

Lecture note on Senior Laboratory in Physics

Fraunhofer diffraction and double-split experiment

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ABSTRACT

The experiment is performed in order to determine the relationship between Fresnel and Fraunhofer diffraction, and then to see how they relate to the Fourier Transform. The set-up consists of a 10m track with movable lenses, transparencies, filters, and screens to create different diffraction patterns. Many different images are examined within a close range and at an infinite distance. When Fraunhofer diffraction is observed, a Fourier Transform is produced optically. As the spatial frequency increases, the images become clearer and more intense. Image-processing is also observed to see “real world” applications.

In this note, we discuss a theoretical background for this experiment. We show that the diffraction pattern of apertures (including a Young’s double-slits and a diffraction grating) is closely related to its power spectrum of Fourier transform. We also show typical power spectra which are calculated using Mathematica 5.2.

1. Principle of Fraunhofer diffraction¹

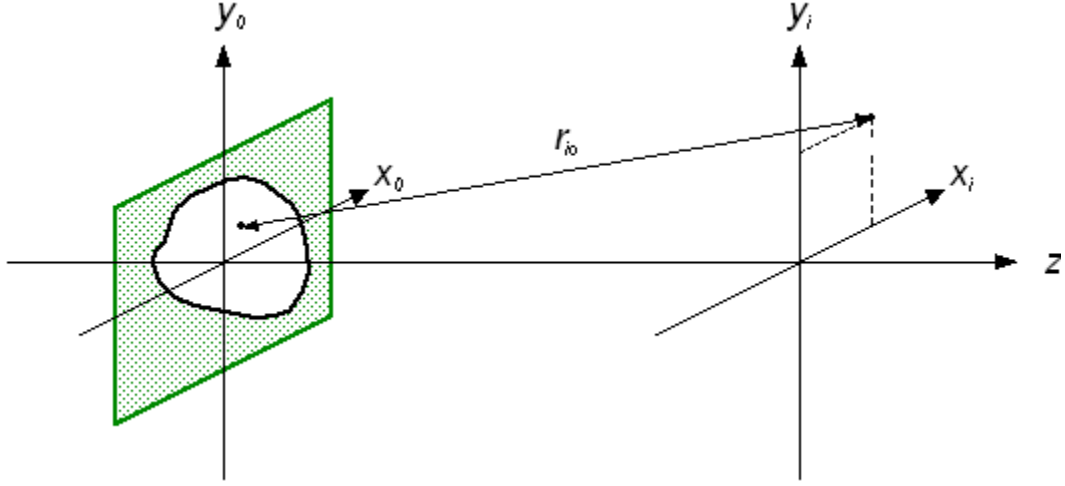


Fig.1 Schematic diagram of the Fraunhofer diffraction

We assume that the aperture acts as the source of a field $E_0(x_0, y_0)$. The field $E_i(x_i, y_i)$ on the screen is given by¹

$$E_i(x_i, y_i) = \frac{\exp(ikz)}{i\lambda z} \iint E_0 \exp\left\{\frac{ik}{2z}[(x_i - x_0)^2 + (y_i - y_0)^2]\right\} dx_0 dy_0. \quad (1)$$

where the integral is a sum of the spherical wave of the field E_0 originating from the object points (x_0, y_0) within the aperture, and k is the wave number ($2\pi/\lambda$). So the surface integral runs over the aperture. Note that the spherical wave diverging from the object point (x_0, y_0) is described by $\exp(ikr_{io})/r_{io}$.

This is known as the Fresnel approximation to the scalar diffraction theory. It is useful when z is very large compared to a wavelength. A further rearrangement of the Fresnel diffraction expression will be computationally convenient.

$$E_i(x_i, y_i) = \frac{\exp(ikz)}{i\lambda z} \iint E_0 \exp\left\{\frac{ik}{2z}[(x_i^2 + x_0^2 - 2x_i x_0) + (y_i^2 + y_0^2 - 2y_i y_0)]\right\} dx_0 dy_0, \quad (2)$$

or

$$E_i(x_i, y_i) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{ik}{2z}(x_i^2 + y_i^2)\right] \iint E_0(x_0, y_0) \exp\left[\frac{ik}{2z}(x_0^2 + y_0^2)\right] \exp\left[-\frac{ik}{z}(x_i x_0 + y_i y_0)\right] dx_0 dy_0.$$

If $z \gg k(x_0^2 + y_0^2)_{\max}$, this equation is approximated as

$$E_i(x_i, y_i) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{ik}{2z}(x_i^2 + y_i^2)\right] \iint E_0(x_0, y_0) \exp\left[-\frac{ik}{z}(x_i x_0 + y_i y_0)\right] dx_0 dy_0. \quad (3)$$

This infinite-distance limit is called the Fraunhofer regime and is the case usually considered in elementary treatments. This is simply the Fourier transform of the aperture illumination. We usually want to know the optical intensity, which is proportional to $|E_i|^2$, so the phase factor in front is irrelevant.

$$\left| \iint E_0(x_0, y_0) \exp[-(k_x x_0 + k_y y_0)] dx_0 dy_0 \right|^2, \quad (4)$$

where $k_x = \frac{kx_i}{2\pi z}$ and $k_y = \frac{ky_i}{2\pi z}$ are wave numbers (so called spatial frequencies), with units of $(\text{length})^{-1}$. So the diffraction pattern is simply the absolute square of the 2D Fourier transform of the aperture. Note that $(x_i, y_i) = \frac{2\pi z}{k}(k_x, k_y)$.

2. Definition of Fourier Transform

The *amplitude* in the diffraction pattern is simply the 2D Fourier transform of the aperture, besides scaling. We will look at intensity, obtained by taking the absolute value of the amplitude. This would be called the **power spectrum** if the independent variable were time instead of the space.

The conventional mathematical representation of an image is a function of two *spatial* variables, $f(x, y)$. The function at a particular location, (x, y) , is the intensity at that point. The term *transform* means an alternative mathematical representation of the image.

A **Fourier Transform** uses a series of complex exponentials (sinusoids) with different frequencies to represent an image. The Fourier Transform has applications in image processing, feature recognition, signal processing etc.

If $f(x, y)$ are spatial variables in the continuous domain the 2D Fourier Transform is defined as:

$$F(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy f(x, y) \exp[-i2\pi(k_x x + k_y y)], \quad (5)$$

where $f(x, y)$ is the light intensity at point (x, y) . (k_x, k_y) are the horizontal and vertical spatial frequencies respectively. Inversely, the Fourier Transform can be transformed back to the spatial domain by the inverse Fourier transform:

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y F(k_x, k_y) \exp[i2\pi(k_x x + k_y y)]. \quad (6)$$

3. Numerical calculation using Fast Fourier transform (FFT)

In numerical calculation we use the program of the FFT (fast Fourier transform) of the Mathematica 5.2.

3.1 Fourier and inverse Fourier transform

There are two kinds of Fourier transforms in Mathematica 5.2.

(a) Fourier transform

`Fourier[{x1, x2, ..., xn}]`
 produces the list { , , ..., } that is the discrete Fourier transform of the list { , , ..., }, defined by .
 Multidimensional data can be transformed by
`Fourier[{ { , , ..., }, { , , ..., }, ..., { , , ..., } }].`

If we input exact data, the data will be numericalized to machine precision.

(b) Inverse Fourier Transform

`InverseFourier[{y1, y2, ..., yn}]`
 produces the list { , , ..., } that is the discrete inverse Fourier transform of { , , ..., }, defined by .
`InverseFourier[{ { , , ..., }, { , , ..., }, ..., { , , ..., } }].`

3.2 Power spectrum

The following discussion is based on the note written by FFT program (Richard G. Palmer)² and the Mathematica programs (Graphics) written by Micahel Trott.³

(a) Shape of the Aperture:

We represent apertures (or *masks*) by arrays of 0's (opaque) and 1's (transparent). We can use **ListDensityPlot[]** to plot them, with the **Mesh→False** option to suppress the grid lines.

(b) Power spectrum of the FFT

Fourier[] from which the FFT of the aperture can be calculated.

Abs[]^2 from which the power spectrum is calculated.

(c) The center position

We need "frequencies" (here called spatial frequencies, or wave numbers, or just k 's) running from $-K$ to K , whereas the FFT gives 0 to $2K$. That applies in both the k_x and k_y direction ($k_x = -K$ to K , $k_y = -K$ to K). We can do both rotations at once with a `RotateLeft[array, {m, n}]`, where $\{m, n\}$ is the center position.

(d) Erasing of the large intensity

There is a problem. The diffraction bands are far too bright. That is because `ListDensityPlot[]` has truncated some of the larger values, just like `Plot[]` and `ListPlot[]` do. `PlotRange->All` tells it to behave, making pure white represent the largest value:

4. Power spectrum of small square aperture

Here we show the power spectrum of small square aperture which is calculated using Mathematica 5.2.

$n = 600$

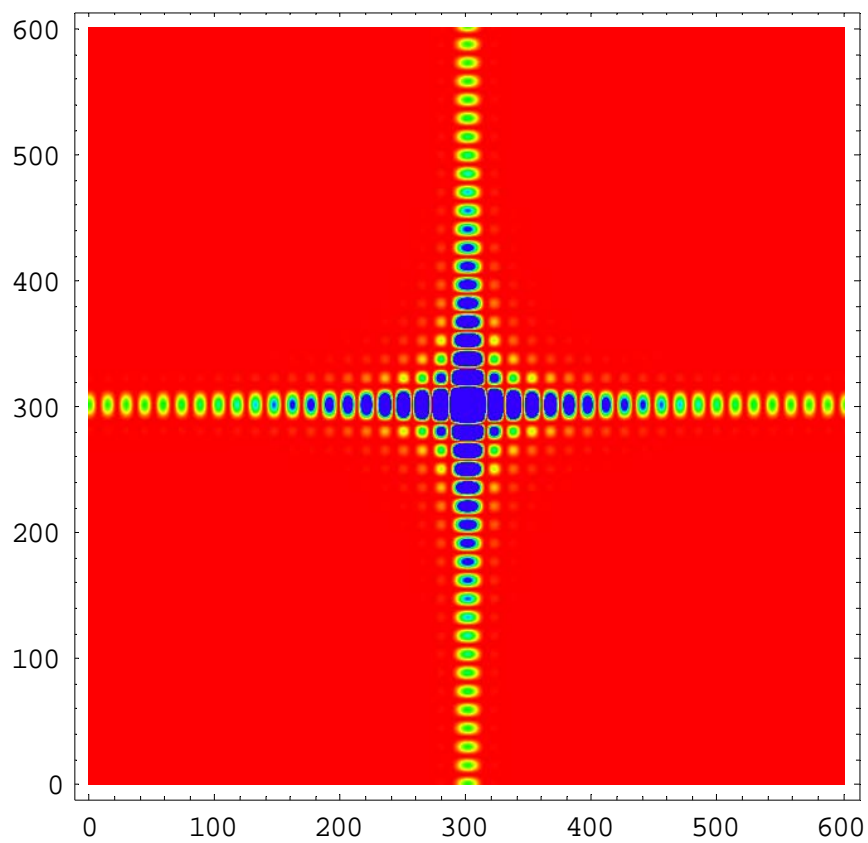
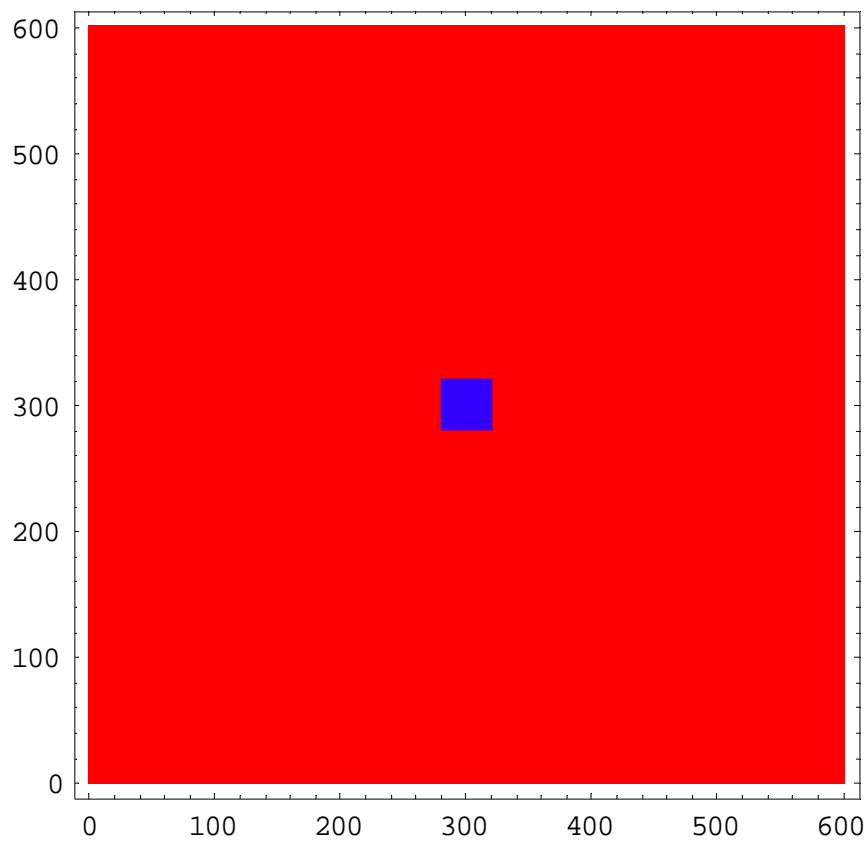
Fig.2a Shape of the aperture (small square)

Fig.2b Power spectrum

((Mathematica 5.2)) I

n=600

```
600
square=Table[If[n/2-20<=x<=n/2+20&& n/2-
20<=y<=n/2+20,1.0,0.0],{y,0,n},{x,0,n}];
ListDensityPlot[square,Mesh->False,ColorFunction->(Hue[0.7#]&
)];fftsq=Fourier[square];
pssq=Abs[fftsq]^2;ldp=ListDensityPlot[RotateLeft[pssq,{n/2,n
/2}],Mesh->False,ColorFunction->(Hue[0.7#]&),PlotRange->{0.000
1,0.01}]
```



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5. Power spectrum of small circular hole aperture

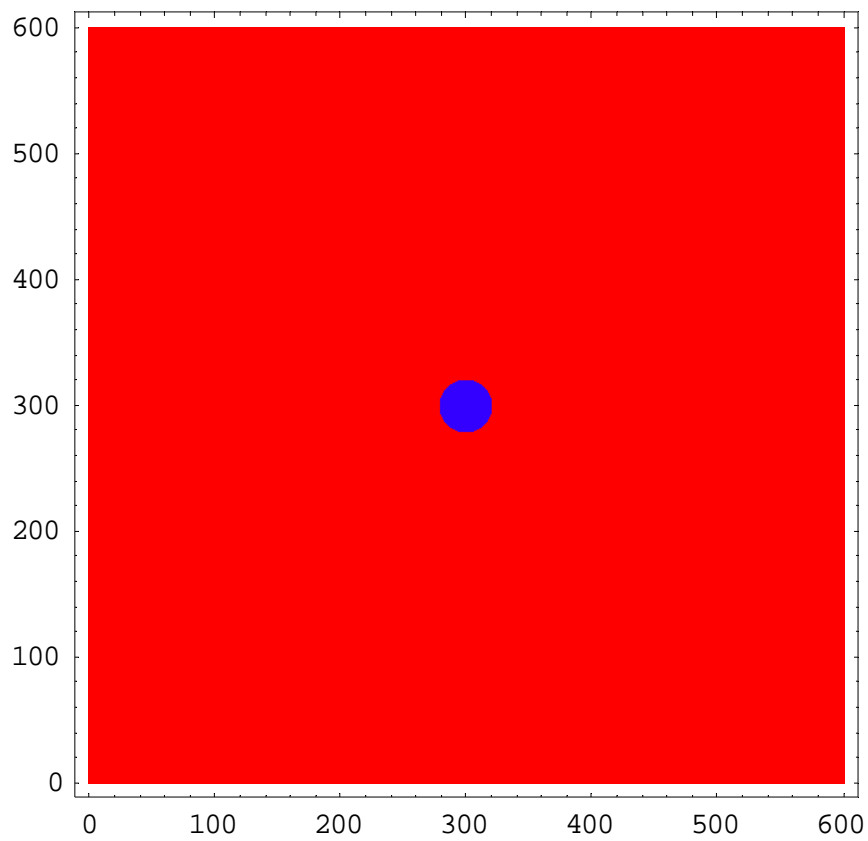
$n = 600$

Fig.3a Shape of the aperture (small circular hole)

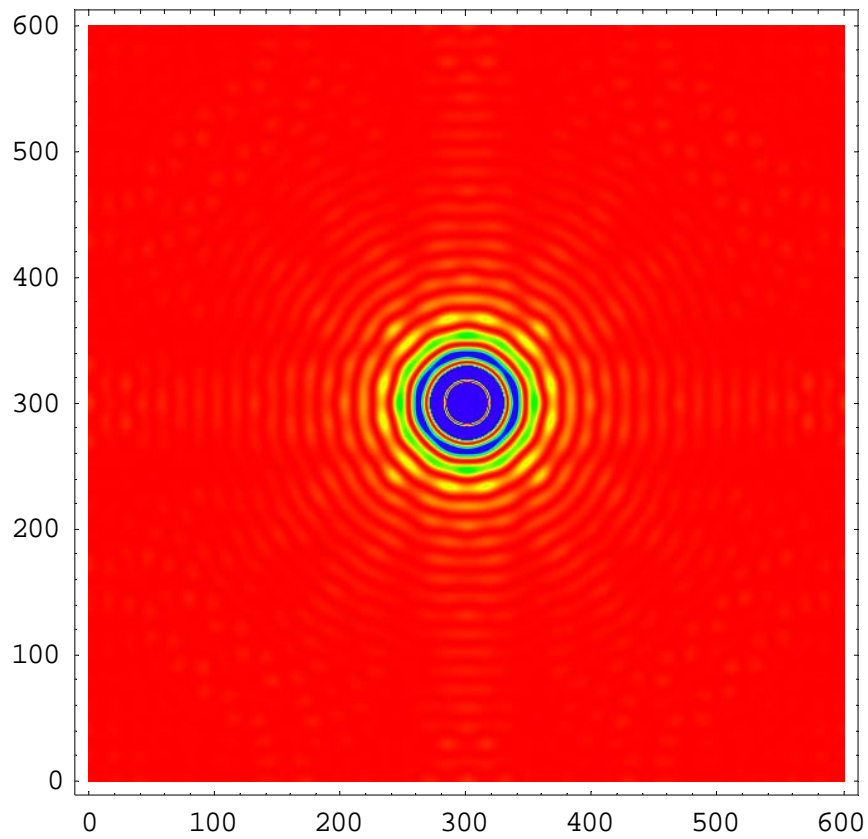
Fig.3b Its power spectrum

((Mathematica 5.2)) II

```
n = 600; (* n x n *)
r = 0.07; (* Radius *)
rr = (r*n*0.5)^2;
circ =
  Table[If[(x - n/2 + 1)^2 + (y - n/2 + 1)^2 < rr, 1.0, 0.0],
    {y, 0, n - 1}, {x, 0, n - 1}];
ListDensityPlot[circ, Mesh → False,
  ColorFunction → (Hue[0.7 #] &)]
pscirc = Abs[Fourier[circ]]^2;
ListDensityPlot[RotateLeft[pscirc, {n/2, n/2}],
  Mesh → False, PlotRange → {0, 0.02},
  ColorFunction → (Hue[0.7 #] &)]
```



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6. Power spectrum of circular hole

$n = 300$

Fig.4a Shape of the aperture (large circular hole)

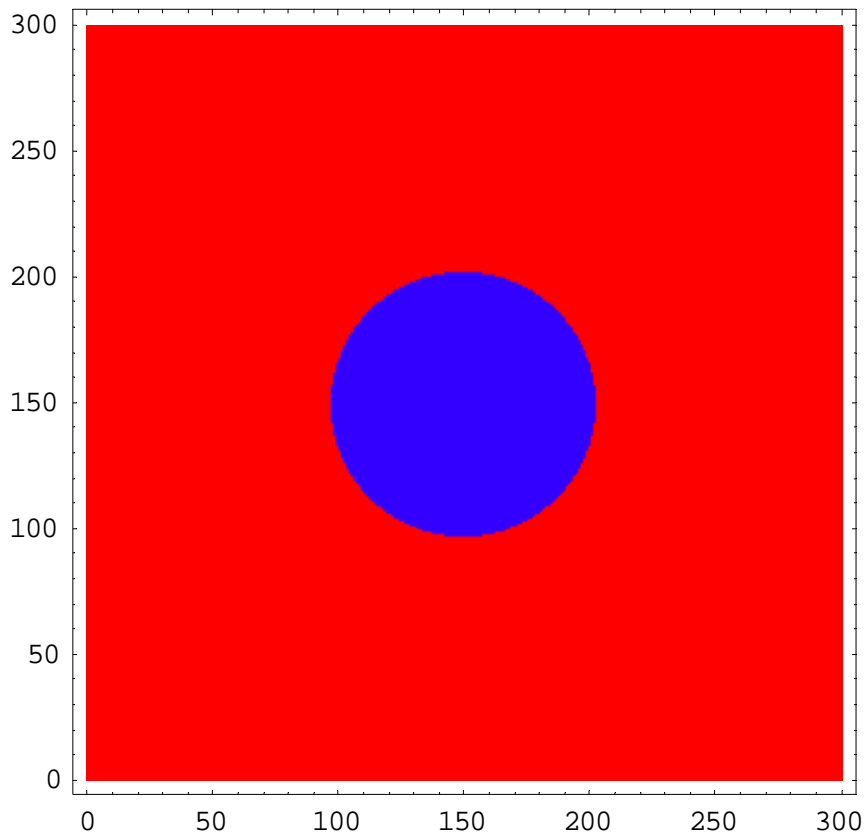
Fig.4b Its power spectrum

((Mathematica 5.2)) III

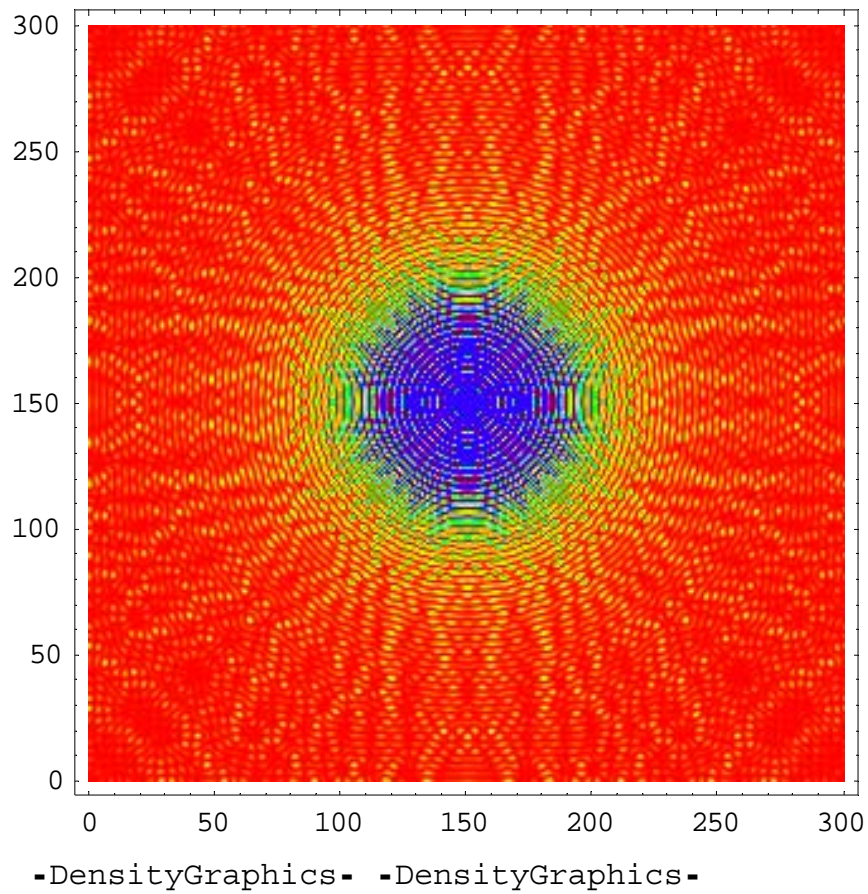
```

n = 300; (* n x n *)
r = 0.35; (* Radius *)
rr = (r*n*0.5)^2;
circ =
  Table[If[(x - n/2 + 1)^2 + (y - n/2 + 1)^2 < rr, 1.0, 0.0],
    {y, 0, n - 1}, {x, 0, n - 1}];
ListDensityPlot[circ, Mesh → False,
  ColorFunction → (Hue[0.7 #] &)]
pscirc = Abs[Fourier[circ]]^2;
ListDensityPlot[RotateLeft[pscirc, {n/2, n/2}],
  Mesh → False, PlotRange → {0, 0.02},
  ColorFunction → (Hue[0.7 #] &)]

```



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7. Power spectrum of a circular ring

$n = 300$

Fig.5a Shape of the aperture (circular ring)

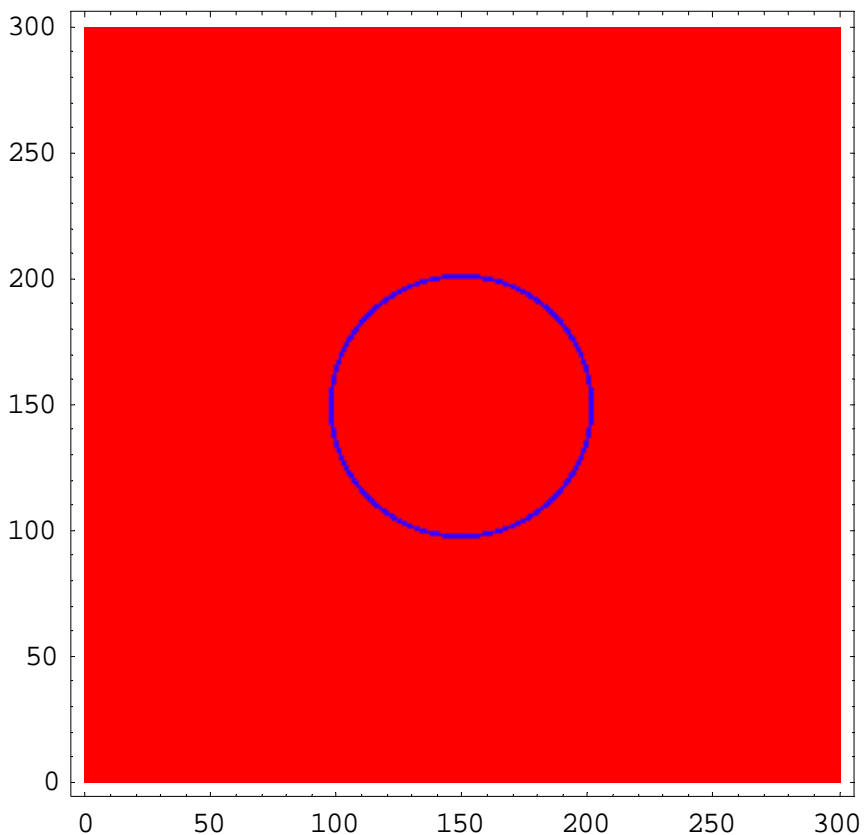
Fig.5b Its power spectrum

((Mathematica 5.2)) IV

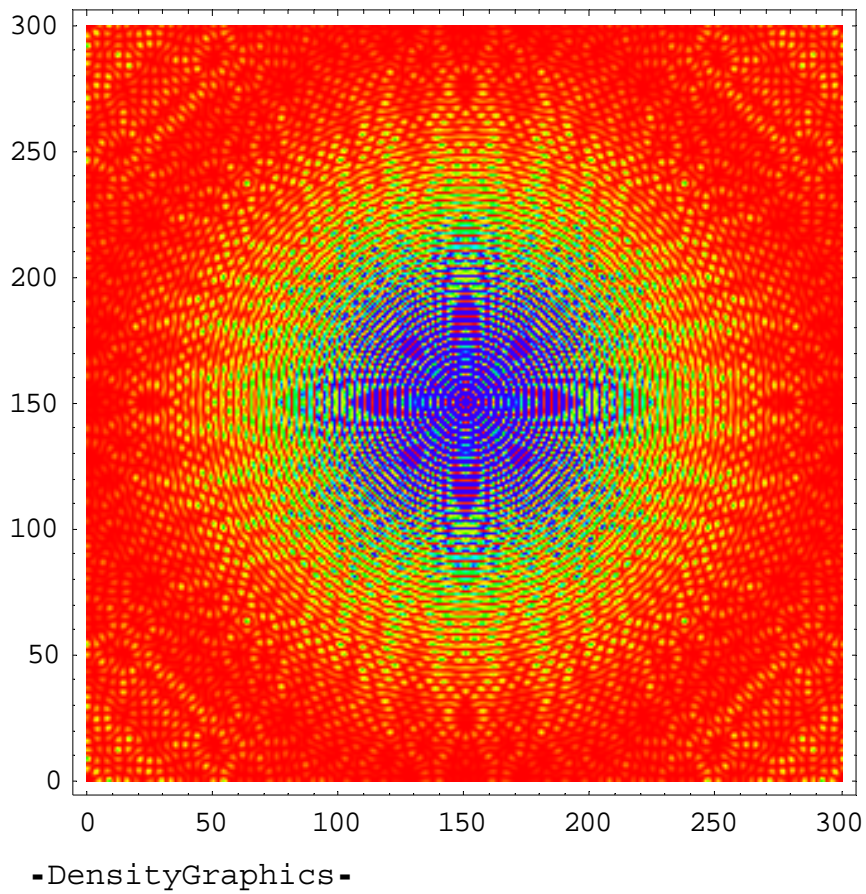
```

n = 300; (* n x n *)
r = 0.35; (* Radius *)
rr = (r * n * 0.5) ^ 2; rrm = (r * (n - 10) * 0.5) ^ 2
circ =
  Table[If[rrm < (x - n / 2 + 1) ^ 2 + (y - n / 2 + 1) ^ 2 < rr,
    1.0, 0.0], {y, 0, n - 1}, {x, 0, n - 1}];
ListDensityPlot[circ, Mesh → False,
  ColorFunction → (Hue[0.7 #] &)]
pscirc = Abs[Fourier[circ]] ^ 2;
ListDensityPlot[RotateLeft[pscirc, {n / 2, n / 2}],
  Mesh → False, PlotRange → {0, 0.03},
  ColorFunction → (Hue[0.7 #] &)]
2575.56

```



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8. Double-slit diffraction

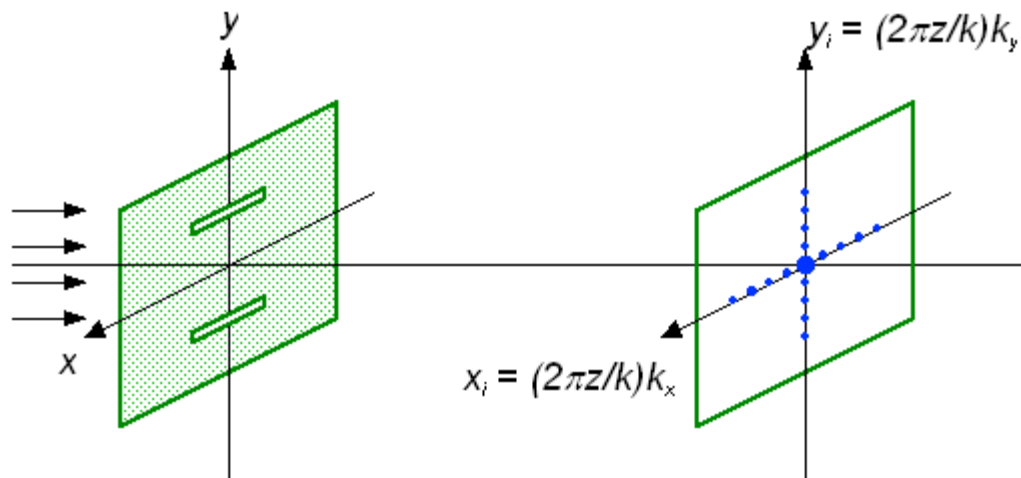


Fig.6 Schematic diagram of the Young's double slits experiment.

$n = 600$

Fig.7a Shape of the aperture (double slits).

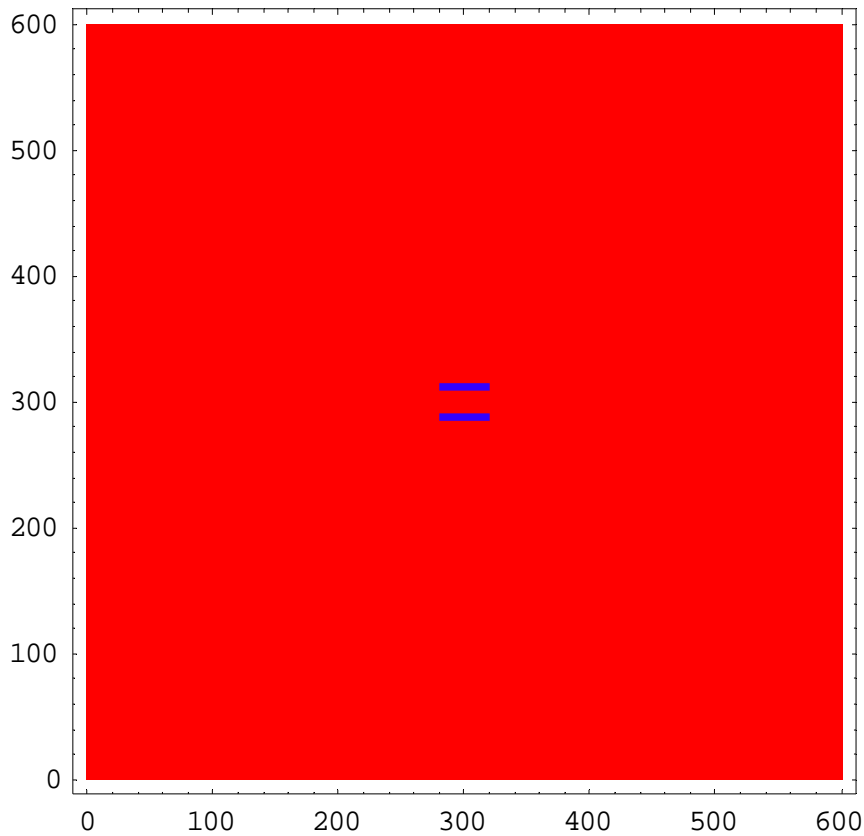
Fig.7b Its power spectrum.

Fig.7c Intensity distribution of the power spectrum along the k_y axis with $k_x = n/2$.

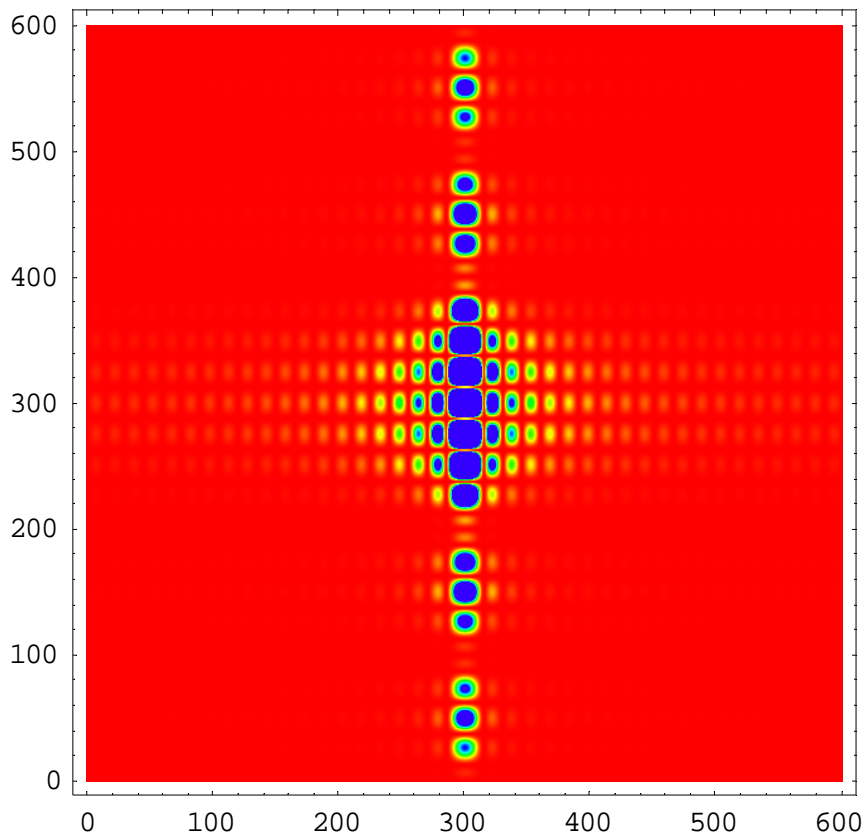
Fig.7d Intensity distribution of the power spectrum along the k_x axis with $k_y = n/2$.

((Mathematica 5.2)) V

```
n=600; (* n x n , n is even*);
doubleslit1=Table[If[(n/2-15)≤y<(n/2-9)&&(n/2-
20)≤x<(n/2+20) || (n/2+9)≤y<(n/2+15)&&(n/2-
20)≤x<(n/2+20),1.0,0.0],{y,0,n-1},{x,0,n-1}];
ListDensityPlot[doubleslit1,Mesh→False,ColorFunction→(Hue[0
.7#]&)]
ps1=Abs[Fourier[doubleslit1]]^2;dh1=RotateLeft[ps1,{n/2,n/2}
];
ListDensityPlot[dh1,Mesh→False,PlotRange→{0,0.01},ColorFunc
tion→(Hue[0.7#]&)];
```

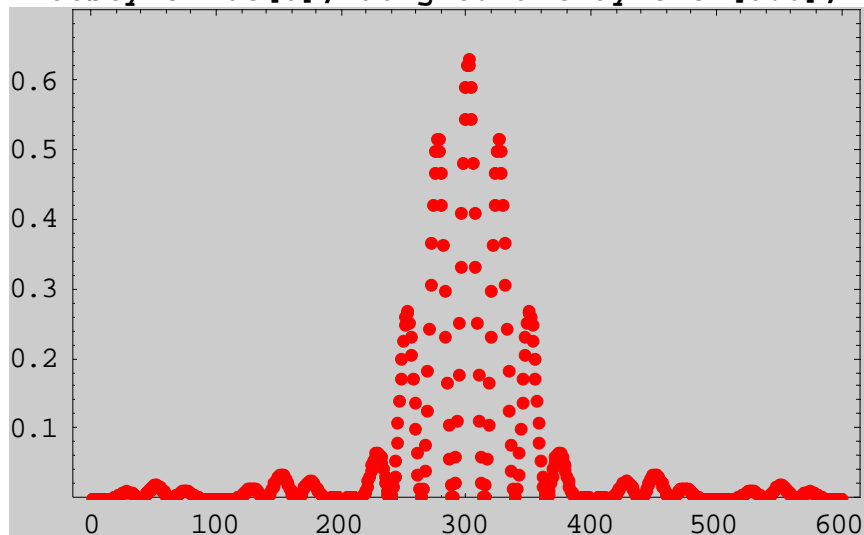


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(*dh1[[m,n/2]] means that the scan is made along the y axis at $x = n/2$ *)

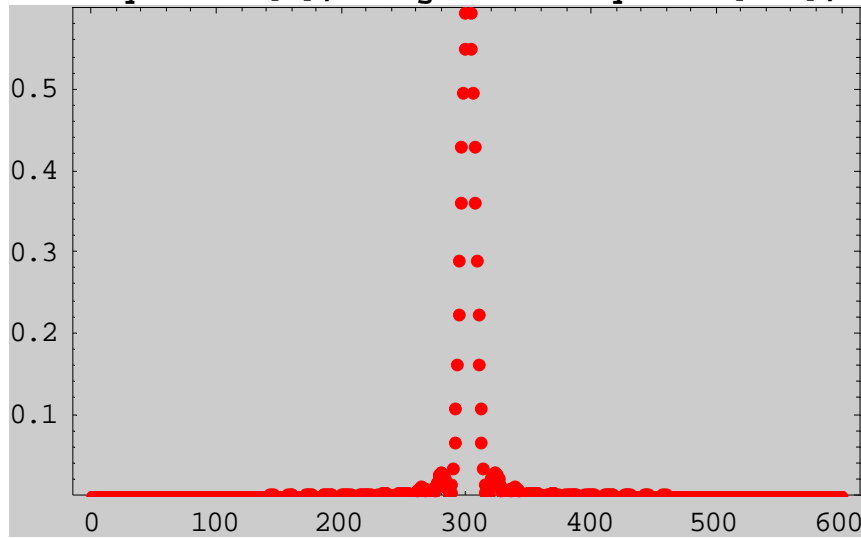
```
ListPlot[Evaluate[Table[dh1[[m,n/2]],{m,1,n}]],
PlotRange→{0,0.7}, Prolog→AbsolutePointSize[4],
PlotStyle→Hue[0],Background→GrayLevel[0.8],Frame→True]
```



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(*dh1[[n/2,m]] means that the scan is made along the x axis at $y = n/2$ *)

```
ListPlot[Evaluate[Table[dh1[[n/2,m]],{m,1,n}]],
PlotRange->{0,0.6}, Prolog->AbsolutePointSize[4],
PlotStyle->Hue[0],Background->GrayLevel[0.8],Frame->True]
```



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8. Power spectrum of the diffraction grating

$n = 600$

Fig.8a Shape of the aperture (diffraction grating)

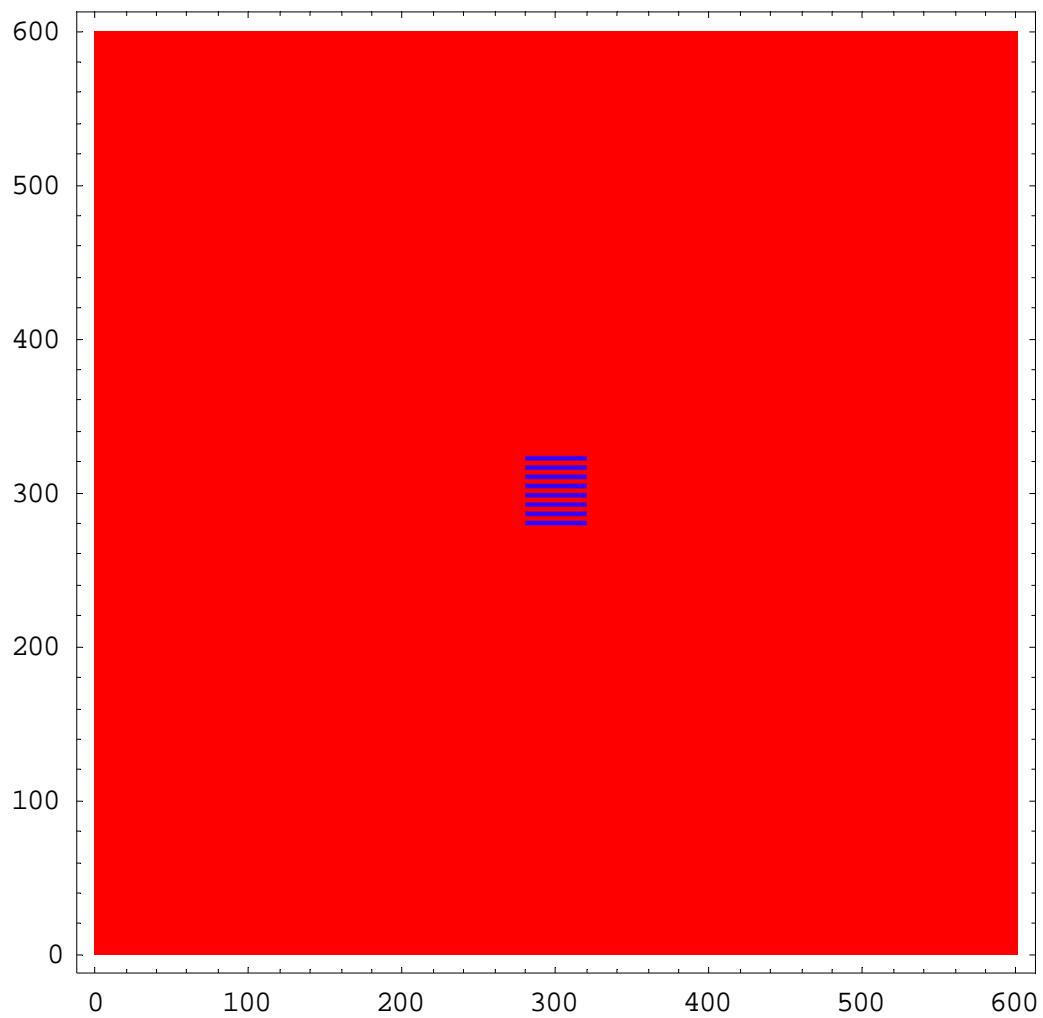
Fig.8b Power spectrum

Fig.8c Intensity distribution of the power spectrum along the k_y axis with $k_x = n/2$,

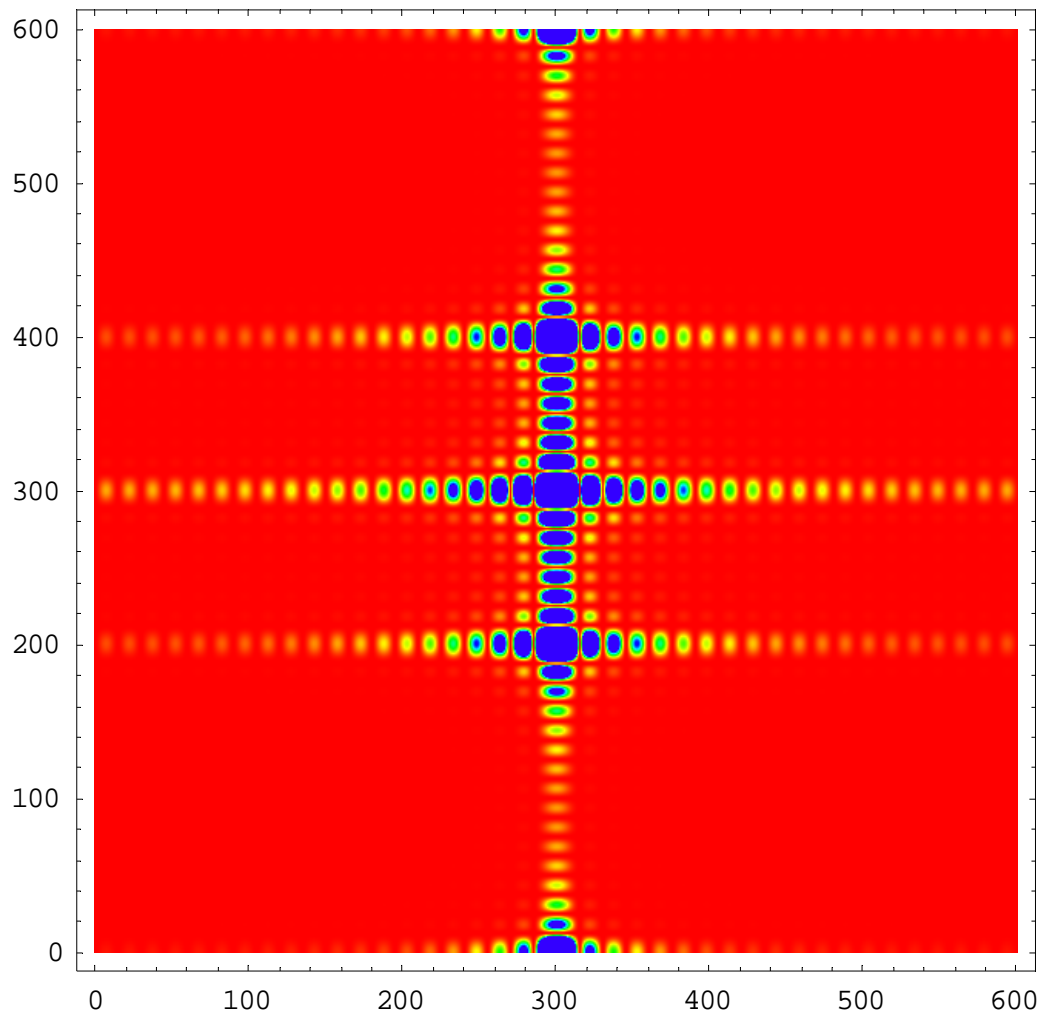
Fig.8d Intensity distribution of the power spectrum along the k_x axis with $k_y = n/2$,

((Mathematica 5.2)) VI

```
n=600; (* n x n *);
slit1=Table[If[(n/2-21)≤y<(n/2-18)&&(n/2-
20)≤x<(n/2+20) | (n/2-15)≤y<(n/2-12)&&(n/2-
20)≤x<(n/2+20) | (n/2-9)≤y<(n/2-6)&&(n/2-
20)≤x<(n/2+20) | (n/2-3)≤y<(n/2)&&(n/2-
20)≤x<(n/2+20) | (n/2+3)≤y<(n/2+6)&&(n/2-
20)≤x<(n/2+20) | (n/2+9)≤y<(n/2+12)&&(n/2-
20)≤x<(n/2+20) | (n/2+15)≤y<(n/2+18)&&(n/2-
20)≤x<(n/2+20) | (n/2+21)≤y<(n/2+24)&&(n/2-
20)≤x<(n/2+20),1.0,0.0],{y,0,n-1},{x,0,n-1}];
ListDensityPlot[slit1,Mesh->False,ColorFunction->(Hue[0.7#]&)]
ps1=Abs[Fourier[slit1]]^2;dh1=RotateLeft[ps1,{n/2,n/2}];
ListDensityPlot[dh1,Mesh->False,PlotRange->{0,0.01},ColorFunc
tion->(Hue[0.7#]&)];
```

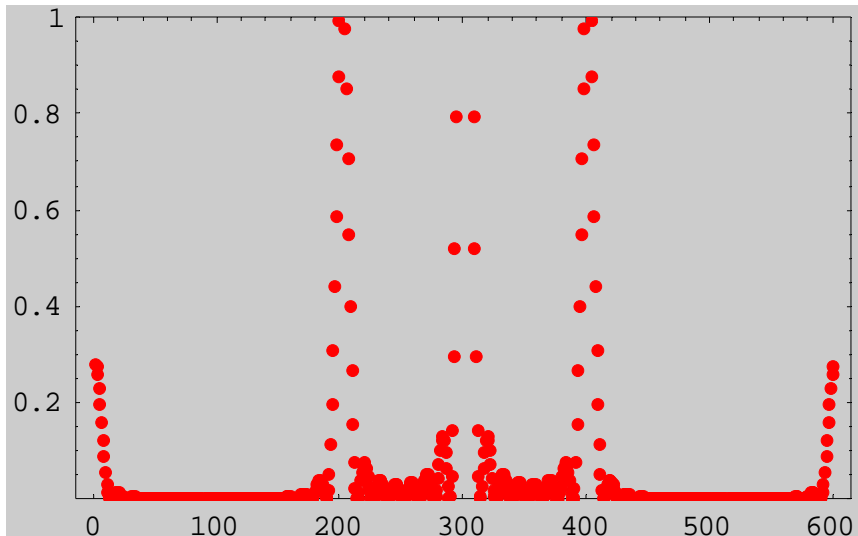



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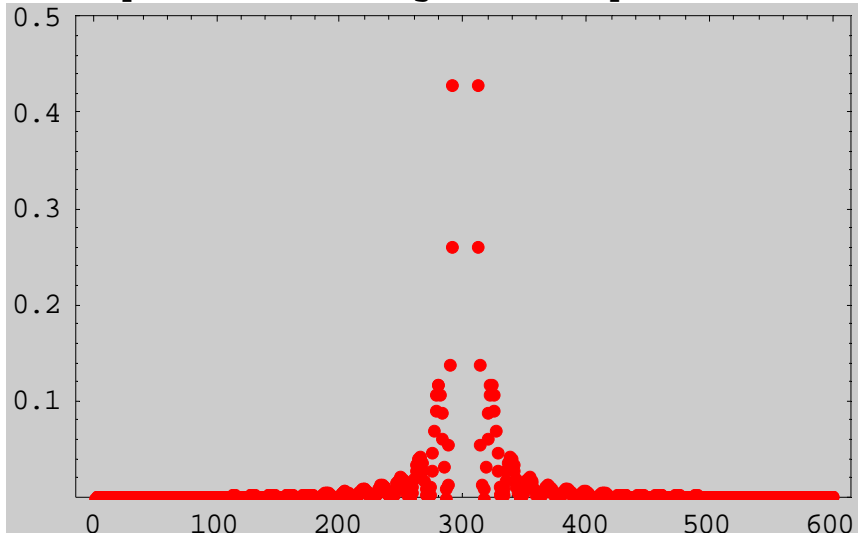


(*dh1[[m,n/2]] means that the scan is made along the y axis
at $x = n/2$ *)

```
ListPlot[Evaluate[Table[ dh1[[m,n/2]],{m,1,n}]],
PlotRange→{0,1}, Prolog→AbsolutePointSize[4],
PlotStyle→Hue[0],Background→GrayLevel[0.8],Frame→True]
```



```
-Graphics-
(*dh1[[n/2,m]] means that the scan is made along the x axis
at y = n/2*)
ListPlot[Evaluate[Table[ dh1[[n/2,m]],{m,1,n}]],
PlotRange→{0,0.5}, Prolog→AbsolutePointSize[4],
PlotStyle→Hue[0], Background→GrayLevel[0.8], Frame→True]
```



```
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```

9. Conclusion

We show that the power spectrum of the aperture in the Fraunhofer regime is well described by the Fourier spectrum of the aperture. One of the Mathematica 5.2 programs is attached to the Appendix, for convenience.

REFERENCES

1. E. Hecht, *Optics* (Addison-Wesley, Reading Massachusetts, 1998).

2. R. G. Palmer, *Mathematica Notebooks on Mathematical Methods for Physics*, Fast Fourier transform.
<http://www.phy.duke.edu/~palmer/notebooks/mmindex.html>
3. M. Trott, *The Mathematica Guide Book for Graphics* (Springer-Verlag, New York, 2004).

Appendix

Mathematica 5.2 program (V) power spectrum of Young's double-slits.