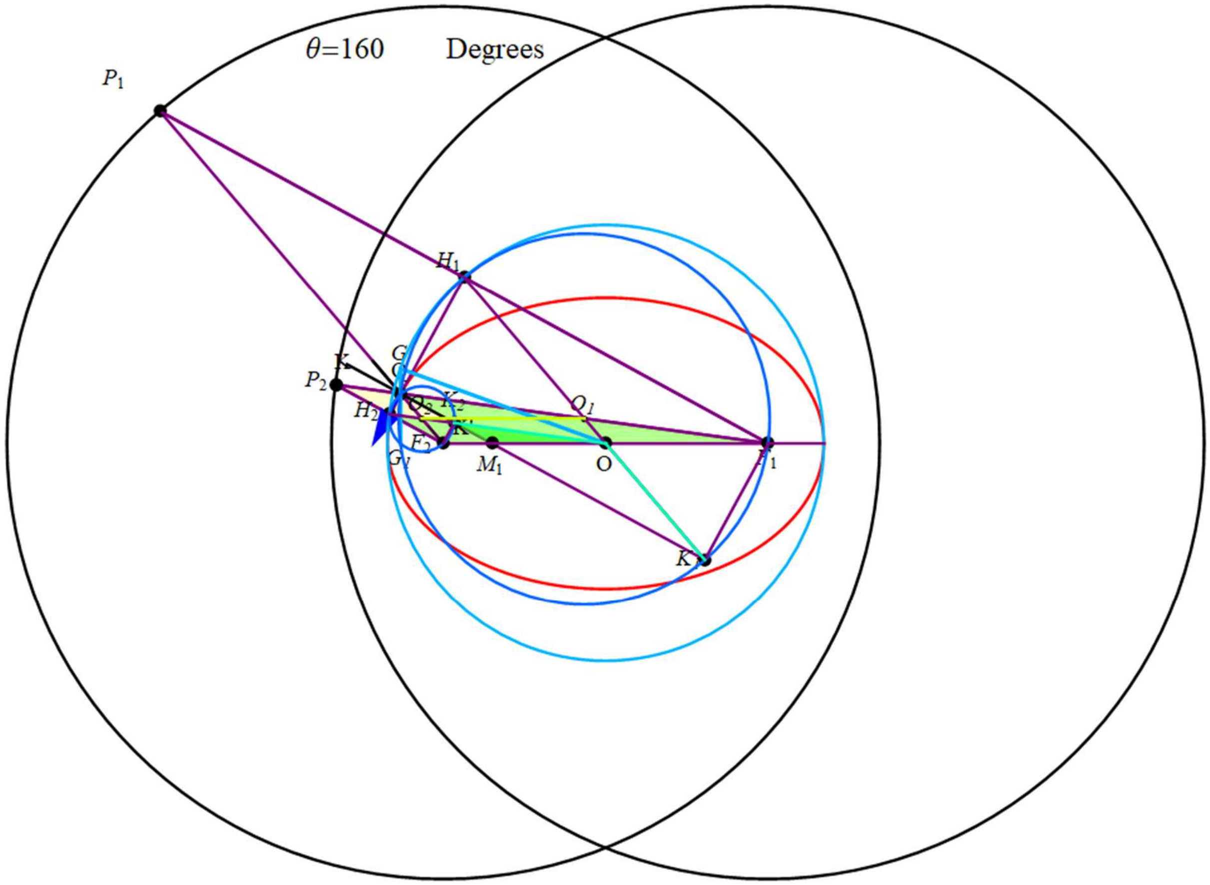


Fig.27(m) $\theta = 160°$.

24. CONCLUSION

In 1916, the circular orbit (Bohr model) was revised by Sommerfeld as an elliptic orbit in the hydrogen atom. From the quantum condition which he assumed, he could show that the total energy has the same form as that predicted from Bohr. The angular momentum are conserved and also discretely quantized. Without the quantum conditions, the Sommerfeld model and are the Kepler central force problem are essentially the same. While studying the spin orbit interactions of alkali metal atoms, we have an opportunity to take a closer a look at the hodograph diagram.

In previous article, we have discussed the geometry of the Kepler problem (Feynman hodograph diagram) in much detail. Unexpectedly we have found a so-called Laplace-Runge-Lenz (LRL) triangles inside the hodograph. We can find four LRL triangles which are similar to each other.

Our discussion starts with a finding of an construction line (auxiliary line) $\overline{QG_1}$, where the line $\overline{QG_1}$ is perpendicular to the x axis where $\mathbf{G}_1 = (a\cos\theta, 0)$. We tried to draw one auxiliary line (in this case, the line QG_1). The line QG_1 is perpendicular to the horizontal axis (the x axis). The point G_1 lies on the x axis. We realize that this line plays an important role in changing our vision of the hodographic solution. We determine the coordinates of each point by using Mathematica program. We realize that the four points Q, H_2, F_2, anf G_1 are on the same circle centered at the point O_2, since both the angle QH_2F_2 and the angle QG_1F_2 are equal to $90°$. We also find that the four points Q, H_1, F_1, and G_1 are on the same circle centered at the point O_1, since both the angle QH_1F_1 and the angle QG_1F_1 are equal to $90°$. From the property of the ellipsoid, the line KK' is the bisector line such that the angle F_2QK' is equal to the angle F_1QK'.

The LRL vector is universally constant as

$$\mathbf{A} = \mathbf{v} \times \mathbf{L} - \frac{k}{r}\mathbf{r} = \frac{k_0 m}{a}\mathbf{F}_1,$$

where $\mathbf{F}_1$ is the position vector of the focus (Sun in the Kepler's model and electron in the Sommerfeld's model). This may correspond to the Kepler's first law. The angular momentum is perpendicular to the 2D plane of motion, and is conserved. This corresponds to the Kepler's second law. The relation

$$A_x^2 = \frac{2E}{m} L_z^2 + k^2,$$

with

$$A_x = k\varepsilon, \qquad L_z^2 = mk\Lambda = mk\frac{b^2}{a}, \qquad E = -\frac{k}{2a}$$

is equivalent to the most familiar expression

$$b^2 = a^2(1 - \varepsilon^2).$$

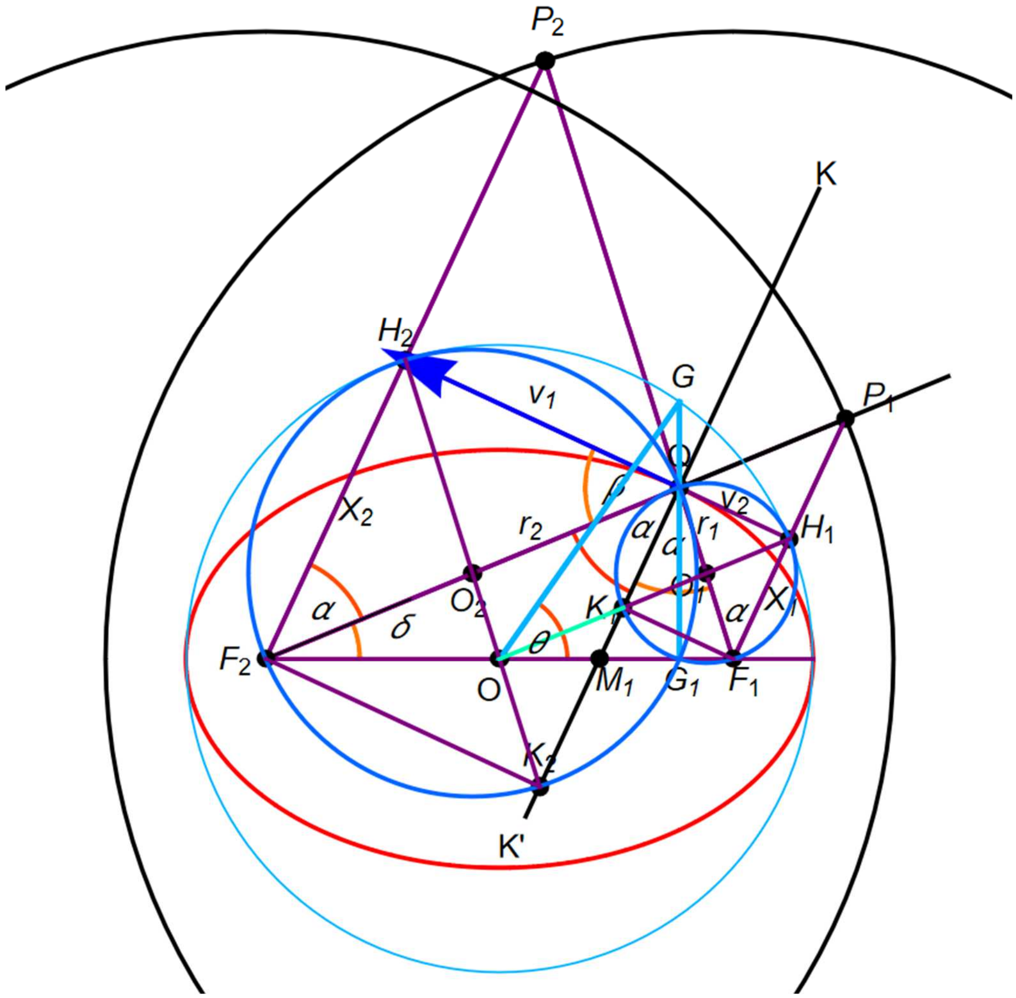


Fig.28 Detail of the Feynman hodograph diagram which is improved here.

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APPENDIX

The notations used in this article are shown for the sake of convenience.

O:	$\mathbf{O} = (0,0)$,	(the origin)
F2;	$\mathbf{F}_2 = (-ae, 0)$	(the focus)
F1:	$\mathbf{F}_1 = (ae, 0)$	(the focus)

$r_{ap} = a(1+\varepsilon)$	aphelion (the farthest distance from the focus)
$r_p = a(1-\varepsilon)$	perihelion (nearest distance from the focus)
$k = k_0 m$	
k_0	(units of cm^3/s^2)
ε	eccentricity
a	semi-major axis; radius of auxiliary circle
b $\quad b = a\sqrt{1-\varepsilon^2}$	semi-minor axis
ϕ	azimuthal angle
θ	eccentricity anomaly
l:	directrix.
Λ $\quad \Lambda = \dfrac{b^2}{a} = a(1-\varepsilon^2)$	semi-latus rectum
$L_z = m\sqrt{\dfrac{k_0 b^2}{a}}$	angular momentum along the z axis (conserved, Kepler's second law)
$L_z^{\ 2} = m^2 \dfrac{k_0 b^2}{a} = mk\Lambda$	
$\lvert E\rvert = \dfrac{k}{2a} = \dfrac{k_0 m}{2a}$	Energy (conserved)
$\dfrac{2\lvert E\rvert}{L_z} = \sqrt{\dfrac{k_0}{ab^2}}$	(units of 1/s)
$\varepsilon = \sqrt{1-\dfrac{L_z^{\ 2}}{mka}}$	
$T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{a^3}{k_0}}$	T, period, Kepler's third law

$\omega_0 t = \theta - \varepsilon \sin\theta$ Kepler's equation

$\omega_0 t$: mean anomaly

θ ; eccentricity anomaly

$v_y{}^c = \sqrt{\frac{k_0 a}{b^2}} \varepsilon$ center of velocity circle

$$r = \frac{\Lambda}{1 + \varepsilon\cos\phi} = a(1 - \varepsilon\cos\theta)$$

$$mr^2\dot{\phi} = L_z$$

$\mathbf{A} = \mathbf{v} \times \mathbf{L} - \frac{k}{r}\mathbf{r} = mk_0\varepsilon\mathbf{e}_x = k\varepsilon\mathbf{e}_x$ Laplace-Runge-Lenz vector

$$\mathbf{A}^2 = \frac{2E}{m}\mathbf{L}^2 + k^2$$

$$\begin{aligned}\mathbf{v} \times \mathbf{L} &= k_0 m \frac{\sqrt{1-\varepsilon^2}}{1 - \varepsilon\cos\theta}(\sqrt{1-\varepsilon^2}\cos\theta, \sin\theta) \\ &= \frac{k_0 m}{a}\overrightarrow{K_2 Q}\end{aligned}$$

$$\mathbf{O}_2 = \frac{1}{2}a(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)$$

$$\begin{aligned}\frac{k}{r}\mathbf{r} &= \frac{k_0 m}{1 - \varepsilon\cos\theta}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta) \\ &= \frac{k_0 m}{a}(\overrightarrow{\mathrm{O_1 Q}} + \overrightarrow{\mathrm{K_2 O_2}})\end{aligned}$$

where

$$\overrightarrow{\mathrm{O_1Q}} + \overrightarrow{\mathrm{K_2O_2}} = (\mathbf{Q} - \mathbf{Q}_1) + (\mathrm{O}_2 - \mathrm{K}_2)$$

$$= \frac{a}{1 - \varepsilon\cos\theta}(\cos\theta - \varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)$$

$$\mathbf{A} = \mathbf{v} \times \mathbf{L} - \frac{k}{r}\mathbf{r}$$

$$= \frac{k_0 m}{a}(\mathbf{O}_1 - \mathbf{O}_2)$$

$$= \frac{k_0 m}{a} a\varepsilon\mathbf{e}_x$$

$$\mathbf{O}_1 = \frac{1}{2}(\mathbf{Q} + \mathbf{F}_1)$$

$$\mathbf{O}_2 = \frac{1}{2}(\mathbf{Q} + \mathbf{F}_2)$$

$$\mathbf{K}_1 = \mathbf{F}_1 + \mathbf{Q} - \mathbf{H}_1$$

$$\overline{F_1H_1}\ \overline{F_2H_2} = X_1X_2 = a^2(1-\varepsilon^2) = b^2$$

$$X_2 = \overline{F_2H_2} = b\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}},$$

$$X_1 = \overline{F_1H_1} = b\sqrt{\frac{1-\varepsilon\cos\theta}{1+\varepsilon\cos\theta}}$$

$$r_1 = a(1-\varepsilon\cos\theta), \qquad r_2 = a(1+\varepsilon\cos\theta)$$

<u>**Velocity**</u>

$$
\begin{aligned}
v_1 &= \frac{2|E|}{L_z}\overline{F_2H_2} \\
&= \sqrt{\frac{k_0}{ab^2}}X_2 \\
&= \sqrt{\frac{k_0}{ab^2}}b\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}} \\
&= \sqrt{\frac{k_0}{a}}\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}}
\end{aligned}
$$

$$
\begin{aligned}
v_2 &= \frac{2|E|}{L_z}\overline{F_1H_1} \\
&= \sqrt{\frac{k_0}{ab^2}}X_1 \\
&= \sqrt{\frac{k_0}{a}}\sqrt{\frac{1-\varepsilon\cos\theta}{1+\varepsilon\cos\theta}}
\end{aligned}
$$

$$
v_1v_2 = \frac{k_0}{a}
$$

Angular momentum:

$$
L_1 = v_1r_1\sin\beta = v_1r_1\sin(\frac{\pi}{2}-\alpha) = v_1r_1\cos\alpha
$$

$$
\begin{aligned}
L_2 &= v_2r_2\sin(\pi-\beta) \\
&= v_2r_2\sin(\beta) \\
&= v_2r_2\sin(\frac{\pi}{2}-\alpha) \\
&= v_2r_2\cos(\alpha)
\end{aligned}
$$

$$
v_1r_1 = \sqrt{ak_0}\sqrt{(1+\varepsilon\cos\theta)(1-\varepsilon\cos\theta)}
$$

$$
v_2r_2 = \sqrt{ak_0}\sqrt{(1+\varepsilon\cos\theta)(1-\varepsilon\cos\theta)}
$$

From the conservation of angular momentum, we have

$$v_1 r_1 = v_2 r_2 \qquad \frac{v_1}{v_2} = \frac{r_2}{r_1}$$

Triangle H_2OF_2 (LRL triangle)

$$\overline{F_2O} = a\varepsilon \,. \qquad \overline{OH_2} = a \,, \qquad \overline{F_2H_2} = X_2 = b\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}}$$

$$\mathbf{M}_1 = (a\varepsilon^2\cos\theta, 0)$$

Triangle QM_1F_1 which is similar to triangle H_2OF_2:

$$\overline{M_1F_1} = a\varepsilon(1-\varepsilon\cos\theta) \,. \qquad \overline{F_1Q} = r_1 = a(1-\varepsilon\cos\theta)$$

$$\begin{aligned}\overline{QM_1} &= X_2(1-\varepsilon\cos\theta) \\ &= b\sqrt{\frac{1+\varepsilon\cos\theta}{1-\varepsilon\cos\theta}}(1-\varepsilon\cos\theta) \\ &= b\sqrt{(1+\varepsilon\cos\theta)(1-\varepsilon\cos\theta)}\end{aligned}$$

<u>Central force:</u>

$$\begin{aligned}\mathbf{F} &= -\frac{k}{r^3}\mathbf{r} \\ &= -\frac{k_0 m}{a^2(1-\varepsilon\cos\theta)^3}(\cos\theta-\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)\end{aligned}$$

with

$$\mathbf{e}_r = \frac{1}{r}\mathbf{r} = \frac{1}{1-\varepsilon\cos\theta}(\cos\theta-\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)$$

<u>Acceleration vector:</u>

$$\begin{aligned} m\mathbf{a}_c &= -\frac{mk_0}{a^2(1-\varepsilon\cos\theta)^3}(\cos\theta-\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta) \\ &= -\frac{mk_0}{a^3(1-\varepsilon\cos\theta)^2}\mathbf{H}_2 \\ &= -\frac{mk_0}{ar_1^2}\mathbf{H}_2 \\ &= \mathbf{F} \end{aligned}$$

where

$$\dot{\theta} = \sqrt{\frac{k_0}{a^3}}\frac{1}{1-\varepsilon\cos\theta}.$$

$$\mathbf{H}_2 = \frac{a}{1-\varepsilon\cos\theta}(\cos\theta-\varepsilon, \sqrt{1-\varepsilon^2}\sin\theta)$$

<u>The angles:</u>

$$\cos\alpha = \frac{X_2}{r_2} = \sqrt{\frac{1-\varepsilon^2}{(1+\varepsilon\cos\theta)(1-\varepsilon\cos\theta)}}$$

$$\alpha+\beta = \frac{\pi}{2}$$

$$\tan\delta = \frac{b\sin\theta}{a(\varepsilon+\cos\theta)} = \frac{\sqrt{1-\varepsilon^2}\sin\theta}{(\varepsilon+\cos\theta)}$$

$$\tan\frac{\phi}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\theta}{2}$$