

# **Lecture Note on Coupled Pendulum**

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(August 24, 2009)

## **Abstract**

One of the authors (M.S.) has been teaching the physics laboratory courses (from the Sophomore Laboratory to the Advanced (Graduate) Laboratory) for many years at the Binghamton University. Undergraduate students spend a lot of time in studying the fundamental physics in the physics courses such as the introductory physics, mechanics, electromagnetism, and so on. Even if they understand the theories in a sufficient depth to be able to apply it readily, they encounter some difficulties in understanding the essence of the experiments in the physics laboratory courses, since the conditions of the experiments are more complicated than the ideal conditions of the physics models. As far as we know, so far there are very few opportunities in discussing the gap between the real physics and ideal physics for the physics laboratory courses. The mathematics which is required for the real physics is much more complicated than the mathematics for the ideal physics.

In nature there are many examples of the coupled pendulum. The nature of the physics is governed mainly by the coupling constant between two pendulums, leading to the complicated physical behaviors. This note is an attempt to provide an account which treats physical examples (coupled pendulum in the present case) in some depth. As an example, we discuss the physics of coupled pendulum in the Sophomore Laboratory and the one dimensional chain in the Junior Laboratory in the Physics Department of the Binghamton University. In this note, first we discuss the motion of coupled pendulum the absence of an external force. The motion is described by nonlinear differential equations even in the limit of small angles. The numerical calculations are carried out using the Mathematica programs. These programs are very useful since the problems can be solved in spite of the complicated nature. It is clearly seen that there are two normal modes, the in-phase mode and the out of phase mode. The motion of the coupled pendulum is discussed by Feynman using very simple model. The motions are expressed by linear differential equations in the limit of small angles. The in-phase mode and out-of-phase modes can be easily derived. The general motion is a superposition of the in-phase mode and the out-of phase mode. Next we discuss the motion of coupled pendulum under an external force with the angular frequency  $\omega$ . When  $\omega$  is close to the angular frequencies of the normal modes, the absorption of the power by the system becomes maximum. We also discuss the motion of a series of masses connected by springs, where the masses are suspended on strings. Experimentally we need to determine the dispersion relation of the 1D chain system for the propagation of the longitudinal and transverse waves. We discuss how to measure the characteristic angular frequency of the particular normal modes.

In writing this note, we read many excellent textbooks and reviews on the simple pendulum and coupled pendulum.<sup>1-8</sup> These references are very useful in our understanding the physics of the coupled pendulum.

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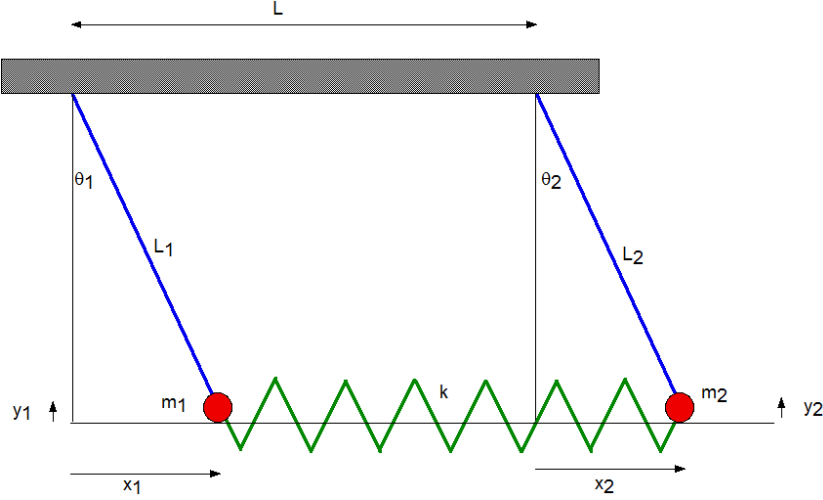
## REFERENCES

## APPENDIX

## 1. Coupled pendulum

### 1.1 Nonlinear differential equations

Two pendulums, connected by a mass-less spring, are allowed to vibrate in the same vertical plane. Here we consider the limiting case of small oscillations. Figure 1 shows the schematic diagram of the system.



**Fig.1** Schematic diagram of the coupled pendulum with mass  $m_1$  and mass  $m_2$ . These masses are connected by a spring with the spring constant  $k$ .  $L_1$  and  $L_2$  are the lengths of the un-stretched spring. The mass  $m_1$  is located at  $(x_1, y_1)$  from the reference vertical line-1. The mass  $m_2$  is located at  $(x_2, y_2)$  from the reference vertical line-2.

The coordinates of the two masses from each reference vertical line are expressed by

$$\begin{aligned} x_1 &= L_1 \sin \theta_1 \\ y_1 &= -L_1 \cos \theta_1, \end{aligned} \quad (1.1)$$

for the mass 1, and

$$\begin{aligned} x_2 &= L_2 \sin \theta_2 \\ y_2 &= L_2 \cos \theta_2, \end{aligned} \quad (1.2)$$

for the mass 2, respectively. The Lagrangian of the system is given by a difference between the kinetic energy and the potential energy,

$$L_T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{y}_2^2 - \frac{1}{2} k (L + x_2 - x_1 - L)^2 - m_1 g y_1 - m_2 g y_2. \quad (1.3)$$

Here we assume that the term of  $m_1 \dot{y}_1^2 / 2 + m_1 \dot{y}_2^2 / 2$  can be neglected compared to the other terms,

$$L_r = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 - \frac{1}{2} k (x_2 - x_1)^2 - m_1 g y_1 - m_2 g y_2, \quad (1.4)$$

where  $L$  is the length of un-stretched spring. This equation can be rewritten as

$$\begin{aligned} L_r = & g L_1 m_1 \cos \theta_1 + g L_2 m_2 \cos \theta_2 - \frac{1}{2} k L_1^2 \sin^2 \theta_1 + k L_1 L_2 \sin \theta_1 \sin \theta_2 \\ & - \frac{1}{2} k L_2^2 \sin^2 \theta_2 + \frac{1}{2} L_1^2 m_1 \dot{\theta}_1^2 \cos^2 \theta_1 + \frac{1}{2} L_2^2 m_2 \dot{\theta}_2^2 \cos^2 \theta_2 \end{aligned} \quad (1.5)$$

The Lagrange's equation given by

$$\frac{d}{dt} \left( \frac{\partial L_r}{\partial \dot{\theta}_1} \right) = \frac{\partial L_r}{\partial \theta_1}, \quad \frac{d}{dt} \left( \frac{\partial L_r}{\partial \dot{\theta}_2} \right) = \frac{\partial L_r}{\partial \theta_2}, \quad (1.6)$$

leads to the differential equations for  $\theta_1$  and  $\theta_2$ ,

$$\ddot{\theta}_1 + \frac{m_1 g \tan \theta_1 + k L_1 \sin \theta_1 - k L_2 \sin \theta_2 - L_1 m_1 \dot{\theta}_1^2 \sin \theta_1}{L_1 m_1 \cos \theta_1} = 0, \quad (1.7)$$

$$\ddot{\theta}_2 - \frac{(m_2 g + k L_2 \cos \theta_2) \tan \theta_2 + k L_1 \sin \theta_1 + L_2 m_2 \dot{\theta}_2^2 \sin \theta_2}{L_2 m_2 \cos \theta_2} = 0. \quad (1.8)$$

For simplicity, we assume that  $|\theta_1|$  and  $|\theta_2|$  are very small angles;

$$\begin{aligned} \cos \theta_1 &= 1, & \cos \theta_2 &= 1, \\ \tan \theta_1 &= \sin \theta_1, & \tan \theta_2 &= \sin \theta_2 \end{aligned}$$

Then we have the nonlinear differential equation

$$\ddot{\theta}_1 + \frac{(m_1 g + k L_1 - L_1 m_1 \dot{\theta}_1^2) \sin \theta_1 - k L_2 \sin \theta_2}{L_1 m_1 \cos \theta_1} = 0, \quad (1.9)$$

$$\ddot{\theta}_2 + \frac{(m_2 g + k L_2 - L_2 m_2 \dot{\theta}_2^2) \sin \theta_2 - k L_1 \sin \theta_1}{L_2 m_2 \cos \theta_2} = 0, \quad (1.10)$$

with the initial conditions given by

$$\begin{aligned} \theta_1(t=0) &= \theta_{10}, & \theta_2(t=0) &= \theta_{20}, \\ v_1(t=0) &= v_{10}, & v_2(t=0) &= v_{20}. \end{aligned} \quad (1.11)$$

((Mathematica))

The derivation of the Lagrange's equation by using the Mathematica (VariationalD program).

```
x1 = L1 Sin[θ1[t]]; x2 = L2 Sin[θ2[t]]; y1 = -L1 Cos[θ1[t]];  
y2 = -L2 Cos[θ2[t]]
```

```
-L2 Cos[θ2[t]]
```

```
LL =  $\frac{1}{2} m_1 D[x_1, t]^2 + \frac{1}{2} m_2 D[x_2, t]^2 - \frac{1}{2} k (L + x_2 - x_1 - L)^2 +$   
-m1 g y1 - m2 g y2 // Expand
```

```
g L1 m1 Cos[θ1[t]] + g L2 m2 Cos[θ2[t]] -  $\frac{1}{2} k L^2 Sin[θ1[t]]^2 +$   
 $k L1 L2 Sin[θ1[t]] Sin[θ2[t]] - \frac{1}{2} k L^2 Sin[θ2[t]]^2 +$   
 $\frac{1}{2} L1^2 m1 Cos[θ1[t]]^2 θ1'[t]^2 + \frac{1}{2} L2^2 m2 Cos[θ2[t]]^2 θ2'[t]^2$ 
```

```
Needs["VariationalMethods`"]
```

```
eq1 = VariationalD[LL, θ1[t], t] / Cos[θ1[t]] // FullSimplify
```

```
-L1 (k L1 Sin[θ1[t]] - k L2 Sin[θ2[t]] + g m1 Tan[θ1[t]] -  
L1 m1 Sin[θ1[t]] θ1'[t]^2 + L1 m1 Cos[θ1[t]] θ1''[t])
```

```
eq2 = VariationalD[LL, θ2[t], t] / Cos[θ2[t]] // FullSimplify
```

```
-L2 (-k L1 Sin[θ1[t]] + k L2 Sin[θ2[t]] + g m2 Tan[θ2[t]] -  
L2 m2 Sin[θ2[t]] θ2'[t]^2 + L2 m2 Cos[θ2[t]] θ2''[t])
```

```
s1 = Solve[eq1 == 0, θ1''[t]]
```

```
{ {θ1''[t] →  
-  $\frac{1}{L1 m1} Sec[θ1[t]] (k L1 Sin[θ1[t]] - k L2 Sin[θ2[t]] +$   
g m1 Tan[θ1[t]] - L1 m1 Sin[θ1[t]] θ1'[t]^2) } }
```

```
s2 = Solve[eq2 == 0, θ2''[t]] // Simplify
```

```
{ {θ2''[t] →  
 $\frac{1}{L2 m2} Sec[θ2[t]] (k L1 Sin[θ1[t]] - (g m2 + k L2 Cos[θ2[t]])$   
Tan[θ2[t]] + L2 m2 Sin[θ2[t]] θ2'[t]^2) } }
```

```
s11 = θ1''[t] - (θ1''[t] /. s1[[1]])
```

```
 $\frac{1}{L1 m1} Sec[θ1[t]] (k L1 Sin[θ1[t]] - k L2 Sin[θ2[t]] +$   
g m1 Tan[θ1[t]] - L1 m1 Sin[θ1[t]] θ1'[t]^2) + θ1''[t]
```

```
s21 = θ2''[t] - (θ2''[t] /. s2[[1]])
```

```
-  $\frac{1}{L2 m2} Sec[θ2[t]]$   
(k L1 Sin[θ1[t]] - (g m2 + k L2 Cos[θ2[t]]) Tan[θ2[t]] +  
L2 m2 Sin[θ2[t]] θ2'[t]^2) + θ2''[t]
```

## 1.2 Numerical calculation of the nonlinear differential equation by Mathematica:

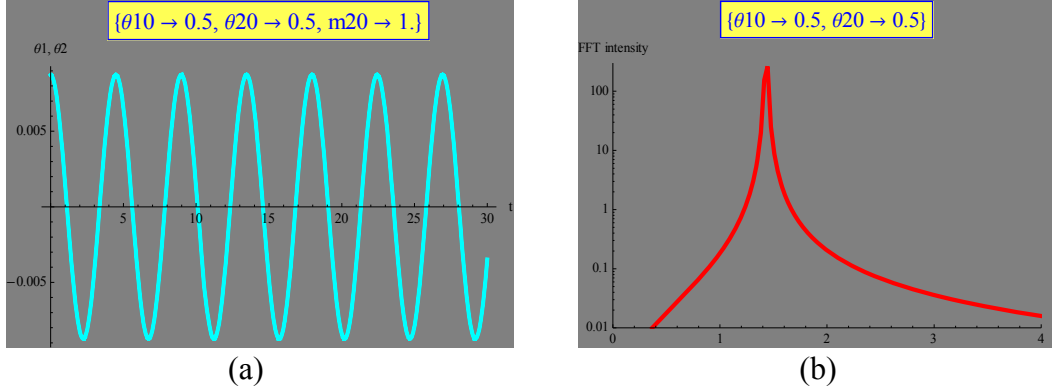
### Plot of $\theta_1(t)$ vs $\theta_2(t)$

The nonlinear differential equation can be solved using the Mathematica (NDSolve program). For simplicity, we assume that  $m = m_1 = m_2 = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ ,  $v_1(t = 0) = v_{10} = 0$ , and  $v_2(t = 0) = v_{20} = 0$ .  $\theta_1(t = 0) = \theta_{10} = 0.5^\circ$ . Here we calculate  $\theta_1(t)$ ,  $\theta_2(t)$ ,  $\theta_1'(t)$ ,  $\theta_2'(t)$ , when  $\theta_2(t = 0) = \theta_{20}$  is changed as a parameter,  $\theta_{20} = -2.5^\circ, -1.5^\circ, -0.5^\circ, 0^\circ, 0.5^\circ, 1.5^\circ$ , and  $2.5^\circ$ . The numerical calculation is carried out for  $t = 0 - 300$ . The wave analysis in  $\theta_1(t)$  thus calculated are carried out using the fast Fourier spectrum (FFT) (Mathematica program). This makes it so easy for one to determine the angular frequencies of the normal modes involved in the wave. The fundamental concept of the normal modes will be discussed in Sec.2.

### 1.2.1 $\theta_1$ vs $t$ and $\theta_2$ vs $t$ and FFT (fast Fourier transform) spectrum

#### (i) In-phase mode: $v_{10} = v_{20} = 0$ . $\theta_{10} = 0.5^\circ$ and $\theta_{20} = 0.5^\circ$

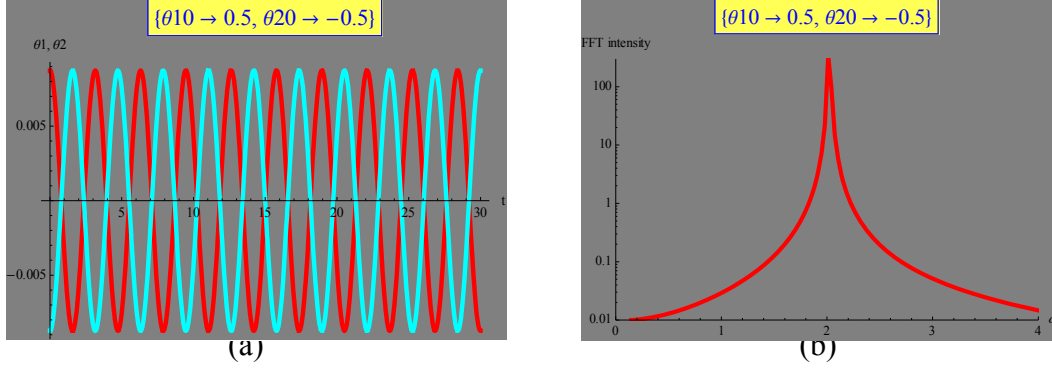
Figure 2(a) shows the plot of  $\theta_1$  and  $\theta_2$  as a function of  $t$ , where  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0.5^\circ$ . It is found that  $\theta_1(t) = \theta_2(t)$  at any  $t$ . Figure 2(b) shows the fast Fourier transform (FFT) of  $\theta_1$  vs  $t$ . This FFT spectrum indicates that there is only one normal mode with the angular frequency  $\omega = \omega_0 = \sqrt{g/L} (= 1.4)$ . This mode corresponds to the in-phase mode (see Sec. 2).



**Fig.2** (a)  $\theta_1$  vs  $t$  (blue) and  $\theta_2$  vs  $t$  (blue).  $m_1 = m_2 = 1$ .  $k = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ . The initial conditions are given by  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0.5^\circ$ . (b) FFT spectrum for  $\theta_1(t)$  vs  $t$  (FFT intensity vs  $\omega$ ). There is one peak at  $\omega \approx 1.4$ .

#### (ii) The out-of-phase mode: $v_{10} = v_{20} = 0$ . $\theta_{10} = 0.5^\circ$ and $\theta_{20} = -0.5^\circ$

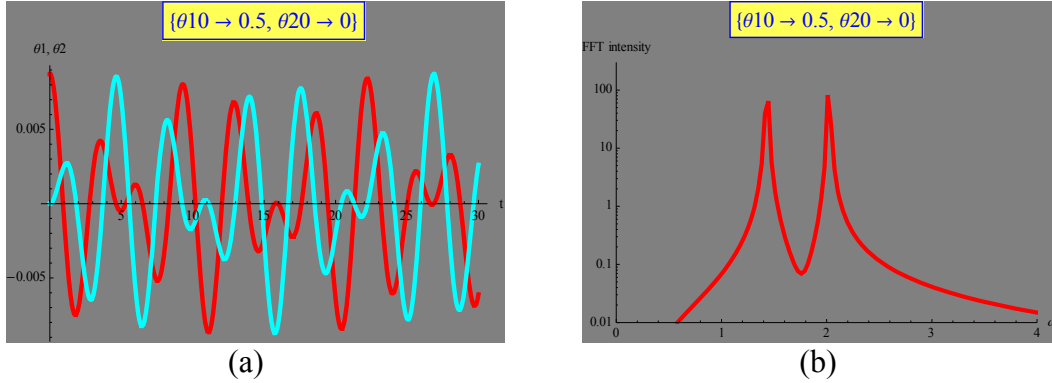
Figure 3(a) shows the plot of  $\theta_1$  and  $\theta_2$  as a function of  $t$ , where  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -0.5^\circ$ . It is found that  $\theta_1(t) = -\theta_2(t)$  at any  $t$ . Figure 3(b) shows the FFT spectrum of  $\theta_1$  vs  $t$ , which indicates that there is only one normal mode with the angular frequency  $\omega \approx 2.0$ . This mode corresponds to the out-of-phase mode (see Sec. 2).



**Fig.3** (a)  $\theta_1$  vs  $t$  (red) and  $\theta_2$  vs  $t$  (blue).  $m_1 = m_2 = 1$ ,  $k = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ . The initial conditions are given by  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -0.5^\circ$ . (b) FFT spectrum for  $\theta_1(t)$  vs  $t$  (FFT intensity vs  $\omega$ ). There is one peak at  $\omega \approx 2.0$  corresponding to the out-of-phase mode.

(iii)  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .

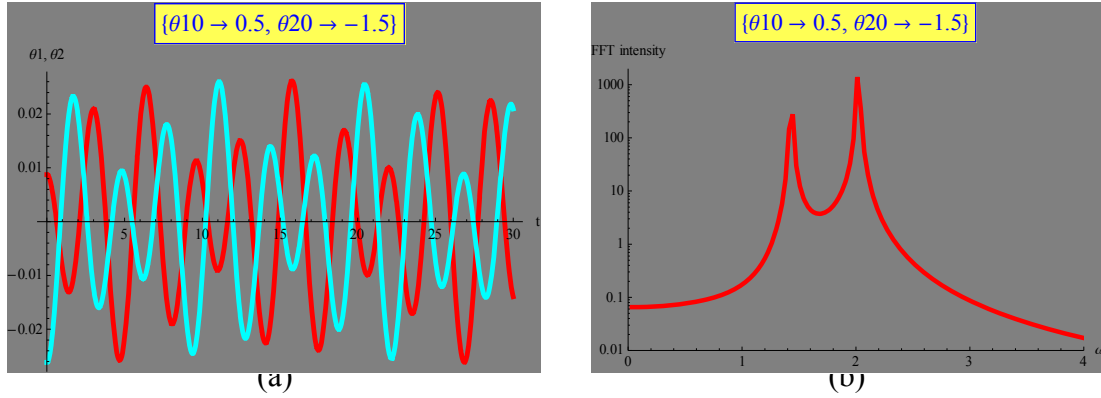
Figure 4(a) shows the plot of  $\theta_1$  and  $\theta_2$  as a function of  $t$ , where  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ . Figure 4(b) shows the FFT spectrum of  $\theta_1$  vs  $t$ , which indicates that there are two normal modes with the angular frequencies  $\omega \approx 2.0$  and  $\omega \approx 1.4$ . The curve of  $\theta_1$  vs  $t$  (also  $\theta_2$  vs  $t$ ) is formed of the superposition of two waves with the normal modes ( $\omega \approx 2.0$  and  $\omega \approx 1.4$ ). The peak height at  $\omega \approx 2.0$  is almost the same as that at  $\omega \approx 1.4$ .



**Fig.4** (a)  $\theta_1$  vs  $t$  (red) and  $\theta_2$  vs  $t$  (blue).  $m_1 = m_2 = 1$ ,  $k = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ .  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ . (b) FFT spectrum for  $\theta_1(t)$  vs  $t$  (FFT intensity vs  $\omega$ ). There are two peaks at  $\omega \approx 2.0$  and  $\omega \approx 1.4$ .

(iv)  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -1.5^\circ$

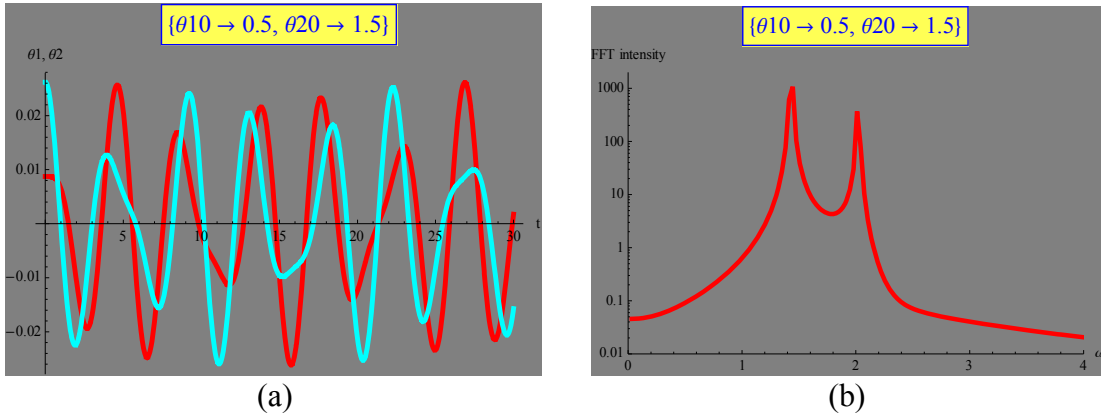
In the case of the initial conditions with  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -1.5^\circ$ , it is found from Figs.5(a) and 5(b) that the curve of  $\theta_1$  vs  $t$  (also  $\theta_2$  vs  $t$ ) are formed of the superposition of two waves with the normal modes ( $\omega \approx 2.0$  and  $\omega \approx 1.4$ ). The peak height at  $\omega \approx 2.0$  is higher than that at  $\omega \approx 1.4$ , indicating the dominance of the normal mode with  $\omega \approx 2.0$ .



**Fig.5** (a)  $\theta_1$  vs  $t$  (red) and  $\theta_2$  vs  $t$  (blue).  $m_1 = m_2 = 1$ ,  $k = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ .  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -1.5^\circ$ . (b) FFT spectrum for  $\theta_1(t)$  vs  $t$  (FFT intensity vs  $\omega$ ). There are two peaks at  $\omega \approx 2.0$  and  $\omega \approx 1.4$ .

(v)  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 1.5^\circ$

In the case of the initial conditions with  $v_{10} = v_{20} = 0$ ,  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 1.5^\circ$ , it is found from Figs.6(a) and 6(b) that the curve of  $\theta_1$  vs  $t$  (also  $\theta_2$  vs  $t$ ) is formed of the superposition of two waves with the normal modes ( $\omega \approx 2.0$  and  $\omega \approx 1.4$ ). The peak height at  $\omega \approx 1.4$  is higher than that at  $\omega \approx 2.0$ , indicating the dominance of the normal mode with  $\omega \approx 1.4$ .



**Fig.6** (a)  $\theta_1$  vs  $t$  (red) and  $\theta_2$  vs  $t$  (blue).  $m_1 = m_2 = 1$ ,  $k = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ .  $v_{10} = v_{20} = 0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 1.5^\circ$ . (b) FFT spectrum for  $\theta_1(t)$  vs  $t$  (FFT intensity vs  $\omega$ ). There are two peaks at  $\omega \approx 2.0$  and  $\omega \approx 1.4$ .



((Mathematica)) The program used in order to get the FFT spectrum shown in Fig.4(b).

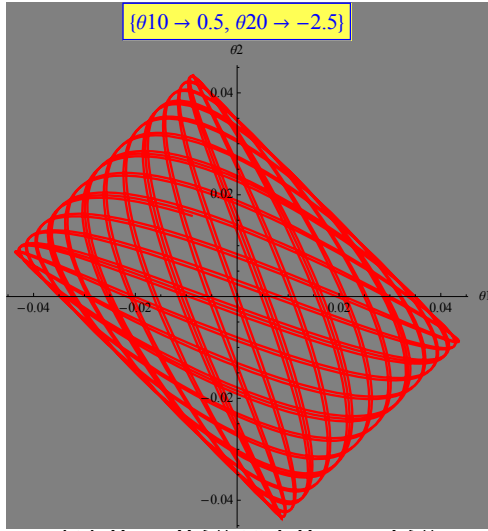
```

m1 = 1; m2 = 1.0; k = 1; l1 = 5; l2 = 5; g = 9.8;
time[{θ10_, v10_, θ20_, v20_}, tmax_, N1_] :=
Module[{numsol, numtable},
  numsol =
  NDSolve[
    {θ1''[t] ==
      ((m1*(l1*θ1'[t]^2 - g) - k*l1)*Sin[θ1[t]] + k*l2*Sin[θ2[t]]) /
      (m1*l1*Cos[θ1[t]]),
      θ2''[t] ==
      ((m2*(l2*θ2'[t]^2 - g) - k*l2)*Sin[θ2[t]] + k*l1*Sin[θ1[t]]) /
      (m2*l2*Cos[θ2[t]]), θ1[0] == θ10, θ1'[0] == v10, θ2[0] == θ20,
      θ2'[0] == v20}, {θ1[t], θ2[t]}, {t, 0, tmax}];
  numtable = Table[{t, Evaluate[ $\frac{180}{\pi}$  θ1[t] /. numsol]}, {t, 0, tmax,  $\frac{tmax}{N1}$ }}];
eq1 = time[{0.5°, 0, 0°, 0}, 200, 8192]; d1 = Dimensions[eq1][[1]];
list1 = Table[eq1[[i, 2]][[1]], {i, 1, d1 - 1}]; list2 = Fourier[list1] // Chop;
list3 = Table[{ $\frac{2\pi}{200}$  n, Abs[list2[[n]]]^2}, {n, 1, 1024}];
s1 = ListLogPlot[list3, Joined → True, PlotRange → {{0, 4}, {0.01, 300}},
  PlotStyle → {Hue[0.0], Thickness[0.01]},
  AxesLabel → {"ω", "FFT intensity"},
  PlotLabel → Style[Framed[{θ10 → 0.5, θ20 → 0}], 16, Blue,
    Background → Lighter[Yellow]], Background → Gray, PlotRange → All,
  Ticks → Automatic, DisplayFunction → Identity];
Show[s1, DisplayFunction → $DisplayFunction]

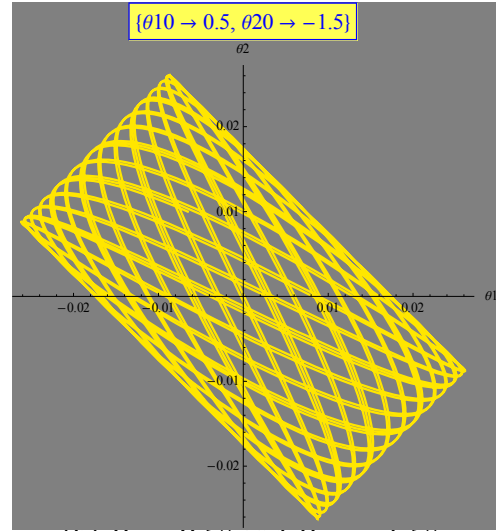
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### 1.2.2 The plot of $\theta_1(t)$ vs $\theta_2(t)$ under various conditions

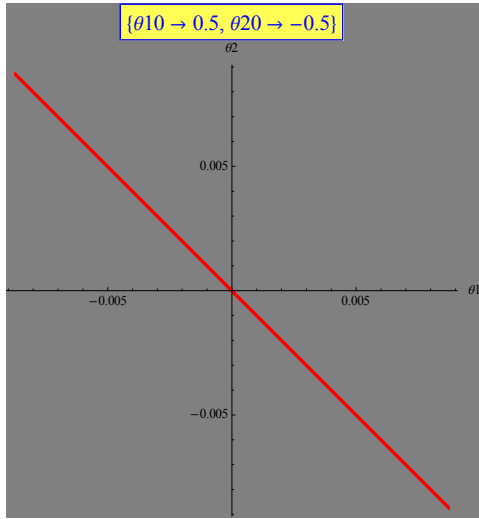
Here we make a plot of  $\theta_1(t)$  vs  $\theta_2(t)$  under various conditions. The results are shown in Figs.7(a) – (g). The in-phase mode is observed under the condition that  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0.5^\circ$ . It is found from Fig.7(e) that  $\theta_1(t) = \theta_2(t)$  any  $t$ , showing a straight line with a positive slope (= 1) in the plane of  $\theta_1(t)$  vs  $\theta_2(t)$ . The out-of-phase mode is observed under the condition that  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -0.5^\circ$ . It is found from Fig.7(c) that  $\theta_1(t) = -\theta_2(t)$  any  $t$ , showing a straight line with a negative slope (= -1) in the plane of  $\theta_1(t)$  vs  $\theta_2(t)$ .



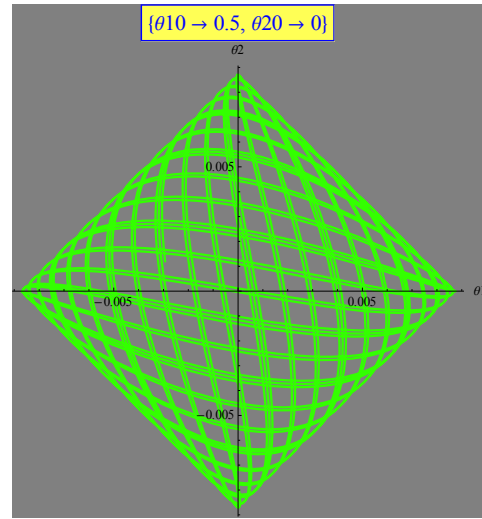
(a)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -2.5^\circ$ .



(b)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -1.5^\circ$ .

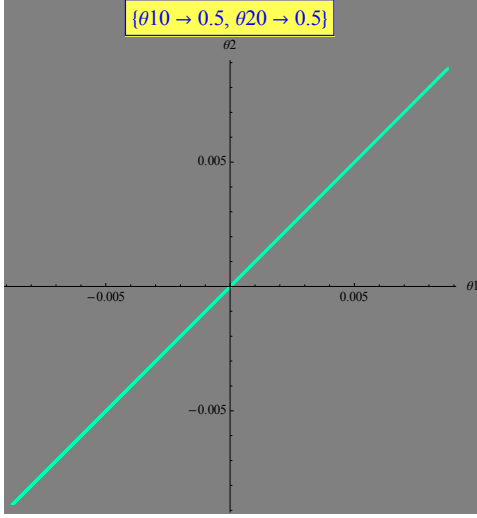


(c) The out-of-phase mode.  
 $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -0.5^\circ$ .

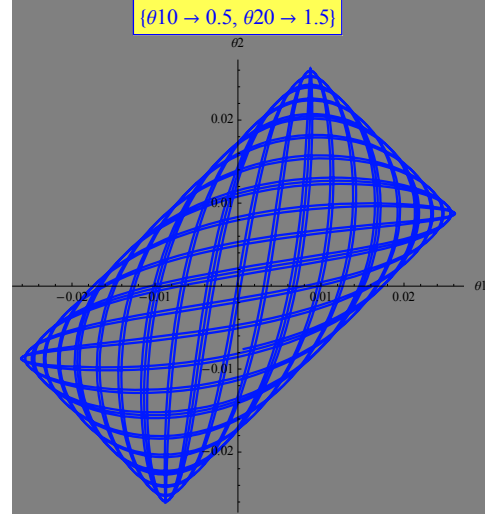


(d)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .

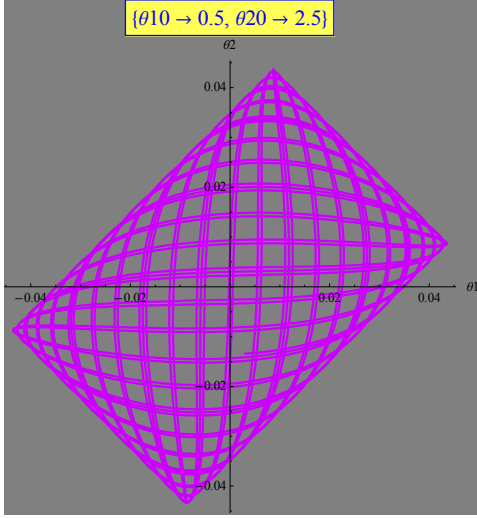
**Fig.7** (a) - (g) Plot of  $\theta_1(t)$  vs  $\theta_2(t)$  when  $\theta_2(t=0) = \theta_{20}$  is changed as a parameter,  $\theta_{20} = -2.5^\circ, -1.5^\circ, -0.5^\circ, 0^\circ, 0.5^\circ, 1.5^\circ$ , and  $2.5^\circ$ .  $\theta_{10} = 0.5^\circ$  is fixed.  $m_1 = m_2 = 1$ .  $k = 1$ .  $L_1 = L_2 = 5$ .  $g = 9.8$ .



(e) The in-phase mode.  
 $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0.5^\circ$ .



(f)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 1.5^\circ$ .



(g)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 2.5^\circ$ .

**Fig.7(a) - (g)** Plot of  $\theta_1(t)$  vs  $\theta_2(t)$  when  $\theta_2(t=0) = \theta_{20}$  is changed as a parameter,  $\theta_{20} = -2.5^\circ, -1.5^\circ, -0.5^\circ, 0^\circ, 0.5^\circ, 1.5^\circ$ , and  $2.5^\circ$ .  $\theta_{10} = 0.5^\circ$  is fixed.  $m_1 = m_2 = 1$ .  $k = 1$ .  $L_1 = L_2 = 5$ .  $g = 9.8$ .

((Mathematica)) The Mathematica program used for Fig.7(a).

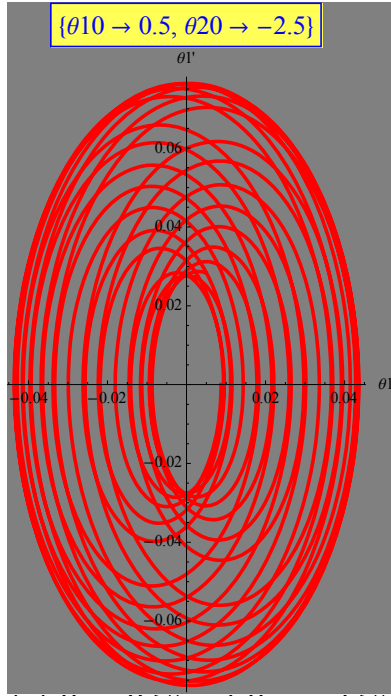
```
Clear["Global`*"]

m1 = 1; m2 = 1; k = 1; l1 = 5; l2 = 5; g = 9.8;
time[{ $\theta_{10}$ _, v10_,  $\theta_{20}$ _, v20_}, tmax_, opts_] :=
Module[{numsol, numgraph1},
  numsol =
  NDSolve[
    { $\theta_1$ ''[t] ==
      ((m1 * (l1 *  $\theta_1$ '[t]2 - g) - k * l1) * Sin[ $\theta_1$ [t]] + k * l2 * Sin[ $\theta_2$ [t]]) /
      (m1 * l1 * Cos[ $\theta_1$ [t]]),
       $\theta_2$ ''[t] ==
      ((m2 * (l2 *  $\theta_2$ '[t]2 - g) - k * l2) * Sin[ $\theta_2$ [t]] + k * l1 * Sin[ $\theta_1$ [t]]) /
      (m2 * l2 * Cos[ $\theta_2$ [t]]),  $\theta_1$ [0] ==  $\theta_{10}$ ,  $\theta_1$ '[0] == v10,  $\theta_2$ [0] ==  $\theta_{20}$ ,
       $\theta_2$ '[0] == v20}, { $\theta_1$ ,  $\theta_2$ }, {t, 0, tmax}];
  numgraph1 = ParametricPlot[Evaluate[{ $\theta_1$ [t],  $\theta_2$ [t]} /. numsol],
    {t, 0, tmax}, opts, DisplayFunction -> Identity];

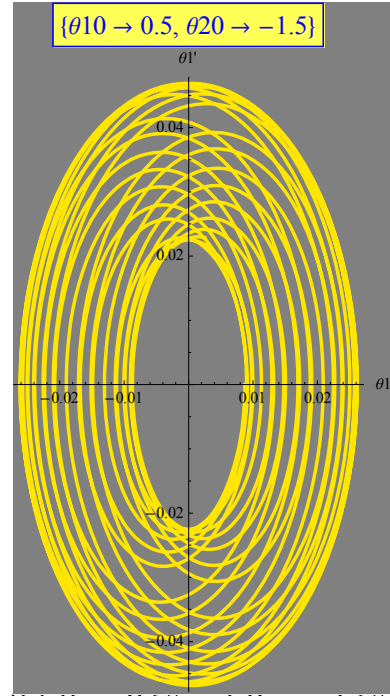
timelist = time[{0.5°, 0, -2.5°, 0}, 100,
  PlotStyle -> {{Hue[0], Thickness[0.005]}, {Hue[0.5], Thickness[0.01]}},
  AxesLabel -> {" $\theta_1$ ", " $\theta_2$ "},
  PlotLabel -> Style[Framed[{ $\theta_{10}$  -> 0.5,  $\theta_{20}$  -> -2.5}], 16, Blue,
  Background -> Lighter[Yellow]], Background -> Gray, Ticks -> Automatic,
  DisplayFunction -> Identity];
```

### 1.2.3 The phase plane of $\theta_1(t)$ vs $\theta_1'(t)$ under various conditions

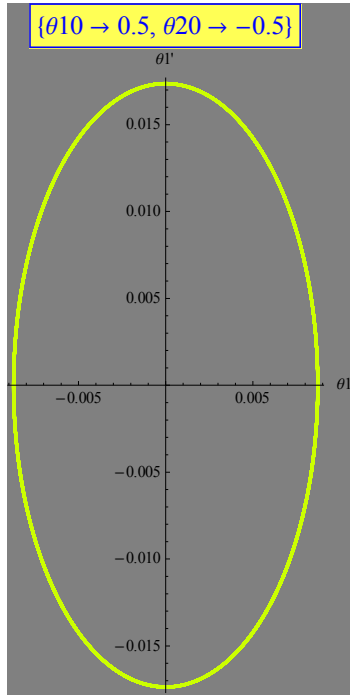
Here we assume that  $m_1 = m_2 = m = 1$ ,  $L_1 = L_2 = 5$ ,  $g = 9.8$ ,  $v_1(t = 0) = v_{10} = 0$ , and  $v_2(t = 0) = v_{20} = 0$ .  $\theta_1(t = 0) = \theta_{10} = 0.5^\circ$ .  $\theta_2(t = 0) = \theta_{20}$  is changed as a parameter.  $\theta_{20} = -2.5^\circ, -1.5^\circ, -0.5^\circ, 0^\circ, 0.5^\circ, 1.5^\circ$ , and  $2.5^\circ$ . We make a plot of the phase plane  $\theta_1(t)$  vs  $\theta_1'(t)$  under various conditions. The results are shown in Figs.8(a) – (g). There are two normal modes (the in-phase mode and the out-of-phase mode). The in-phase mode is observed under the condition that  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0.5^\circ$ , showing a simple ellipse in the phase plane of  $\theta_1(t)$  vs  $\theta_1'(t)$ . The out-of-phase mode is observed under the condition that  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -0.5^\circ$ , showing a simple ellipse in the phase plane of  $\theta_1(t)$  vs  $\theta_1'(t)$ . In the case when the curve of  $\theta_1(t)$  vs  $\theta_1'(t)$  [or  $\theta_2(t)$  vs  $\theta_2'(t)$ ], the shape of the phase plane is deviated from the simple ellipse, becomes a kind of spiral depending on the value of  $\theta_{20}$ . The results are shown in Figs.8 (a) – (g)



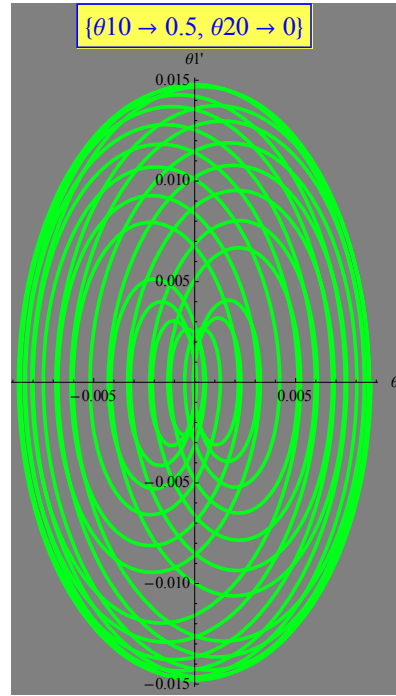
(a)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -2.5^\circ$ .



(b)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -1.5^\circ$ .

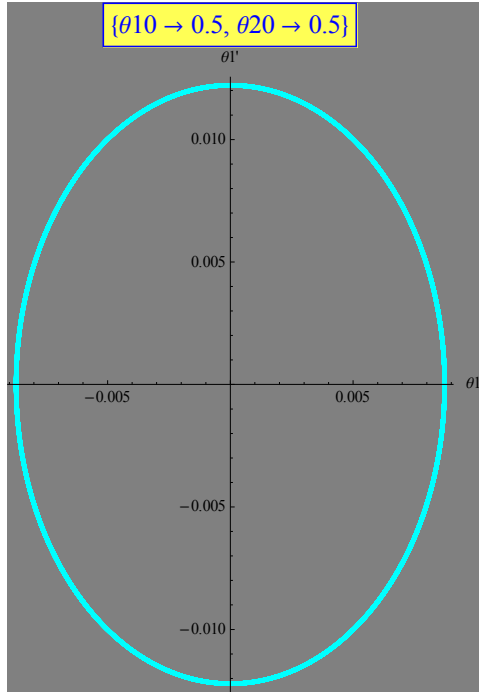


(c) The out-of phase mode.  
 $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = -0.5^\circ$ .

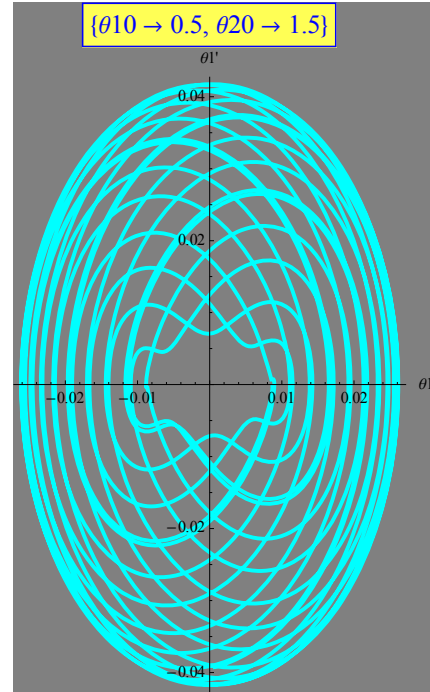


(d)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .

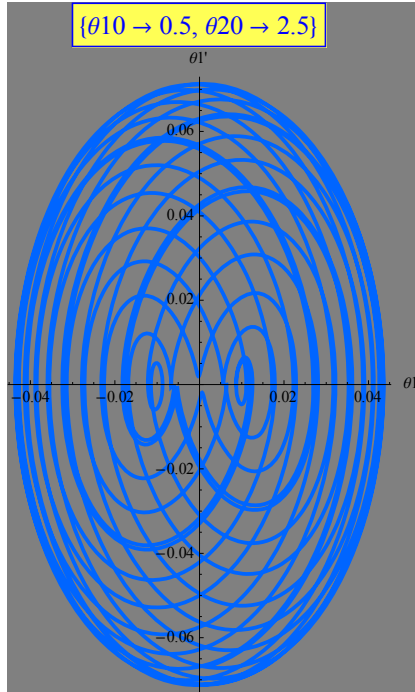
**Fig.8(a) - (g)** Plot of the phase plane  $\theta_1(t)$  vs  $\theta_1'(t)$ , when  $\theta_2(t = 0) = \theta_{20}$  is changed as a parameter,  $\theta_{20} = -2.5^\circ, -1.5^\circ, -0.5^\circ, 0^\circ, 0.5^\circ, 1.5^\circ$ , and  $2.5^\circ$ .  $\theta_{10} = 0.5^\circ$  is fixed.  $m_1 = m_2 = 1$ .  $k = 1$ .  $L_1 = L_2 = 5$ .



(e) The in-phase mode (normal mode)  
 $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0.5^\circ$ .



(f)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 1.5^\circ$ .



(g)  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 2.5^\circ$ .

**Fig.8(a) - (g)** Plot of the phase plane  $\theta_1(t)$  vs  $\theta_1'(t)$ , when  $\theta_2(t=0) = \theta_{20}$  is changed as a parameter,  $\theta_{20} = -2.5^\circ, -1.5^\circ, -0.5^\circ, 0^\circ, 0.5^\circ, 1.5^\circ$ , and  $2.5^\circ$ .  $\theta_{10} = 0.5^\circ$  is fixed.  $m_1 = m_2 = 1$ .  $k = 1$ .  $L_1 = L_2 = 5$ .

((Mathematica)) The programs to derive Fig.8(a).

```
Clear["Global`*"]

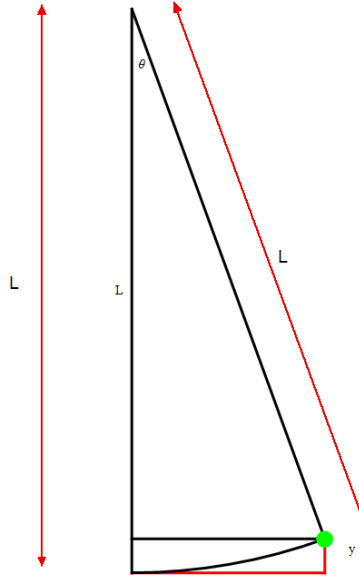
m1 = 1; m2 = 1; k = 1; l1 = 5; l2 = 5; g = 9.8;
time[{ $\theta_{10}$ _, v10_,  $\theta_{20}$ _, v20_}, tmax_, opts_] :=
Module[{numsol, numgraph1},
  numsol =
  NDSolve[
    { $\theta_1$ ''[t] ==
      ((m1 * (l1 *  $\theta_1$ '[t]^2 - g) - k * l1) * Sin[ $\theta_1$ [t]] + k * l2 * Sin[ $\theta_2$ [t]]) /
      (m1 * l1 * Cos[ $\theta_1$ [t]]),
       $\theta_2$ ''[t] ==
      ((m2 * (l2 *  $\theta_2$ '[t]^2 - g) - k * l2) * Sin[ $\theta_2$ [t]] + k * l1 * Sin[ $\theta_1$ [t]]) /
      (m2 * l2 * Cos[ $\theta_2$ [t]]),  $\theta_1$ [0] ==  $\theta_{10}$ ,  $\theta_1$ '[0] == v10,  $\theta_2$ [0] ==  $\theta_{20}$ ,
       $\theta_2$ '[0] == v20}, { $\theta_1$ ,  $\theta_2$ }, {t, 0, tmax}];
  numgraph1 = ParametricPlot[Evaluate[{ $\theta_1$ [t],  $\theta_1$ '[t]} /. numsol],
    {t, 0, tmax}, opts, DisplayFunction -> Identity]

timelist = time[{0.5°, 0, -2.5°, 0}, 75,
  PlotStyle -> {{Hue[0.], Thickness[0.01]}, {Hue[0.5], Thickness[0.01]}},
  AxesLabel -> {" $\theta_1$ ", " $\theta_1$ '"},
  PlotLabel -> Style[Framed[{ $\theta_{10}$  -> 0.5,  $\theta_{20}$  -> -2.5}], 16, Blue,
    Background -> Lighter[Yellow]], Background -> Gray, PlotRange -> All,
  Ticks -> Automatic, DisplayFunction -> Identity];
Show[timelist, DisplayFunction -> $DisplayFunction]
```

## 2. Normal modes in the coupled pendulum (from the book of Feynman lectures<sup>1</sup>)

### 2.1 Linear differential equations

The solution of the coupled pendulum is shown by Feynman as follows (see the book of Feynman Lectures on Physics volume-1, chapter 48). This method is very useful for one in understanding the concept of the normal modes.



**Fig.9** Geometry of the pendulum. The mass is located at the co-ordinate  $(x, y)$  from the vertical reference line.

Using the geometry, we have

$$\begin{aligned} x_1 &= L_1 \sin \theta_1 = 2L_1 \sin \frac{\theta_1}{2} \cos \frac{\theta_1}{2} \approx 2L_1 \sin \frac{\theta_1}{2} \\ y_1 &= L_1(1 - \cos \theta_1) = 2L_1 \sin^2 \frac{\theta_1}{2} = 2L_1 \left(\frac{x_1}{2L_1}\right)^2 = \frac{x_1^2}{2L_1}, \end{aligned} \quad (2.1)$$

for the mass-1 and

$$\begin{aligned} x_2 &= L_2 \sin \theta_2 = 2L_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_2}{2} \approx 2L_2 \sin \frac{\theta_2}{2} \\ y_2 &= L_2(1 - \cos \theta_2) = 2L_2 \sin^2 \frac{\theta_2}{2} = 2L_2 \left(\frac{x_2}{2L_2}\right)^2 = \frac{x_2^2}{2L_2}, \end{aligned} \quad (2.2)$$

for the mass-2. The mass  $m_1$  and  $m_2$  are located at the  $x$ -coordinates  $x_1$  and  $x_2$  from each vertical reference line. The Lagrangian can be given by

$$\begin{aligned} L_T &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 - \frac{1}{2} k(x_2 - x_1)^2 - m_1 g y_1 - m_2 g y_2 \\ &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 - \frac{1}{2} k(x_2 - x_1)^2 - m_1 g \frac{x_1^2}{2L} - m_2 g \frac{x_2^2}{2L}. \end{aligned} \quad (2.3)$$

(i) The Lagrange's equation

$$\frac{d}{dt} \frac{\partial L_T}{\partial \dot{x}_1} = \frac{\partial L_T}{\partial x_1},$$



leads to

$$m_1 \ddot{x}_1 = k(x_2 - x_1) - \frac{m_1 g}{L} x_1 = -k(x_1 - x_2) - \frac{m_1 g}{L} x_1 \quad (2.4)$$

(ii) The Lagrange's equation

$$\frac{d}{dt} \frac{\partial L_T}{\partial \dot{x}_2} = \frac{\partial L_T}{\partial x_2},$$

leads to

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) - \frac{m_2 g}{L} x_2 \quad (2.5)$$

((Note)) The standard derivation of Eqs (2.4) and (2.5) will be shown in Sec.5 without using the Lagrangian method.

Then we get the two linear differential equations,

$$\begin{aligned} \ddot{x}_1 &= -\frac{k}{m_1}(x_1 - x_2) - \frac{g}{L} x_1 \\ \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) - \frac{g}{L} x_2 \end{aligned} \quad (2.6)$$

## 2.2 Steady-state solutions

For simplicity, we assume that  $m_1 = m_2 = m$ .

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2(x_1 - x_2) - \omega_0^2 x_1 \\ \ddot{x}_2 &= -\omega_1^2(x_2 - x_1) - \omega_0^2 x_2, \end{aligned} \quad (2.7)$$

where the characteristic angular frequencies are defined by

$$\begin{aligned} \omega_1 &= \sqrt{\frac{k}{m}} \\ \omega_0 &= \sqrt{\frac{g}{L}} \end{aligned} \quad (2.8)$$

In order to get steady solutions, we assume that the solutions are given by the forms,

$$\begin{aligned} x_1 &= \text{Re}(X_1 e^{i\omega t}) \\ x_2 &= \text{Re}(X_2 e^{i\omega t}) \end{aligned} \quad (2.9)$$

Then we have

$$\begin{aligned}\omega^2 X_1 &= \omega_1^2 (X_1 - X_2) + \omega_0^2 X_1 \\ \omega^2 X_2 &= \omega_1^2 (X_2 - X_1) + \omega_0^2 X_2,\end{aligned}$$

or

$$\begin{pmatrix} \omega_0^2 + \omega_1^2 & -\omega_1^2 \\ -\omega_1^2 & \omega_0^2 + \omega_1^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad (2.10)$$

with  $\lambda = \omega^2$ . This is an eigenvalue problem.  $\lambda$  is the eigenvalue. This eigenvalue problem can be easily solved by using Mathematica.

((**Mathematica**)) The solution of eigenvalue problem

$$\begin{aligned}\mathbf{A1} &= \begin{pmatrix} \omega_0^2 + \omega_1^2 & -\omega_1^2 \\ -\omega_1^2 & \omega_0^2 + \omega_1^2 \end{pmatrix} \\ \{ \{ \omega_0^2 + \omega_1^2, -\omega_1^2 \}, \{ -\omega_1^2, \omega_0^2 + \omega_1^2 \} \} \end{aligned}$$

**Eigensystem[A1]**

$$\{ \{ \omega_0^2, \omega_0^2 + 2\omega_1^2 \}, \{ \{ 1, 1 \}, \{ -1, 1 \} \} \}$$

We find that there are two normal modes; the in-phase mode and the out-of phase mode.

(i) In-phase mode with  $\omega^{(1)} = \omega_0$

The eigenvector corresponding to this mode is given by

$$\mathbf{u}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (2.11)$$

where this vector is normalized to the unity.

(ii) Out-of-phase mode  $\omega^{(2)} = \sqrt{\omega_0^2 + 2\omega_1^2}$

The frequency of the out-of-phase mode is higher than that of the in-phase mode. The eigenvector corresponding to this mode is

$$\mathbf{u}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (2.12)$$

The spring is never stretched and both masses oscillate at the frequency  $\omega^{(1)}$ , as if the spring were absent. The spring contributes a restoring force and raises the frequency. The spring is periodically stretched and shrunk and therefore the spring constant  $k$  contributes to the expression for the frequency  $\omega^{(2)}$ .

Here we note that

$$\omega^{(1)} = \omega_0 = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{5}} = 1.40$$

$$\omega^{(2)} = \sqrt{\omega_0^2 + 2\omega_1^2} = \sqrt{\frac{g}{L} + 2\frac{k}{m}} = \sqrt{1.96 + 2} = 1.98997 \approx 2$$

when  $m = m_1 = m_2 = 1$ ,  $L = L_1 = L_2 = 5$ ,  $g = 9.8$ . These values are in good agreement with the results derived from Sec.1.2.

Since the differential equations are linear, the general form is given by any superposition of  $\mathbf{u}^{(1)}$  and  $\mathbf{u}^{(2)}$ ,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = C_1 \mathbf{u}^{(1)} + C_2 \mathbf{u}^{(2)} = \frac{C_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{C_2}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} C_1 + C_2 \\ C_1 - C_2 \end{pmatrix} \quad (2.13)$$

where  $C_1$  and  $C_2$  are complex constants and are determined from the initial conditions. Then we have

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[(C_1 + C_2)e^{i\omega t}] & \dot{x}_1(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[i\omega(C_1 + C_2)e^{i\omega t}] \\ x_2(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[(C_1 - C_2)e^{i\omega t}] & \dot{x}_2(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[i\omega(C_1 - C_2)e^{i\omega t}] \end{aligned} \quad (2.14)$$

and the initial conditions,

$$\begin{aligned} x_1(t=0) &= \frac{1}{\sqrt{2}} \operatorname{Re}[(C_1 + C_2)] & \dot{x}_1(0) &= \frac{1}{\sqrt{2}} \operatorname{Re}[i\omega(C_1 + C_2)] \\ x_2(t=0) &= \frac{1}{\sqrt{2}} \operatorname{Re}[(C_1 - C_2)] & \dot{x}_2(0) &= \frac{1}{\sqrt{2}} \operatorname{Re}[i\omega(C_1 - C_2)] \end{aligned} \quad (2.15)$$

**(i) Initial condition for the observation of the in-phase mode**

To this end, it is necessary to have  $C_2 = 0$ . Then we get

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[C_1 e^{i\omega t}] \\ x_2(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[C_1 e^{i\omega t}] \end{aligned} \quad (2.16)$$

which leads to  $x_1(t) = x_2(t)$ . In other words, if we have the initial condition such that

$$x_1(t=0) = \frac{1}{\sqrt{2}} \operatorname{Re}(C_1) = x_2(t=0) \quad \dot{x}_1(0) = \frac{1}{\sqrt{2}} \operatorname{Re}[i\omega(C_1)] = \dot{x}_2(0) \quad (2.17)$$

Then the in-phase mode can be realized experimentally.

**(ii) Initial condition for the observation of the out-of-phase mode**

To this end, it is necessary to have  $C_1 = 0$ . Then we get

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2}} \operatorname{Re}[C_2 e^{i\omega t}] \\ x_2(t) &= -\frac{1}{\sqrt{2}} \operatorname{Re}[C_2 e^{i\omega t}] \end{aligned} \quad (2.18)$$

which leads to  $x_1(t) = -x_2(t)$ . In other words, if we have the initial condition such that

$$x_1(t=0) = -x_2(t=0) = \frac{1}{\sqrt{2}} \operatorname{Re}(C_2) \quad \dot{x}_1(0) = -\dot{x}_2(0) = \frac{1}{\sqrt{2}} \operatorname{Re}[i\omega(C_2)] \quad (2.19)$$

Then the out-of-phase mode can be realized experimentally.

### 2.3 Steady-state solutions in the case of $m_1 \neq m_2$ .

We consider the steady-state solution when  $m_1 \neq m_2$ ,  $L_1 = L_2 = L$ . The equations of motion is given by

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2(x_1 - x_2) - \omega_0^2 x_1 \\ \ddot{x}_2 &= -\omega_2^2(x_2 - x_1) - \omega_0^2 x_2 \end{aligned} \quad (2.20)$$

with

$$\begin{aligned} \omega_0 &= \sqrt{\frac{g}{L}} \\ \omega_1 &= \sqrt{\frac{k}{m_1}} \\ \omega_2 &= \sqrt{\frac{k}{m_2}} \end{aligned} \quad (2.21)$$

In order to get steady solutions, we assume that the solutions are given by the forms,

$$\begin{aligned} x_1 &= \operatorname{Re}(X_1 e^{i\omega t}) \\ x_2 &= \operatorname{Re}(X_2 e^{i\omega t}) \end{aligned} \quad (2.22)$$

Then we have

$$\begin{pmatrix} \omega_0^2 + \omega_1^2 & -\omega_1^2 \\ -\omega_2^2 & \omega_0^2 + \omega_2^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (2.23)$$

with  $\lambda = \omega^2$ . This is an eigenvalue problem.  $\lambda$  is the eigenvalue. This eigenvalue problem can be easily solved by using Mathematica.

((**Mathematica**)) The solution of the eigenvalue problem using Mathematica

$$\mathbf{A1} = \begin{pmatrix} \omega_1^2 + \omega_0^2 & -\omega_1^2 \\ -\omega_2^2 & \omega_2^2 + \omega_0^2 \end{pmatrix}$$

$$\{\{\omega_0^2 + \omega_1^2, -\omega_1^2\}, \{-\omega_2^2, \omega_0^2 + \omega_2^2\}\}$$

$$\mathbf{eq1} = \mathbf{Eigensystem}[\mathbf{A1}]$$

$$\{\{\omega_0^2, \omega_0^2 + \omega_1^2 + \omega_2^2\}, \{\{1, 1\}, \{-\frac{\omega_1^2}{\omega_2^2}, 1\}\}\}$$

(i) **In-phase mode with**  $\omega^{(1)} = \omega_0$

The eigenvector corresponding to this mode is given by

$$\mathbf{u}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (2.24)$$

where this vector is not normalized to the unity. In this case we have the in-phase mode,

$$x_2(t) = x_1(t)$$

(ii) **Out-of-phase mode**  $\omega^{(2)} = \sqrt{\omega_0^2 + \omega_1^2 + \omega_2^2}$

The frequency of the out-of-phase mode is higher than that of the in-phase mode. The eigenvector corresponding to this mode is

$$\mathbf{u}^{(2)} = \begin{pmatrix} -\omega_1^2 / \omega_2^2 \\ 1 \end{pmatrix}, \quad (2.25)$$

where this vector is not normalized to the unity. In this case we have the out-of-phase mode,

$$x_1(t) = -\frac{\omega_1^2}{\omega_2^2} x_2(t) = -\frac{m_2}{m_1} x_2(t). \quad (2.26)$$

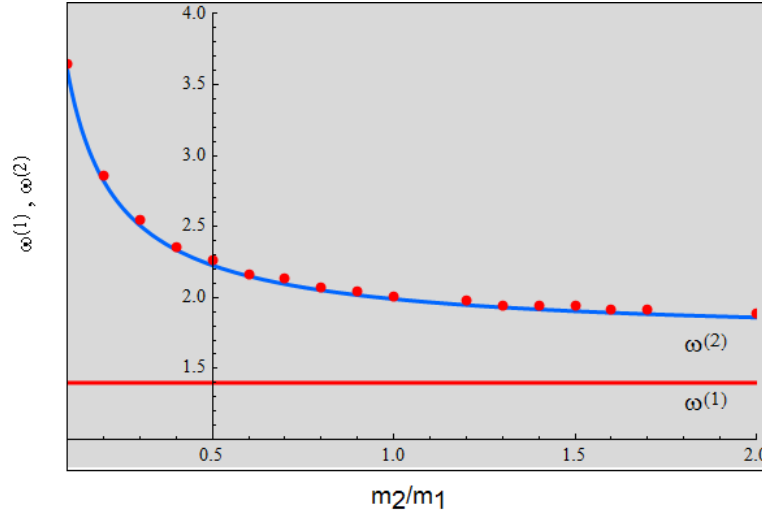
Figure 10 show the plot of  $\omega^{(1)}$  and  $\omega^{(2)}$  as a function of the mass ratio  $m_2/m_1$ , where  $m_1 = 1$ ,  $L = L_1 = L_2 = 1$ ,  $k = 1$ , and  $g = 9.8$ .  $\omega^{(1)}$  is independent of the mass ratio  $m_2/m_1$ ;

$$\omega^{(1)} = \omega_0 = \sqrt{g/L} = 1.4. \quad (2.27)$$

In contrast,  $\omega^{(2)}$  is strongly dependent on the mass ratio  $m_2/m_1 = \xi$ , for  $\xi < 1$ ,

$$\omega^{(2)} = \sqrt{\omega_0^2 + \omega_1^2 + \omega_2^2} = \sqrt{\frac{g}{L} + \frac{k}{m_1} + \frac{k}{m_2}} = \sqrt{2.96 + \frac{1}{\xi}}. \quad (2.28)$$

In the limit of  $\xi \rightarrow \infty$ ,  $\omega^{(2)}$  tends to  $\sqrt{2.96} = 1.72$ .



**Fig.10** The angular frequencies of the in-phase mode,  $\omega^{(1)}$ , and the out-of-phase mode,  $\omega^{(2)}$  as a function of the mass ratio  $m_2/m_1$ .  $m_1 = 1$ .  $k = 1$ .  $L_1 = L_2 = 5$ .  $g = 9.8$ . The values of  $\omega^{(2)}$  derived from the FFT spectrum are also plotted as a function of  $m_2/m_1$  (see Sec.3).

### 3. Numerical calculation in the case when $m_1$ is not equal to $m_2$ for the coupled pendulum

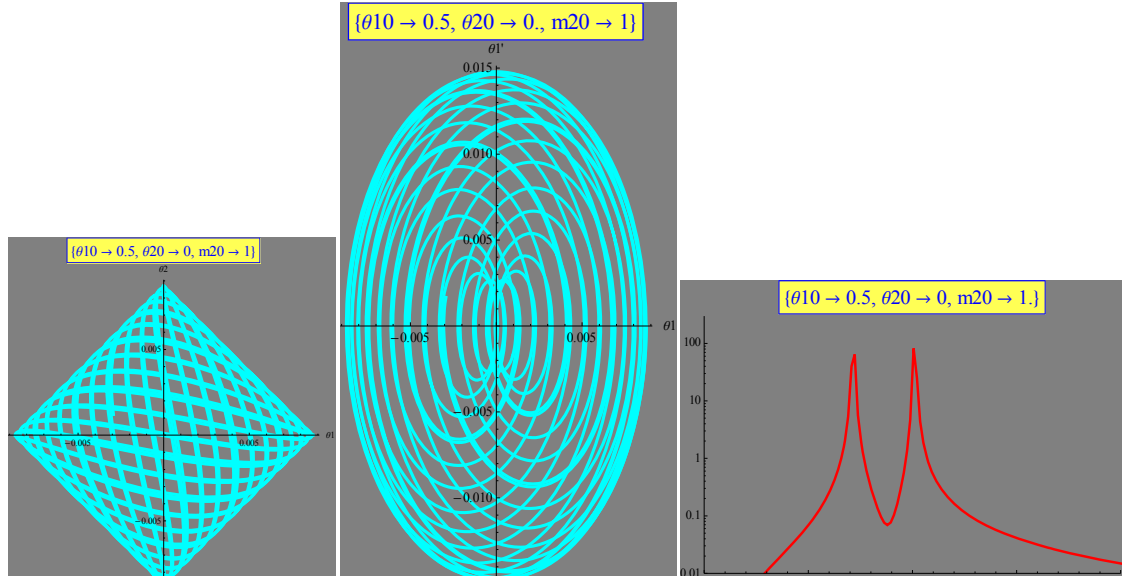
Here we proceed the further discussion of Sec.2 for the case of  $m_1 \neq m_2$ . Using the Mathematica, numerical calculations is made where  $m_1 = 1$ ,  $L_1 = L_2 = 5$ ,  $g = 9.8$ , and the mass  $m_2$  is changed as a parameter ( $0.1 \leq m_2 \leq 2.0$ ). For simplicity we adopt the initial conditions such that

$$\theta_1(t=0) = \theta_{10} = 0.5^\circ \quad \theta_2(t=0) = \theta_{20} = 0^\circ$$

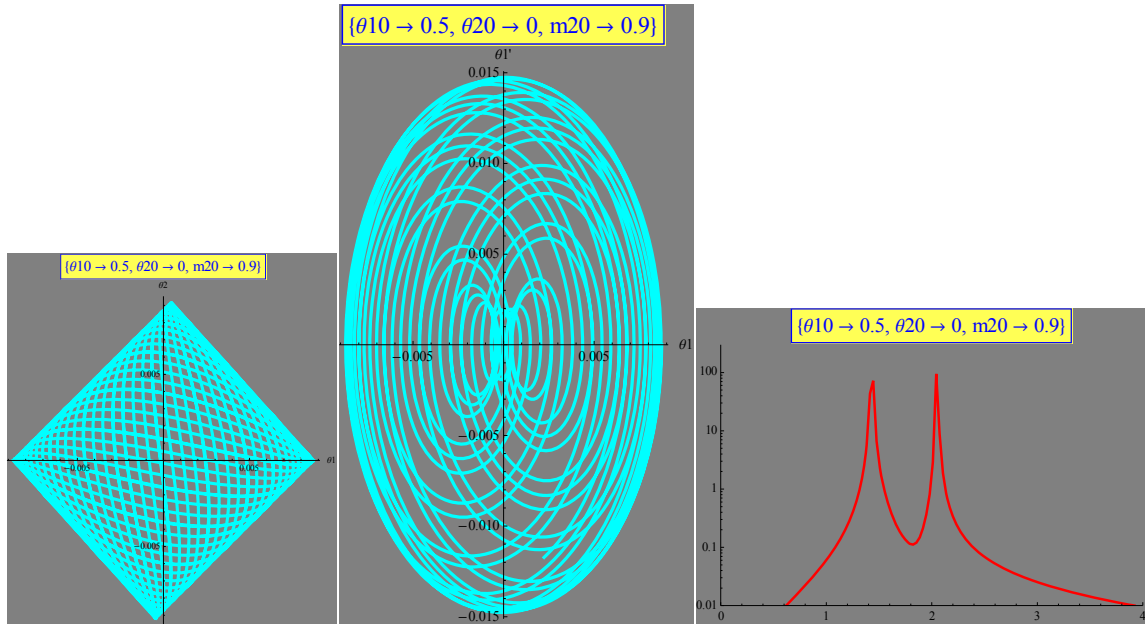
$$v_1(t=0) = v_{10} = 0, \quad v_2(t=0) = v_{20} = 0.$$

The results are shown in Fig.11, (i) the plot of the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase plane [ $\theta_1(t)$  vs  $\theta_1'(t)$ ], and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . As shown in the FFT spectrum, we find that there are two normal modes; the in-phase mode with  $\omega^{(1)}$  and the out-of-phase mode  $\omega^{(2)}$ . The angular frequency  $\omega^{(1)}$  is independent of the mass  $m_2$ , while  $\omega^{(2)}$  is strongly dependent on the mass  $m_2$  for  $m_2 < 1.0$ . The values of  $\omega^{(2)}$  obtained from the FFT spectrum for each mass are plotted in Fig.10 as red points. These points fall very well on the theoretical curve given by Eq.(2.28). The phase plane ( $\theta_1(t)$  vs  $\theta_1'(t)$ ) exhibits complicated patterns depending on the mass  $m_2$ . When  $m_2 = 0.69$ , the phase plane becomes very simple, consisting of big ellipse, medium ellipse, and small ellipse. The time dependence of  $\theta_1(t)$  and  $\theta_2(t)$  is shown in Fig.11(h).

We can conclude that the change of mass  $m_2$  does not give rise to any significant change in the motion of coupled pendulum since the differential equations are still linear. The situation may be different for the large angle cases where the differential equations becomes nonlinear. We do not discuss this problem in this note in spite of much interest.

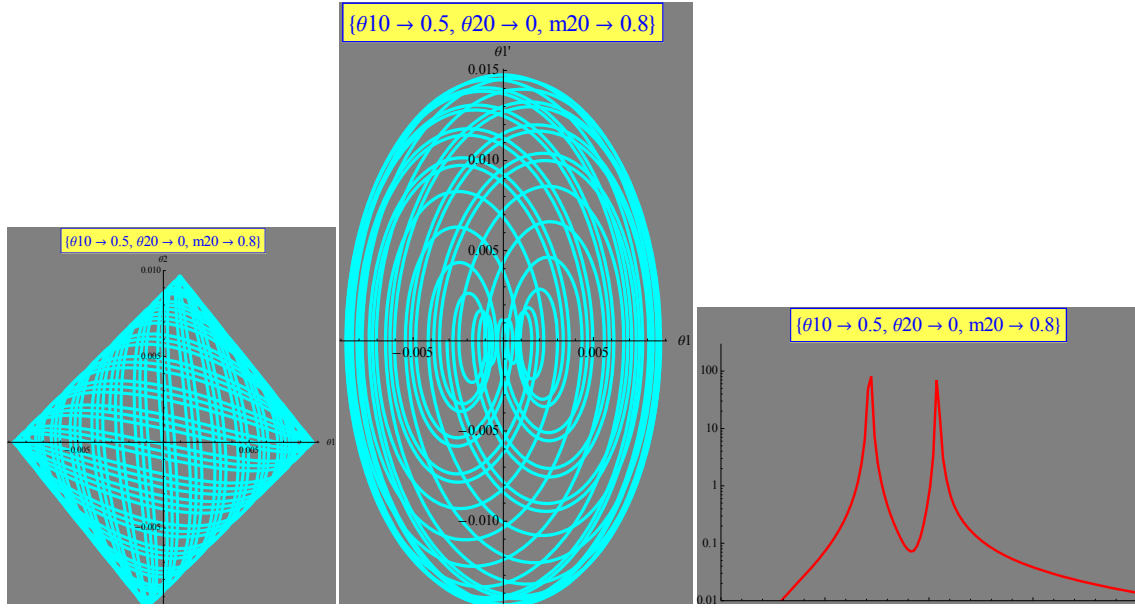


- (a)  $m_1 = 1, m_2 = 1.0$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.01$ .

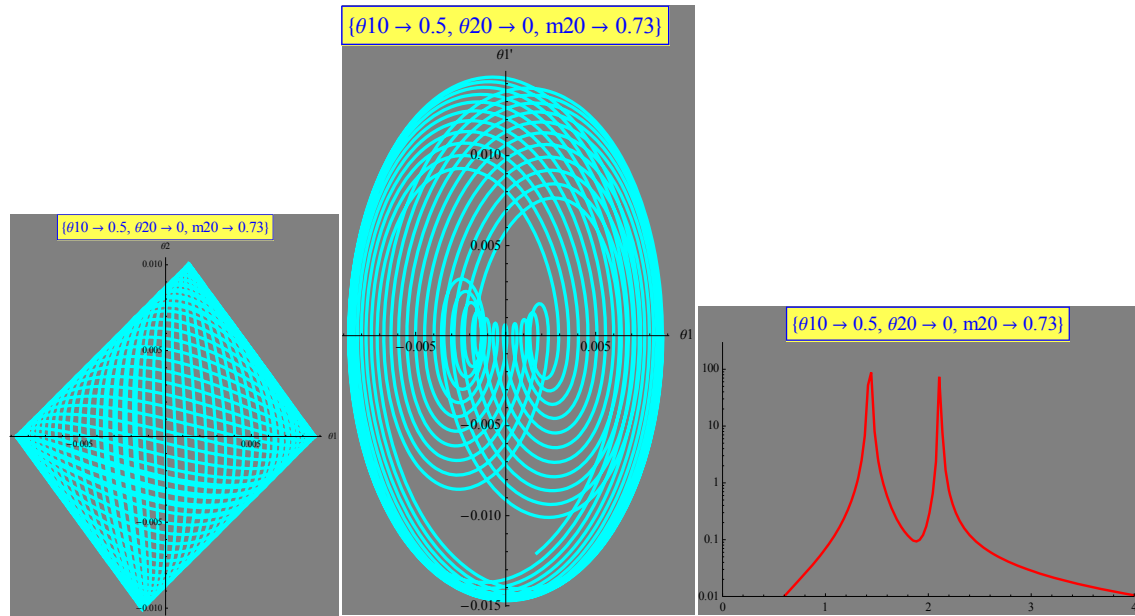


- (b)  $m_1 = 1, m_2 = 0.9$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.04$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,



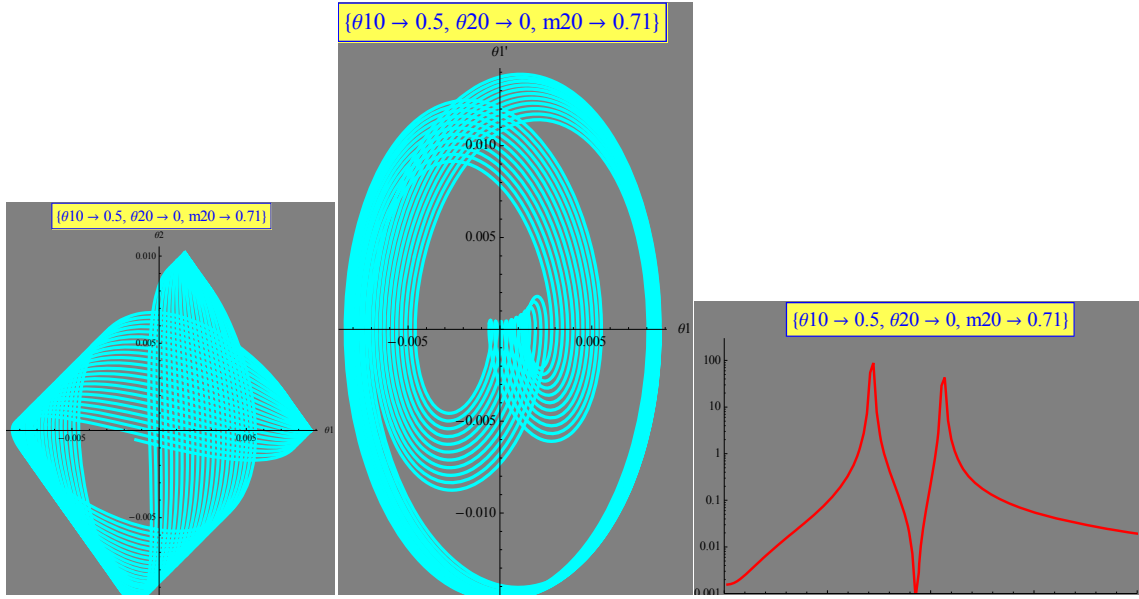
(c)  $m_1 = 1, m_2 = 0.8$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.07$ .



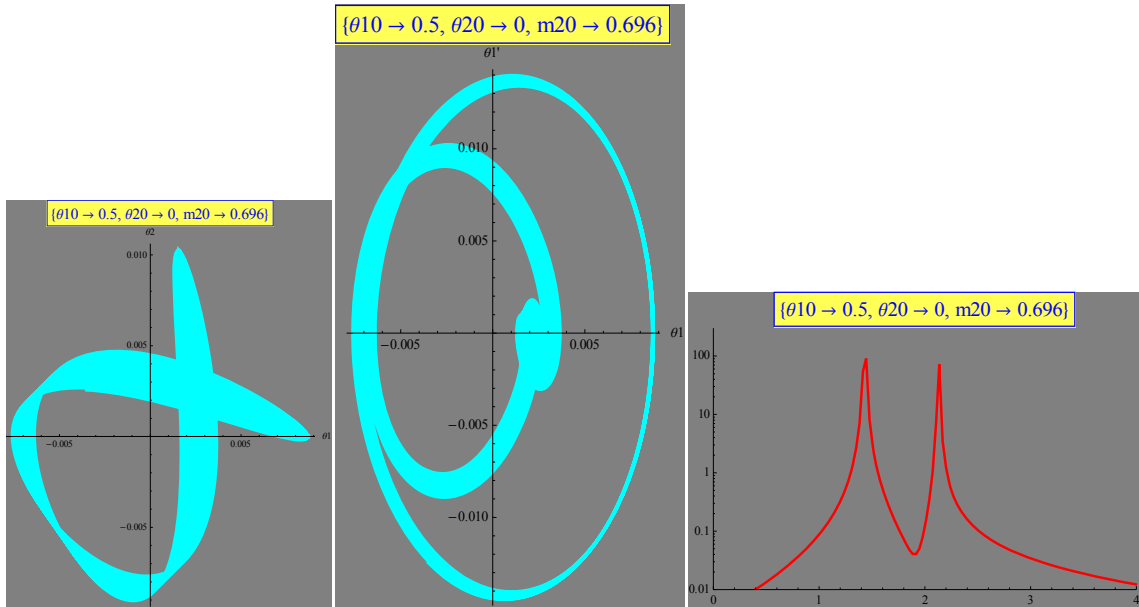
(d)  $m_1 = 1, m_2 = 0.73$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.10$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,



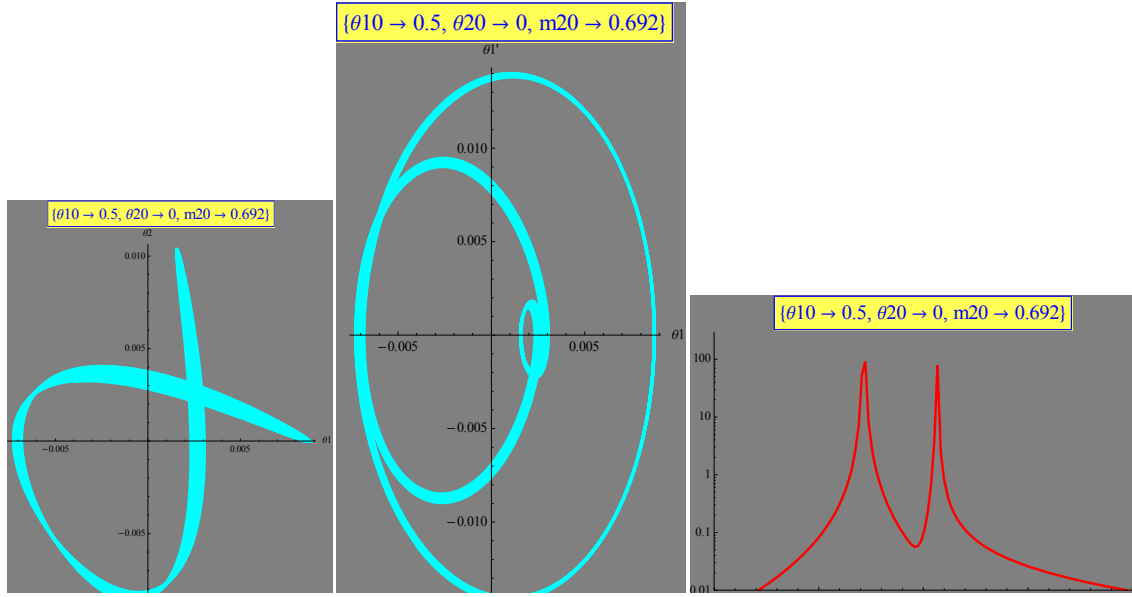


(e)  $m_1 = 1, m_2 = 0.71$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .



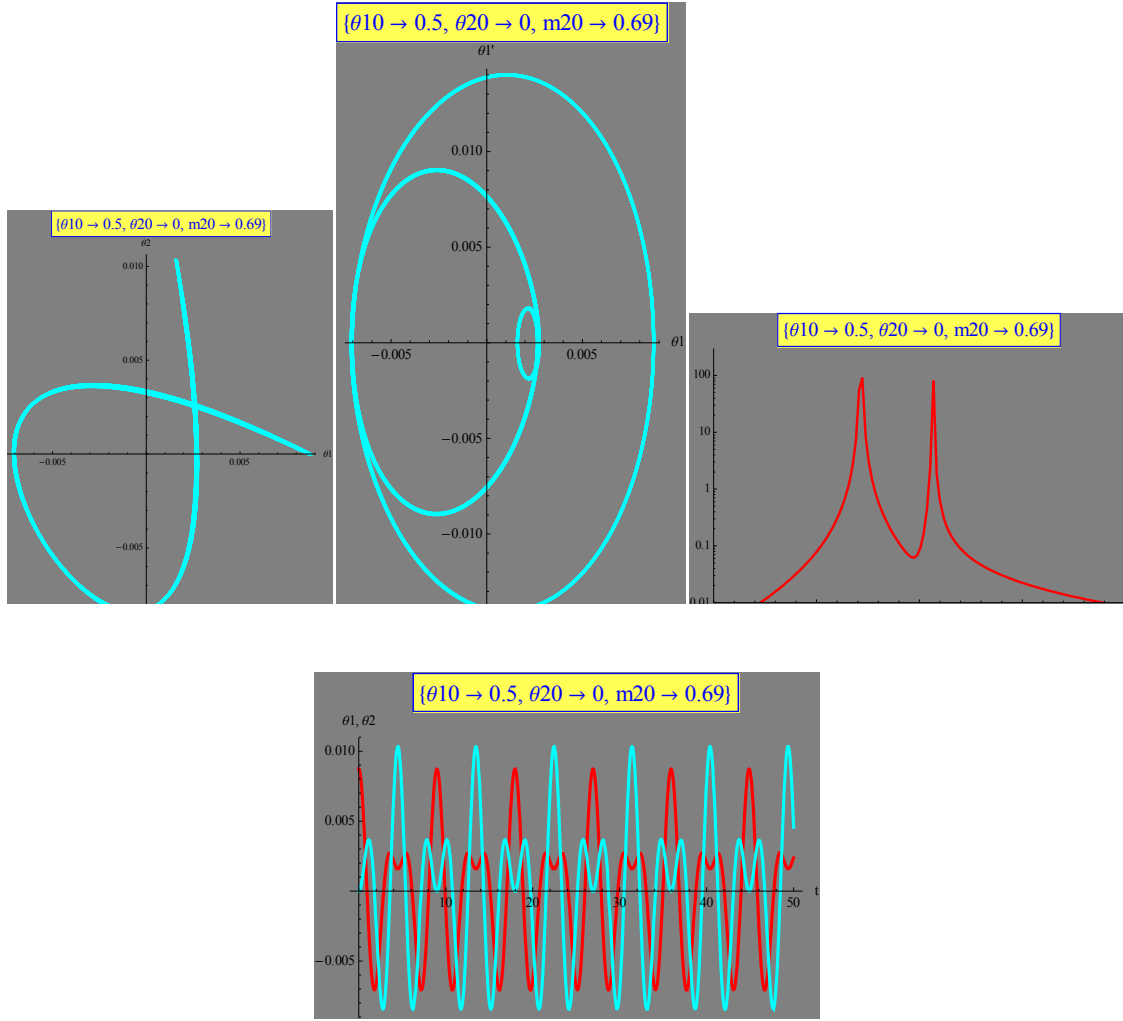
(f)  $m_1 = 1, m_2 = 0.696$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,



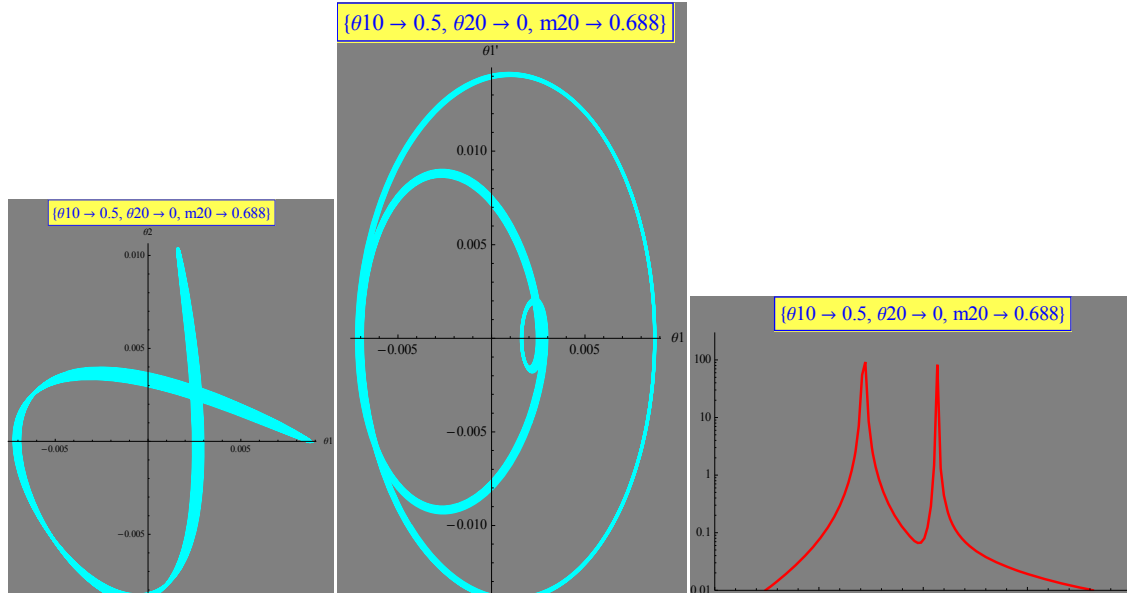
- (g)  $m_1 = 1, m_2 = 0.692$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,

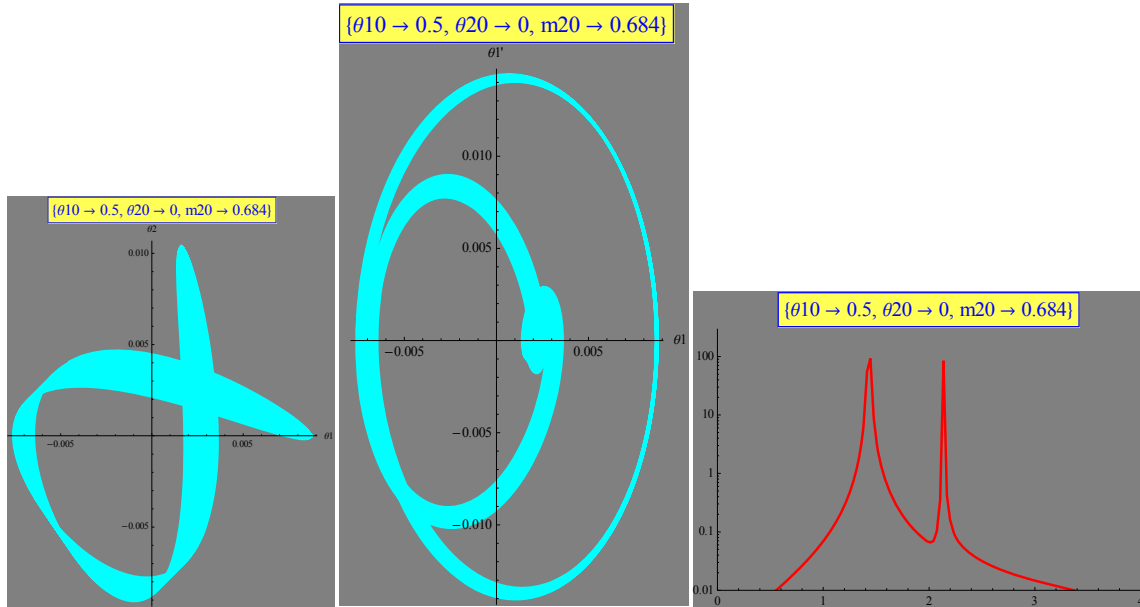


- (h)  $m_1 = 1$ ,  $m_2 = 0.69$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ . (iv) time dependence of  $\theta_1$  and  $\theta_2$  for  $0 \leq t \leq 50$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,

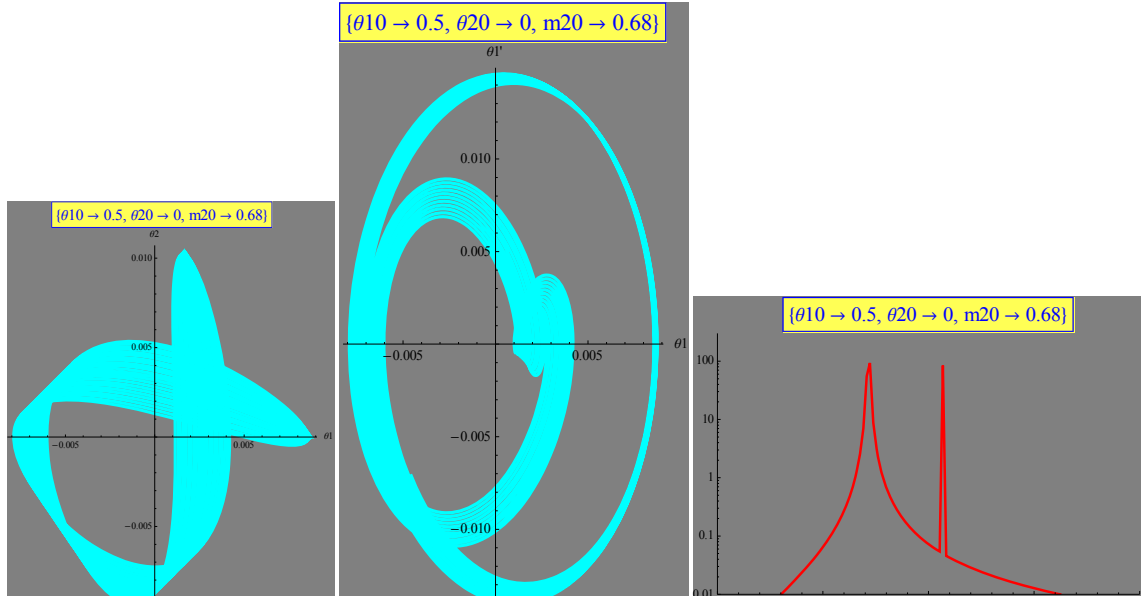


- (i)  $m_1 = 1, m_2 = 0.688$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

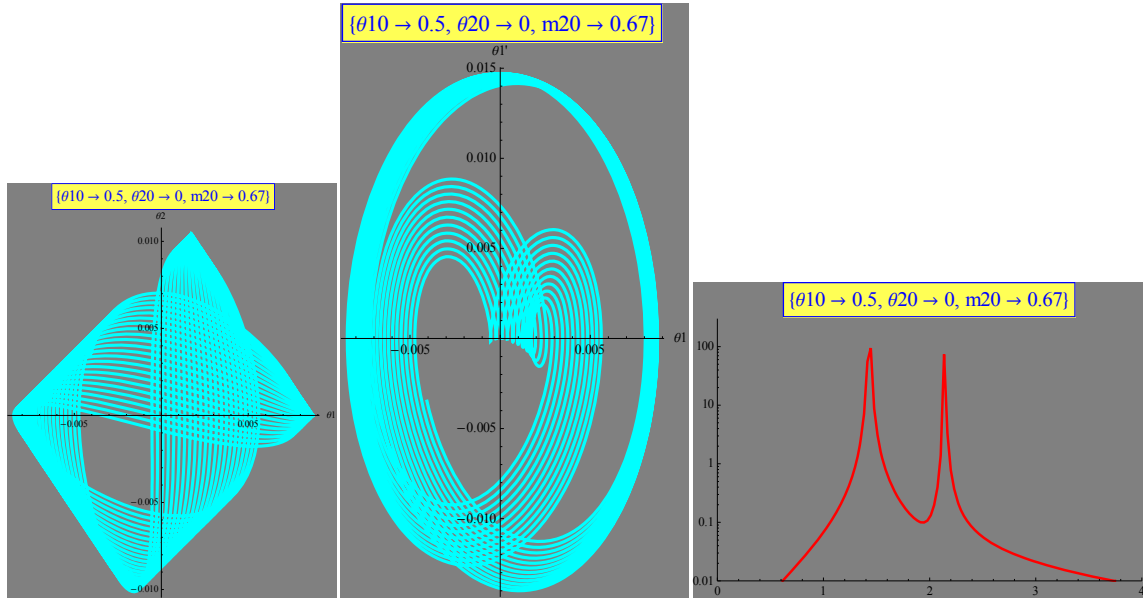


- (j)  $m_1 = 1, m_2 = 0.684$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,

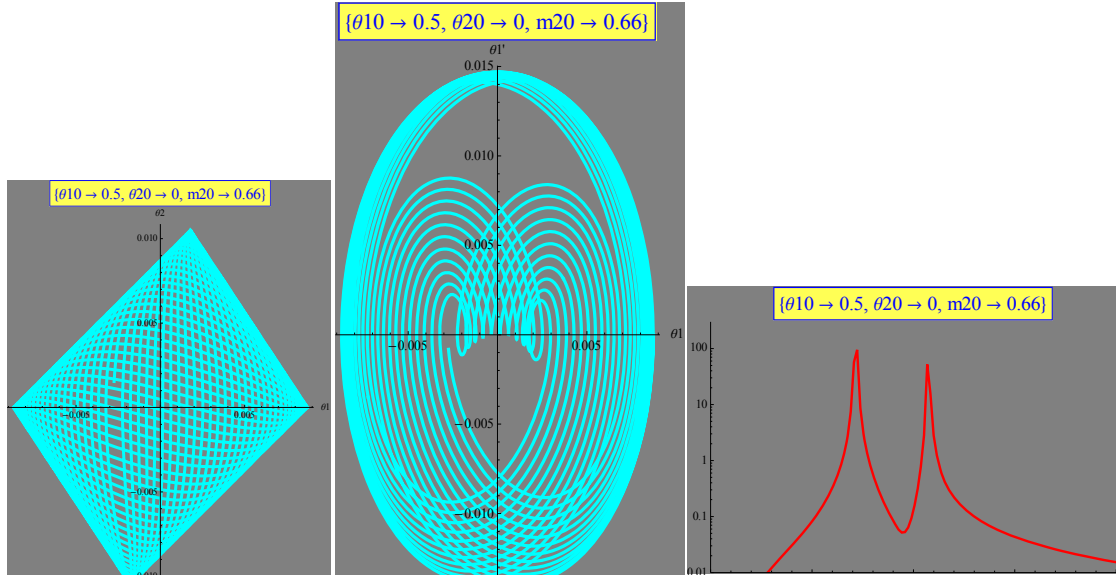


- (k)  $m_1 = 1$ ,  $m_2 = 0.68$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

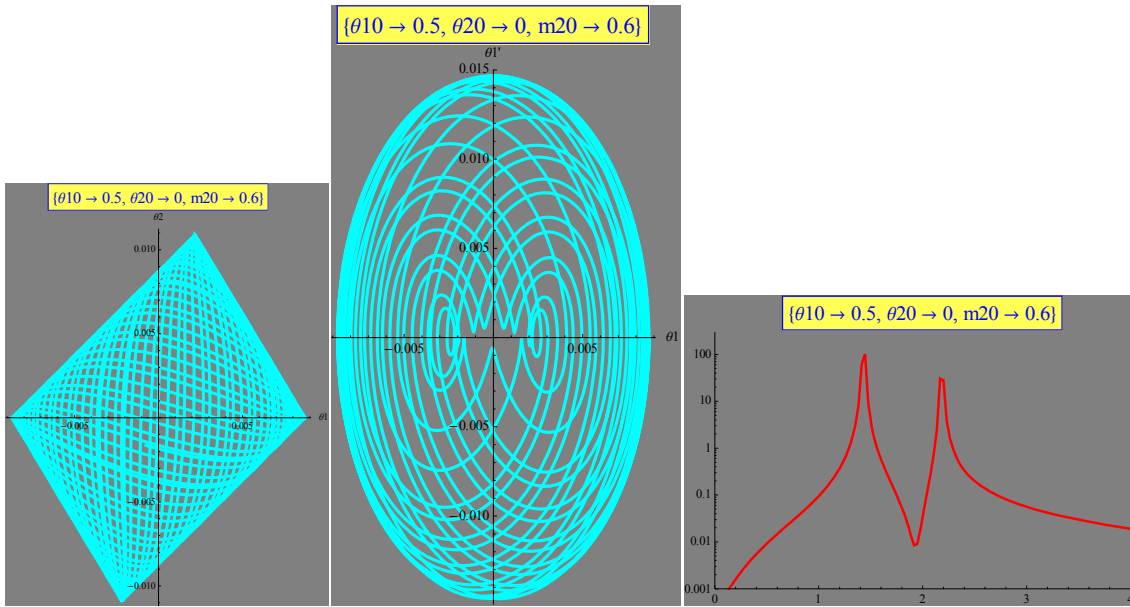


- (l)  $m_1 = 1$ ,  $m_2 = 0.67$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,

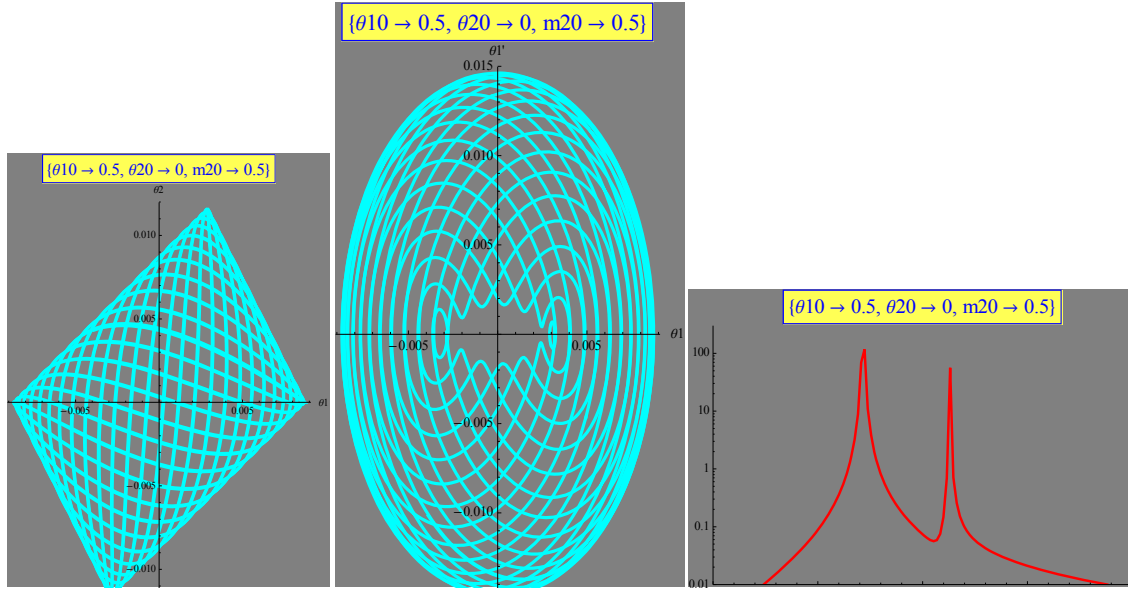


(m)  $m_1 = 1, m_2 = 0.66$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.14$ .

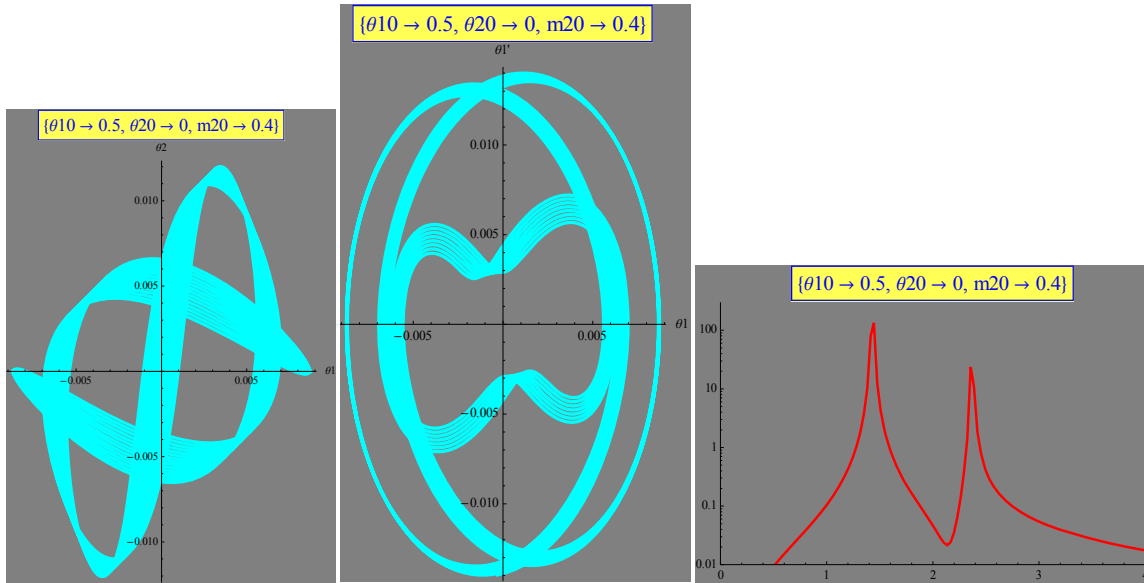


(n)  $m_1 = 1, m_2 = 0.6$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.17$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,

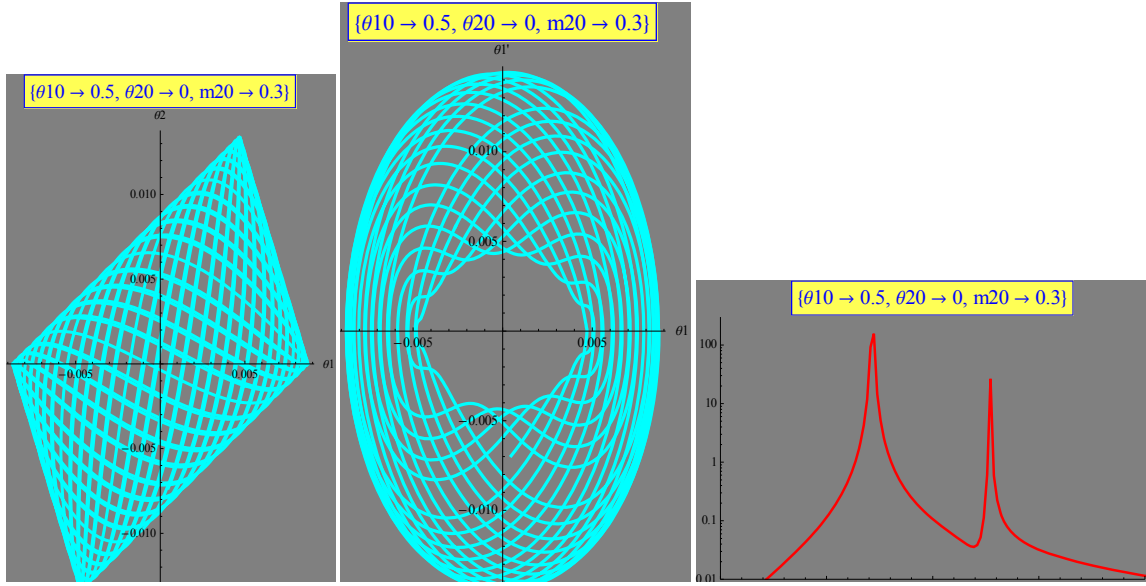


- (o)  $m_1 = 1, m_2 = 0.5$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.26$ .

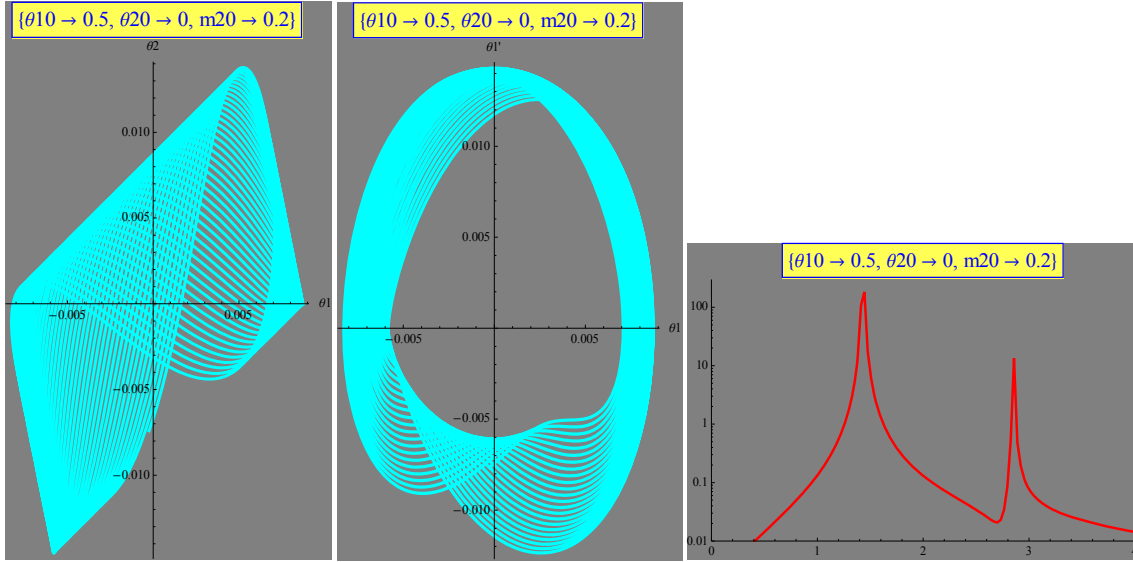


- (p)  $m_1 = 1, m_2 = 0.4$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.36$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,



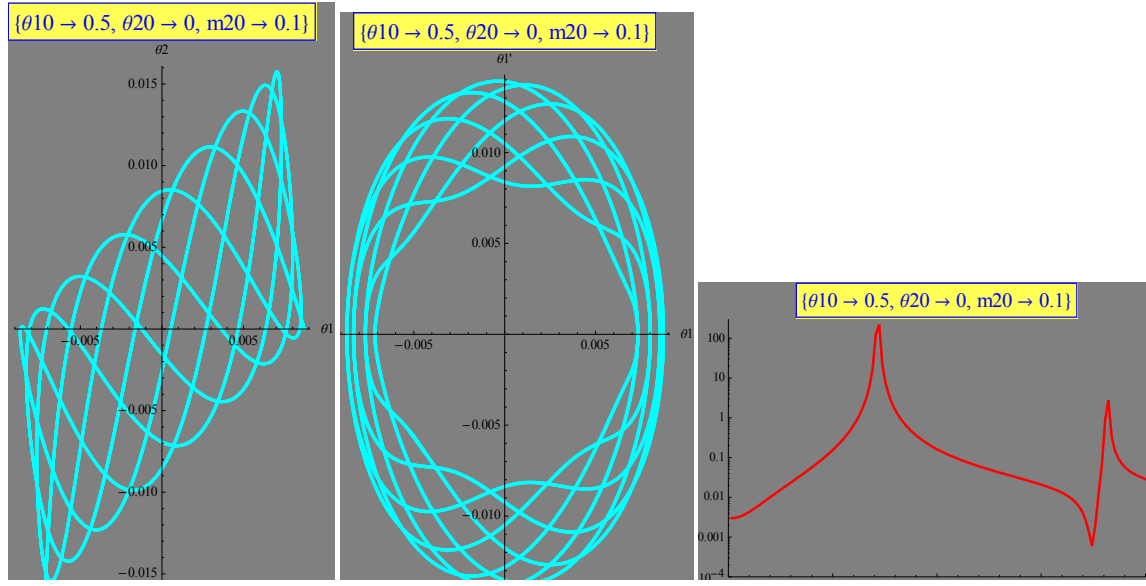
- (q)  $m_1 = 1, m_2 = 0.3$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0^\circ$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.54$ .



- (r)  $m_1 = 1, m_2 = 0.2$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 2.86$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,



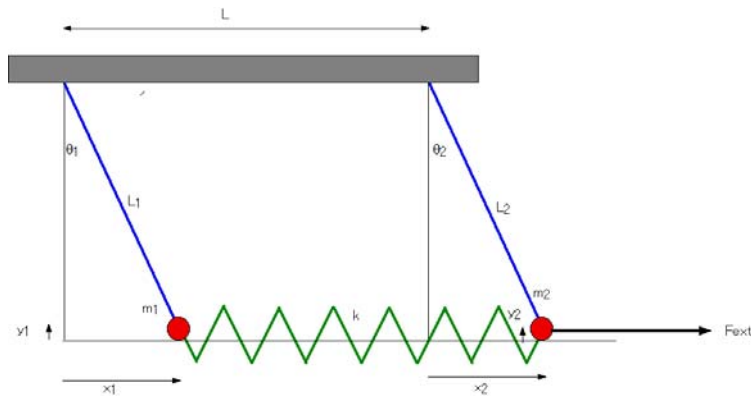


(s)  $m_1 = 1, m_2 = 0.1$ .  $\theta_{10} = 0.5^\circ$  and  $\theta_{20} = 0$ .  $v_{10} = v_{20} = 0$ . (i)  $\theta_1$  vs  $\theta_2$ , (ii) the phase plane  $\theta_1$  vs  $\theta_1'$  and (iii) the FFT spectrum (FFT intensity vs  $\omega$ ).  $\omega^{(1)} = 1.4$ .  $\omega^{(2)} = 3.64$ .

**Fig.11** (a) - (s) The plots of plot of (i) the  $\theta_1(t)$  vs  $\theta_2(t)$ , (ii) the phase space  $[\theta_1(t)$  vs  $\theta_1'(t)]$ , and (iii) the FFT spectrum of  $\theta_1(t)$  vs  $t$  for each mass  $m_2$ . The values of  $\theta_1(t)$  and  $\theta_2(t)$  are calculated as a function of the time  $t$  for  $t = 0 - 200$  (typically).  $m_2 = 1, 0.9, 0.8, 0.73, 0.71, 0.696, 0.692, 0.690, 0.688, 0.684, 0.68, 0.67, 0.66, 0.60, 0.50, 0.40, 0.30$ ,

#### 4. Forced oscillations of coupled pendulum

##### 4.1 Linear differential equations



**Fig.12** Schematic diagram of coupled pendulum with mass  $m_1$  and mass  $m_2$ . An external force  $F_{\text{ext}}$  with the angular frequency  $\omega$  is applied to the mass  $m_2$ .

We consider the motion of coupled pendulum where the external driving force ( $= F_0 \cos(\omega t)$ ) is applied to a mass  $m_2$ . We assume that each pendulum has the same damping constant  $\gamma$ .

$$\begin{aligned} m_1 \ddot{x}_1 &= -k(x_1 - x_2) - m_1 \gamma \dot{x}_1 - \frac{m_1 g}{L} x_1 \\ m_2 \ddot{x}_2 &= -k(x_2 - x_1) - \frac{m_2 g}{L} x_2 - m_2 \gamma \dot{x}_2 + F_0 \cos(\omega t) \end{aligned} \quad (4.1)$$

where  $m_1 = m_2 = m$ ,

$$\begin{aligned} \omega_1 &= \sqrt{\frac{k}{m}} \\ \omega_0 &= \sqrt{\frac{g}{L}} \end{aligned} \quad (4.2)$$

Then we have

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2 (x_1 - x_2) - \omega_0^2 x_1 - \gamma \dot{x}_1 \\ \ddot{x}_2 &= -\omega_1^2 (x_2 - x_1) - \omega_0^2 x_2 - \gamma \dot{x}_2 + \frac{F_0}{m} \cos(\omega t). \end{aligned} \quad (4.3)$$

## 4.2 Steady-state solution

Here we assume that

$$\begin{aligned} x_1 &= \text{Re}[X_1 e^{i\omega t}] \\ x_2 &= \text{Re}[X_2 e^{i\omega t}] \\ \frac{F_0}{m} \cos(\omega t) &= \frac{F_0}{m} \text{Re}[e^{i\omega t}] \end{aligned} \quad (4.4)$$

or

$$\begin{aligned} -\omega^2 X_1 &= -\omega_1^2 (X_1 - X_2) - i\gamma \omega X_1 - \omega_0^2 X_1 \\ -\omega^2 X_2 &= -\omega_1^2 (X_2 - X_1) - \omega_0^2 X_2 - i\gamma \omega X_2 + \frac{F_0}{m} \end{aligned}$$

or

$$\begin{pmatrix} \omega_0^2 + \omega_1^2 - \omega^2 + i\gamma\omega & -\omega_1^2 \\ -\omega_1^2 & \omega_0^2 + \omega_1^2 - \omega^2 + i\gamma\omega \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{F_0}{m} \end{pmatrix} \quad (4.5)$$

Then we have

$$\begin{aligned}
\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \begin{pmatrix} \omega_0^2 + \omega_1^2 - \omega^2 + i\gamma\omega & -\omega_1^2 \\ -\omega_1^2 & \omega_0^2 + \omega_1^2 - \omega^2 + i\gamma\omega \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{F_0}{m} \end{pmatrix} \\
&= \frac{F_0}{2m} \begin{pmatrix} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} - \frac{1}{\omega_0^2 + 2\omega_1^2 - \omega^2 + i\gamma\omega} \\ \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} + \frac{1}{\omega_0^2 + 2\omega_1^2 - \omega^2 + i\gamma\omega} \end{pmatrix} \\
&= \frac{F_0}{2m} \begin{pmatrix} \frac{1}{\omega^{(1)2} - \omega^2 + i\gamma\omega} - \frac{1}{\omega^{(2)2} - \omega^2 + i\gamma\omega} \\ \frac{1}{\omega^{(1)2} - \omega^2 + i\gamma\omega} + \frac{1}{\omega^{(2)2} - \omega^2 + i\gamma\omega} \end{pmatrix}
\end{aligned} \tag{4.6}$$

where  $\omega^{(1)} = \omega_0$  is the angular frequency of the in-phase mode and  $\omega^{(2)} = \sqrt{\omega_0^2 + 2\omega_1^2}$  is the angular frequency of the out-of-phase mode.

((**Mathematica-1**)) Calculation of the inverse matrix

$$\text{Inverse} \left[ \begin{pmatrix} \omega_0^2 + \omega_1^2 - \omega^2 + i\gamma\omega & -\omega_1^2 \\ -\omega_1^2 & \omega_0^2 + \omega_1^2 - \omega^2 + i\gamma\omega \end{pmatrix} \right] \cdot \begin{pmatrix} 0 \\ \frac{F_0}{m} \end{pmatrix} // \text{Apart}$$

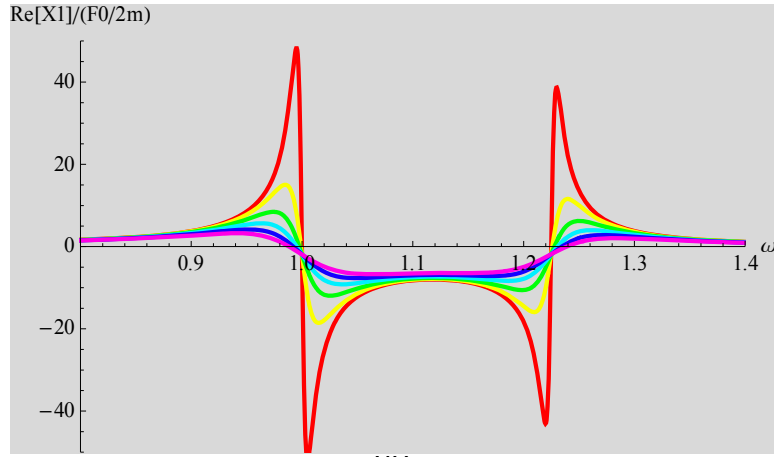
$$\left\{ \left\{ \frac{F_0}{2m (\gamma\omega - \omega^2 + \omega_0^2)} - \frac{F_0}{2m (\gamma\omega - \omega^2 + \omega_0^2 + 2\omega_1^2)} \right\}, \right.$$

$$\left. \left\{ \frac{F_0}{2m (\gamma\omega - \omega^2 + \omega_0^2)} + \frac{F_0}{2m (\gamma\omega - \omega^2 + \omega_0^2 + 2\omega_1^2)} \right\} \right\}$$

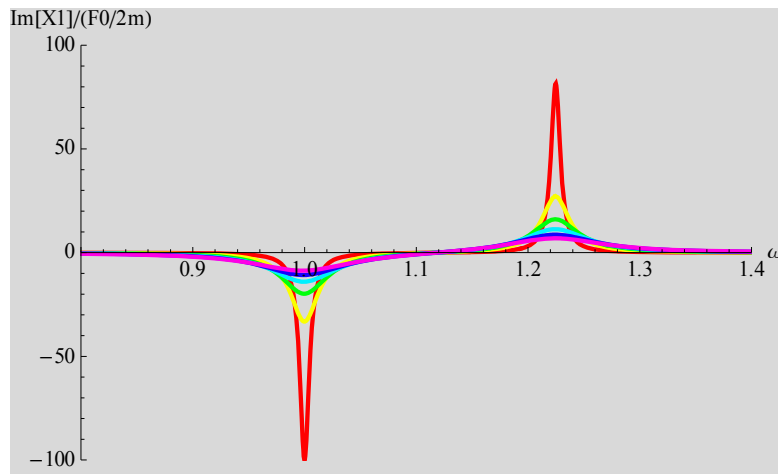
((**Mathematica-2**)) Plot of  $\text{Re}[X_1]$ ,  $\text{Im}[X_1]$ ,  $\text{Re}[X_2]$ , and  $\text{Im}[X_2]$  normalized by  $F_0/2m$ , as a function of  $\omega$ .

$$\begin{aligned}
\frac{X_1}{F_0/2m} &= \frac{1}{\omega^{(1)2} - \omega^2 + i\gamma\omega} - \frac{1}{\omega^{(2)2} - \omega^2 + i\gamma\omega} \\
\frac{X_2}{F_0/2m} &= \frac{1}{\omega^{(1)2} - \omega^2 + i\gamma\omega} + \frac{1}{\omega^{(2)2} - \omega^2 + i\gamma\omega},
\end{aligned} \tag{4.7}$$

where  $\omega_0 = 1$ ,  $\omega_1 = 0.5$ , and  $\gamma$  is changed as a parameter.

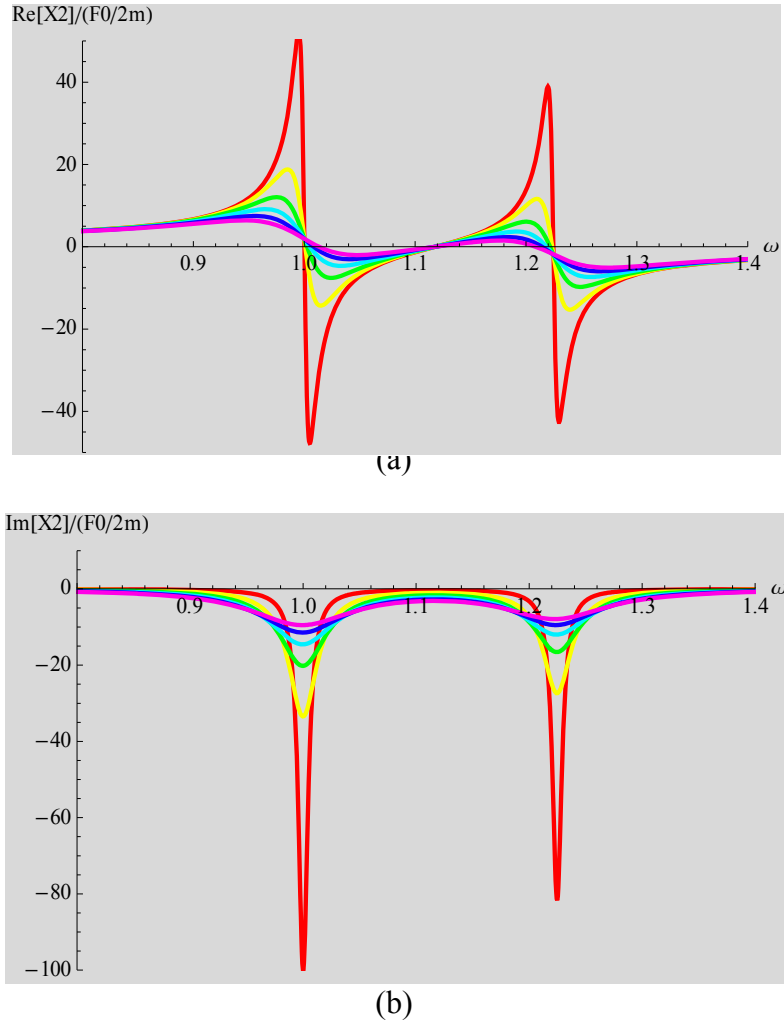


(a)



(b)

**Fig.13**(a) and (b) Plot of  $\text{Re}[X_1]/(F_0/2m)$  and  $\text{Im}[X_1]/(F_0/2m)$  as a function of  $\omega$ , where  $\gamma = 0.01$  (red), 0.03, ..., 0.09, and 0.11 is changed as a parameter.  $\omega_0 = 1$ .  $\omega_1 = 1/2$ .

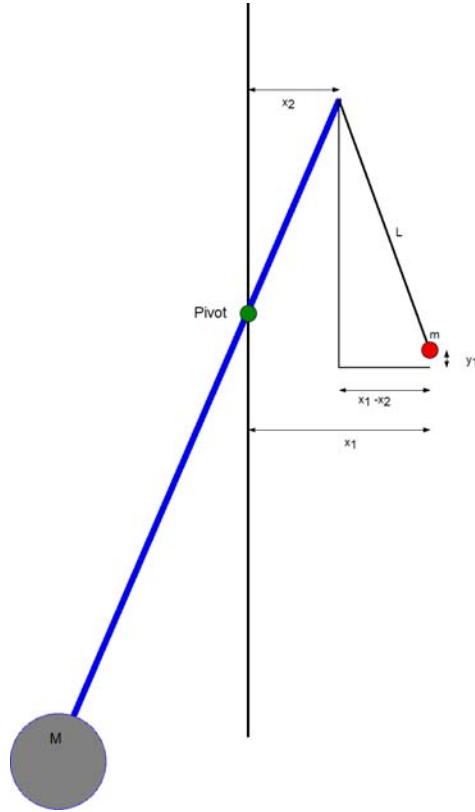


**Fig.14** (a) and (b) Plot of  $\text{Re}[X_2]/(F_0/2m)$  and  $\text{Im}[X_2]/(F_0/2m)$  as a function of  $\omega$ , where  $\gamma = 0.01$  (red), 0.03, ..., 0.09, and 0.11 is changed as a parameter.  $\omega_0 = 1$ .  $\omega_1 = 1/2$ .

## 5. Coupled pendulum (Sophomore laboratory of the Physics Department, Binghamton University)

### 5.1 Linear differential equation

The system consists of two pendulums, having a physical pendulum with large mass  $M$  (the driving pendulum) and a simple pendulum with small mass  $m$  (the system under investigation). The amplitudes of the motion of the pendulums are assumed to be small. So we can consider the motion as taking place along the horizontal line.



**Fig.15** Schematic diagram. The system consists of coupled pendulum (or pendula) having a physical pendulum with a large mass  $M$  (the driving pendulum) and a simple pendulum with small mass  $m$ . We are interested in the motion of the mass  $m$ . Since  $M \gg m$ , The motion of the large mass  $M$  is not influenced by that of small mass. In other words, the coordinate  $x_2$  changes with time  $t$  as  $X_{20} \cos(\omega t)$ , where  $X_{20}$  is a constant amplitude and  $\omega$  is the angular frequency. The movement of the point at  $x_2$  leads to the external force on the pendulum with the small mass  $m$ .

The Lagrangian  $L_T$  is given by

$$\begin{aligned}
 L_T &= \frac{1}{2} m \dot{x}_1^2 - mgy_1 \\
 &= \frac{1}{2} m \dot{x}_1^2 - mg \frac{(x_1 - x_2)^2}{2L} .
 \end{aligned}
 \tag{5.1}$$

where the expression for  $y_1$  in terms of  $x_1$ ,  $x_2$ , and  $L$  is derived in Sec.2 (see Eq.(2.1)). The Lagrange's equation:

$$\frac{d}{dt} \frac{\partial L_T}{\partial \dot{x}_1} = \frac{\partial L_T}{\partial x_1} ,$$

leads to the differential equation

$$\frac{d}{dt}(m\dot{x}_1) = -\frac{mg}{L}(x_1 - x_2), \quad (5.2)$$

or

$$\ddot{x}_1 + \frac{g}{L}x_1 = \frac{g}{L}x_2. \quad (5.3)$$

When  $\omega_0$  is defined as

$$\omega_0 = \sqrt{\frac{g}{L}}. \quad (5.4)$$

We have the linear differential equation given by

$$\ddot{x}_1 + \omega_0^2 x_1 = \omega_0^2 x_2. \quad (5.5)$$

Note that in the limit of  $x_2 \rightarrow 0$ , this differential equation expresses a simple harmonics. When the damping force ( $-m\gamma\dot{x}_1$ ) is newly added to the original differential equation, we have

$$\ddot{x}_1 + \gamma\dot{x}_1 + \omega_0^2 x_1 = \omega_0^2 x_2. \quad (5.6)$$

Note that  $F_{\text{ext}} = m\omega_0^2 x_2$  is the external driving force with the angular frequency  $\omega$ .

## 5.2 Steady-state solution

We assume that the form of the steady state solutions is given by

$$\begin{aligned} x_1 &= \text{Re}[X_1(\omega)e^{i\omega t}] \\ x_2 &= \text{Re}[X_{20}e^{i\omega t}] \end{aligned}, \quad (5.7)$$

where  $X_1(\omega)$  is the complex amplitude and  $X_{20}$  is a given constant (real) amplitude. Then we get

$$\begin{aligned} X_1(\omega) &= \frac{X_{20}\omega_0^2}{(-\omega^2 + i\omega\gamma + \omega_0^2)} = \frac{X_{20}\omega_0^2(-\omega^2 - i\omega\gamma + \omega_0^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \\ &= \frac{X_{20}\omega_0^2 e^{i\phi}}{\sqrt{(-\omega^2 + \omega_0^2)^2 + \gamma^2\omega^2}} \end{aligned} \quad (5.8)$$

where

$$\phi = \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right) = \arctan\left(\frac{\frac{\omega}{\omega_0} \frac{\gamma}{\omega_0}}{1 - \frac{\omega^2}{\omega_0^2}}\right). \quad (5.9)$$

The real part  $\text{Re}[X_1(\omega)]$  (so-called the dispersion) and the imaginary part  $\text{Im}[X_1(\omega)]$  (so-called the absorption) are obtained as

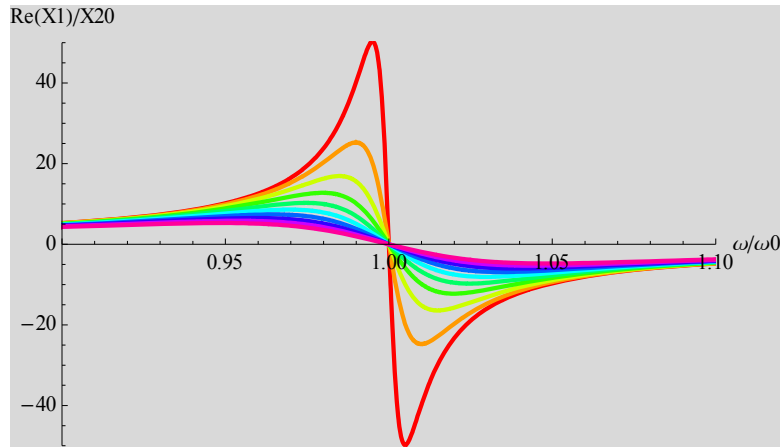
$$\text{Re}[X_1(\omega)] = \frac{X_{20}\omega_0^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} = \frac{X_{20}(1 - \frac{\omega^2}{\omega_0^2})}{(1 - \frac{\omega^2}{\omega_0^2})^2 + \frac{\omega^2}{\omega_0^2} \frac{\gamma^2}{\omega_0^2}} \quad (5.10 \text{ a})$$

$$-\text{Im}[X_1(\omega)] = \frac{X_{20} \frac{\omega}{\omega_0} \frac{\gamma}{\omega_0}}{(1 - \frac{\omega^2}{\omega_0^2})^2 + \frac{\omega^2}{\omega_0^2} \frac{\gamma^2}{\omega_0^2}}$$

Note that the magnetic susceptibility for the magnetic system is similar to the so-called susceptibility  $\chi(\omega)$  defined by

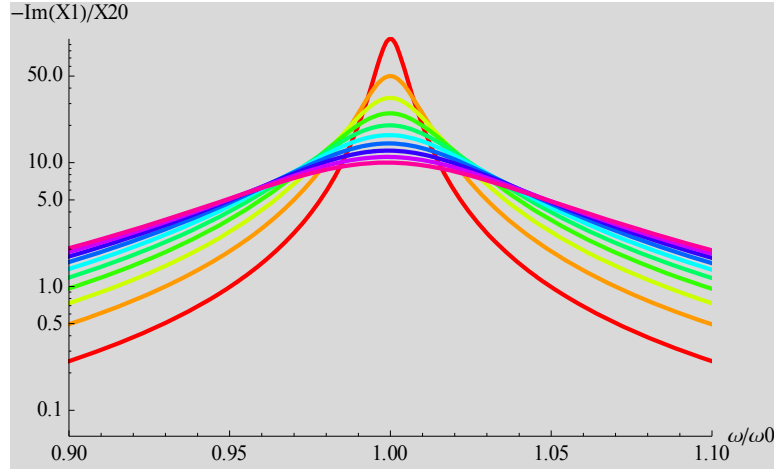
$$\chi(\omega) = \text{Re}[X_1(\omega)] - i \text{Im}[X_1(\omega)], \quad (5.10 \text{ b})$$

where we use the negative sign (-1) instead of the plus sign before the imaginary part.



**Fig.16** Plot of  $\text{Re}[X_1(\omega)]/X_{20}$  as a function of  $\omega/\omega_0$ .  $\gamma/\omega_0 = 0.01, 0.02, \dots, 0.09$ , and  $0.10$  from the top to the bottom at  $\omega/\omega_0 = 1$ .





**Fig.17** Plot of  $-\text{Im}[X_1(\omega)]/X_{20}$  as a function of  $\omega/\omega_0$ , where  $\gamma/\omega_0 = 0.01, 0.02, \dots, 0.09$ , and  $0.10$  (from the top to the bottom at  $\omega/\omega_0 = 1$  are changed as a parameter).

We now make a numerical calculation of the relative amplitude  $|X_1(\omega)|/X_2$ , given by

$$I(\omega) = \frac{|X_1(\omega)|}{X_2} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{\gamma^2}{\omega_0^2} \left(\frac{\omega}{\omega_0}\right)^2}}, \quad (5.11)$$

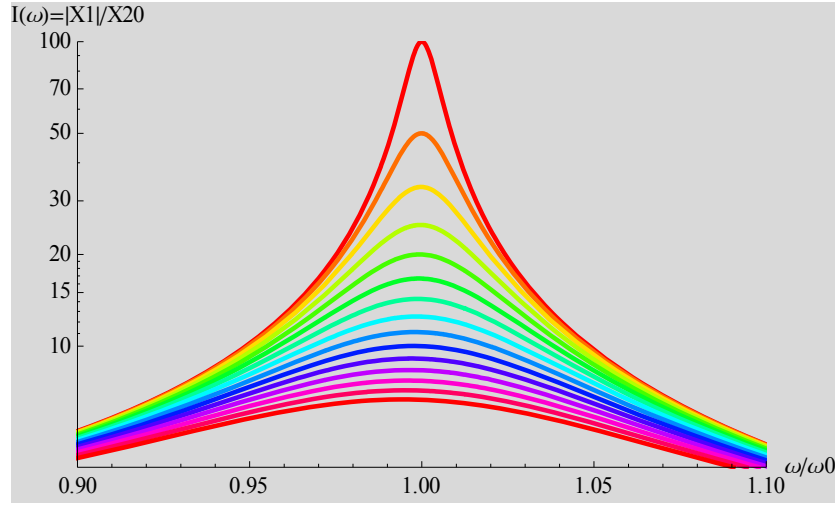
where  $X_{20}$  is a positive constant. This ratio shows a sharper peak at  $\omega = \omega_0$ . The peak value ( $= \omega_0/\gamma$ ) increases as  $\gamma$  decreases. The function  $I(\omega)$  can be expressed by

$$I(\omega) = \frac{1}{2\sqrt{\left(1 - \frac{\omega}{\omega_0}\right)^2 + \frac{\gamma^2}{4\omega_0^2}}}, \quad (5.12)$$

in the vicinity of  $\omega = \omega_0$ . On the other hand, in the limit of  $\omega/\omega_0 \ll 1$ , we also have

$$I(\omega) \approx 1 + \left(1 - \frac{\gamma^2}{2\omega_0^2}\right) \frac{\omega^2}{\omega_0^2}. \quad (5.13)$$

The relative amplitude approaches to unity in the limit of  $\omega/\omega_0 \ll 1$ .



**Fig.18** Amplitude  $|X_1(\omega)|/X_2$  as a function of  $\omega/\omega_0$ , where  $\gamma/\omega_0 = 0.01, 0.02, \dots, 0.14$ , and  $0.15$  (from the top to the bottom) is changed as a parameter.

### 4.3 Absorption of energy by the system

We now evaluate the energy absorbed in the system from the external force. The averaged energy absorbed per period  $T (= 2\pi/\omega)$  is given by

$$P_{avg}(\omega) = \frac{1}{T} \int_0^T F_{ext}(t) v_1(t) dt \quad (5.14)$$

where  $T (= 2\pi/\omega)$  is the period.  $F_{ext}(t)$  is the external force,

$$F_{ext}(t) = \text{Re}[F_0 e^{i\omega t}] = \text{Re}[(m\omega_0^2 X_{20}) e^{i\omega t}] \quad (5.15)$$

and the velocity of the mass  $m$  is given by

$$v_1(t) = \dot{x}_1(t) = \text{Re}[V_1 e^{i\omega t}] = \text{Re}[(i\omega X_1) e^{i\omega t}] \quad (5.16)$$

where  $F_0 = m\omega_0^2 X_{20}$  (real in the present case), and  $V_1 = i\omega X_1$  (the complex number in the present case). Here we note that

$$\begin{aligned} F_{ext}(t) &= \text{Re}[F_0 e^{i\omega t}] = \frac{F_0 e^{i\omega t} + F_0^* e^{-i\omega t}}{2} \\ v_1(t) &= \text{Re}[V_1 e^{i\omega t}] = \frac{V_1 e^{i\omega t} + V_1^* e^{-i\omega t}}{2} \end{aligned} \quad (5.17)$$

where  $F_0^*$  and  $V_1^*$  are the complex conjugates of  $F_0$  and  $V_1$ , respectively. Then we have

$$\begin{aligned}
P_{avg}(\omega) &= \frac{2\pi}{\omega} \int_0^{2\pi/\omega} \left( \frac{F_0 e^{i\omega t} + F_0^* e^{-i\omega t}}{2} \right) \left( \frac{V_1 e^{i\omega t} + V_1^* e^{-i\omega t}}{2} \right) dt \\
&= \frac{2\pi}{\omega} \frac{1}{4} \int_0^{2\pi/\omega} (F_0 V_1 e^{2i\omega t} + F_0 V_1^* + F_0^* V_1 + F_0^* V_1^* e^{-2i\omega t}) dt \\
&= \frac{1}{4} (F_0 V_1^* + F_0^* V_1) = \frac{1}{2} \text{Re}[F_0 V_1^*] = \frac{1}{2} \text{Re}[F_0^* V_1]
\end{aligned} \tag{5.18}$$

Using this expression (very convenient one), we can calculate the averaged power  $P(\omega)$  as

$$\begin{aligned}
P_{avg}(\omega) &= \frac{1}{2} \text{Re}[(m\omega_0^2 X_{20})(i\omega X_1)^*] \\
&= \frac{1}{2} \text{Re}[(m\omega_0^2 X_{20})(-i\omega X_1^*)]
\end{aligned} \tag{5.19}$$

Noting that

$$X_1^*(\omega) = \frac{X_{20}\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\gamma} = \frac{X_{20}\omega_0^2(\omega_0^2 - \omega^2 + i\omega\gamma)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}, \tag{5.20}$$

we get the expression of the averaged power as

$$\begin{aligned}
P_{avg}(\omega) &= \frac{1}{2} m\omega\omega_0^2 X_{20} \text{Re}\left[-i \frac{X_{20}\omega_0^2(\omega_0^2 - \omega^2 + i\omega\gamma)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\right] \\
&= \frac{1}{2} \frac{m\gamma\omega^2(\omega_0^2 X_{20})^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}
\end{aligned} \tag{5.21}$$

$P_{avg}(\omega)$  has a peak at  $\omega = \omega_0$ . The peak value is given by  $P_{\max} = m(\omega_0^2 X_{20})^2 / (2\gamma)$ . This peak increases with decreasing  $\gamma$ . Then  $P_{avg}(\omega)/P_{\max}$  can be expressed by

$$\frac{P_{avg}(\omega)}{P_{\max}} = \frac{\gamma^2 \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} = \frac{\frac{\gamma^2}{\omega_0^2} \frac{\omega^2}{\omega_0^2}}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{\omega_0^2} \frac{\omega^2}{\omega_0^2}} \tag{5.22}$$

This function is expressed by a Lorentzian form in the vicinity of  $\omega/\omega_0 = 1$ , since

$$\begin{aligned}
\frac{P_{avg}(\omega)}{P_{max}} &= \frac{\frac{\gamma^2}{\omega_0^2} \frac{\omega^2}{\omega_0^2}}{(1 + \frac{\omega}{\omega_0})^2 (1 - \frac{\omega}{\omega_0})^2 + \frac{\gamma^2}{\omega_0^2} \frac{\omega^2}{\omega_0^2}} \approx \frac{\frac{\gamma^2}{\omega_0^2} \frac{\omega_0^2}{\omega_0^2}}{(1 + \frac{\omega_0}{\omega_0})^2 (1 - \frac{\omega}{\omega_0})^2 + \frac{\gamma^2}{\omega_0^2} \frac{\omega_0^2}{\omega_0^2}} \\
&= \frac{\frac{\gamma^2}{4}}{(\omega - \omega_0)^2 + \frac{\gamma^2}{4}}
\end{aligned} \tag{5.23}$$

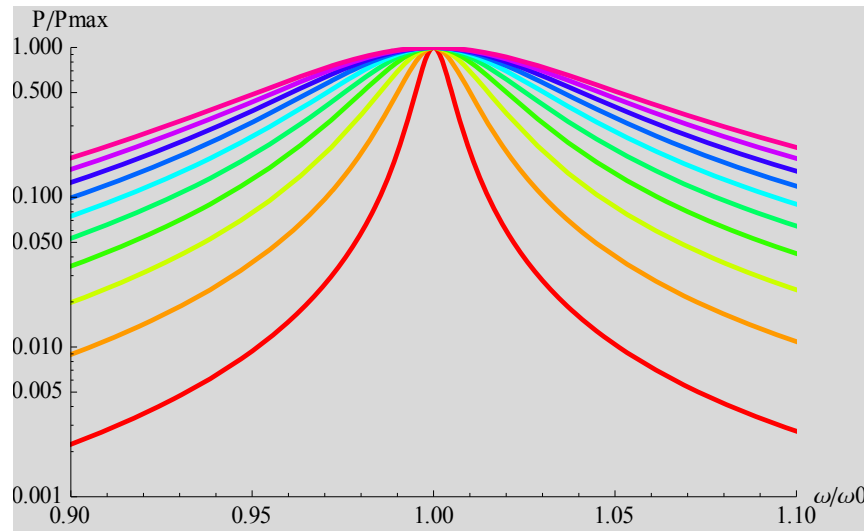
$P_{avg}(\omega)/P_{max}$  becomes 1/2 when  $\omega_u = \omega_0 + \gamma/2$  and  $\omega_l = \omega_0 - \gamma/2$  around  $\omega = \omega_0$ . Then the Full-width at half maximum (FWHM)  $\Delta\omega$  is obtained as  $\Delta\omega = \omega_u - \omega_l = \gamma$ . The FWHM become small as  $\gamma$  is reduced to zero. The  $Q$ -value is defined by

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\gamma}, \tag{5.24}$$

which provides a measure for the narrowness of the width for the resonance curve. It is noted that  $P_{avg}(\omega)$  is related to  $\text{Im}[X_1(\omega)]$  through

$$P_{avg}(\omega) = \frac{1}{2} F_0 \omega (-\text{Im}[X_1(\omega)]) = \frac{1}{2} F_0 \omega \text{Im}[\chi(\omega)] \tag{5.25}$$

So  $P_{avg}(\omega)$  is proportional to the absorption [imaginary part of  $X_1(\omega)$ ].



**Fig.19** Plot of  $P/P_{max}$  as a function of  $\omega/\omega_0$ . where  $\gamma/\omega_0 = 0.01, 0.02, \dots$ , and  $0.10$  (from the bottom to the top) is changed as a parameter.

## 5.4 The power dissipated by a friction force

Where is the average power absorbed by the external force? We show that the average power absorbed by the system is dissipated by the friction force given by

$$f_{friction}(t) = -m\gamma v_1(t) = -m\gamma \dot{x}_1(t) = \text{Re}[(-m\gamma i\omega X_1)e^{i\omega t}] \quad (5.26)$$

In fact, the averaged power dissipated by the friction force during the period  $T (= 2\pi/\omega)$  is

$$\begin{aligned} P_{dissipation} &= \frac{1}{T} \int_0^T [-f_{friction}(t)]v_1(t)dt = \frac{1}{2} \text{Re}[(m\gamma i\omega)^* X_1^* i\omega X_1] \\ &= \frac{1}{2} m\gamma \omega^2 |X_1|^2 \\ &= \frac{1}{2} \frac{m\gamma \omega^2 (\omega_0^2 X_{20})^2}{(-\omega^2 + \omega_0^2)^2 + \gamma^2 \omega^2} \end{aligned} \quad (5.27)$$

where

$$|X_1| = \frac{X_{20}\omega_0^2}{\sqrt{(-\omega^2 + \omega_0^2)^2 + \gamma^2 \omega^2}} \quad (5.28)$$

So we find that

$$P_{dissipation} = P_{avg} \quad (5.29)$$

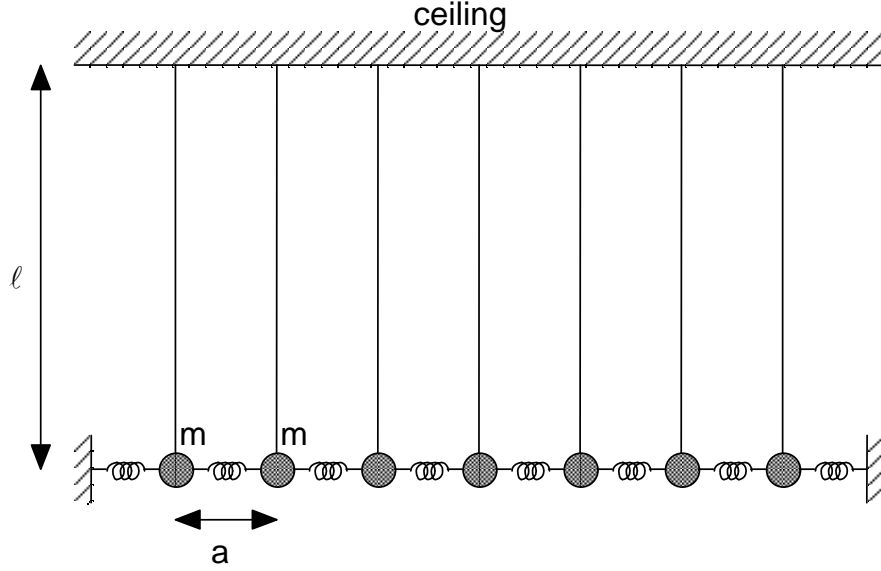
## 6. Coupled pendulum (Junior laboratory of the Physics Department, Binghamton University)

### 6.1 Differential equation

Discussions of thermal properties of solids often begin with a treatment that is essentially, or sometimes explicitly, one dimensional (1D). The model is that of a series of masses connected by springs. This experiment attempts to duplicate this model, the one complication being that the masses are suspended on strings. This means that when there is a displacement of the mass from equilibrium, this pendulum-type restoring force must be included. For example, Newton's law written for the displacement  $x_n$  of the  $n$ -th mass is: for longitudinal displacements

$$m \frac{d^2}{dt^2} x_n = -\frac{mg}{l} x_n - K_L(x_n - x_{n-1}) + K_L(x_{n+1} - x_n) \quad (6.1)$$

where  $K_L$  is the spring constant,  $l$  is the distance between the ceiling and the masses. The second and the third terms are due to the springs.



**Fig.20** Coupled pendulums consisting of  $N$  masses (mass  $m$ ). Masses are connected by spring constant  $K$ .

((Note))

The first term of this equation is due to the gravitational force. This is derived as follows. In Fig.9, we have

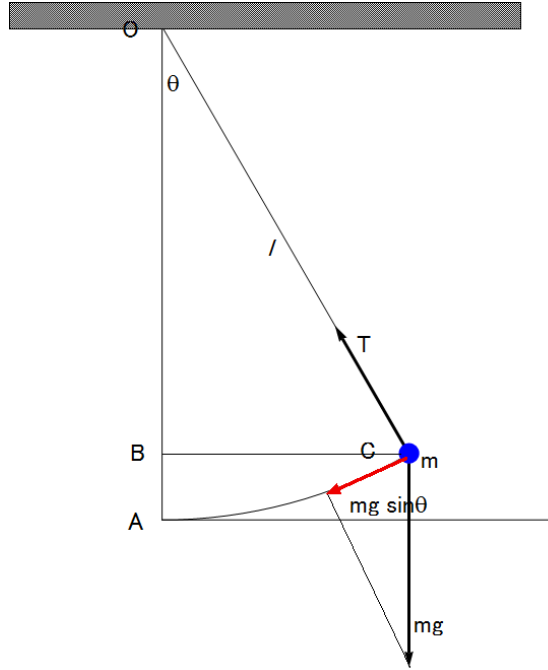
$$\begin{aligned} s &= l\theta \\ x &= l \sin \theta \approx l\theta, \end{aligned} \tag{6.2}$$

where  $x$  is the  $x$ -coordinate of the mass and  $s$  is the distance of arc AC. The equation of motion for the mass  $m$  is expressed by

$$m \frac{d^2 s}{dt^2} = mL \frac{d^2 \theta}{dt^2} = -mg \sin \theta \approx -mg \theta \tag{6.3}$$

Using the relation ( $x = L\theta$ ), we have

$$\begin{aligned} m \frac{d^2 \theta}{dt^2} &= -\frac{mg}{L} \theta \\ m \frac{d^2 x}{dt^2} &= -\frac{mg}{L} x \end{aligned} \tag{6.4}$$



**Fig.21** Simple pendulum with a mass ( $m$ ).  $T$  is the tension.  $mg$  is the gravitational force directed downward. The length of arc AC is  $s = l\theta$ .

## 6.2 Dispersion relation

We assume that  $x_n$  can be expressed by the form

$$x_n = \text{Re}[A \sin(kna) e^{i\omega t}] = A \sin(kna) \cos(\omega t), \quad (6.5)$$

with  $n = 1, 2, \dots, N$ , where  $A$  is a real amplitude,  $a$  is the length of the unstretched spring, and  $k$  is the wave number. The validity of this assumption is given in the Lecture Note on Oscillations and Waves. From the boundary condition that  $x_{N+1} = 0$ , we have

$$k = k_j = \frac{j\pi}{(N+1)a}, \quad (6.6)$$

where  $j = 1, 2, 3, \dots, N$ . The substitution of the form of  $x_n$  into the original equation of motion leads to the dispersion relation,

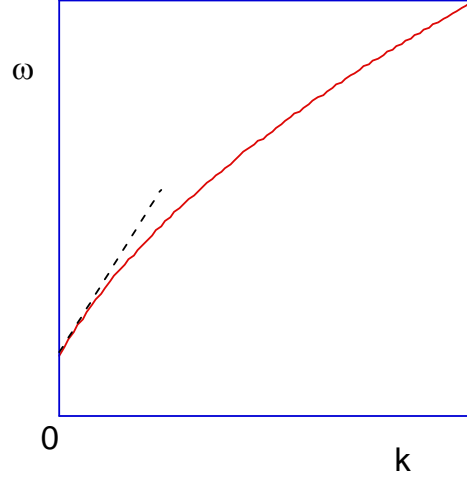
$$\omega_j^2 = \frac{g}{l} + \frac{4K_L}{m} \sin^2\left(\frac{ak_j}{2}\right) = \frac{g}{l} + \frac{4K_L}{m} \sin^2\left[\frac{j\pi}{2(N+1)}\right], \quad (6.7)$$

for a longitudinal waves, and a similar equation results for transverse waves

$$\omega_j^2 = \frac{g}{l} + \frac{4K_T}{m} \sin^2\left(\frac{ak_j}{2}\right) = \frac{g}{l} + \frac{4K_T}{m} \sin^2\left[\frac{j\pi}{2(N+1)}\right]. \quad (6.8)$$

In the long-wave-length limit ( $k \rightarrow 0$ ) the slope of the  $\omega$  versus  $k$  curve is the speed of wave propagation. From this one can determine the spring constants  $K_T$  and  $K_L$  except that in the

situation here the term  $g/l$  means that the slope approaches zero as  $k$  approaches zero. One can thus get an estimate for  $K_L$  or for  $K_T$ , but the data near  $k = 0$  must be disregarded. A bit of thought will enable you to plot the data as straight lines and use least squares fits to make appropriate estimates.



**Fig.22** Dispersion relation for either the longitudinal wave or the transverse wave. The slope of  $\omega$  vs.  $k$  at  $k = 0$  (denoted by the dotted line) corresponds to the velocity. The velocity (corresponding to the slope in  $\omega$  vs  $k$ ) is different between the longitudinal wave and the transverse wave.

### 6.3. Initial conditions for the normal modes for the transverse wave

We consider the method how to measure the dispersion relation for the transverse waves, where the displacement of the masses is perpendicular to the chain axis. We can apply similar method to the longitudinal wave, where the displacement of the masses is parallel to the chain axis. The position of the  $n$ -th mass at the equilibrium position is given by

$$x_{nj}(t) = A \sin(k_j n a) \cos(\omega_j t), \quad (6.9)$$

for the normal mode  $j$ . The velocity of the  $n$ -th mass at the equilibrium position is obtained as

$$\dot{x}_{nj}(t) = -A \omega \sin(k_j n a) \sin(\omega_j t). \quad (6.10)$$

From these two Eqs. we get the initial conditions to get the characteristic angular frequency of the normal mode  $j$  experimentally.

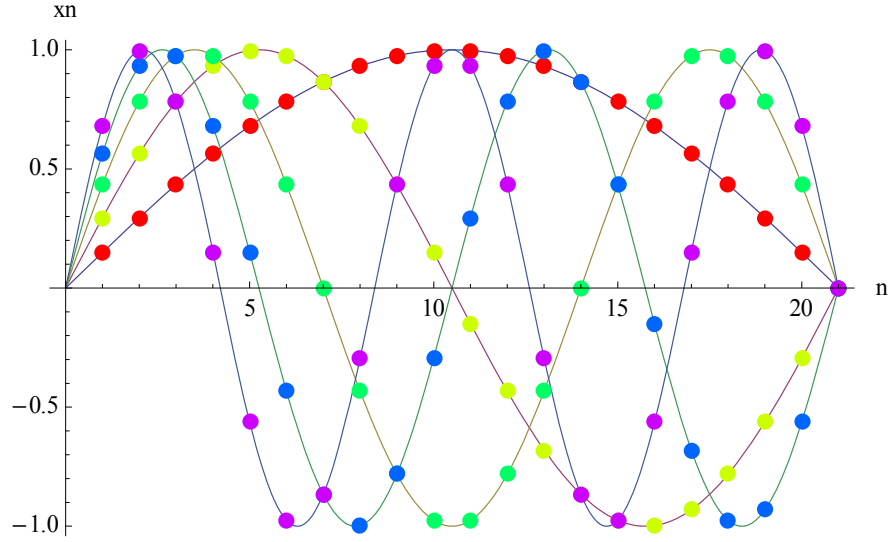
$$\begin{aligned} x_{nj}(t=0) &= A \sin(k_j n a) = A \sin\left[\frac{n j \pi}{N+1}\right] \\ \dot{x}_{nj}(t=0) &= 0 \end{aligned} \quad (6.11)$$

The pattern for the positions ( $x_{nj}$ ) of the masses for  $j = 1, 2, 3, 4$ , and  $5$  at  $t = 0$  for  $N = 20$  are shown in Fig. All the masses stay at rest at  $t = 0$ . The initial pattern for each  $j$  is set up at  $t = 0$ . At  $t > 0$ , these masses are released at the same time. Since



$$x_{nj}(t + T_j) = x_{nj}(t), \quad (6.12)$$

the initial pattern periodically appears at the period given by  $T_j (=2\pi/\omega_j)$ , for the  $j$ -mode. Note that it is very difficult to do such an experiment for  $j > 5$ , since the initial pattern becomes complicated. So the experimental data of the dispersion relation is limited only for  $j < 5$ .



**Fig.23** The initial pattern for the positions ( $x_{nj}$ ) of the masses for  $j = 1, 2, 3, 4$ , and  $5$ .  $N = 20$ .  $x_n$  is the displacement for the transverse wave. For the longitudinal wave, the displacement is parallel to the chain axis.

## 6.4 Experimental Procedure

- (i) Measure  $\omega$  versus  $k$  experimentally. (This is the dispersion relation). Take the data of  $\omega_j$  vs  $k_j$  for the mode  $j = 1, 2, 3, 4$ , and  $5$ , where  $k_j = j\pi/(N+1)a$ , where  $(N+1)a$  is the total length of the chain. Before the measurements, the initial pattern for each mode  $j$  needs to be formed. Note that the initial pattern for the longitudinal waves is different from that for the transverse waves.
- (ii) Obtain the constant  $K_L$  (N/m), from the analysis by the least-squares fitting.
- (iii) Obtain the constant  $K_T$  in units of N/m. Note that  $K_T$  should be much smaller than  $K_L$ .
- (iv) Exhibit "experimental" and "theoretical"  $\omega$  versus  $k$  data.

## CONCLUSION

The motion of coupled pendulum is mainly expressed by nonlinear differential equations. In the limiting case where the differential equations become linear differential equations, the normal modes can be derived from the eigenvalue problems. The general solution can be expressed by a superposition of such normal modes. We discuss the case when an external force

is newly applied to the system with the normal mode. When the angular frequency of the external force is very close to that for one of the normal modes, there occurs absorption by the system from the external force. The dispersion relation is derived for the one dimensional chain of the masses, where each mass is kept by string coming from ceiling. This is the simple model of the 1D lattice system, which is closely related to the lattice of the real crystals.

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## APPENDIX

We use the following Mathematica 7.0 programs in the present notes.

### 1. VariationalD

#### VariationalD

`VariationalD[f, u[x], x]`  
returns the variational derivative of the integral  $\int f dx$  with respect to  $u[x]$ , where the integrand  $f$  is a function of  $u$ , its derivatives, and  $x$ .

`VariationalD[f, u[x, y, ...], {x, y, ...}]`  
returns the variational derivative of the multiple integral  $\int f dx dy \dots$  with respect to  $u[x, y, \dots]$ , where  $f$  is a function of  $u$ , its derivatives and the coordinates  $x, y, \dots$

`VariationalD[f, {u[x, y, ...], v[x, y, ...], ...}, {x, y, ...}]`  
gives a list of variational derivatives with respect to  $u, v, \dots$

#### MORE INFORMATION

- To use VariationalD, you first need to load the [Variational Methods Package](#) using `Needs["VariationalMethods`"]`.

## 2. NDSolve

### NDSolve

Updated in 7  
Show changes

`NDSolve[eqns, y, {x, xmin, xmax}]`  
finds a numerical solution to the ordinary differential equations *eqns* for the function *y* with the independent variable *x* in the range *xmin* to *xmax*.

`NDSolve[eqns, y, {x, xmin, xmax}, {t, tmin, tmax}]`  
finds a numerical solution to the partial differential equations *eqns*.

`NDSolve[eqns, {y1, y2, ...}, {x, xmin, xmax}]`  
finds numerical solutions for the functions *y<sub>i</sub>*.

► [MORE INFORMATION](#)

## 3. Fourier

### Fourier

`Fourier[list]`  
finds the discrete Fourier transform of a list of complex numbers.

[MORE INFORMATION](#)

- The discrete Fourier transform  $v_s$  of a list  $u_r$  of length  $n$  is by default defined to be  $\frac{1}{\sqrt{n}} \sum_{r=1}^n u_r e^{2\pi i (r-1)(s-1)/n}$ . »
- Note that the zero frequency term appears at position 1 in the resulting list.
- Other definitions are used in some scientific and technical fields.
- Different choices of definitions can be specified using the option `FourierParameters`.
- With the setting `FourierParameters -> {a, b}` the discrete Fourier transform computed by `Fourier` is  $\frac{1}{n(1-a)/2} \sum_{r=1}^n u_r e^{2\pi i b (r-1)(s-1)/n}$ . »
- Some common choices for  $\{a, b\}$  are  $\{0, 1\}$  (default),  $\{-1, 1\}$  (data analysis),  $\{1, -1\}$  (signal processing).
- The setting  $b = -1$  effectively corresponds to conjugating both input and output lists.
- To ensure a unique inverse discrete Fourier transform,  $|b|$  must be relatively prime to  $n$ . »
- The list of data supplied to `Fourier` need not have a length equal to a power of two.
- The *list* given in `Fourier[list]` can be nested to represent an array of data in any number of dimensions.
- The array of data must be rectangular.
- If the elements of *list* are exact numbers, `Fourier` begins by applying `N` to them.
- `Fourier` can be used on `SparseArray` objects.

## 4. ParametricPlot

### ParametricPlot

`ParametricPlot[{fx, fy}, {u, umin, umax}]`

generates a parametric plot of a curve with  $x$  and  $y$  coordinates  $f_x$  and  $f_y$  as a function of  $u$ .

`ParametricPlot[{fx, fy}, {gx, gy}, ..., {u, umin, umax}]`

plots several parametric curves.

`ParametricPlot[{fx, fy}, {u, umin, umax}, {v, vmin, vmax}]`

plots a parametric region.

`ParametricPlot[{fx, fy}, {gx, gy}, ..., {u, umin, umax}, {v, vmin, vmax}]`

plots several parametric regions.