Lecture note on Solid State Physics
Josephson junction and DC SQUID

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(May 16, 2006)

Abstract
In the Senior Laboratory (Phys 427) in our Physics Department, we introduced an equipment of Mr. SQUID (superconducting quantum interference device) (Star Cryoelectronics, LLC, 25 Bisbee Court, Suite A, Santa Fe, NM 87508) in the Spring Semester, 2006. It is a DC SQUID magnetometer system incorporating a high-temperature superconductor thin film SQUID sensor chip. This equipment allows one to observe unique features of superconductivity (using liquid nitrogen cooling) such as \( I-V \) curve and \( V-\Phi \) curves. This also allows one to learn about the operation of SQUID by following a series of experiments. This lecture note is intended as a substitute for textbook on Josephson junction, superconductivity, electronics, and related topics. We think that with this lecture note a great deal of background could be provided for undergraduate and graduate students who do not have enough knowledge on the Josephson junction. We use the Mathematica 5.2 for the simulation. All the programs we made are presented here. The Mathematica program is useful to both graduate students and undergraduate students who want to understand the principle of the RSI model in the Josephson junctions and the DC SQUID model.

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1 Introduction

For superconducting tunnel junctions with extremely thin insulating layers (10 – 15 Å) (weak link between the superconductors), the electron pair correlations extend through the insulating barrier. In this situation, it has been predicted by Josephson that paired electrons (Cooper pairs) can tunnel without dissipation from one superconductor to the other superconductor on the opposite side of the insulating layer [B.D. Josephson, Phys. Lett. 1, 251 (1962). The direct supercurrent of pairs, for currents less that $I_c$, flows with zero-voltage drop across the junction (DC Josephson effect). The width of the insulating barrier of the junction limits the maximum that can flow across the junction, but introduce no resistance in the flow. Josephson also predicted that in the case a constant finite voltage $V$ is established across the junction, an alternating supercurrent $I_c \sin(\omega t + \phi_0)$ flows with frequency $\omega = 2eV/\hbar$ (AC Josephson effect). We solve a nonlinear differential equation for the phase $\phi$ using the Mathematica 5.2. The calculations are made using ND (solving the differential equations numerically under appropriate initial conditions). The use of Mathematica 5.2 is essential to our understanding the nonlinear behavior of the Josephson junction.

2 Josephson junction

Tunneling if Cooper pairs form a superconductor through a layer of insulator into another superconductor. Such a junction is called a weak link.

(i) DC Josephson effect
A DC current flows across the junction in the absence of any electric or magnetic field.

(ii) AC Josephson effect
A DC voltage applied across the junction causes rf (radio frequency) current oscillation across the junction.

(iii) Macroscopic long range quantum interference
A DC magnetic field applied through a superconducting circuit containing two junctions causes the maximum supercurrent to show interference effects as a function of magnetic field intensity.
Josephson, Pippard’s graduate student at Cambridge, attending Philip Anderson’s lectures there in 1961 to 1962, became fascinated by the concept of the phase of the BCS-GL order parameter as a manifestation of the quantum theory on a macroscopic scale. Playing with the theory of Giaver tunneling, Josephson found a phase-dependent term in the current; he then worked out all the consequences in a series of papers, private letters, and a privately circulated fellowship thesis. In particular, Josephson predicted that a direct current should flow, without any applied voltage, between two superconductors separated by a thin insulating layer. This current would come as a consequence of the tunneling of electron pairs between the superconductors, and the current would be proportional to the sine of the phase difference between the superconductors. At a finite applied voltage $V$, an alternating supercurrent of frequency $2eV/h$ should flow between the superconductors. Josephson’s work established the phase as a fundamental variable in superconductivity. (Book edited by Hoddeson et al.12).

2.1 DC Josephson effect

Let $\psi_1$ be the probability amplitude of electron pairs on one side of a junction. Let $\psi_2$ be the probability amplitude of electron pairs on the other side. For simplicity, let both superconductors be identical.

\[
\frac{i\hbar}{\partial t} \psi_1 = \hbar \psi_2,
\]

\[
\frac{i\hbar}{\partial t} \psi_2 = \hbar \psi_1
\]
where \( hT \) is the effect of the electron-pair coupling or (transfer interaction across the insulator). \( T(1/s) \) is the measure of the leakage of \( \psi_1 \) into the region 2, and of \( \psi_2 \) into the region 1.

Let

\[
\psi_1 = \sqrt{n_1} e^{i\theta_1}, \\
\psi_2 = \sqrt{n_2} e^{i\theta_2}
\]

where

\[
|\psi_1|^2 = n_1 \\
|\psi_2|^2 = n_2
\]

Then we have

\[
\delta \theta = \theta_2 - \theta_1. 
\]

Now equate the real and imaginary parts of Eqs.(3) and (4),

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((Mathematica5.2)) Program-1

```mathematica
eq1 = \[\hat{\hbar}\] D[\[\psi_1\][t], t] == \[\hbar\] T \[\psi_2\][t]
\[\hat{\hbar}\] \[\psi_1\'][[t]] == T \[\hbar\] \[\psi_2\][[t]]

eq2 = \[\hat{\hbar}\] D[\[\psi_2\][t], t] == \[\hbar\] T \[\psi_1\][t]
\[\hat{\hbar}\] \[\psi_2\'][[t]] == T \[\hbar\] \[\psi_1\][[t]]

rule1 = \{\[\psi_1\] \[\to\] \[\text{Exp}[i \theta_1][#]] & \}
\{\[\psi_1\] \[\to\] \[\sqrt{n_1}[#1] e^{i \theta_1}[#1] & \}

rule2 = \{\[\psi_2\] \[\to\] \[\text{Exp}[i \theta_2][#]] & \}
\{\[\psi_2\] \[\to\] \[\sqrt{n_2}[#1] e^{i \theta_2}[#1] & \}

eq3 = eq1 /. rule1 /. rule2 // Simplify
\[i \text{Exp}[i \theta_1][t] \hbar (n_1'[t] + 2 i n_1[t] \theta_1'[t])\] == \[\text{Exp}[i \theta_2[t]] T \hbar \sqrt{n_2[t]} \]

2 \[\sqrt{n_1[t]}\]

eq4 = eq2 /. rule1 /. rule2 // Simplify
\[i \text{Exp}[i \theta_2[t] \hbar (n_2'[t] + 2 i n_2[t] \theta_2'[t])\] == \[\text{Exp}[i \theta_1[t]] T \hbar \sqrt{n_1[t]} \]

2 \[\sqrt{n_2[t]}\]
```
\[
\frac{\partial n_1}{\partial t} = 2T \sqrt{n_1 n_2} \sin \delta \\
\frac{\partial n_2}{\partial t} = -2T \sqrt{n_1 n_2} \sin \delta \\
\frac{\partial \theta_1}{\partial t} = -T \sqrt{\frac{n_2}{n_1}} \cos \delta \\
\frac{\partial \theta_2}{\partial t} = -T \sqrt{\frac{n_1}{n_2}} \cos \delta
\]

If \( n_1 \approx n_2 \) as for identical superconductors 1 and 2, we have
\[
\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \quad \text{or} \quad \frac{\partial (\theta_2 - \theta_1)}{\partial t} = 0.
\]

The current flow from the superconductor S1 and to the superconductor S2 is proportional to \( \frac{\partial n_2}{\partial t} \). \( J \) is the current of superconductor pairs across the junction
\[
J = J_0 \sin(\theta_2 - \theta_1),
\]
where \( J_0 \) is proportional to \( T \) (transfer interaction).
\[
I = I_0 \sin \phi.
\]

(5)

2.2 **AC Josephson effect**

![Schematic diagram for experiment of AC Josephson effect. A finite DC voltage is applied across both the ends.](image)

Let a dc voltage \( V \) be applied across the junction. An electron pair experiences a potential energy difference \( qV \) on passing across the junction (\( q = -2e \)). We can say that a pair on one side is at \(-eV\) and a pair on the other side is at \(eV\).
\[ i\hbar \frac{\partial \psi_1}{\partial t} = hT \psi_2 - eV \psi_1, \]
\[ i\hbar \frac{\partial \psi_2}{\partial t} = hT \psi_1 + eV \psi_2, \]
or
\[ \frac{1}{2} \frac{\partial n_1}{\partial t} + in_1 \frac{\partial \theta_1}{\partial t} = \frac{ieVn_1}{\hbar} - iT \sqrt{n_1n_2} e^{i\delta}, \]
\[ \frac{1}{2} \frac{\partial n_2}{\partial t} + in_2 \frac{\partial \theta_2}{\partial t} = -\frac{ieVn_2}{\hbar} - iT \sqrt{n_1n_2} e^{-i\delta}. \]

This equation breaks up into the real part and imaginary part,
\[ \frac{\partial n_1}{\partial t} = 2T \sqrt{n_1n_2} \sin \delta \]
\[ \frac{\partial n_2}{\partial t} = -2T \sqrt{n_1n_2} \sin \delta \]
\[ \frac{\partial \theta_1}{\partial t} = \frac{eV}{\hbar} - T \sqrt{\frac{n_2}{n_1}} \cos \delta \]
\[ \frac{\partial \theta_2}{\partial t} = -\frac{eV}{\hbar} - T \sqrt{\frac{n_1}{n_2}} \cos \delta \]

From these two equations with \( n_1 = n_2 \),
\[ \frac{\partial (\theta_2 - \theta_1)}{\partial t} = \frac{\partial \delta}{\partial t} = -\frac{2eV}{\hbar} . \]
\[ J = J_0 \sin[\delta(t)]. \]

with
\[ \delta(t) = \delta(0) - \frac{2e}{\hbar} \int V dt . \]

When \( V = V_0 \) = constant, we have
\[ \delta(t) = \delta(0) - \frac{2e}{\hbar} V_0 t . \]
\[ J = J_0 \sin[\delta(0) - \frac{2e}{\hbar} V_0 t]. \]

The current oscillates with frequency
\[ \omega_0 = \frac{2e}{\hbar} V . \]

A DC voltage of 1 \( \mu V \) produces a frequency of 483.5935 MHz.
\[ \phi = \frac{2e}{\hbar} V. \]

((Note))
Suppose that $V = V_0 = 1 \mu V$. The corresponding frequency $\nu_0$ is estimated from the relation,

$$\frac{2eV_0}{\hbar} = 2\pi \nu_0,$$

or

$$\nu_0 = \frac{2eV_0}{2\pi \hbar} = \frac{2 \times (1.60219 \times 10^{-12}) \times 10^{-6}}{2\pi \times 1.05459 \times 10^{-34}} = 483.5935 \text{MHz}.$$

3 I-V characteristic of Josephson junction\(^{5,7,8}\)

We now consider the $I$-$V$ characteristic of the Josephson tunneling junction where a insulating layer is sandwiched between two superconducting layers (the same type). A capacitor is formed by these two superconductors. In this type of Josephson junctions, one can see the quasiparticle $I$-$V$ curve which is different with increasing voltage and decreasing voltage (hysteresis). There are two voltage states, 0 V and $2\Delta/e$, where $\Delta$ is an energy gap of each superconductor. The $I$-$V$ curve is characterized by (i) maximum Josephson tunneling current of Cooper pairs at $V = 0$ and (ii) Quasi-particle tunneling current ($V > 2\Delta/e$).

![Fig.3 Schematic diagram of quasiparticle I-V characteristic (usually observed in a S-I-S Josephson tunneling-type). Josephson current (up to a maximum value $I_c$) flowsat $V = 0$. $\Delta$ is an energy gap of the superconductor. The DC Josephson supercurrent flows under $V = 0$. For $V > 2\Delta/e$ the quasiparticle tunneling current is seen.](image)

The strong nonlinearity in the quasiparticle $I$-$V$ curve of a tunneling junction is not an appropriate to the application to the SQUID element. This nonlinearity can be removed by the use of thin normal film deposited across the electrodes. In this effective resistance is a parallel combination of the junction. The $I$-$V$ characteristic has no hysteresis. Such behavior is often observed in the bridge-type Josephson junction where two superconducting thin films are bridged by a very narrow superconducting thin film.
Fig. 4 Schematic diagram of $I$-$V$ characteristic of a Josephson junction (usually observed in *bridge-type junction*), which is reversible on increasing and decreasing $V$. A Josephson supercurrent flows up to $I_c$ at $V = 0$. A transition occurs from the $V = 0$ state to a finite voltage state for $I > I_c$. Above this voltage the $I$-$V$ characteristic exhibits an Ohm’s law with a finite resistance of the Junction. The current has a oscillatory component of angular frequency $\omega (= 2eV/\hbar)$ (the AC Josephson effect).

4. RSJ (Resistively shunted junction) model: Josephson junction circuit application

Here we discuss the $I$-$V$ characteristics of a Josephson tunneling junction using an equivalent circuit shown below. This circuit includes the effect of various dissipative processes and the distributed capacity with so-called lumped circuit parameters (connection of $R$ and $C$ in parallel).

4.1 Fundamental equation

Fig. 5 Equivalent circuit of a real Josephson junction with a current noise source. (RSJ model). J.J. stands for the Josephson junction.
We now consider an equivalent circuit for the Josephson junction, which is described above. $J_J$ stands for the Josephson junction.

\[ I_c \sin \phi + \frac{V}{R} + C \dot{V} + I_N(t) = I, \]  

(11)

where the first term is a Josephson current, the second term is an ohmic current, the third term is a displacement current, and $I_N(t)$ is the noise current source. Since

\[ \dot{\phi} = \frac{2e}{\hbar} V, \]

we get a second-order differential equation for the phase $\phi$

\[ \frac{\hbar C}{2 e} \ddot{\phi} + \frac{\hbar}{2eR} \dot{\phi} = I - I_c \sin \phi - I_N(t) = -\frac{2e}{\hbar} \frac{\partial U(\phi)}{\partial \phi} - I_N(t), \]  

(12)

where $U(\phi)$ is an equivalent potential and is defined by

\[ U(\phi) = -\frac{\Phi_0}{2\pi c} (I \phi + I_c \cos \phi) = -\frac{\Phi_0}{2\pi c} \frac{I}{I_c} (\phi + \cos \phi) = -E_J (\kappa \phi + \cos \phi), \]

with $\kappa = I/I_c$, where $\Phi_0 = (2\pi \hbar c/2e = 2.06783372\times10^{-7}$ Gauss cm$^2$) is a magnetic quantum flux and $E_J = \Phi_0 I_c/(2\pi)$ is the Josephson coupling constant. If $\phi$ is regarded as the coordinate $x$, the above equation corresponds to the equation of motion of a mass with $(\hbar/2e)^2 C$ in the presence of the potential $U(\phi)$. The second term of the left-hand side is a friction proportional to the velocity. When $I_N(t) = 0$ and the second term is neglected, the above equation can be rewritten as

\[ \frac{1}{2} \left( \frac{\hbar}{2e} \right)^2 C \dot{\phi}^2 + U(\phi) = E = \text{const}, \]  

(13)

where $E$ is the total energy and is constant. For $\kappa > 1$ (or $I > I_c$), $U(\phi)$ monotonically decreases with increasing $\phi$: $d\phi/dt \neq 0$ (finite voltage-state). For $\kappa < 1$ (or $I < I_c$), $U(\phi)$ has local minima. There are two solutions: $d\phi/dt = 0$ (no voltage state) or $d\phi/dt \neq 0$ (finite voltage-state) for large $C$. In this case, the $I$-$V$ characteristic has a hysteresis.

(((Mathematica 5.2)) Program-2)

```
Ul=- ( \kappa \phi + Cos[\phi] ); Plot[Evaluate[Table[Ul,\{\kappa,0,1,0.05\}\}],\{\phi,0,6 \pi\}, PlotStyle\rightarrow Table[Hue[0.1 i],\{i,0,10\}\],Prolog\rightarrow AbsoluteThickness[2],Background\rightarrow GrayLevel[0.7], AxesLabel\rightarrow\{"\phi\","\{U(\phi)\}\"\}],PlotRange\rightarrow\{"0,6 \pi\","\{-20,1\}\"\}]
```
Fig. 6  Equivalent potential energy $U(\phi)$ with a parameter $\kappa = I/I_c$. (a) $0 < \kappa < 1$. $U(\phi)$ has local maxima and local minima. (b) $1 < \kappa < 3$. $U(\phi)$ increases with increasing $\phi$.

Analogy: Simple rigid pendulum
We consider a simple rigid pendulum light stiff rod of length $l$ with a bob of mass $m$.

The pendulum can rotate freely about the pivot $P$. The equation of motion is given by

$$ I \ddot{\theta} = T - mgl \sin \theta - \eta \dot{\theta}, $$

where $T$ is an external torque, $I$ is a moment of inertia around the pivot $P$ and the third term of the right-hand side is a viscosity of air. There is an analogy between this pendulum and the Josephson junction,

$$ T = I \ddot{\theta} + \eta \dot{\theta} + mgl \sin \theta \quad \text{(pendulum)}; \quad I = \frac{\hbar C}{2e} \dot{\phi} + \frac{\hbar}{2eR} \ddot{\phi} + I_c \sin \phi \quad \text{(Josephson)}. $$

(14)  

(15)
When a small torque is applied, the pendulum finally settles down at a constant angle of deflection $\theta$. No angular velocity ($\omega = 0$) corresponds to no voltage across a junction ($V = 0$). The junction is superconducting. $x = l \sin \theta$.

If the torque is gradually increased, the pendulum deflects to a greater but steady angle. We can pass more current through a junction without any voltage appearing. Critical torque ($T_c = mgl$). This is the torque which deflects the pendulum through a right angle so that it is horizontal.

For $T > T_c$ the pendulum cannot remain at rest but rotates continuously. As the pendulum rotates, the horizontal deflection $x$ oscillates. The angular velocity is always in the same direction. This corresponds to the case of $I > I_c$ and $V \neq 0$. A DC voltage will appear across the junction if the current passed through it exceeds a critical value.

On the half-cycle during which the bob is rising the rotation decelerates because gravity opposes the applied torque, but on the following half-cycle the bob is accelerated by gravity as it falls.

$$v = \frac{1}{2\pi} \left( \frac{d\theta}{dt} \right) = \frac{2e}{2\pi \hbar} V_{dc}.$$ The phase is in one-to one correspondence with an angle of rotation of a damped pendulum, driven by a constant torque, in a constant gravitational field. The regime $I < I_c$ corresponds to a situation in which the applied torque is less than a critical torque necessary to raise the pendulum to an angle $\pi/2$. For $I > I_c$ the pendulum rotates in a manner such that the average energy dissipated per rotation is equal to the average work per rotation.

### 4.2 Differential equation for $\phi$

For the sake of simplicity, we use the dimensionless quantities. Here we assume that
\[ \eta = \frac{V}{I_c R}, \quad \kappa = \frac{I}{I_c}, \quad \tau = \omega_j t, \]

where \( \omega_j \) is the Josephson plasma frequency and is defined by
\[ \omega_j = \left( \frac{2e I_c}{hC} \right)^{1/2}, \]
\[ \frac{d\phi}{dt} = \frac{d\phi}{d\tau} = \frac{d\phi}{d\tau} \omega_j. \]

Similarly we have
\[ \frac{d^2\phi}{dt^2} = \omega_j^2 \frac{d^2\phi}{d\tau^2}. \]

Then we get
\[ \frac{hC}{2eI_c} \omega_j^2 \frac{d^2\phi}{d\tau^2} + \frac{h}{2eRI_c} \omega_j \frac{d\phi}{d\tau} + \sin \phi = \kappa - \kappa_n(\tau), \]
or
\[ \frac{d^2\phi}{d\tau^2} + \beta_j \frac{d\phi}{d\tau} + \sin \phi = \kappa - \kappa_n(\tau), \quad (16) \]
where
\[ \beta_j = \frac{h \omega_j}{2eRI_c} = \frac{h \omega_j^2}{2eRI_c \omega_j} = \frac{h}{2eRI_c} = \frac{1}{\omega_j RC}. \]

The normalized voltage is described by
\[ \eta(\tau) = \frac{V}{I_c R} = \frac{h}{2e I_c R} \frac{d\phi}{dt} = \frac{h \omega_j}{2eRI_c} \frac{d\phi}{d\tau} = \beta_j \frac{d\phi}{d\tau}, \]
since
\[ \frac{\phi}{h} = \frac{2e}{h} V. \]

We are interested in the DC current-voltage characteristic so we need to determine the time averaged voltage
\[ \langle V \rangle = \left\langle \frac{h}{2e} \frac{d\phi}{dt} \right\rangle = \left\langle I_c R \beta_j \frac{d\phi}{d\tau} \right\rangle = I_c R \beta_j \left\langle \frac{d\phi}{d\tau} \right\rangle, \]
or
\[ <\eta > = \left\langle \frac{V}{I_c R} \right\rangle = \beta_j \left\langle \frac{d\phi}{d\tau} \right\rangle. \quad (17) \]

Note that the McCumber parameter \( \beta_c \) is sometimes used in stead of \( \beta_j \), where \( \beta_c = 1/\beta_j^2 \).

4.3 \textbf{I-V characteristic for } \beta_j \gg 1

For the special case \( \beta_j \gg 1 \) (small capacitance limit), the above equation is reduced to
\[ \beta_j \frac{d\phi}{d\tau} + \sin \phi = \kappa, \]
\[ \langle V \rangle = \left\langle I_c R \beta_j \frac{d\phi}{d\tau} \right\rangle = I_c R \beta_j \frac{1}{T} \int_0^T \frac{d\phi}{d\tau} d\tau = 2\pi I_c R \beta_j \frac{1}{T}, \]

where
where $T$ is a period;

$$T = \frac{2\pi}{\sqrt{\kappa^2 - 1}} \beta_j.$$ 

Here

$$T = \int_0^T d\tau = \int_0^{2\pi} \frac{d\phi}{\kappa - \sin \phi} = \beta_j \int_0^{2\pi} \frac{d\phi}{\kappa - \sin \phi},$$

or

$$\frac{T}{\beta_j} = \int_0^{2\pi} \frac{d\phi}{\kappa - \sin \phi} = \int_0^\pi \frac{d\phi}{\kappa - \sin \phi} + \int_0^\pi \frac{d\phi}{\kappa + \sin \phi} = \int_0^\pi \left( \frac{1}{\kappa - \sin \phi} + \frac{1}{\kappa + \sin \phi} \right) d\phi,$$

or

$$\frac{T}{\beta_j} = \frac{2\pi}{\sqrt{\kappa^2 - 1}} \text{ for } \kappa > 1,$$

$$\frac{T}{\beta_j} = 0 \text{ for } \kappa < 1,$$

So we get

$$\langle V \rangle = 2\pi I R \frac{\beta_j}{T} = I_c R \sqrt{\kappa^2 - 1},$$

or

$$\frac{\langle V \rangle}{I_c R} = \sqrt{\kappa^2 - 1},$$

(18)

or

$$\kappa = \sqrt{1 + \left( \frac{\langle V \rangle}{I_c R} \right)^2}.$$  

(19)

((Mathematica 5.2)) Program-3

```
K1[x_] := \(2 \pi \frac{1}{\sqrt{1 + x^2}} - 1 \) \(\frac{x}{\sin(x)} + \frac{1}{x + \sin(x)}\) d\phi; Y1 = K1[x]; Y2 = Simplify[Y1, x > 1]

eq1 = V1 == \(2 \pi I_c R \) \(Y2\); eq2 = Solve[eq1, x]  
\{\{x \to -\sqrt{1 + \frac{V1^2}{I_c^2 R^2}}\}, \{x \to \sqrt{1 + \frac{V1^2}{I_c^2 R^2}}\}\}

I1 = \(\sqrt{1 + \frac{V1^2}{I_c^2 R^2}}\) / \{R \to 1, I_c \to 1\}

\[\sqrt{1 + V1^2}\]

Plot[{I1, V1}, \{V1, 0, 4\}, PlotStyle\to\{Hue[0], Hue[0.4]\}, Prolog\to Ab
```
5 Phase plane analysis

We now consider a equation
\[
\frac{d^2 \phi}{d\tau^2} + \beta J \frac{d\phi}{d\tau} + \sin \phi = \kappa ,
\]
(20)
or
\[
\frac{d\Omega}{d\tau} = \kappa - \beta J - \sin \phi ,
\]
(21)
where
\[
\Omega(\tau) = \frac{d\phi}{d\tau},
\]
(22)
is proportional to voltage \([\eta = \beta J \frac{d\phi}{d\tau}]\). Then we have
\[
\frac{d^2 \phi}{d\tau^2} = \frac{d\Omega}{d\tau} = \frac{d\Omega}{d\phi} \frac{d\phi}{d\tau} = \frac{d\Omega}{d\phi} \Omega = \frac{d}{d\phi} \Omega^2 .
\]
The above equation can be rewritten as
\[
\frac{d}{d\phi} \frac{\Omega^2}{2} + \beta J \Omega + \sin \phi = \kappa .
\]
(23)

The state of the system is represented at any time by a particular point in the \((\Omega, \phi)\) plane. As the time \(\tau\) varies, this point describes a trajectory. Each particular trajectory depends on the initial conditions. Thus for a fixed value of \((\Omega, \phi)\) plane, the system is represented by a set of possible paths in the \((\Omega, \phi)\) plane. Such a plot is often called a phase space diagram.

We begin by discussing the orbits when \(\kappa = 0\) and \(\beta J = 0\).
\[ \frac{d^2 \Omega^2}{d\phi^2} + \sin \phi = 0. \]

This equation can be integrated as
\[
\frac{\Omega^2}{2} - \cos \phi = a = \text{const}. \tag{24}
\]

Open orbits require that \( a \) always be larger than 2.

((Mathematica 5.2)) Program-4
<<Graphics`ImplicitPlot`; eq1 = \[\frac{1}{2} a^2 - \cos \phi;\]
pt1 = ImplicitPlot[eq1, \[\phi, -2\pi, 2\pi\], \(\Omega, -2\pi, 2\pi\), PlotPoints \[\to 100,\]
Contours \[\to 50,\] PlotStyle \[\to\] \{Hue[0.7], Thickness[0.006]\},
DisplayFunction \[\to Identity,\] PlotRange \[\to All\];
Show[pt1, DisplayFunction \[\to$$DisplayFunction]

**Fig.10** The \( \Omega \) vs \( \phi \) plane trajectories. \( \Omega^2/2 - \cos \phi = a \) where \( a \) is changed as a parameter. The closed orbits for \( a<1 \) (\( \Omega^2(\phi=0)/2 < 2 \)) and the open orbits for \( a>1 \) (\( \Omega^2(\phi=0)/2 > 2 \)).

6 Numerical calculation

In the plane of the parameters \( \beta_j \) and \( \kappa \) the situation can be summarized as follows.\(^6\,7\)
(a) For \( \kappa>1 \) and arbitrary \( \beta_j \) value no equilibrium point exists; there is only a periodic solution of the second kind. Therefore the junction will be in the finite voltage state.
(b) For \( \kappa<1 \), the situation is more complicated. The behavior depends on the particular value of \( \beta_j \).
Fig. 11 Critical line for $k_c(\beta J)$ (denoted by red line) in the $\beta J$ vs $\kappa$ plane. The blue line denotes the expression given by $k_c(\beta J) = \frac{4}{\pi} \beta J$. The system undergoes stable oscillations when $\kappa > k_c(\beta J)$ for fixed $\beta J$, in addition to the zero-voltage state.

For $\beta J < 0.2$, the simple relation holds: $k_c(\beta J) = \frac{4}{\pi} \beta J$.

A curve can be identified, denoted by $k_c(\beta J)$, which divides the plane into two regions corresponding to one or two stable state solutions, respectively.

We now solve the differential equation by using Mathematica 5.2

\[ \Omega(\tau) = \frac{d\phi}{d\tau}, \]

and

\[ \frac{d\Omega(\tau)}{d\tau} + \beta J \Omega(\tau) + \sin(\phi(\tau)) = \kappa. \]

Initial condition at $\tau = 0$ (or $t = 0$):

\[ \Omega(\tau = 0) = v_0 \text{ and } \phi(\tau = 0) = \phi_0. \]

We calculate the $\tau$ dependence of $\Omega(\tau)$ and $\phi(\tau)$ for $0 \leq \tau \leq \tau_{\text{max}}$ by using Mathematica 5.2 [NDSolve], where $\beta J$ and $\kappa$ are changed as parameters.

(i) Curve of $\Omega(\tau)$ vs $\phi(\tau)$

(ii) The direction of the curves of $\Omega(\tau)$ vs $\phi(\tau)$, when $\tau$ increases [field vector]

(iii) The $\tau$ dependence of $\Omega(\tau)$ and $\phi(\tau)$.

(iv) The determination of the maximum and minimum values of $\Omega(\tau)$ in the long $\tau$-region where $\Omega(\tau)$ is a well-defined oscillatory function of $\tau$. 
6.1 Simulation-1

((Mathematica 5.2)) Program-5

Phase space $\Omega$ vs $\phi$ with vector field

$\beta_J = 0.6$ is fixed. The current ratio $\kappa$ is changed around the critical value $\kappa_c = 0.6965$. We show the phase diagram of the voltage $\phi(\tau)$ vs $\Omega(\tau)$ for various initial conditions. $\phi(0) = 0$, $\Omega(0) = -10 - 10$. We also show the vector field.

```
Clear["Global`*"]
<<Graphics`Graphics`
<<Graphics`PlotField`
(*Subroutine, ParametricPlot in the phase space*)

phase[{{\phi}_0_, v0_}, {\beta}_J_, \kappa_}, {\tau}_max_, opts__] := Module[{numsol, numgraph},
numsol = NDSolve[{
\Omega'[\tau] + \beta_J \Omega[\tau] + \sin[\phi[\tau]] == \kappa, \phi'[\tau] == \Omega[\tau], \phi[0] == \phi_0, \Omega[0] == v0}, 
{\Omega[\tau], \phi[\tau]},
{\tau, 0, \tau}_max
]/Flatten;
numgraph = ParametricPlot[
{\phi[\tau], \Omega[\tau]} /. numsol,
{\tau, 0, \tau}_max, opts, 
DisplayFunction -> Identity]
];

field[{\beta}_J_, \kappa_}, {xmin_, xmax_}, {ymin_, ymax_}, opts__] :=
PlotVectorField[
{y, -\beta_J y - \sin[x] + \kappa}, 
{x, xmin, xmax}, {y, ymin, ymax}, opts];

phlist = phase[{{0}_0, 0.6, 0.5}, 100, PlotStyle -> Hue[0.1 (# + 6)], AxesLabel -> {"\phi", "\Omega"}, Prolog -> AbsoluteThickness[2], Background -> GrayLevel[0.5], PlotRange -> All, Ticks -> {\pi Range[-10, 10], Range[-6, 6]}, DisplayFunction -> Identity] &/@Range[-10, 10, 1];
f1 = field[{0.6, 0.5}, {-8 \pi, 12 \pi}, {-6, 6}, PlotPoints -> 20, ScaleFunction -> (0.4#&), ScaleFactor -> None, DisplayFunction -> Identity];
Show[phlist, f1, DisplayFunction -> $DisplayFunction]
```

Fig.12 The phase-plane trajectories in the $\Omega$ vs $\phi$. $\beta_J = 0.6$. $\kappa$ is changes as a parameter, $\kappa = 0.5 - 0.7$.

(1) $\beta_J = 0.6$ and $\kappa = 0.5$
(2) $\beta_{j} = 0.6$ and $\kappa = 0.6$

(3) $\beta_{j} = 0.6$ and $\kappa = 0.65$
(4) \( \beta_i = 0.6 \) and \( \kappa = 0.67 \)

(5) \( \beta_i = 0.6 \) and \( \kappa = 0.69 \)

(6) \( \beta_i = 0.6 \) and \( \kappa = 0.696 \)
\( \beta_i = 0.6 \) and \( \kappa = 0.6962 \)

A stable periodic solution appears. The states of zero and finite voltage are both possible.
(10) $\beta_j = 0.6$ and $\kappa = 0.697$

(11) $\beta_j = 0.6$ and $\kappa = 0.699$
\( \beta_I = 0.6 \) and \( \kappa = 0.7 \)

6.2 Simulation-2

((Mathematica 5.2)) Program-6

\( \beta_I = 0.6 \) is fixed. The current ratio \( \kappa \) is changed around the critical value \( \kappa_c = 0.6965 \). We show the \( \tau \) dependence of the voltage \( \Omega(\tau) \) for various initial conditions. Note that the normalized DC voltage \( \langle V \rangle / I_R \) is defined by \( < \eta > = \beta_J \langle \Omega \rangle \). For \( \kappa > \kappa_c(\beta_J) \), \( \Omega(\tau) \) is a sum of time-independent term \( \langle \Omega \rangle \) and a periodically oscillating function of \( \tau \). The average voltage corresponds to \( < \eta > = \beta_J \langle \Omega \rangle \), where \( \langle \Omega \rangle \) is the average of the maximum and minimum values of \( \Omega(\tau) \) in the long \( \tau \)-region, where \( \Omega(\tau) \) is a well-defined periodically oscillating function of \( \tau \). The method to find the maximum and minimum values of \( \Omega(\tau) \) will be shown in Sec.5.5 for convenience.

\[
\text{phlist=phase[{0,#},{0.6,0.693},100, PlotStyle\rightarrow Hue[0.1 (#+6)], AxesLabel\rightarrow{"\tau","\Omega"},Prolog\rightarrow AbsoluteThickness[2], Background\rightarrow GrayLevel[0.5],PlotRange\rightarrow{{0,30 \pi},{-1,3}},Ticks\rightarrow\{\pi \text{ Range}[0,50,10], \text{ Range}[-6,6]},DisplayFunction\rightarrow Identity\}&/@Range[-10,10,1];Show[phlist,DisplayFunction\rightarrow$DisplayFunction]
\]

Fig.13 \( \Omega \) vs \( \tau \) for \( \beta_I = 0.6 \). The parameter \( \kappa \) is varied as a parameter, \( \kappa = 0.693 - .1.20 \).

(1) \( \beta_I = 0.6 \) and \( \kappa = 0.693 \)
(2) $\beta = 0.6$ and $\kappa = 0.694$

(3) $\beta = 0.6$ and $\kappa = 0.695$
(4) $\beta_J = 0.6$ and $\kappa = 0.6960$

![Graph showing oscillatory behavior with $\beta_J = 0.6$ and $\kappa = 0.6960$.]

(5) $\beta_J = 0.6$ and $\kappa = 0.6962$

The average voltage $\eta = \beta_J \langle \Omega(\tau) \rangle V/RI_c$ is equal to 0, where $\beta_J = 0.6$ and $\kappa = I/I_c = 0.6962$, independent of the initial condition $\Omega(\tau = 0)$.

![Graph showing oscillatory behavior with $\beta_J = 0.6$ and $\kappa = 0.6962$.]

(6) $\beta_J = 0.6$ and $\kappa = 0.6965$

The average voltage $\eta = \beta_J \langle \Omega(\tau) \rangle V/RI_c$ is nearly equal to 0.6x0.95 = 0.57, where $\beta_J = 0.6$ and $\kappa = I/I_c = 0.6965$.

![Graph showing oscillatory behavior with $\beta_J = 0.6$ and $\kappa = 0.6965$.]
(7) $\beta_j = 0.6$ and $\kappa = 0.6970$

(8) $\beta_j = 0.6$ and $\kappa = 0.698$
(9) \( \beta_i = 0.6 \) and \( \kappa = 0.70 \)

\[
\begin{array}{c}
\Omega \\
\tau \\
-1 \\
1 \\
2 \\
3
\end{array}
\]

(10) \( \beta_i = 0.6 \) and \( \kappa = 0.80 \)

The average voltage \( \eta (= \beta_j \langle \Omega(\tau) \rangle) \) is equal to 0.6x1.22656 = 0.7359 and 0.6x0 = 0 where \( \beta_i = 0.6 \) and \( \kappa (= I/I_c) = 0.8 \), depending on the initial condition \( \Omega(\tau = 0) \).

\[
\begin{array}{c}
\Omega \\
\tau \\
-1 \\
1 \\
2 \\
3
\end{array}
\]

(11) \( \beta_i = 0.6 \) and \( \kappa = 0.90 \)

The average voltage \( \eta (= \beta_j \langle \Omega(\tau) \rangle) \) is nearly equal to 0.6x1.429 = 0.8574 and 0.6x0 = 0 where \( \beta_i = 0.6 \) and \( \kappa (= I/I_c) = 0.9 \), depending on the initial condition \( \Omega(\tau = 0) \). This implies the existence of the hysteresis behavior. The \( I-V \) curve with increasing \( V \) is different from that with decreasing \( V \).
(12) $\beta = 0.6$ and $\kappa = 1.0$

The average voltage $\eta = \beta J \langle \Omega(\tau) \rangle$ is equal to $0.6 \times 1.61579 = 0.9695$, where $\beta = 0.6$ and $\kappa = I/I_c = 1.0$, independent of the initial condition $\Omega(\tau = 0)$. This implies no hysteresis behavior.

(13) $\beta = 0.6$ and $\kappa = 1.2$

The average voltage $\eta = \beta J \langle \Omega(\tau) \rangle$ is equal to $0.6 \times 1.97065 = 1.1824$, where $\beta = 0.6$ and $\kappa = I/I_c = 1.2$, independent of the initial condition $\Omega(\tau = 0)$. This implies no hysteresis behavior.
6.3 Simulation-3
((Mathematica 5.2)) Program-7

$\beta_J = 0.2$ is fixed. The current ratio $\kappa$ is changed around the critical value $\kappa_c = 0.253$. We show the $\tau$ dependence of the voltage $\Omega(\tau)$ for various initial conditions.

Clear["Global`*"]
<<Graphics`Graphics`
<<Graphics`PlotField`
(*Subroutine, ParametricPlot in the phase space*)

phase[{φ0_,v0_},{βJ_,κ_},tmax,opts___]:=Module[{numsol,numgraph},
numsol=NDSolve[{Ω'[τ]+βJ Ω[τ]+Sin[φ[τ]]==κ,φ'[τ]==Ω[τ],φ[0]==φ0,Ω[0]==v0},{Ω[τ],φ[τ]},{τ,0,tmax}]//Flatten;
umgraph=Plot[Ω[τ]/.numsol,{τ,0,tmax},opts,DisplayFunction→Identity]

phlist=phase[{0,#},{βJ,κ},100,PlotStyle→Hue[0.1(#+6)],AxesLabel→{"τ","Ω"},Prolog→AbsoluteThickness[2],Background→GrayLevel[0.5],PlotRange→{{0,30 π},{-1,3}},Ticks→{π Range[0,50,10], Range[-6,6]},DisplayFunction→Identity]&/@Range[-10,10,1];Show[phlist,DisplayFunction→$DisplayFunction]

Fig.14 $\Omega$ vs $\tau$ for $\beta_J = 0.2$. The parameter $\kappa$ is varied as a parameter, $\kappa = 0.24 - 0.50$.

(1) $\beta_J = 0.2$ and $\kappa = 0.24$
(2) $\beta_J = 0.2$ and $\kappa = 0.245$

(3) $\beta_J = 0.2$ and $\kappa = 0.250$
The average voltage \( \eta (= \beta_j \Omega(\tau) = V/RI_c) \) is equal to 0.2x1.0269 = 0.20538 and 0.2x0 = 0, where \( \beta_j = 0.2 \) and \( \kappa (= I/I_c) = 0.253 \), depending on the initial condition \( \Omega(\tau = 0) \).
(7) $\beta_j = 0.2$ and $\kappa = 0.254$

(8) $\beta_j = 0.2$ and $\kappa = 0.255$
(9) \[ \beta = 0.2 \text{ and } \kappa = 0.256 \]

(10) \[ \beta = 0.2 \text{ and } \kappa = 0.30 \]

(11) \[ \beta = 0.2 \text{ and } \kappa = 0.4 \]
\( \beta_J = 0.2 \) and \( \kappa = 0.5 \)

The average voltage \( \eta = \beta_J \langle \Omega(\tau) \rangle = V / R I_c \) is nearly equal to \( 0.2 \times 2.48381 = 0.49676 \) and \( 0.2 \times 0 = 0 \), where \( \beta_J = 0.2 \) and \( \kappa = I / I_c = 0.5 \), depending on the initial condition \( \Omega(\tau = 0) \). This implies the existence of the hysteresis behavior. The \( I-V \) curve with increasing \( V \) is different from that with decreasing \( V \).

\[ \Omega(\tau) = 0.9 \]

\( \kappa = 0.90 \) is fixed. The current ratio \( \kappa \) is changed around the critical value \( \kappa_c = 0.9197 \).

We show the \( \tau \) dependence of the voltage \( \Omega(\tau) \) for various initial conditions: \( \phi(0) = 0 \), \( \Omega(0) = -10 - 10 \).

\[
\text{Clear}[\text{"Global`*"}]
\text{<<Graphics`Graphics`}
\text{<<Graphics`PlotField`}
(*Subroutine, ParametricPlot in the phase space*)
\]

\[
\text{phase}[\{\phi_0_, v_0_, \beta J_, \kappa_, \tau_{max}, \text{opts}_\text{__}\}] := \text{Module}[\{\text{numsol, numgraph}\}, \text{numsol} = \text{NDSolve}[\{\Omega'[\tau] + \beta J \\
\Omega[\tau] + \sin[\phi[\tau]] = \kappa, \phi'[\tau] = \Omega[\tau], \phi[0] = 0, \Omega[0] = v_0\}, \{\Omega[\tau], \phi[\tau]\}, \{\tau, 0, \tau_{max}\}] // \text{Flatten}; \text{numgraph} = \text{Plot}[\Omega[\tau] /. \text{numsol}, \{\tau, 0, \tau_{max}\}, \text{opts}, \text{DisplayFunction} \to \text{Identity}]]
\]

\[
\text{phlist} = \text{phase}[\{0, #\}, \{0.9, 0.90\}, \text{100}, \text{PlotStyle} \to \text{Hue}[0.1 \text{(#+6)}], \text{AxesLabel} \to \{"\tau", \"\Omega\"\}, \text{Prolog} \to \text{AbsoluteThickness}[2], \text{Background} \to \text{GrayLevel}[0.5], \text{PlotRange} \to \{\{0, 30 \pi\}, \{-1, 3\}\}, \text{Ticks} \to \{\pi \text{ Range}[0, 50, 10], \text{Range}[-6, 6]\}, \text{DisplayFunction} \to \text{Identity}] \& /@ \text{Range}[-10, 10, 1]; \text{Show}[\text{phlist}, \text{DisplayFunction} \to \$\text{DisplayFunction}]
\]

\[ \Omega \text{ vs } \tau \text{ for } \beta_J = 0.9. \text{ The parameter } \kappa \text{ is varied as a parameter, } \kappa = 0.90 - .20. \]
(1) $\beta = 0.90$ and $\kappa = 0.90$

(2) $\beta = 0.90$ and $\kappa = 0.91$

(3) $\beta = 0.90$ and $\kappa = 0.915$
(4) $\beta_I = 0.90$ and $\kappa = 0.918$

![Graph](image)

(5) $\beta_I = 0.90$ and $\kappa = 0.919$

![Graph](image)

(6) $\beta_I = 0.90$ and $\kappa = 0.9195$
(7) \( \beta_J = 0.90 \) and \( \kappa = 0.9197 \)

(8) \( \beta_J = 0.90 \) and \( \kappa = 0.91975 \)
(9) \( \beta_i = 0.90 \) and \( \kappa = 0.9198 \)

(10) \( \beta_i = 0.90 \) and \( \kappa = 0.92 \)

(11) \( \beta_i = 0.90 \) and \( \kappa = 0.94 \)
(12) $\beta_J = 0.90$ and $\kappa = 1$

The average voltage $<\eta> (= \beta_J \langle \Omega(\tau) \rangle )$ is equal to $0.9 \times 0.98318 = 0.8849$ and $0.9 \times 0 = 0$, where $\beta_J = 0.9$ and $\kappa (= I/I_c) = 1.0$, depending on the initial condition $\Omega(\tau = 0)$. This implies the existence of hysteresis behavior.

(13) $\beta_J = 0.90$ and $\kappa = 1.1$

The average voltage $<\eta> (= \beta_J \langle \Omega(\tau) \rangle )$ is equal to $0.9 \times 1.12581 = 1.0132$, where $\beta_J = 0.9$ and $\kappa (= I/I_c) = 1.1$, independent of the initial condition $\Omega(\tau = 0)$. This implies no hysteresis behavior.

(14) $\beta_J = 0.90$ and $\kappa = 2.0$

The average voltage $<\eta> (= \beta_J \langle \Omega(\tau) \rangle )$ is equal to $0.9 \times 2.2029 = 1.9826$, where $\beta_J = 0.9$ and $\kappa (= I/I_c) = 2$, which is independent of the initial condition $\Omega(\tau = 0)$. 

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6.5 Simulation: the relation of $<\eta> = \beta_j <\Omega>$ vs $\kappa$

Here we show how to determine the average voltage $<\eta>$ as a function of the current $\kappa$, where $\beta_j$ is changed as a parameter.

(1) Using the following Mathematica 5.2 program, we find the maximum and minimum of $\Omega(\tau)$ in the long-time region where $\Omega(\tau)$ periodically oscillates with $\tau$.

(2) The average $<\Omega>$ is calculated as (maximum+minimum)/2. The average voltage $<\eta> = \beta_j <\Omega>$ is plotted as a function of $\kappa$ for each $\beta_j (= 0.2 – 1.2)$.

((Mathematica 5.2)) Program-9
Clear["Global`*"]
<<Graphics`Graphics`
(*Subroutine, to find Maximum and minimum*)
phase1[{φ0_,v0_},{βJ_,κ_},τmax_,opts__]:=Module[{numso1,numgraph},numso1=NDSolve[{
Ω'[τ]+βJ Ω[τ]+Sin[φ[τ]]==κ,φ'[τ]==Ω[τ],φ[0]==φ0,Ω[0]==v0},{Ω[τ],φ[τ]},{
τ,0,τmax}]/Flatten;numgraph=Plot[
Ω[τ]/.numso1,{τ,0,τmax},opts,DisplayFunction→Identity];Max1[{φ0_,v0_},{βJ_,κ_},{τmin_,τmax_}]:=Module[{numsol},numsol=NDSolve[{
Ω'[τ]+βJ Ω[τ]+Sin[φ[τ]]==κ,φ'[τ]==Ω[τ],φ[0]==φ0,Ω[0]==v0},{Ω[τ],φ[τ]},{
τ,0,τmax}]/Flatten;maximum=FindMaximum[βJ Ω[τ]/.numso1,{τ,τmin,τmax}]];
Min1[{φ0_,v0_},{βJ_,κ_},{τmin_,τmax_}]:=Module[{numsol},numsol=NDSolve[{
Ω'[τ]+βJ Ω[τ]+Sin[φ[τ]]==κ,φ'[τ]==Ω[τ],φ[0]==φ0,Ω[0]==v0},{Ω[τ],φ[τ]},{
τ,0,τmax}]/Flatten;minimum=FindMinimum[βJ Ω[τ]/.numso1,{τ,τmin,τmax}]];Sei[βJ_,κ_]:=Module[{A1,B1,av1,list1},A1=Max1[{0,#},{βJ_,κ_},{20 π,30 π}]&/@Range[-10,10,1];B1=Min1[{0,#},{βJ_,κ_},{20 π,30 π}]&/@Range[-10,10,1];av1=(A1+B1)/2;list1=Table[{κ,av1[[k,1]]},{k,1,21}];Nat1[βJ_]:=Flatten[Table[Sei[βJ_,κ_],{κ,0,3,0.01}],1];Saw1
\( [\beta J_\land] := \text{ListPlot}\left[ \text{Nat1}[\beta J], \text{PlotStyle} \to \{\text{Hue[0.7]}, \text{PointSize[0.015]}\}, \text{AxesLabel} \to \{"\kappa", "<\eta>"\}, \text{PlotLabel} \to \text{NumberForm}[\beta J], \text{PlotRange} \to \{\{0,3\}, \{0,3\}\} \right] \)

Table\left[ \text{Saw1}[\beta J], \{\beta J, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2\} \right]

Fig.16 \( <\eta> = \beta J <\Omega > \) vs \( \kappa \) for \( \beta J = 0.2, 0.4, 0.6, 0.8, 1.0, \) and 1.2. The number denoted in each curve is the value of \( \beta J \). For \( \beta J \leq 1 \), \( <\eta> \) takes two values with \( <\eta> = 0 \) and finite value of \( <\eta> \). Note that for \( \beta J = 1 \) and 1.2, \( <\eta> \) takes three values (multi-valued function) near \( \kappa = 1 \).

1. \( \beta J = 0.2 \)

2. \( \beta J = 0.4 \)

3. \( \beta J = 0.6 \)
6.6 Result on $\kappa$ vs $\langle \eta \rangle$ from simulations

Figure 17 shows the $\langle \eta \rangle$ vs $\kappa$ curve for $\beta_I = 0.1 - 1.2$. For $\beta_I = 0.6$, no voltage drop develops until the value of $\kappa$ reaches 1. At the point ($\langle \eta \rangle = 0$ and $\kappa = 1$) there occurs a transition from the zero-voltage state ($\langle \eta \rangle = 0$) to the finite-voltage state ($\langle \eta \rangle \neq 0$). The $\langle \eta \rangle$ vs $\kappa$ curve approaches the straight line denoted by $\langle \eta \rangle = \kappa$ with further increasing $\langle \eta \rangle$. With decreasing $\langle \eta \rangle$ from the high $\langle \eta \rangle$-side, in turn, the $\langle \eta \rangle$ vs $\kappa$ curve starts to deviate from the straight line $\langle \eta \rangle = \kappa$. The transition occurs from the finite-voltage state
to the zero-voltage state at $\kappa = 0.6965$. Similar hysteresis behaviors are also seen for the cases of $\beta_J = 0.12 - 0.9$.

Fig. 17 The $I$-$V$ curve ($\kappa$ vs $\langle \eta \rangle$) for $\beta_J = 0.1 - 1.2$.

Fig. 18 Schematic diagram of the $I$-$V$ ($\kappa$ vs $\langle \eta \rangle$) trajectories as $\langle \eta \rangle$ changes. $\beta_J = 0.6$. At $\langle \eta \rangle = 0$, $\kappa$ changes from 0 to 1. At $\kappa = 1$, $\langle \eta \rangle$ changes from 0 to a value above 1. With further increasing $\langle \eta \rangle$, the relation $\kappa = \langle \eta \rangle$ holds valid (reversible). With decreasing $\langle \eta \rangle$, in turn, the relation $\kappa = \langle \eta \rangle$ still holds valid. There is a transition from this state to the zero-voltage state ($\langle \eta \rangle = 0$) at $\kappa = \kappa_c = 0.6965$. 
7 DC SQUID (superconducting quantum interference device)

7.1 Current density and flux quantization

In quantum mechanics, the current density is defined as

\[ J = \frac{q \hbar}{2m^2} \left[ \psi^* \nabla \psi - \psi \nabla^* \psi \right] - \frac{q^2 |\psi|^2}{mc} A, \]

where \( q = -2e, e > 0 \) is a charge for electron pairs, \( m \) is a mass, \( A \) is a vector potential, and \( \psi \) is a wavefunction. When the wavefunction is given by the amplitude \( |\psi(r)| \) and the phase \( \theta(r) \) as

\[ \psi = |\psi(r)| e^{i\theta(r)}, \]

then \( J \) can be rewritten as

\[ J = \frac{q \hbar}{m} |\psi|^2 (\nabla \theta - \frac{q}{c \hbar} A). \]

Note that this current density is invariant under the gauge transformation. \( A' = A + \nabla \chi \) and \( \theta' = \theta + \frac{q \chi}{c \hbar} \),

\[ J' = \frac{q \hbar}{m} |\psi|^2 \left( \nabla \theta' - \frac{q}{c \hbar} A' \right) = \frac{q \hbar}{m} |\psi|^2 \left( \nabla \theta' - \frac{q}{c \hbar} A \right) = \frac{q \hbar}{m} |\psi|^2 \left( \nabla \theta - \frac{q}{c \hbar} A \right), \]

where \( \psi'(r) = e^{i\theta(r) \frac{q \chi}{c \hbar}} \psi(r) = |\psi(r)| e^{i[\theta(r) + \frac{q \chi}{c \hbar}]} \).

If we consider now a cylinder which may become superconductor in an external magnetic field and if we take a path from a surface at a distance which is larger than the penetration depth \( \lambda \), then \( J = 0 \). When \( q = -2e \), we have

\[ J = -\frac{2e \hbar}{m} |\psi|^2 (\nabla \theta + \frac{2e}{c \hbar} A) = 0, \]

or

\[ \nabla \theta = -\frac{2e}{c \hbar} A, \]

\[ \oint \nabla \theta \cdot dl = -\frac{2e}{c \hbar} \oint A \cdot dl = -\frac{2e}{c \hbar} \oint \nabla \times A \cdot da = -\frac{2e}{c \hbar} \oint B \cdot da = -\frac{2e}{c \hbar} \Phi = -2\pi \frac{\Phi}{\Phi_0}, \]

where \( \Phi \) is the magnetic flux inside the ring and \( \Phi_0 = \frac{2\pi \hbar c}{2e} \approx 2.06783372 \times 10^{-7} \) Gauss cm\(^2\) is a quantum fluxoid. In the last equation we apply the Stoke’s theorem.

((Note))

The current flows along the ring. However, this current flows only on the surface boundary (region from the surface to the penetration depth \( \lambda \)). Inside of the system (region far from the surface boundary), there is no current since \( \nabla \times H = \frac{4\pi J}{c} \) and \( H = 0 \) (Meissner effect).

7.2 DC SQUID (double junctions): quantum mechanics

DC SQUID consists of two points contacts in parallel, forming a ring. Each contact forms a Josephson junctions of superconductor 1, insulating layer, and superconductor 2 (S\(_1\)-I-S\(_2\)). Suppose that a magnetic flux \( \Phi \) passes through the interior of the loop.
Fig. 19 Schematic diagram of superconducting quantum interference device. $\delta_1$ and $\delta_2$ refer to two point-contact weak links. The rest of the circuit is strongly superconducting.

Here we have

$$\oint \nabla \cdot d\mathbf{l} = \theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b}.$$

or

$$\theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b} = 2\pi \frac{\Phi}{\Phi_0},$$

or

$$\delta_1 - \delta_2 = 2\pi \frac{\Phi}{\Phi_0},$$

where $\delta_1 = (\theta_{1b} - \theta_{1a})$ is the phase difference between the superconductors $a$ and $b$ through the junction 1 and $\delta_2 = (\theta_{2b} - \theta_{2a})$ are the phase difference between the superconductors $a$ and $b$ through the junction 2.

When $B = 0$ (or $\Phi = 0$), we have $\delta_1 - \delta_2 = 0$. In general, we put the form

$$\delta_1 = \delta_0 + \frac{e}{\hbar c} \Phi,$$

$$\delta_2 = \delta_0 - \frac{e}{\hbar c} \Phi.$$

The total current is given by

$$I = I_1 + I_2 = I_c [\sin(\delta_1) + \sin(\delta_2)]$$

$$= I_c [\sin(\delta_0 + \frac{e}{\hbar c} \Phi) + \sin(\delta_0 - \frac{e}{\hbar c} \Phi)]$$

$$= 2I_c \sin(\delta_0) \cos(\frac{e}{\hbar c} \Phi)$$

or

$$I = 2I_c \sin(\delta_0) \cos(\pi \frac{\Phi}{\Phi_0}).$$

(30)

The current varies with $\Phi$ and has a maximum of $2I_c$ when $\frac{e}{\hbar c} \Phi = s\pi$ ($s$: integers), or
The simple two point contact device corresponds to a two-slit interference pattern, for which the physically interesting quantity is the modulus of the amplitude rather than the square modulus, as it is for optical interference patterns.

### 7.3 Analogy of the diffraction with double slits and single slit

![Diagram of diffraction effect of Josephson junction.](image)

Fig. 20 Diffraction effect of Josephson junction. A magnetic field $\mathbf{B}$ along the $z$ direction, which is penetrated into the junction (in the normal phase).

We consider a junction (1) of rectangular cross section with magnetic field $\mathbf{B}$ applied in the plane of the junction, normal to an edge of width $w$.

\[
J = J_0 \sin[\delta_1 + \frac{q}{\hbar c} \int \mathbf{A} \cdot d\mathbf{l}],
\]

with $q = -2e$. We use the vector potential $\mathbf{A}$ given by

\[
\mathbf{A} = \frac{1}{2} (\mathbf{B} \times \mathbf{r}) = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right),
\]

\[
\mathbf{A'} = \mathbf{A} + \nabla \chi = \left(-By, 0, 0\right),
\]

where

\[
\chi = -\frac{Bxy}{2}.
\]

Then we have

\[
J = J_0 \sin[\delta_1 + \frac{q}{\hbar c} \int_{x_1}^{x_2} (-By)dx] = J_0 \sin[\delta_1 - \frac{qB}{\hbar c} yW],
\]

\[
dI_1 = JLdy = J_0 L \sin[\delta_1 - \frac{qB}{\hbar c} yW]dy,
\]

or
\[ I_1 = J_0 L \int_{-\tau/2}^{\tau/2} \sin[\delta_i - \frac{qB}{\hbar c} yW] dy. \]

Then we have
\[ I_1 = J_0 L \frac{2\hbar c}{BWq} \sin(\delta_i) \sin\left(\frac{qB \tau W}{2\hbar c}\right). \]

Here we introduce the total magnetic flux passing through the area \( Wt \) (\( \Phi_w = BWt \)), \( I_c = J_0Lt \), and
\[ \frac{\Phi_w}{\Phi_0} = \frac{BWt}{2\pi\hbar c} = \frac{eBWt}{\pi\hbar c}, \quad \text{or} \quad \frac{\Phi_w}{\Phi_0} = \frac{eBWt}{\hbar c}. \]

Therefore we have
\[ I_1 = I_c \sin(\delta_i) \left( \frac{\pi \Phi_w}{\Phi_0} \right). \]

The total current is given by
\[ I = I_1 + I_1 = I_c \left[ \sin(\delta_i) + \sin(\delta_i) \right] \left( \frac{\pi \Phi_w}{\Phi_0} \right), \]

or
\[ I = I_1 + I_1 = 2I_c \sin(\delta_i) \cos(\pi \frac{\Phi}{\Phi_0}) \left( \frac{\pi \Phi_w}{\Phi_0} \right). \]

The short period variation is produced by interference from the two Josephson junctions, while the long period variation is a diffraction effect and arises from the finite dimensions of each junction. The interference pattern of \( |I|^2 \) is very similar to the intensity of the Young’s double slits experiment. If the slits have finite width, the intensity must be multiplied by the diffraction pattern of a single slit, and for large angles the oscillations die out.

((Young’s double slit experiment))

We consider the Young’s double slits (the slits are separated by \( d \)). Each slit has a finite width \( a \).
Fig. 21 Geometric construction for describing the Young’s double-slit experiment (not to scale).

\((\text{double slits})\)

\(E\) is the electric field of a light with the wavelength \(\lambda\). \(d\) is the separation distance between the centers of the slits.

Fig. 22 A reconstruction of the resultant phasor \(E_R\) which is the combination of two phasors \((E_0)\).

\[ E_R = 2E_0\cos\frac{\alpha}{2}. \]

The intensity \(I \propto E_R^2 = 4E_0^2\cos^2\frac{\alpha}{2} = 2E_0^2(1 + \cos \alpha).\]

where the phase difference \(\alpha\) is given by

\(\alpha = \frac{2\pi d}{\lambda}\sin \theta.\)

\((\text{single slit})\)

We assume that each slit has a finite width \(a.\)
Fig. 23 Phasor diagram for a large number of coherent sources. All the ends of phasors lie on the circular arc of radius \( R \). The resultant electric field magnitude \( E_R \) equals the length of the chord.

\[
E_0 = R\beta ,
\]

\[
E_R = 2R\sin\frac{\beta}{2} = 2\frac{E_0}{\beta} \sin\frac{\beta}{2} = E_0 \frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}.
\]

where the phase difference \( \beta \) is given by \( \beta = \frac{2\pi a}{\lambda} \sin \theta \). Then the resultant intensity \( I \) for the double slits (the distance \( d \) (each slit has a finite width \( a \) is given by

\[
I = I_m \cos^2 \left( \frac{\sin \left( \frac{\beta}{2} \right)}{\frac{\beta}{2}} \right)^2 = I_m \frac{1}{2}(1 + \cos \alpha)(\frac{\sin \left( \frac{\beta}{2} \right)}{\frac{\beta}{2}})^2.
\]

''((Mathematica 5.2))'' Program-10

\[
f[\alpha_, \beta_] := (1 + \cos[\alpha]) \frac{\sin[\frac{\beta}{2}]^2}{\left(\frac{\beta}{2}\right)^2}
\]

\[
\text{Plot}[\text{Evaluate}[\text{Table}[f[\alpha, N \alpha], \{N, 20, 20\}], \{\alpha, -15 \pi, 15 \pi\}]], \text{PlotPoints} \rightarrow 200, \text{PlotStyle} \rightarrow \text{Table}[\text{Hue}[0.3 i], \{i, 0, 10\}], \text{PlotRange} \rightarrow \{-6 \pi, 6 \pi\}, \{0, 0.002\}], \text{Prolog} \rightarrow \text{AbsoluteThickness}[1.2], \text{Background} \rightarrow \text{GrayLevel}[0.5]]
\]
Fig. 24 The combined effects of two-slit and single-slit interference. The pattern consists of a diffraction envelope and interference fringes.

7.4 DC SQUID Junction based on the RSJ model
7.4.1 Formulation

The DC SQUID consists of two Josephson junctions connected in parallel on a superconducting loop of inductance $L$.

Fig. 25 Simple notation for the DC SQUID consisting of two Josephson junctions (J.J.) in parallel.
As shown in Fig. 26, the total current is given by
\[ I_B = I_1 + I_2. \]
The total magnetic flux is given by \( \Phi = \Phi_{\text{ext}} + LI_z \), where \( L \) is the total self-inductance (\( L = L_1 + L_2 \) and \( L_1 = L_2 = L/2 \) in this case)
\[ I_z = \frac{I_2 - I_1}{2}, \]
where \( I_z \) is the loop (circulating) current.
\[
\begin{align*}
I_z \sin \phi_1 + \frac{V_1}{R} + CV_1 &= I_1, \\
I_z \sin \phi_2 + \frac{V_2}{R} + CV_2 &= I_2,
\end{align*}
\]
\[
\begin{align*}
\dot{\phi}_1 &= \frac{2e}{h} V_1, \\
\dot{\phi}_2 &= \frac{2e}{h} V_2.
\end{align*}
\]
Since
\[
V = V_1 + \frac{L}{2} \frac{dI_1}{dt}, \quad V = V_2 + \frac{L}{2} \frac{dI_2}{dt}, \quad \text{and} \quad I_B = I_1 + I_2,
\]
and the total voltage \( V \) is given by the simple form,
\[
V = \frac{1}{2} (V_1 + V_2) + \frac{L}{4} \frac{dI_B}{dt} = \frac{1}{2} (V_1 + V_2) = \frac{h}{4e} \left( \frac{d\phi_1}{dt} + \frac{d\phi_2}{dt} \right)
\]
since \( dI_B / dt = 0 \) (or \( I_B \) is independent of \( t \)).

For the sake of simplicity, we use the dimensionless quantities. Here we assume that
\[
\eta = \frac{V}{I_1 R}, \quad \kappa = \frac{I}{I_c}, \quad \tau = \omega_j t.
\]
Then we have
\[
\frac{hC}{2eI_c} \omega_j^2 \frac{d^2 \phi_1}{d \tau^2} + \frac{h}{2eRI_c} \omega_j \frac{d\phi_1}{d \tau} + \sin \phi_1 = \kappa_1,
\]
or
\[
\frac{d^2 \phi_1}{d\tau^2} + \beta_j \frac{d\phi_1}{d\tau} + \sin \phi_1 = \kappa_1,
\]
where
\[
\beta_j = \frac{1}{\omega_C R}.
\]
Similarly,
\[
\frac{d^2 \phi_2}{d\tau^2} + \beta_j \frac{d\phi_2}{d\tau} + \sin \phi_2 = \kappa_2.
\]
We also have
\[
\kappa_1 + \kappa_2 = \kappa_B,
\]
\[
\kappa_2 - \kappa_1 = 2\kappa_s,
\]
or
\[
2\kappa_1 = \kappa_B - 2\kappa_s,
\]
where
\[
\kappa_B = \frac{I_B}{I_c}, \quad \kappa_1 = \frac{I_1}{I_c}, \quad \kappa_2 = \frac{I_2}{I_c}, \quad \kappa_s = \frac{I_s}{I_c} = \frac{\kappa_2 - \kappa_1}{2}
\]
Then we have
\[
\frac{d^2 \phi_1}{d\tau^2} + \beta_j \frac{d\phi_1}{d\tau} + \sin \phi_1 = \frac{\kappa_B}{2} - \kappa_s.
\]
(33)
Similarly we have
\[
\frac{d^2 \phi_2}{d\tau^2} + \beta_j \frac{d\phi_2}{d\tau} + \sin \phi_2 = \frac{\kappa_B}{2} + \kappa_s.
\]
(34)
The phases are related to the external magnetic flux by
\[
\phi_1 - \phi_2 = 2\pi \frac{\Phi}{\Phi_0} = 2\pi \left( \frac{LI_s}{\Phi_0} + \Phi_{\text{ext}} \right) = 2\pi \left( \frac{\beta}{2} \kappa_s + \Phi_{\text{ext}} \right),
\]
or
\[
\kappa_s = \frac{2}{\beta} \left( \frac{\phi_1 - \phi_2}{2\pi} - \Phi_{\text{ext}} \right),
\]
(35)
where
\[
\beta = \frac{2I_c L}{\Phi_0}.
\]
The normalized voltage \( \eta \) is given by
\[
\eta = \frac{1}{2} \beta_j \left( \frac{d\phi_1}{d\tau} + \frac{d\phi_2}{d\tau} \right).
\]
(36)

7.4.2 Two-dimensional (2D) SQUID potential

Equations (33) and (34) describing the DC SQUID dynamics can be regarded as an equation of motion of a point mass in a field of force with a 2D SQUID potential \( U(\phi_1, \phi_2) \).
\[
\frac{d^2 \phi_1}{d\tau^2} + \beta_j \frac{d\phi_1}{d\tau} = -\sin \phi_1 + \frac{\kappa_B}{2} - \kappa_s = -\frac{2e}{\hbar I_c} \frac{\partial U(\phi_1, \phi_2)}{\partial \phi_1}
\]
and
\[
\frac{d^2 \phi_2}{d^2 \tau} + \beta \frac{d \phi_2}{d \tau} = -\sin \phi_2 + \frac{\kappa_B}{2} + \kappa_\gamma = -\frac{2e}{\hbar I_c} \frac{\partial U(\phi_1, \phi_2)}{\partial \phi_2}
\]
or
\[
\frac{\partial U(\phi_1, \phi_2)}{\partial \phi_1} = \frac{\hbar I_c}{2e} (\sin \phi_1 - \frac{\kappa_B}{2} + \kappa_\gamma) = \frac{\Phi_0}{2\pi} I_c \sin \phi_1 - \frac{2\pi}{\beta} (\frac{\phi_1 - \phi_2}{2} - \pi - \frac{\Phi_{ext}}{\Phi_0})
\]
(37)
\[
\frac{\partial U(\phi_1, \phi_2)}{\partial \phi_2} = \frac{\hbar I_c}{2e} (\sin \phi_2 - \frac{\kappa_B}{2} - \kappa_\gamma) = \frac{\Phi_0}{2\pi} I_c \sin \phi_2 - \frac{2\pi}{\beta} (\frac{\phi_1 - \phi_2}{2} - \pi - \frac{\Phi_{ext}}{\Phi_0})
\]
(38)
Thus the normalized 2D SQUID potential \( \tilde{U}(\phi_1, \phi_2) \) \([= U(\phi_1, \phi_2)/(2E_j)]\) is obtained as
\[
\tilde{U}(\phi_1, \phi_2) = \frac{1}{\beta} (\phi_1 - \phi_2 - \pi - \frac{\Phi_{ext}}{\Phi_0})^2 - \cos(\phi_1 - \phi_2) \cos(\phi_1 + \phi_2) - \frac{\kappa_B}{2} (\phi_1 + \phi_2)
\]
(39)
where \( E_j = \Phi_0 I_c / (2\pi) \) is the Josephson coupling constant. It is convenient to introduce new variable
\[
x = \frac{\phi_1 + \phi_2}{2\pi}, \quad y = \frac{\phi_1 - \phi_2}{2\pi}
\]
The loop current \( \kappa_\gamma \) is related to \( y \) as
\[
\kappa_\gamma = \frac{2}{\beta} (y - \frac{\Phi_{ext}}{\Phi_0})
\]
Then the 2D SQUID potential \( \tilde{U}(\phi_1, \phi_2) \) is rewritten as
\[
\tilde{U}(x, y) = \frac{\pi}{\beta} (y - \frac{\Phi_{ext}}{\Phi_0})^2 - \frac{\kappa_B}{2\pi} \cos(\pi x) \cos(\pi y)
\]
Here we make a contour plot of \( U(\phi_1, \phi_2) \) in the \( \phi_1-\phi_2 \) plane, as \( \beta, \Phi_{ext}/\Phi_0, \) and \( \kappa_B \) are changed as parameters. The parameters \( \beta = 1 \) and \( \Phi_{ext}/\Phi_0 = 0.25 \) are fixed. The current \( \kappa_B \) is changed as a parameter. As will be shown in Fig.32, the critical current \( (\kappa_B)_c \) is equal to 1.628 for \( \beta = 1 \) and \( \Phi_{ext}/\Phi_0 = 0.25 \). In Fig.27 we show the contour plot of \( U(\phi_1, \phi_2) \) in the \( \phi_1-\phi_2 \) plane, where \( \beta = 1 \) and \( \Phi_{ext}/\Phi_0 = 0.25 \). For \( \kappa_B = 0.4 \), \( U(\phi_1, \phi_2) \) has multiple metastable state separated by saddle points on the \( \phi_2 = \phi_1 \) line. With increasing \( \kappa_B \), these saddle points gradually disappear. At \( \kappa^\ast(\kappa_B)_c \), it seems that all the saddle points disappear, suggesting no stable state corresponding to local minima of the potential energy.

((Mathematica 5.2)) Program 11
(*2D SQUID potential, \( \beta = 1 - 2, \kappa_B = 1 - 3; n0 = \Xi/\Phi_0 \ (0 - 0.5), pp = points*)
\[
F[\phi1, \phi1_, \beta_, \kappa_B_, n0_] :=
\frac{\pi}{\beta} (\phi1 - \phi2 - n0)^2 - \frac{\kappa_B}{2\pi} \cos(\pi(\phi1 + \phi2 - n0) - \phi1 + \phi2 - n2) - \cos(\pi(\phi1 + \phi2 - n0) - \phi1 + \phi2 - n2) - \cos(\pi(\phi1 + \phi2 - n0) - \phi1 + \phi2 - n2)
\]
\[
mp[pp_, \beta_, \kappaB_, n0_] := Module[{ss1, ss2}, ss1 = ContourPlot[F[\phi1, \phi1, \beta, \kappaB, n0], \{\phi1, -4 \pi, 4 \pi\}, \{\phi2, -4 \pi, 4 \pi\}, PlotPoints -> pp, ContourLines -> True, PlotRange -> All, ColorFunction -> Hue,}]
53
\(\text{AspectRatio}\rightarrow\text{Automatic},\) 
\(\text{Compiled}\rightarrow\text{False}];\) 
\(\text{ss2}=\text{ListContourPlot}[\text{ss1[1]}],\) 
\(\text{Contours}\rightarrow(\#[[2]][/\text{Partition}[\text{Sort}[\text{Flatten}[\text{ss1[1]]],\text{pp}}] \), 
\(\text{ColorFunction}\rightarrow(\text{Hue}[2\#]&),\text{ContourLines}\rightarrow\text{False},\) 
\(\text{MeshRange}\rightarrow\{[-4\pi,4\pi],[-4\pi,4\pi]\},\) 
\(\text{DisplayFunction}\rightarrow\text{Identity}];\) 
\(\text{Show}[\text{ss2},\) 
\(\text{DisplayFunction}\rightarrow\$\text{DisplayFunction},\text{FrameTicks}\rightarrow\text{True},\) 
\(\text{AspectRatio}\rightarrow\text{Automatic}])\]

\(\text{Table}[\text{mp[100,1.0,κB,0.25]},\{\text{xB},0,2.0,0.2}\}]\)

**Fig. 27** The contour plot of \(U(\phi_1,\phi_2)\) in the \(\phi_1-\phi_2\) plane: \(\phi_1\) is \(x\)-axis and \(\phi_2\) is the \(y\) axis. \(\kappa_B = 0.4, 0.8, 1.2, 1.6,\) and 1.8. \(\beta = 1.\) \(\Phi_{\text{ext}}/\Phi_0 = 0.25.\) \((\kappa_B)_{\text{c}} = 1.628.\)

1. \(\kappa_B = 0.4, \beta = 1.0\) and \(\Phi/\Phi_0 = 0.25\)

![Contour Plot](image1)

2. \(\kappa_B = 0.8, \beta = 1.0\) and \(\Phi/\Phi_0 = 0.25\)

![Contour Plot](image2)
(3) $\kappa_B = 1.2$, $\beta = 1.0$ and $\Phi/\Phi_0 = 0.25$

(4) $\kappa_B = 1.6$, $\beta = 1.0$ and $\Phi/\Phi_0 = 0.25$
(5) \( \kappa_B = 1.8, \beta = 1.0 \) and \( \Phi / \Phi_0 = 0.25 \)

7.5 Simple case: \( \beta_0 \gg 1 \) and \( \beta = 0 \)

For simplicity we assume that \( \beta_0 \gg 1 \). This assumption is appropriate for the operation of DC SQUID.

First we consider the critical current at \( V = 0 \). We also assume that \( \beta = 0 \).

\[
I_B = I_1 + I_2 = I_c (\sin \phi_1 + \sin \phi_2) = I_c [\sin \phi_1 + \sin (\phi_1 - 2\pi \frac{\Phi_{\text{ext}}}{\Phi_0})],
\]

since

\[
\phi_2 = \phi_1 - 2\pi \frac{\Phi_{\text{ext}}}{\Phi_0}.
\]

Then we have

\[
I_B = 2I_c \sin (\phi_1 - \pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \cos (\pi \frac{\Phi_{\text{ext}}}{\Phi_0}).
\]
The maximum of \( I_B \) is

\[
I_{\text{max}} = 2I_c \left| \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \right|. \tag{40}
\]

The critical current is a periodic function of the external magnetic flux.

**Fig. 28** Ideal case for the \( I_B/I_c \) vs \( \Phi_{\text{ext}}/\Phi_0 \) curve in the DC SQUID, where \( I_B \) is the maximum supercurrent. \( I_B = 2I_c \) when \( \Phi_{\text{ext}}/\Phi_0 = n \) (integer) and \( I_B = 0 \) for \( \Phi_{\text{ext}}/\Phi_0 = n + 1/2 \).

We consider the general case (but still \( L = 0 \) and \( \beta_J \gg 1 \))

\[
\beta_J \frac{d\phi_1}{d\tau} + \sin \phi_1 = \frac{\kappa_B}{2} - \kappa_s, \tag{41}
\]

\[
\beta_J \frac{d\phi_2}{d\tau} + \sin \phi_2 = \frac{\kappa_B}{2} + \kappa_s, \tag{42}
\]

\[
\phi_2 = \phi_1 - 2\pi \frac{\Phi_{\text{ext}}}{\Phi_0}. \tag{43}
\]

From the addition of Eqs.(41) and (42) with the help of the relation Eq.(43), we have

\[
\beta_J \frac{d}{d\tau} \left( \phi_1 - \pi \frac{\Phi_{\text{ext}}}{\Phi_0} \right) + \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \sin(\phi_1 - \pi \frac{\Phi_{\text{ext}}}{\Phi_0}) = \frac{\kappa_B}{2}. \tag{44}
\]

When we introduce a new parameter

\[
\phi_i = \phi_1 - \pi \frac{\Phi_{\text{ext}}}{\Phi_0},
\]

we have the final form

\[
\beta_J \frac{d\phi_i}{d\tau} + \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \sin \phi_i = \frac{\kappa_B}{2}. \tag{44}
\]

We are interested in the DC current-voltage characteristic so we need to determine the time averaged voltage

\[
V = \left\langle \frac{\hbar}{2e} \frac{d\phi_i}{dt} \right\rangle = \left( I_c R \beta_J \frac{d\phi_i}{d\tau} \right) = \frac{I_c R \beta_J}{T} \int_0^T \frac{d\phi_i}{d\tau} d\tau = I_c R \beta_J \frac{2\pi}{\tau}. \tag{44}
\]

Here
\[
\tau = 2\pi \frac{d\Phi_1}{d\tau} = 2\pi \frac{\beta_j d\Phi_1}{\kappa_B - \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \sin \varphi_1} = \frac{\beta_j}{\kappa_B - \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0})} \int_0^\pi \frac{d\phi}{\kappa' - \sin \varphi_1} = \frac{1}{\cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0})} \tau',
\]

where

\[
\kappa' = \frac{\kappa_B}{2\cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0})},
\]

\[
\tau' = \frac{2\pi}{\sqrt{\kappa'^2 - 1}} \quad \text{for } \kappa > 1,
\]

\[
\tau' = 0 \quad \text{for } \kappa < 1.
\]

Then we have

\[
V = 2\pi I_c R \beta_j \frac{1}{\tau} = \frac{2\pi I_c R}{\tau'} \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) = I_c R \sqrt{\kappa'^2 - 1} \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}),
\]

or

\[
\frac{V}{I_c R} = \left| \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \right| \sqrt{\kappa'^2 - 1},
\]

or

\[
|\kappa'| = \sqrt{1 + \left( \frac{V}{I_c R \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0})} \right)^2} = \frac{1}{2 \cos(\pi \frac{\Phi_{\text{ext}}}{\Phi_0})} \frac{I_B}{I_c},
\]

or

\[
\frac{I_B}{2I_c} = \frac{1}{\sqrt{\cos^2(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) + \left( \frac{V}{RI_c} \right)^2}},
\]

or

\[
\frac{<V>}{RI_c} = \sqrt{\left( \frac{I_B}{2I_c} \right)^2 - \cos^2(\pi \frac{\Phi_{\text{ext}}}{\Phi_0})},
\]

or

\[
<\frac{V}{R} > = \frac{R}{2} \sqrt{\left( \frac{I_B^2}{2I_c} - 4I_c^2 \cos^2(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}) \right)}. \tag{45}
\]

When this equation for the voltage is compared with that for one Josephson junction with \(\beta_j \gg 1\)

\[
\langle V \rangle = R \sqrt{I^2 - I_c^2}.
\]
We find that the critical current is $2I_c |\cos(\pi \Phi_{\text{ext}} / \Phi_0)|$. This means that the critical current is $2I_c$ for $\Phi_{\text{ext}} / \Phi_0 = n \ (\text{integer})$ and zero for $\Phi_{\text{ext}} / \Phi_0 = n + 1/2$. In other words, the critical current is a periodic function of $\Phi$ with a period of $\Phi_0$. However, the actual critical current does not oscillate between 0 and $2I_c$ because of the finite self-inductance $L$. In the above model, $L$ (or $\beta = 0$) is assumed to be zero. The critical current varies between $2I_c$ and finite value depending on the value of $\beta$ (see the detail in Sec.8).

When the total current $I_B$ is constant, the voltage across the DC SQUID periodically changes with the external magnetic flux. This is the phenomenon one exploit to create the most sensitive magnetic field detection.

((Note)) Figure 29 is obtained from the Instruction manual of Mr. SQUID.9

Fig.29 Detected voltage vs the magnetic flux $\Phi/\Phi_0$. The current $I_B$ is kept at fixed value which is a little larger than $2I_c$. The detected voltage shows a maximum for $\Phi = (n+1/2)\Phi_0$, and a minimum for $\Phi = n\Phi_0$. The detected voltage is a periodic function of $\Phi$ with a period of $\Phi_0$. (This figure is copied from the User Guide of Mr SQUID9).

7.6 More general case: $\beta J \gg 1$ and finite $\beta$

In order to avoid hysteresis in the $I$-$V$ curve one usually choose over-damped junction. Here we start with the differential equations given by

$$
\beta J \frac{d\phi_1}{d\tau} + \sin \phi_1 = \frac{\kappa_B}{2} - \kappa_s, \tag{46}
$$

$$
\beta J \frac{d\phi_2}{d\tau} + \sin \phi_2 = \frac{\kappa_B}{2} + \kappa_s, \tag{47}
$$

$$
\phi_1 - \phi_2 = 2\pi \left( \frac{\beta}{2} \kappa_s + \frac{\Phi_{\text{ext}}}{\Phi_0} \right). \tag{48}
$$

or
\[ \kappa_s = \frac{2}{\beta} \left( \frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{\text{ext}}}{\Phi_0} \right). \]

Thus we have two differential equations

\[ \beta J \frac{d\phi_1}{d\tau} + \sin \phi_1 = \frac{\kappa_B}{2} \frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{\text{ext}}}{\Phi_0}, \]

\[ \beta J \frac{d\phi_2}{d\tau} + \sin \phi_2 = \frac{\kappa_B}{2} \frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{\text{ext}}}{\Phi_0}, \]

\[ \eta = \frac{1}{2} \beta J \left( \frac{d\phi_1}{d\tau} + \frac{d\phi_2}{d\tau} \right) = \frac{1}{2} \left( \kappa_B - \sin \phi_1 - \sin \phi_2 \right), \]

where the initial conditions \([\phi_1(0)\) and \(\phi_2(0)\)] are chosen appropriately

8 Simulation

The differential equations for \(\phi_1(\tau)\) and \(\phi_2(\tau)\) are numerically solved by using the Mathematica 5.2. We show our calculation on the \(\tau\) dependence of \(\eta\). The parameters \(\kappa_B, \beta, \beta\) and \(\Phi_{\text{ext}}/\Phi_0\) are appropriately changed for our calculations.

8.1 Relation of \(<\eta>\) vs \(\kappa_B\) with \(\Phi_{\text{ext}}/\Phi_0\) as a parameter

We calculate the relation \(<\eta>\) vs \(\kappa_B\) where \(\Phi_{\text{ext}}/\Phi_0 = 0, 0.05, 0.1, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45,\) and \(0.5\). We choose \(\beta J = 10\) for the over-damped case. So that no hysteresis is seen in the \(I-V\) curve. The parameter \(\beta\) is changed as a parameter: \(\beta = 0.02 - 3\). The voltage \(<\eta>\) suddenly increases from zero to a finite value at the critical current which is dependent on the magnetic flux \(\Phi_{\text{ext}}/\Phi_0\) and \(\beta\).

(i) Using the following Mathematica 5.2 program, we find the maximum and minimum of \(\Omega(\tau)\) in the long time region where \(\Omega(\tau)\) periodically oscillates with \(\tau\).

(ii) The average \(<\Omega>\) is calculated as \((\text{maximum}+\text{minimum})/2\). The average voltage \(<\eta> = \beta J <\Omega>\) is plotted as a function of \(\kappa\) for the fixed \(\beta J (= 10)\) and \(\beta\), where the magnetic flux \(\Phi_{\text{ext}}/\Phi_0\) is changed as a parameter.

(iii) The voltage is equal to zero for \(\kappa<\kappa_B)\). It suddenly increase with increasing \(\kappa\) above \((\kappa_B)\). We determine the critical current \((\kappa_B)\) as a function of the magnetic flux where \(\beta\) is changed as a parameter.

((Mathematica 5.2)) Program-12

Clear["Global`*"]
<<Graphics`Graphics`
(*Subroutine, DC SQUID beta=1 betaJ=10 voltage vs magnetic flux*)
DCSQ[\{\phi_1, \phi_2\}, \{\beta J, \kappa B, \beta, N_0\}, \tau_{\text{max}}, \text{opts}\} :=
Module[\{\text{numso1}, \text{numgraph}\},
\text{numso1} = \text{NDSolve}[\{\beta J \phi_1'[\tau] + \sin[\phi_1[\tau]] = \kappa B - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N_0 \right), \beta J \phi_2'[\tau] + \sin[\phi_2[\tau]] = \kappa B - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N_0 \right), \phi_1[0] = \phi_0, \phi_2[0] = \phi_0\}, \{\tau, 0, \tau_{\text{max}}\}];
\text{numgraph} = \text{Plot}[\left( \frac{\kappa B - \sin[\phi_1[\tau]] - \sin[\phi_2[\tau]]}{2} \right) /. \text{numso1}, \{\tau, 0, \tau_{\text{max}}\}, \text{opts, DisplayFunction} \to \text{Identity}];
\text{Max1}[\{\phi_0, \phi_0\}, \{\beta J, \kappa B, \beta, N_0\}, \{\tau_{\text{min}}, \tau_{\text{max}}\}] :=
Module[\{\text{numso1}\},
\text{numso1} = \text{NDSolve}[\{\beta J \phi_1'[\tau] + \sin[\phi_1[\tau]] = \kappa B - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N_0 \right), \beta J \phi_2'[\tau] + \sin[\phi_2[\tau]] = \kappa B - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N_0 \right), \phi_1[0] = \phi_0, \phi_2[0] = \phi_0\}, \{\tau, 0, \tau_{\text{max}}\}];
\text{Maxim} = \text{FindMaximum}[\left( \frac{\kappa B - \sin[\phi_1[\tau]] - \sin[\phi_2[\tau]]}{2} \right) /. \text{numso1}, \{\tau, \tau_{\text{min}}, \tau_{\text{max}}\}];
\text{Min1}[\{\phi_0, \phi_0\}, \{\beta J, \kappa B, \beta, N_0\}, \{\tau_{\text{min}}, \tau_{\text{max}}\}] :=
Module[\{\text{numso1}\},
\text{numso1} = \text{NDSolve}[\{\beta J \phi_1'[\tau] + \sin[\phi_1[\tau]] = \kappa B - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N_0 \right), \beta J \phi_2'[\tau] + \sin[\phi_2[\tau]] = \kappa B - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N_0 \right), \phi_1[0] = \phi_0, \phi_2[0] = \phi_0\}, \{\tau, 0, \tau_{\text{max}}\}];
\text{Minim} = \text{FindMinimum}[\left( \frac{\kappa B - \sin[\phi_1[\tau]] - \sin[\phi_2[\tau]]}{2} \right) /. \text{numso1}, \{\tau, \tau_{\text{min}}, \tau_{\text{max}}\}];
\text{Sei}[\beta J, \beta, N_0] := \text{Module}[\{\text{A1}, \text{B1}, \text{list1}, \text{avel}\}, \text{A1} = \text{Max1}[\{0, 0\}, \{\beta J, \beta, N_0\}, \{400 \pi, 800 \pi\}] \&/@\text{Range}[0, 3, 0.01];
\text{B1} = \text{Min1}[\{0, 0\}, \{\beta J, \beta, N_0\}, \{400 \pi, 800 \pi\}] \&/@\text{Range}[0, 3, 0.01]; \text{avel} = (\text{A1} + \text{B1})/2;
\text{list1} = \text{Table}[0.01 (k - 1), \text{avel}[[k, 1]]}, [k, 1, 301]];\]
\text{h}[\text{n0}] := \text{ListPlot}[\text{Evaluate}[\text{Sei}[10, 1, n0]], \text{PlotStyle} \to \{\text{Hue}[2 n0], \text{PointSize}[0.01]\}], \text{AxesLabel} \to \{"\kappa B", \"\eta\"\}, \text{PlotLabel} \to \text{NumberForm}[n0]]
\text{f1} = \text{Table}[\text{h}[p], \{p, 0, 0.5, 0.05\}]

\text{Fig.30} \quad \eta = \beta J, \Omega vs \kappa (\beta_J = 10 \text{ and } \beta = 1) \text{ for } \Phi_{\text{ext}}/\Phi_0 = 0 - 0.5. \text{ The number denoted in each curve is the value of } \Phi_{\text{ext}}/\Phi_0. \text{ Note that } \eta \text{ takes several}
values at the same $\kappa$ around $\Phi_{ext}/\Phi_0 = 0.4$. All solutions are plotted in the figures. Some values are unphysical. We do not understand why so many values appear. Some solutions may correspond to metastable states.

(1) $\Phi_{ext}/\Phi_0 = 0. \beta = 1$ and $\beta_J = 10$

(2) $\Phi_{ext}/\Phi_0 = 0.1. \beta = 1$ and $\beta_J = 10$

(3) $\Phi_{ext}/\Phi_0 = 0.2. \beta = 1$ and $\beta_J = 10$

(4) $\Phi_{ext}/\Phi_0 = 0.3. \beta = 1$ and $\beta_J = 10$
(5) $\Phi_{ext}/\Phi_0 = 0.4$. $\beta = 1$ and $\beta_0 = 10$

(6) $\Phi_{ext}/\Phi_0 = 0.45$. $\beta = 1$ and $\beta_0 = 10$

(7) $\Phi_{ext}/\Phi_0 = 0.5$. $\beta = 1$ and $\beta_0 = 10$
Fig. 31 \( <\eta> = \beta_j <\Omega> \) vs \( \kappa (\beta_\eta = 10) \) and \( (\beta = 0.3) \) for \( \Phi_{ext}/\Phi_0 = 0 - 0.5 \). The number denoted in each curve is the value of \( \Phi_{ext}/\Phi_0 \). Note that \( <\eta> \) takes several values at the same \( \kappa \) around \( \Phi_{ext}/\Phi_0 = 0.4 \) (multi-valued function). All solutions are plotted in the figures. Some values are unphysical. We do not understand why so many values appear. Some solutions correspond to metastable states.

(1) \( \Phi_{ext}/\Phi_0 = 0. \beta = 0.3 \) and \( \beta_\eta = 10 \)

(2) \( \Phi_{ext}/\Phi_0 = 0.1. \beta = 0.3 \) and \( \beta_\eta = 10 \)
(3) \( \Phi_{\text{ext}} / \Phi_0 = 0.2. \beta = 0.3 \) and \( \beta_j = 10 \)

(4) \( \Phi_{\text{ext}} / \Phi_0 = 0.3. \beta = 0.3 \) and \( \beta_j = 10 \)

(4) \( \Phi_{\text{ext}} / \Phi_0 = 0.4. \beta = 0.3 \) and \( \beta_j = 10 \)

(5) \( \Phi_{\text{ext}} / \Phi_0 = 0.45. \beta = 0.3 \) and \( \beta_j = 10 \)
From the above simulation we find that the critical current \((\kappa_B)_c\) decreases with increasing the magnetic flux from 2 at \(\Phi_{\text{ext}}/\Phi_0 = 0\) to some finite value (but not zero) at \(\Phi_{\text{ext}}/\Phi_0 = 0.5\) because of the finite value of \(\beta\) (finite inductance). First we estimate the critical current analytically based on an approximation \(\sin\phi \approx 2\phi/\pi\) for \(|\phi|<\pi/2\) (one can easily prove this using the Mathematica 5.2). We use the following approximations,

\[
\kappa_1 = \sin \phi_1 = \frac{2}{\pi} \phi_1, \quad \kappa_2 = \sin \phi_2 = \frac{2}{\pi} \phi_2, \quad \kappa_s = \frac{\kappa_2 - \kappa_1}{2}, \quad \kappa_B = \kappa_1 + \kappa_2, \quad \phi_1 - \phi_2 = 2\pi \left(\frac{\beta}{2} \kappa_s + \frac{\Phi_{\text{ext}}}{\Phi_0}\right) = \frac{\pi}{2} (\kappa_1 - \kappa_2) = -\pi \kappa_s \text{ or } \beta \kappa_s + 2\frac{\Phi_{\text{ext}}}{\Phi_0} = -\kappa_s.
\]

From this relation we have

\[
\kappa_s = -\frac{\Phi_{\text{ext}}}{\Phi_0} \frac{2}{\beta + 1} = \frac{\kappa_2 - \kappa_1}{2},
\]

\[
\kappa_1 + \kappa_2 = \kappa_B, \quad \kappa_1 - \kappa_2 = \frac{\Phi_{\text{ext}}}{\Phi_0} \frac{4\kappa_B}{(\beta + 1)},
\]

or

\[
\kappa_1 = \frac{\kappa_B}{2} + \frac{2}{\beta + 1} \frac{\Phi_{\text{ext}}}{\Phi_0}, \quad (49)
\]
\[ \kappa_2 = \frac{\kappa_B}{2} - \frac{2}{\beta + 1} \frac{\Phi_{ext}}{\Phi_0}. \]

The value of \( \kappa_1 \) has a maximum at \( \Phi_{ext}/\Phi_0 = 1/2 \). The condition for the critical current is that the maximum of \( \kappa_1 \) should be equal to 1. As \( \Phi_{ext}/\Phi_0 \) changes from zero to 1/2, the value of \( \kappa_1 \) changes from \( \kappa_1 = \kappa_B/2 \) to

\[ \kappa_1 = \frac{\kappa_B}{2} + \frac{1}{\beta + 1}. \]

Since the critical current of \( \kappa_1 \) is equal to 1, the critical current of \( \kappa_B \) should be equal to

\[ (\kappa_1)_c = \left(\frac{\kappa_B}{2}\right)_c + \frac{1}{\beta + 1} = 1 \]

or

\[ (\kappa_B)_c = 2\left(1 - \frac{1}{\beta + 1}\right) = \frac{2\beta}{\beta + 1}. \] (50)

Note that the change in the SQUID voltage is approximated by

\[ \Delta V = R_N I_c \Delta \kappa_1 = R_N I_c \frac{1}{\beta + 1} = \frac{R_N I_c}{\beta + 1}, \] (51)

where \( \Delta \kappa_1 = 1/(\beta + 1) : \kappa_1 = \kappa_B/2 \) at \( \Phi_{ext}/\Phi_0 = 0 \) and \( \kappa_1 = (\kappa_B/2) + 1/(\beta + 1) \). We assume that the normal-state resistance of the DC SQUID is \( R_N/2 \): the slope of \( I-V \) curve is given by \( R_N/2 \), but not by \( R_N \). So the resistance of each Josephson junction is \( R_N \) since the parallel configuration. In our Mr. SQUID, we have \( R_N/2 = 1.44 \Omega \) and \( 2I_c = 66mA \).

In Figs.30 and 31, we show the plot of \( <\eta> \) vs \( \kappa_B \). The zero-voltage state \( (<\eta> = 0) \) is stable for \( \kappa_B \leq (\kappa_B)_c \), where \( (\kappa_B)_c \) is the critical current. Figure 32 shows the critical current \( (\kappa_B)_c \) as a function of \( \Phi_{ext}/\Phi_0 \), where \( \beta \) is changed as a parameter and \( \beta_2 = 10 \). We find that \( (\kappa_B)_c \) decreases with increasing the magnetic flux \( \Phi_{ext}/\Phi_0 \). There occurs a transition at \( \kappa_B = (\kappa_B)_c \) from the zero-voltage state \( (<\eta> = 0) \) to a finite-voltage state \( (<\eta> \approx \kappa_B) \). In Fig.33 we show the plot of \( (\kappa_B)_c \) as a function of \( \beta \) at \( \Phi_{ext}/\Phi_0 = 1/2 \), where \( \beta_2 = 10 \). The data point fall well on the solid line denoted by \( (\kappa_B)_c = 2\beta/(\beta + 1) \).
Fig. 32 The critical current \( (\kappa_B)_c \) as a function of \( \Phi_{\text{ext}}/\Phi_0 \), where \( \beta \) is changed as a parameter. \( \beta_0 = 10 \).
Fig. 33 The critical current \((\kappa_B)_c\) at \(\Phi_{\text{ext}}/\Phi_0 = 1/2\) as a function of \(\beta\). \(\beta = 10.\) The solid line denotes the expression given by \((\kappa_B)_c = 2\beta/(1 + \beta)\).

8.3 Relation of \(<\eta>\) vs \(\Phi_{\text{ext}}/\Phi_0\) with \(\kappa_B\) as a parameter

We calculate the magnetic flux \(\Phi_{\text{ext}}/\Phi_0\) dependence on the average voltage, where \(\kappa_B\) is changed as a parameter. When \(\kappa_B\) is fixed, the average voltage \(<\eta>\) periodically changes with \(\Phi_{\text{ext}}/\Phi_0\) [the periodicity \(\Delta(\Phi_{\text{ext}}/\Phi_0) = 1\)]. In Fig. 34, we show our calculation for \(\kappa_B = 1.5 - 2.9\). We find that \(<\eta>\) is a multivalued function of \(\Phi_{\text{ext}}/\Phi_0\). We think that the lowest curve may be a stable solution. This curve has a maximum at \(\Phi_{\text{ext}}/\Phi_0 = 1/2\) and 0 near \(\Phi_{\text{ext}}/\Phi_0 = 0\) and 1. Note that we do not take into account of the effect of Johnson noise. This is a principle of the DC SQUID. The element plays a role of the transformation between the voltage and the magnetic flux.

\begin{verbatim}
((Mathematica 5.2)) Program-13
Clear["Global`*"]
<<Graphics`Graphics`
(*Subroutine, DC SQUID beta=1 betaJ=10 voltage vs magnetic flux*)
\end{verbatim}
\text{DCSQ}[[\phi_{01}, \phi_{02}], \{\beta J, \kappa B, \beta, NO\}, t_{\text{max}}, \text{opts}]] := \\
\text{Module}[\{\text{numsol}, \text{numgraph}\}, \\
\text{numsol} = \\
\text{NDSolve}[
\{\beta J \phi_1'[t] + \sin[\phi_1[t]] = \frac{x B}{2} - \frac{2}{\beta} \left( \frac{\phi_1[t] - \phi_2[t]}{2\pi} - NO \right), \\
\beta J \phi_2'[t] + \sin[\phi_2[t]] = \frac{x B}{2} + \frac{2}{\beta} \left( \frac{\phi_1[t] - \phi_2[t]}{2\pi} - NO \right), \\
\phi_1[0] = \phi_{01}, \phi_2[0] = \phi_{02}, \{\phi_1[t], \phi_2[t]\}, \{t, 0, t_{\text{max}}\}] // \\
\text{Flatten}; \\
\text{numgraph} = \text{Plot}[\left( \frac{\phi_2[t] - \sin[\phi_1[t]] - \sin[\phi_2[t]]}{2} \right) /. \text{numsol}, \\
\{t, 0, t_{\text{max}}\}, \text{opts}, \text{DisplayFunction} \to \text{Identity}]; \\
\text{Max}1[[\phi_{01}, \phi_{02}], \{\beta J, \kappa B, \beta, NO\}, \{t_{\text{min}}, \text{tmax}\}] := \\
\text{Module}[\{\text{numsol}\}, \\
\text{numsol} = \\
\text{NDSolve}[
\{\beta J \phi_1'[t] + \sin[\phi_1[t]] = \frac{x B}{2} - \frac{2}{\beta} \left( \frac{\phi_1[t] - \phi_2[t]}{2\pi} - NO \right), \\
\beta J \phi_2'[t] + \sin[\phi_2[t]] = \frac{x B}{2} + \frac{2}{\beta} \left( \frac{\phi_1[t] - \phi_2[t]}{2\pi} - NO \right), \\
\phi_1[0] = \phi_{01}, \phi_2[0] = \phi_{02}, \{\phi_1[t], \phi_2[t]\}, \{t, 0, \text{tmax}\}] // \\
\text{Flatten}; \\
\text{maximum} = \\
\text{FindMaximum}[\left( \frac{\phi_2[t] - \sin[\phi_1[t]] - \sin[\phi_2[t]]}{2} \right) /. \text{numsol}, \\
\{t, t_{\text{min}}, \text{tmax}\}]; \\
\text{Min}1[[\phi_{01}, \phi_{02}], \{\beta J, \kappa B, \beta, NO\}, \{t_{\text{min}}, \text{tmax}\}] := \\
\text{Module}[\{\text{numsol}\}, \\
\text{numsol} = \\
\text{NDSolve}[
\{\beta J \phi_1'[t] + \sin[\phi_1[t]] = \frac{x B}{2} - \frac{2}{\beta} \left( \frac{\phi_1[t] - \phi_2[t]}{2\pi} - NO \right), \\
\beta J \phi_2'[t] + \sin[\phi_2[t]] = \frac{x B}{2} + \frac{2}{\beta} \left( \frac{\phi_1[t] - \phi_2[t]}{2\pi} - NO \right), \\
\phi_1[0] = \phi_{01}, \phi_2[0] = \phi_{02}, \{\phi_1[t], \phi_2[t]\}, \{t, 0, \text{tmax}\}] // \\
\text{Flatten}; \\
\text{maximum} = \\
\text{FindMinimum}[\left( \frac{\phi_2[t] - \sin[\phi_1[t]] - \sin[\phi_2[t]]}{2} \right) /. \text{numsol}, \\
\{t, t_{\text{min}}, \text{tmax}\}]]

\text{Sawako}[\beta_-, \beta, \kappa B] := \text{Module}[\{A1, B1, list1, avel\}, A1 = \text{Max}1[[0, 0] \\
, \{\beta J, \kappa B, \beta, \#\}, \{400 \pi, 800 \pi\}]\&/@\text{Range}[0, 1, 0.005]; \\
B1 = \text{Min}1[[0, 0], \{\beta J, \kappa B, \beta, \#\}, \{400}
\[ \pi, 800 \pi \] \&/@Range[0, 1, 0.005]; avel = (A1 + B1)/2;
list1 = Table[{0.005 (k - 1), avel[[k, 1]]}, {k, 1, 201}]

g[\kappa B_] := ListPlot[Evaluate[Sawako[10, 1.5, \kappa B]], PlotStyle -> Hue[0.7], PointSize[0.015], AxesLabel -> {"\(\Phi/\Phi_0\)", "<\eta>"}, PlotLabel -> NumberForm[\kappa B], PlotRange -> {{0, 1}, {0, 1.3}}]
f1 = Table[g[\kappa B], {\kappa B, 0.5, 3, 0.1}]

Fig. 34 The average voltage \(<\eta>\) vs \(\Phi_B/\Phi_0\), where \(\kappa_B\) is changed as a parameter. \(\beta_J = 10\). \(\beta = 1.5\).

(1) \(\kappa_B = 1.5, \beta_J = 10, \beta = 1.5\)

(2) \(\kappa_B = 1.7, \beta_J = 10, \beta = 1.5\)

(3) \(\kappa_B = 1.9, \beta_J = 10, \beta = 1.5\)

(4) \(\kappa_B = 2.2, \beta_J = 10, \beta = 1.5\)
8.4 Loop current

Here we discuss how the loop current changes with the time $\tau$, depending on the total current $\kappa_B$ and the external magnetic flux $\Phi_{\text{ext}}$. The loop current is given by

$$\kappa_s = \frac{2}{\beta} \left( \frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{\text{ext}}}{\Phi_0} \right).$$

Here we consider one typical case: $\beta_J = 10$, $\beta = 1$, and $\Phi_{\text{ext}}/\Phi_0$ being changed as a parameter. We choose the initial condition that $\phi_1(0) = \phi_2(0) = 0$. 

((Mathematica 5.2)) Program-14

Clear["Global"^*]
\begin{verbatim}
<<Graphics`Graphics`
(*Subroutine, DC SQUID Magnetic flux dependence of loop current*)
DCSQ[\{\phi_01, \phi_02\}, \{\beta_J, \kappa_B, \beta_\kappa, N0\}, \tau_{max}, opts__] :=
Module[\{numsol, numgraph\},

numsol =
NDSolve[\{\beta_J \phi_1'[\tau] + \sin[\phi_1[\tau]] = \frac{\kappa_B}{2} - \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N0 \right),
\beta_J \phi_2'[\tau] + \sin[\phi_2[\tau]] = \frac{\kappa_B}{2} + \frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N0 \right), \phi_1[0] = \phi_01,
\phi_2[0] = \phi_02\}, \{\phi_1[\tau], \phi_2[\tau]\}, \{\tau, 0, \tau_{max}\}] // Flatten;

numgraph = Plot[\frac{2}{\beta} \left( \frac{\phi_1[\tau] - \phi_2[\tau]}{2 \pi} - N0 \right) /. numsol, \{\tau, 0, \tau_{max}\},
opts, DisplayFunction -> Identity];

phlist = DCSQ[\{0, 0\}, \{10, 1, 1, #\}, 3000, PlotStyle -> Hue[1.4 (# + 5)], AxesLabel -> {\"\tau\", \"<\kappa_s>\"}, Prolog -> AbsoluteThickness[2], Background -> GrayLevel[0.5], PlotRange -> {{0, 800}, {-1, 1}}, Ticks -> {\pi Range[0, 200, 100], Range[-1, 2]}, DisplayFunction -> Identity] & /@ Range[0, 0.5, 0.1]; Show[phlist, DisplayFunction -> $DisplayFunction]

Fig.35 \kappa_s vs \tau where \Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4, and 0.5. \kappa_B is changed as a parameter. \beta = 1. \beta_J = 10. Note that in the figures the y axis should be \kappa_s, but not <\kappa_s>.

(1) \kappa_B = 1.0 and \Phi_{ext}/\Phi_0 = 0, (red), 0.1, 0.2, 0.3, 0.4, and 0.5 (purple) from the top to the bottom. \beta = 1. \beta_J = 10.

(2) \kappa_B = 1.1 and \Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4, and 0.5. \beta = 1. \beta_J = 10.
\end{verbatim}
(3) $\kappa_B = 1.4$ and $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5$. $\beta = 1. \beta_J = 10$. The loop current starts to oscillate with time only for $\Phi_{ext}/\Phi_0 = 0.4 \text{ and } 0.5$.

(4) $\kappa_B = 1.6$ and $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5$. $\beta = 1. \beta_J = 10$. The loop current starts to oscillate with time only for $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4 \text{ and } 0.5$.

(5) $\kappa_B = 2.0$ and $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5$. $\beta = 1. \beta_J = 10$. 
9 CONCLUSION

We have discussed the physics of the Josephson junction and the principle of the DC SQUID. We do not discuss the rf SQUID (consisting of only one Josephson junction) the principle of the SQUID magnetometer. The SQUID magnetometer is the most sensitive measurement device. It can measure magnetic flux on the order of one flux quantum. The magnetic properties of magnetic systems including spin glass, superspin glass, and superparamagnet are studied using the SQUID magnetometer (MPMS XT-5, Quantum Design) in our Laboratory. Finally we want to say that a book (in Japanese) written by Ohtsuka\textsuperscript{12} is very useful for our understanding of the principle of the RSI model and DC SQUID model.

REFERENCES
This book was very useful for us in writing this lecture note. However, unfortunately this book was written in Japanese.

APPENDIX

For convenience, the following programs of Mathematica 5.2 are presented.

1) Program-5 (Fig.12): Phase space $\Omega$ vs $\phi$ with vector field. $\beta_l = 0.60$. $\kappa$ is changed as a parameter.

2) Program-6 (Fig.13): $\Omega$ vs $\tau$ for $\beta_l = 0.60$. $\kappa$ is changed as a parameter.

3) Program-9 (Fig.16): average voltage vs current

4) Program-13 (Fig.34) average voltage vs magnetic flux