Scattering of identical particles Masatsugu Sei Suzuki Department of Physics, SUNY at Binghamton (Date: April 06, 2015)

If the particles which are scattered are identical particles, we have to take into account the symmetry properties of the total wave functions, symmetric nature for the fermions and ant-symmetric nature for the boson, under the interchange of co-ordinates. The differential cross section for the identical particles is discussed.

1. Wave function of identical particles

We consider the two particles (denoted by particle 1 and particle 2) located at r_1 and r_2 , respectively. We assume a Hamiltonian of two particles at r_1 and r_2 . p_1 and p_2 are the momentum of particles 1 and 2, respectively. $V(|\hat{r}_1 - \hat{r}_2|)$ is the interaction between two particles with mass m_1 and m_2 . This is so-called the central field problem.

We discuss the scattering of these two identical particles which exhibits effect of the symmetry of their wave function. The wave function of these two identical particles ($m_1 = m_2 = m$) can be written as

$$|\psi\rangle = |\boldsymbol{p}_G\rangle|E_r\rangle = |\boldsymbol{p}_G\rangle|\psi_r\rangle.$$

In the representation of the basis $|\mathbf{r}_{G}, \mathbf{r}\rangle$, the wave function can be rewritten as

$$\langle \boldsymbol{r}_{G}, \boldsymbol{r} | \boldsymbol{\psi} \rangle = \langle \boldsymbol{r}_{G} | \boldsymbol{p}_{G} \rangle \langle \boldsymbol{r} | \boldsymbol{\psi}_{r} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar} \boldsymbol{p}_{G} \cdot \boldsymbol{r}_{G}) \langle \boldsymbol{r} | \boldsymbol{\psi}_{r} \rangle$$

Here we have the following definition.

(i) The relative co-ordinate operator:

$$\hat{\boldsymbol{r}}=\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2,$$

(ii) The relative momentum operator:

$$\hat{\boldsymbol{p}} = \frac{m_2 \hat{\boldsymbol{p}}_1 - m_1 \hat{\boldsymbol{p}}_2}{m_1 + m_2} = \frac{1}{2} (\hat{\boldsymbol{p}}_1 - \hat{\boldsymbol{p}}_2).$$

(iii) The co-ordinate operator for the center of mass:

$$\hat{\mathbf{r}}_{G} = \frac{m_{1}\hat{\mathbf{r}}_{1} + m_{2}\hat{\mathbf{r}}_{2}}{m_{1} + m_{2}} = \frac{1}{2}(\hat{\mathbf{r}}_{1} + \hat{\mathbf{r}}_{2}).$$

(iv) The momentum operator for the center of mass:

$$\hat{\boldsymbol{p}}_G = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2.$$

2. Symmetry of the wave function

The wave function can be rewritten as

$$\langle \mathbf{r}_{G}, \mathbf{r} | \psi \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(\frac{i}{\hbar} \mathbf{p}_{G} \cdot \mathbf{r}_{G}) \langle \mathbf{r} | \psi_{r} \rangle$$
$$= \frac{1}{(2\pi\hbar)^{3/2}} \exp[\frac{i}{2\hbar} \mathbf{p}_{G} \cdot (\mathbf{r}_{1} + \mathbf{r}_{2})] \psi_{r}(\mathbf{r})$$

where

$$\langle \boldsymbol{r} | \boldsymbol{\psi}_r \rangle = \boldsymbol{\psi}_r(\boldsymbol{r})$$

with the relative co-ordinate,

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$$

and the center of mass co-ordinate,

$$\boldsymbol{r}_G = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$$

The wave function $\exp[\frac{i}{2\hbar} \mathbf{p}_G \cdot (\mathbf{r}_1 + \mathbf{r}_2)]$, of the center of mass is obviously symmetric under the interchange of the particles \mathbf{r}_1 and \mathbf{r}_2 . Thus, in order that $\psi(-\mathbf{r}) = \psi(\mathbf{r})$, we must have

$$\psi_r(-\mathbf{r}) = \psi_r(\mathbf{r}).$$

This wave function can be decomposed into a radial part and a spherical-harmonics part, i.e.

$$\psi_r(\mathbf{r}) = \langle \mathbf{r} | n, l, m \rangle = R_{nl}(\mathbf{r}) Y_l^m(\theta, \phi), \qquad \hat{\pi} | l, m \rangle = (-1)^l | l, m \rangle$$

Note that exchanging the two particles is equivalent to inverting the vector $\mathbf{r}_1 - \mathbf{r}_2$ (i.e. changing its sign). With such an inversion, the spherical harmonics undergo the transformation

$$Y_l^m(\theta,\phi) \rightarrow (-1)^l Y_l^m(\theta,\phi)$$

So the wave function $\psi_r(\mathbf{r})$ is an even function (even parity) for l = even and an odd function (odd parity).

((Note))

The permutation operator is equivalent to the parity operator in this case.

$$\hat{\pi}|l,m\rangle = (-1)^l|l,m\rangle$$

or

$$\langle \boldsymbol{n} | \hat{\boldsymbol{\pi}} | l, m \rangle = (-1)^{l} \langle \boldsymbol{n} | l, m \rangle, \qquad \langle -\boldsymbol{n} | l, m \rangle = (-1)^{l} \langle \boldsymbol{n} | l, m \rangle$$

where

$$\hat{\pi}|n\rangle = |-n\rangle$$

3. Scattering of identical particles



Fig. Two indistinguishable processes in the scattering of identical particles

Neglecting the symmetry we can write the relative wave function $\psi_r(\mathbf{r})$ asymptotically as

$$e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$

(i) The symmetric wave function $\psi_r(\mathbf{r})$ is obtained as

$$\psi_{rS}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r} + e^{-i\mathbf{k}\cdot\mathbf{r}} + f(\pi-\theta)\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$
$$= (e^{i\mathbf{k}\cdot\mathbf{r}} + e^{-i\mathbf{k}\cdot\mathbf{r}}) + [f(\theta) + f(\pi-\theta)]\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$

where under the interchange of the two particles, $r \rightarrow -r$, we have

$$\theta \to \pi - \theta$$
, $\phi \to \phi + \pi$, $r \to r$

The co-efficient of $\frac{e^{ikr}}{r}$ is the scattering amplitude,

$$f_{s}(\theta) = f(\theta) + f(\pi - \theta)$$

(ii) The anti-symmetric wave function is obtained as

$$\psi_{rA}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r} - e^{-i\mathbf{k}\cdot\mathbf{r}} - f(\pi-\theta)\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$
$$= (e^{i\mathbf{k}\cdot\mathbf{r}} - e^{-i\mathbf{k}\cdot\mathbf{r}}) + [f(\theta) - f(\pi-\theta)]\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}$$

The co-efficient of $\frac{e^{ikr}}{r}$ is the scattering amplitude,

$$f_A(\theta) = f(\theta) - f(\pi - \theta)$$



We now consider the case of two electrons with spin 1/2. The addition of two spins yields the triplet spin state (s = 1, symmetric) and singlet state (s = 0, anti-symmetric). The total wave function should be anti-symmetric because of fermion.

(i) Singlet spin state (anti-symmetric) The scattering is given by

The scattering is given by

$$\begin{pmatrix} \frac{d\sigma}{d\Omega} \end{pmatrix}_{\sin glet} = |f_s|^2 = |f(\theta) + f(\pi - \theta)|^2 = |f(\theta)|^2 + |f(\pi - \theta)|^2 + 2\operatorname{Re}[f^*(\theta)f(\pi - \theta)]$$

(ii) Triplet spin state (symmetric state)

The scattering is given by

$$\begin{pmatrix} \frac{d\sigma}{d\Omega} \end{pmatrix}_{triplet} = |f_A|^2$$

= $|f(\theta) - f(\pi - \theta)|^2$
= $|f(\theta)|^2 + |f(\pi - \theta)|^2 - 2\operatorname{Re}[f^*(\theta)f(\pi - \theta)]$

(iii) Averaged differential cross section

In most scattering experiments, the particles can form either the singlet or the triplet state. Here we define a spin averaged cross section as

$$\begin{pmatrix} \frac{d\sigma}{d\Omega} \\ _{av} \end{pmatrix}_{av} = \frac{1}{4} |f_{s}|^{2} + \frac{3}{4} |f_{A}|^{2}$$

$$= \frac{1}{4} (|f(\theta)|^{2} + |f(\pi - \theta)|^{2} + 2\operatorname{Re}[f^{*}(\theta)f(\pi - \theta)])$$

$$+ \frac{3}{4} (|f(\theta)|^{2} + |f(\pi - \theta)|^{2} - 2\operatorname{Re}[f^{*}(\theta)f(\pi - \theta)])$$

$$= |f(\theta)|^{2} + |f(\pi - \theta)|^{2} - \operatorname{Re}[f^{*}(\theta)f(\pi - \theta)])$$

There are 4 states (3 states are symmetric spin state and 1 state is antisymmetric**4.** Expression of the cross section by partial wave expansion

Here we use the partial wave expansion

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta),$$

with

$$f_l(k) = \frac{1}{k} e^{i\delta_l} \sin(\delta_l)$$

For the triplet state, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{triplet} = \left|f_A\right|^2 = \left|f(\theta) - f(\pi - \theta)\right|^2$$

where

$$f(\theta) - f(\pi - \theta) = \sum_{l=0}^{\infty} (2l+1) [P_l(\cos\theta) - P_l(-\cos\theta)] f_l(k)$$
$$= \sum_{l=0}^{\infty} (2l+1) [1 + (-1)^{l+1}] P_l(\cos\theta) f_l(k)$$
$$= 2 \sum_{l=odd}^{\infty} (2l+1) P_l(\cos\theta) f_l(k)$$

Note that

$$P_l(-x) = (-1)^l P_l(x)$$
.

For the singlet state, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\sin glet} = \left|f_A\right| \left|f(\theta) + f(\pi - \theta)\right|^2$$

where

$$f(\theta) + f(\pi - \theta) = \sum_{l=0}^{\infty} (2l+1) [P_l(\cos\theta) + P_l(-\cos\theta)] f_l(k)$$
$$= \sum_{l=0}^{\infty} (2l+1) [1 + (-1)^l] P_l(\cos\theta) f_l(k)$$
$$= 2 \sum_{l=even}^{\infty} (2l+1) P_l(\cos\theta) f_l(k)$$

5. Feynman's thought experiment on the scattering of un-polarized electrons (Feynman's lecture on physics)

We consider the thought experiment of two electrons with spin 1/2 (one bombarding electron and an electron as a target). We assume that the spins of electrons are un-polarized. There are four types of possibilities for spin states with equal fraction (1/4).

Bombarding electron	electron as target	fraction
$ +\rangle$	$ +\rangle$	1/4
$ +\rangle$	$\left -\right\rangle$	1/4
$\left -\right\rangle$	$ +\rangle$	1/4
$\left -\right\rangle$	$\left -\right\rangle$	1/4

(i)

Suppose, however, the "bombarding" spin is up and the "target" spin is up. The electron entering counter 1 can have spin up. In this case we cannot tell whether it came from the bombarding beam or from the target. The two possibilities are shown in Figs (a) and (b) below; they are indistinguishable in principle, and hence there will be an interference of the two probabilities. Then probability of detecting an electron at the detector D_1 is given by

$$P_1 = \frac{1}{4} \left| f(\theta) - f(\pi - \theta) \right|^2$$

where 1/4 is a fraction of case for the un-polarized electron experiment.



(a)



Fig. The scattering of electrons on electrons. If the incoming electrons have parallel spins, the processes (a) and (b) are indistinguishable.

(ii)

Suppose that the "bombarding" spin is up and the "target" spin is down. The electron entering counter 1 can have spin up or spin down, and by measuring this spin we can tell whether it came from the bombarding beam or from the target. The two possibilities are shown in Figs (a) and (b) below; they are distinguishable in principle, and hence there will be no interference merely an addition of the two probabilities.

The probability of detecting a spin-up electron at the detector D_1 is given by

$$P_2 = \frac{1}{4} \left| f(\theta) \right|^2$$

The probability of detecting a spin-down electron at the detector D_1 is given by







(b)

Fig. The scattering of electrons with antiparallel spins. The processes (a) and (b) are distinguishable.

(iii)

Suppose that the "bombarding" spin is down and the "target" spin is up. The electron entering counter 1 can have spin up or spin down, and by measuring this spin we can tell whether it came from the bombarding beam or from the target. The two possibilities are shown in Figs (a) and (b) below; they are distinguishable in principle, and hence there will be no interference merely an addition of the two probabilities.

The probability of detecting a spin-down electron at the detector D_1 is given by

$$P_4 = \frac{1}{4} \left| f(\theta) \right|^2.$$

The probability of detecting a spin-up electron at the detector D_1 is given by







(b)

Fig. The scattering of electrons with antiparallel spins. The processes (a) and (b) are distinguishable.

(iv)

Suppose, however, the "bombarding" spin is down and the "target" spin is down. The electron entering counter 1 can have spin down. In this case we cannot tell whether it came from the bombarding beam or from the target. The two possibilities are shown in Figs (a) and (b) below; they are indistinguishable in principle, and hence there will be an interference of the two probabilities. Then probability of detecting an electron at the detector D_1 is given by

$$P_6 = \frac{1}{4} \left| f(\theta) - f(\pi - \theta) \right|^2,$$





Fig. The scattering of electrons on electrons. If the incoming electrons have parallel spins, the processes (a) and (b) are indistinguishable.

Fraction of cases	Spin of particle 1	Spin of particle 2	Spin at D_1	$\begin{array}{c} {\rm Spin} \ {\rm at} \\ D_2 \end{array}$	Probability	
$\frac{1}{4}$	up	up	up	up	$ f(\theta)-f(\pi-\theta) ^2$	
$\frac{1}{4}$	down	down	down	down	$ f(\theta)-f(\pi-\theta) ^2$	
$\frac{1}{4}$ up	up	down 5	up	down	$ f(\theta) ^2$	
	laowin (down	up	$ f(\pi-\theta) ^2$		
$\frac{1}{4}$ down	down	up ∫	up	down	$ f(\pi-\theta) ^2$	
	uown	۳۶ ک	down	up	$ f(\theta) ^2$	
Total probability $= \frac{1}{2} f(\theta) - f(\pi - \theta) ^2 + \frac{1}{2} f(\theta) ^2 + \frac{1}{2} f(\pi - \theta) ^2$						

Now if electrons are completely un-polarized, the results for this experiment are best calculated by listing all of the various possibilities as we have done in Table. A separate *probability* is computed for each distinguishable alternative. The total probability is then the sum of all the separate probabilities.

The total probability is given by

$$P_{tot} = P_1 + P_2 + P_3 + P_4 + P_5 + P_6$$

= $2\frac{1}{4} |f(\theta) - f(\pi - \theta)|^2 + 2\frac{1}{4} |f(\theta)|^2 + 2\frac{1}{4} |f(\pi - \theta)|^2$
= $\frac{1}{2} |f(\theta) - f(\pi - \theta)|^2 + \frac{1}{2} |f(\theta)|^2 + \frac{1}{2} |f(\pi - \theta)|^2$
= $|f(\theta)|^2 + |f(\pi - \theta)|^2 - \operatorname{Re}[f(\theta)f^*(\pi - \theta)]$

6. Fermion with s = 1/2 and boson with s = 1

(i) Two particles (fermion) with s = 1/2

We consider the scattering of two identical spin-1/2 fermions, such as electrons. For the triplet spin state (s = 1, symmetric spin state, degeneracy 3) the spatial function should be antisymmetric. For the singlet spin state (s = 0, antisymmetric spin state, degeneracy 1) the spatial function is symmetric. If the spins of electrons are un-polarized, the average cross section can be expressed as

$$\frac{3}{4}\left|f(\theta) - f(\pi - \theta)\right|^2 + \frac{1}{4}\left|f(\theta) + f(\pi - \theta)\right|^2 = \left|f(\theta)\right|^2 + \left|f(\pi - \theta)\right|^2 - \operatorname{Re}[f^*(\theta)f(\pi - \theta)]$$

((Note))

CG (Clebsch-Gordan) values:
$$s_1 = 1/2$$
, $s_2 = 1/2$

 $D_{1/2} \ge D_{1/2} = D_1 + D_0$

(i) S = 1 (m = 1, 0, -1). symmetric spin state

$$\begin{vmatrix} ++ \rangle & (m=1) \\ \\ \frac{|+-\rangle+|-+\rangle}{\sqrt{2}} & (m=0) \\ |--\rangle & (m=-1) \end{aligned}$$

(ii) S = 0 (m = 0). antisymmetric spin state.

$$\frac{\left|+-\right\rangle-\left|-+\right\rangle}{\sqrt{2}} \qquad (m=0)$$

(ii) Two particles (boson) with s = 1

The particle with spin s = 1 is a boson. So the wavefunction should be symmetric under the exchange of particles. There are 9 states for two spins with s = 1. There are 6 symmetric spin states and 3 antisymmetric spin states. Then the total cross section is obtained as

$$\frac{6}{9}\left|f(\theta) + f(\pi - \theta)\right|^2 + \frac{3}{9}\left|f(\theta) - f(\pi - \theta)\right|^2 = \left|f(\theta)\right|^2 + \left|f(\pi - \theta)\right|^2 + \frac{2}{3}\operatorname{Re}[f^*(\theta)f(\pi - \theta)]$$

((Note) CG (Clebsch-Gordan) values: $s_1 = 1, s_2 = 1$

$$D_1 \times D_1 = D_2 + D_1 + D_0$$

(i) $S = 2(|m| \le 2)$ symmetric spin states

$$|1,1\rangle|1,1\rangle$$
 (m=2)

$$\frac{|1,1\rangle|1,0\rangle + |1,0\rangle|1,1\rangle}{\sqrt{2}} \qquad (m=1)$$

$$\frac{|1,1\rangle|1,-1\rangle+2|1,0\rangle|1,0\rangle+|1,-1\rangle|1,1\rangle}{\sqrt{6}} \qquad (m=0)$$

$$\frac{|1,0\rangle|1,-1\rangle+|1,-1\rangle|1,0\rangle}{\sqrt{2}} \qquad (m=-1)$$

$$|1,-1\rangle |1,-1\rangle \qquad (m=-2)$$

(**ii**)

$S = 1 (|m| \le 1)$

antisymmetric spin state

$$\frac{|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle}{\sqrt{2}} \qquad (m=1)$$

$$\frac{|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle}{\sqrt{2}} \qquad (m=0)$$

$$\frac{|1,0\rangle|1,-1\rangle-|1,-1\rangle|1,0\rangle}{\sqrt{2}} \qquad (m=-1)$$

(iii)
$$S = 0 \ (m = 0)$$
 symmetric spin state

$$\frac{|1,1\rangle|1,-1\rangle - |1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{3}} \qquad (m = 0)$$

7. General case (Schiff, Schwinger)

The spin states of the two particles, each of spin *s*, can be separated into symmetrical states and antisymmetrical states. We know that for s = 1/2, the $(2s+1)^2 = 4$ states consists of three symmetrical states and one anti-symmetrical one. In general, if you have two variables, each taking on *n* values, the number of antisymmetrical combinations is $\frac{n(n-1)}{2}$, and the number of symmetrical one is $\frac{n(n-1)}{2} + n^2 = \frac{n(n+1)}{2}$ correctly adding to n^2 . Thus the fraction of spin states that are symmetrical or antisymmetrical is (n = 2s + 1).

Symmetrical fraction:
$$\frac{n(n+1)}{2n^2} = \frac{n+1}{2n} = \frac{s+1}{2s+1} > \frac{1}{2}$$
,
Antisymmetrical fraction: $\frac{n(n-1)}{2n^2} = \frac{n-1}{2n} = \frac{s}{2s+1} < \frac{1}{2}$,

As a check we put $s = \frac{1}{2}$ and get the respective fractions of 3/4 and 1/4.

In a collision with all spin states equally probable, the fraction of symmetrical spin states will have the scattering amplitude

$$f(\theta) \pm f(\pi - \theta),$$

for the respective Bose-Einstein (BE)/Fermi-Dirac (FD) statistics, whereas the spin antisymmetrical fraction will have the scattering amplitude

$$f(\theta) \mp f(\pi - \theta)$$

So

$$\frac{d\sigma}{d\Omega} = \frac{s+1}{2s+1} \left| f(\theta) \pm f(\pi-\theta) \right|^2 + \frac{s}{2s+1} \left| f(\theta) \mp f(\pi-\theta) \right|^2$$
$$= \left| f(\theta) \right|^2 + \left| f(\pi-\theta) \right|^2 \pm \frac{2}{2s+1} \operatorname{Re}[f^*(\theta)f(\pi-\theta)]$$

and, for $\theta = \frac{\pi}{2}$,

$$\frac{(d\sigma/d\Omega)_{QM}}{(d\sigma/d\Omega)_{Cl}}|_{\theta=\pi/2} = 1 + \frac{1}{2s+1} = \frac{s+1}{s+\frac{1}{2}} > 1, \qquad \text{for BE statistics}$$

$$\frac{(d\sigma/d\Omega)_{QM}}{(d\sigma/d\Omega)_{Cl}}|_{\theta=\pi/2} = 1 - \frac{1}{2s+1} = \frac{s}{s+\frac{1}{2}} < 1, \qquad \text{for FD statistics}$$

which shows how, in principle, the statistics and the spin can be determined. It is an empirical fact, one now understood theoretically, that there is a connection between spin and statistics:

BE stastistics: s = 0, 1, 2,FD stastistics: s = 1/2, 3/2, 5/2,

so, in fact the possibilities are

$$\frac{\left(\frac{d\sigma}{d\Omega}\right)_{QM}}{\left(\frac{d\sigma}{d\Omega}\right)_{Cl}}|_{\theta=\pi/2}=2, 4/3, 6/5, \dots \qquad \text{for BE statistics}$$

$$\frac{\left(\frac{d\sigma}{d\Omega}\right)_{QM}}{\left(\frac{d\sigma}{d\Omega}\right)_{Cl}}|_{\theta=\pi/2}=1/2, 3/4, 5/6, \dots \qquad \text{for FD statistics}$$

((Note)) General formula for the scattering cross section for two particles with s

In general, for an un-polarized beam of particles with spin *s*, the system can be in $(2s+1)^2$ spin states that are distributed with equal probabilities. For the total number of possibilities, (2s+1) spin states are antisymmetric. Thus the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 + \frac{2}{2s+1} (-1)^{2s} \operatorname{Re}[f^*(\theta) f(\pi - \theta)].$$

where *s* is an half integer for the fermions and an integer for the boson (Schiff L.I. Quantum Mechanics).

(i)

When s = 1/2 (fermion)

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 - \left(\operatorname{Re}[f^*(\theta) f(\pi - \theta)] \right)$$

(ii)

When s = 1 (boson)

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 + \frac{2}{3} \operatorname{Re}[f^*(\theta) f(\pi - \theta)]$$

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APPENDIX: Two identical particles with s = 3/2

CG (Clebsch-Gordan) values: $s_1 = 3/2$, $s_2 = 3/2$ (2s+1 = 4).

$$D_{3/2} \times D_{3/2} = D_3 + D_2 + D_1 + D_0$$

There are 4 x 4 = 16 states. The number of symmetric states is 7+3 = 10. The number of antisymmetric states is 5+1 = 6 states.

$$1+2+3+4 = \frac{1}{2}4 \times 5 = 10$$
 (for symmetric states)

 $1+2+3 = \frac{1}{2}3 \times 4 = 6$ (for antisymmetric states)

(i)
$$S = 3 \ (m = 3, 2, 1, 0, -1, -2, -3)$$
 symmetric spin states (7 states)

$$\begin{vmatrix} \frac{3}{2}, \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2}, \frac{3}{2}$$

(ii) $S = 2 \ (m = 2, 1, 0, -1, -2)$ antisymmetric spin state (5 states) $\frac{\left|\frac{3}{2}, \frac{3}{2}\right\rangle \left|\frac{3}{2}, \frac{1}{2}\right\rangle - \left|\frac{3}{2}, \frac{1}{2}\right\rangle \left|\frac{3}{2}, \frac{3}{2}\right\rangle}{\sqrt{2}} \qquad (m = 2)$ $\frac{\left|\frac{3}{2}, \frac{3}{2}\right\rangle \left|\frac{3}{2}, -\frac{1}{2}\right\rangle - \left|\frac{3}{2}, -\frac{1}{2}\right\rangle \left|\frac{3}{2}, \frac{3}{2}\right\rangle}{\sqrt{2}} \qquad (m = 1)$

$$\frac{\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{3}{2},-\frac{3}{2}\right\rangle + \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{3}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{3}{2},\frac{3}{2}\right\rangle}{2} \qquad (m=0)$$

$$\frac{\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{3}{2},-\frac{3}{2}\right\rangle - \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{3}{2},\frac{1}{2}\right\rangle}{\sqrt{2}} \qquad (m=-1)$$

$$\frac{\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{3}{2},-\frac{3}{2}\right\rangle - \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{3}{2},-\frac{1}{2}\right\rangle}{\sqrt{2}} \qquad (m=-2)$$

(ii)
$$S = 1 \ (m = 1, 0, -1)$$
 symmetric spin state (3 states)

$$\frac{\sqrt{3} \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{3}{2}, -\frac{1}{2} \right\rangle - 2 \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle + \sqrt{3} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{3}{2}, \frac{3}{2} \right\rangle}{\sqrt{10}} \qquad (m = 1)$$

$$\frac{3 \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{3}{2}, -\frac{3}{2} \right\rangle - \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{3}{2}, -\frac{1}{2} \right\rangle - \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle + 3 \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left| \frac{3}{2}, \frac{3}{2} \right\rangle}{2\sqrt{5}} \qquad (m = 0)$$

$$\frac{\sqrt{3} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{3}{2}, -\frac{3}{2} \right\rangle - 2 \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{3}{2}, -\frac{1}{2} \right\rangle + \sqrt{3} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle}{\sqrt{10}} \qquad (m = -1)$$

(iv)
$$S = 0 \ (m = 0)$$
 antisymmetric spin state (1 state)
 $\frac{\left|\frac{3}{2}, \frac{3}{2}\right| \left|\frac{3}{2}, -\frac{3}{2}\right\rangle - \left|\frac{3}{2}, \frac{1}{2}\right\rangle \left|\frac{3}{2}, -\frac{1}{2}\right\rangle + \left|\frac{3}{2}, -\frac{1}{2}\right\rangle \left|\frac{3}{2}, \frac{1}{2}\right\rangle - \left|\frac{3}{2}, -\frac{3}{2}\right\rangle \left|\frac{3}{2}, \frac{3}{2}\right\rangle}{2}$