Lecture note on Solid State Physics
Josephson junction and DC SQUID

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Abstract
In the Senior Laboratory (Phys 427) in our Physics Department, we introduced an equipment of Mr. SQUID (superconducting quantum interference device) (Star Cryoelectronics, LLC, 25 Bisbee Court, Suite A, Santa Fe, NM 87508) in the Spring Semester, 2006. It is a DC SQUID magnetometer system incorporating a high-temperature superconductor thin film SQUID sensor chip. This equipment allows one to observe unique features of superconductivity (using liquid nitrogen cooling) such as *I*-*V* curve and *V*-Φ curves. This also allows one to learn about the operation of SQUID by following a series of experiments. This lecture note is intended as a substitute for textbook on Josephson junction, superconductivity, electronics, and related topics. We think that with this lecture note a great deal of background could be provided for undergraduate and graduate students who do not have enough knowledge on the Josephson junction. We use the Mathematica 5.2 for the simulation. All the programs we made are presented here. The Mathematica program is useful to both graduate students and undergraduate students who want to understand the principle of the RSI model in the Josephson junctions and the DC SQUID model.

Content

1 Introduction

For superconducting tunnel junctions with extremely thin insulating layers (10 – 15 Å) (weak link between the superconductors), the electron pair correlations extend through the insulating barrier. In this situation, it has been predicted by Josephson that paired electrons (Cooper pairs) can tunnel without dissipation from one superconductor to the other superconductor on the opposite side of the insulating layer [B.D. Josephson, Phys. Lett. 1, 251 (1962). The direct supercurrent of pairs, for currents less that I_c, flows with zero-voltage drop across the junction (DC Josephson effect). The width of the insulating barrier of the junction limits the maximum that can flow across the junction, but introduce no resistance in the flow. Josephson also predicted that in the case a constant finite voltage V is established across the junction, an alternating supercurrent $I_c \sin(\omega_J t+\phi_0)$ flows with frequency $\omega_J = 2eV/\hbar$ (AC Josephson effect). We solve a nonlinear differential equation for the phase ϕ using the Mathematica 5.2. The calculations are made using ND (solving the differential equations numerically under appropriate initial conditions). The use of Mathematica 5.2 is essential to our understanding the nonlinear behavior of the Josephson junction.

2 Josephson junction[1-10]

Tunneling if Cooper pairs form a superconductor through a layer of insulator into another superconductor. Such a junction is called a weak link.

(i) DC Josephson effect

A DC current flows across the junction in the absence of any electric or magnetic field.

(ii) AC Josephson effect

A DC voltage applied across the junction causes rf (radio frequency) current oscillation across the junction.

(iii) Macroscopic long range quantum interference

A DC magnetic field applied through a superconducting circuit containing two junctions causes the maximum supercurrent to show interference effects as a function of magnetic field intensity.

((Brian D. Josephson))
Josephson, Pippard's graduate student at Cambridge, attending Philip Anderson's lectures there in 1961 to 1962, became fascinated by the concept of the phase of the BCS-GL order parameter as a manifestation of the quantum theory on a macroscopic scale. Playing with the theory of Giaver tunneling, Josephson found a phase-dependent term in the current; he then worked out all the consequences in a series of papers, private letters, and a privately circulated fellowship thesis. In particular, Josephson predicted that a direct current should flow, without any applied voltage, between two superconductors separated by a thin insulating layer. This current would come as a consequence of the tunneling of electron pairs between the superconductors, and the current would be proportional to the sine of the phase difference between the superconductors. At a finite applied voltage V, an alternating supercurrent of frequency 2eV/h should flow between the superconductors. Josephson's work established the phase as a fundamental variable in superconductivity.(Book edited by Hoddeson et al.[12]).

2.1 DC Josephson effect[3]

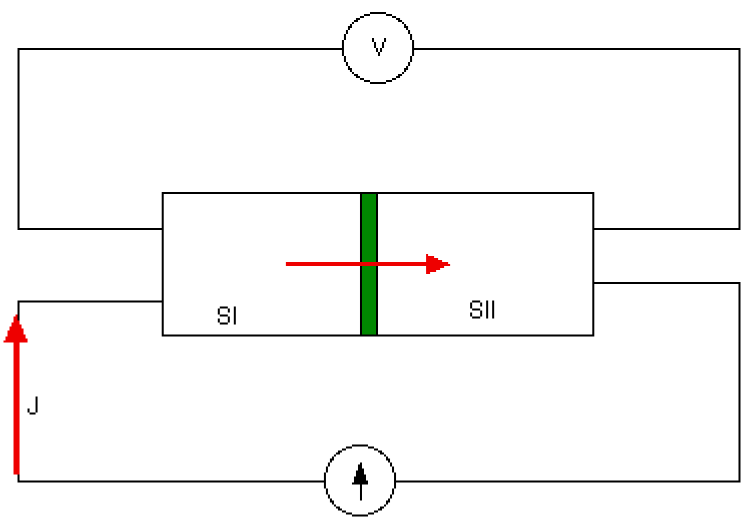


Fig.1 Schematic diagram for experiment of DC Josephson effect. Two superconductors SI and SII (the same metals) are separated by a very thin insulating layer (denoted by green). A DC Josphson supercurrent (up to a maximum value I_c) flows without dissipation through the insulating layer.

Let ψ_1 be the probability amplitude of electron pairs on one side of a junction. Let ψ_2 be the probability amplitude of electron pairs on the other side. For simplicity, let both superconductors be identical.

$$i\hbar\frac{\partial\psi_1}{\partial t} = \hbar T\psi_2,$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \hbar T\psi_1$$

where $\hbar T$ is the effect of the electron-pair coupling or (transfer interaction across the insulator). T(1/s) is the measure of the leakage of ψ_1 into the region 2, and of ψ_2 into the region 1.

Let

$$\psi_1 = \sqrt{n_1} e^{i\theta_1},$$

$$\psi_2 = \sqrt{n_2} e^{i\theta_2}$$

where

$$|\psi_1|^2 = n_1$$

$$|\psi_2|^2 = n_2$$

((Mathematica5.2)) Program-1

```
eq1=ⅈ ℏ D[ψ1[t],t]==ℏ T ψ2[t]
 ⅈ ℏ ψ1′[t] == T ℏ ψ2[t]
 eq2=ⅈ ℏ D[ψ2[t],t]==ℏ T ψ1[t]
 ⅈ ℏ ψ2′[t] == T ℏ ψ1[t]
 rule1 = {ψ1 → (√n1[#] Exp[ⅈ θ1[#]] &)}
 {ψ1 → (√n1[#1] ⅇ^(ⅈ θ1[#1]) &)}
 rule2 = {ψ2 → (√n2[#] Exp[ⅈ θ2[#]] &)}
 {ψ2 → (√n2[#1] ⅇ^(ⅈ θ2[#1]) &)}
 eq3=eq1/.rule1/.rule2//Simplify
```

$$\frac{i\, e^{i\,\theta1[t]}\, \hbar\, (n1'[t] + 2\, i\, n1[t]\, \theta1'[t])}{2\sqrt{n1[t]}} == e^{i\,\theta2[t]}\, T\, \hbar \sqrt{n2[t]}$$

```
 eq4=eq2/.rule1/.rule2//Simplify
```

$$\frac{i\, e^{i\,\theta2[t]}\, \hbar\, (n2'[t] + 2\, i\, n2[t]\, \theta2'[t])}{2\sqrt{n2[t]}} == e^{i\,\theta1[t]}\, T\, \hbar \sqrt{n1[t]}$$

Then we have

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} = -iT\sqrt{n_1 n_2} e^{i\delta} \qquad (3)$$

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + i n_2 \frac{\partial \theta_2}{\partial t} = -iT\sqrt{n_1 n_2} e^{-i\delta} \qquad (4)$$

where

$$\delta = \theta_2 - \theta_1 .$$

Now equate the real and imaginary parts of Eqs.(3) and (4),

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1 n_2}\sin\delta$$

$$\frac{\partial n_2}{\partial t} = -2T\sqrt{n_1 n_2}\sin\delta$$

$$\frac{\partial \theta_1}{\partial t} = -T\sqrt{\frac{n_2}{n_1}}\cos\delta$$

$$\frac{\partial \theta_2}{\partial t} = -T\sqrt{\frac{n_1}{n_2}}\cos\delta$$

If $n_1 \approx n_2$ as for identical superconductors 1 and 2, we have

$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \qquad \text{or} \qquad \frac{\partial(\theta_2 - \theta_1)}{\partial t} = 0\,.$$

$$\frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t}$$

The current flow from the superconductor S_1 and to the superconductor S_2 is proportional to $\frac{\partial n_2}{\partial t}$. J is the current of superconductor pairs across the junction

$$J = J_0 \sin(\theta_2 - \theta_1)\,,$$

where J_0 is proportional to T (transfer interaction).

$$I = I_0 \sin\phi\,. \tag{5}$$

2.2 AC Josephson effect[3]

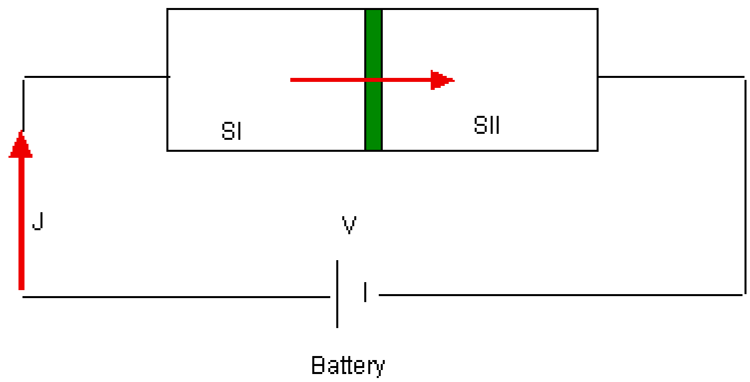


Fig.2 Schematic diagram for experiment of AC Josephson effect. A finite DC voltage is applied across both the ends.

Let a dc voltage V be applied across the junction. An electron pair experiences a potential energy difference qV on passing across the junction (q = -2e). We can say that a pair on one side is at $-eV$ and a pair on the other side is at eV.

$$i\hbar\frac{\partial\psi_1}{\partial t} = \hbar T\psi_2 - eV\psi_1$$
,
$$i\hbar\frac{\partial\psi_2}{\partial t} = \hbar T\psi_1 + eV\psi_2$$

or

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + in_1\frac{\partial\theta_1}{\partial t} = \frac{ieVn_1}{\hbar} - iT\sqrt{n_1 n_2}e^{i\delta}, \qquad (6)$$

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + in_2\frac{\partial\theta_2}{\partial t} = -\frac{ieVn_2}{\hbar} - iT\sqrt{n_1 n_2}e^{-i\delta}. \qquad (7)$$

This equation breaks up into the real part and imaginary part,

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1 n_2}\sin\delta$$

$$\frac{\partial n_2}{\partial t} = -2T\sqrt{n_1 n_2}\sin\delta$$

$$\frac{\partial\theta_1}{\partial t} = \frac{eV}{\hbar} - T\sqrt{\frac{n_2}{n_1}}\cos\delta$$
.
$$\frac{\partial\theta_2}{\partial t} = -\frac{eV}{\hbar} - T\sqrt{\frac{n_1}{n_2}}\cos\delta$$

From these two equations with $n_1 = n_2$,

$$\frac{\partial(\theta_2-\theta_1)}{\partial t} = \frac{\partial\delta}{\partial t} = -\frac{2eV}{\hbar}.$$

$$J = J_0\sin[\delta(t)].$$

with

$$\delta(t) = \delta(0) - \frac{2e}{\hbar}\int V dt.$$

When $V = V_0$ = constant, we have

$$\delta(t) = \delta(0) - \frac{2e}{\hbar}V_0 t.$$

$$J = J_0\sin[\delta(0) - \frac{2e}{\hbar}V_0 t]. \qquad (8)$$

The current oscillates with frequency

$$\omega_0 = \frac{2e}{\hbar}V. \qquad (9)$$

A DC voltage of 1 μV produces a frequency of 483.5935 MHz.

$$\dot{\phi} = \frac{2e}{\hbar}V. \qquad (10)$$

((Note))

Suppose that $V = V_0 = 1\ \mu V$. The corresponding frequency ν_0 is estimated from the relation,

$$\frac{2eV_0}{\hbar} = 2\pi\nu_0 ,$$

or

$$\nu_0 = \frac{2eV_0}{2\pi\hbar} = \frac{2\times(1.60219\times10^{-12})\times10^{-6}}{2\pi\times1.05459\times10^{-27}} = 483.5935 MHz .$$

3 *I-V* characteristic of Josephson junction[5,7,8]

We now consider the *I-V* characteristic of the Josephson tunneling junction where a insulating layer is sandwiched between two superconducting layers (the same type). A capacitor is formed by these two superconductors. In this type of Josephson junctions, one can see the quasiparticle *I-V* curve which is different with increasing voltage and decreasing voltage (hysteresis). There are two voltage states, 0 V and 2Δ/e, where Δ is an energy gap of each superconductor. The *I-V* curve is characterized by (i) maximum Josephson tunneling current of Cooper pairs at $V = 0$ and (ii) Quasi-particle tunneling current ($V>2\Delta/e$).

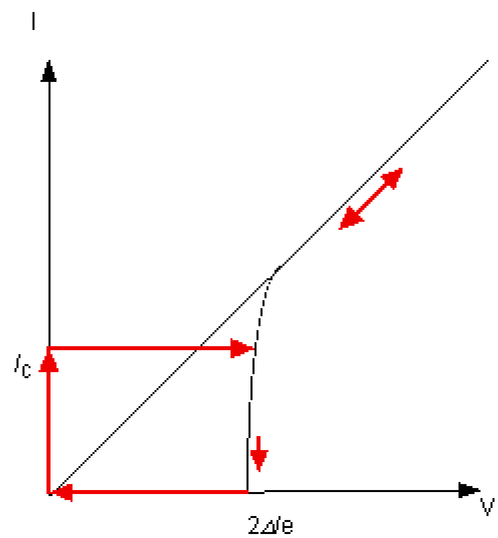


Fig.3 Schematic diagram of quasiparticle *I-V* characteristic (usually observed in a S-I-S Josephson *tunneling-type*). Josephson current (up to a maximum value I_c) flowsat $V = 0$. Δ is an energy gap of the superconductor . The DC Josepson supercurrent flows under $V = 0$. For $V>2\Delta$/e the quasiparticle tunneling current is seen.

The strong nonlinearity in the quasiparticle *I-V* curve of a tunneling junction is not an appropriate to the application to the SQUID element. This nonlineraity can be removed by the use of thin normal film deposited across the electrodes. In this effective resistance is a parallel combination of the junction. The *I-V* characteristic has no hysteresis. Such behavior is often observed in the bridge-type Josephson junction where two superconducting thin films are bridged by a very narrow superconducting thin film.

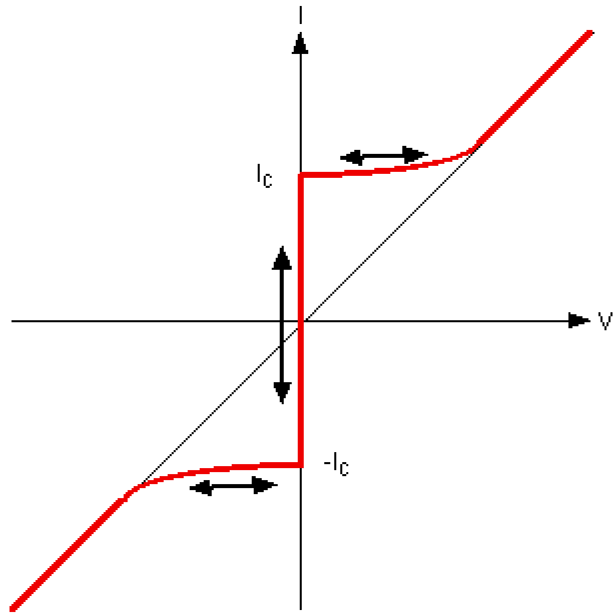


Fig.4 Schematic diagram of *I*-*V* characteristic of a Josephson junction (usually observed in *bridge-type junction*), which is reversible on increasing and decreasing V. A Josephson supercurrent flows up to I_c at $V = 0$. A transition occurs from the $V = 0$ state to a finite voltage state for $I>I_c$.. Above this voltage the *I*-*V* characteristic exhibits an Ohm's law with a finite resistance of the Junction. The current has a oscillatory component of angular frequency ω (= $2eV/\hbar$) (the AC Josephson effect).

4. RSJ (Resistively shunted junction) model: Josephson junction circuit application[7]

Here we discuss the *I*-*V* characteristics of a Josephson tunneling junction using an equivalent circuit shown below. This circuit includes the effect of various dissipative processes and the distributed capacity with so-called lumped circuit parameters (connection of R and C in parallel).

4.1 Fundamental equation

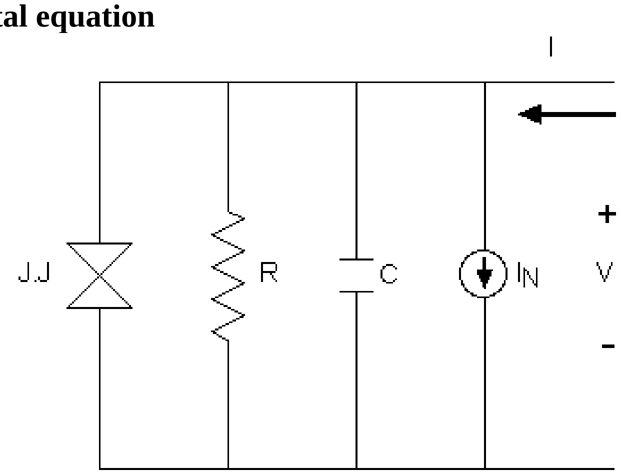


Fig.5 Equivalent circuit of a real Josephson junction with a current noise source. (RSJ model). J.J. stands for the Josephson junction.

We now consider an equivalent circuit for the Josephson junction, which is described above. *J.J.* stands for the Josephson junction.

$$I_c \sin\phi + \frac{V}{R} + C\dot{V} + I_N(t) = I\,, \tag{11}$$

where the first term is a Josephson current, the second term is an ohmic current, the third term is a displacement current, and $I_N(t)$ is the noise current source. Since

$$\dot{\phi} = \frac{2e}{\hbar}V\,,$$

we get a second-order differential equation for the phase ϕ

$$\frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} = I - I_c \sin\phi - I_N(t) = -\frac{2e}{\hbar}\frac{\partial U(\phi)}{\partial \phi} - I_N(t)\,, \tag{12}$$

where $U(\phi)$ is an equivalent potential and is defined by

$$U(\phi) = -\frac{\Phi_0}{2\pi c}(I\phi + I_c \cos\phi) = -\frac{\Phi_0}{2\pi c}I_c(\frac{I}{I_c}\phi + \cos\phi) = -E_J(\kappa\phi + \cos\phi)\,,$$

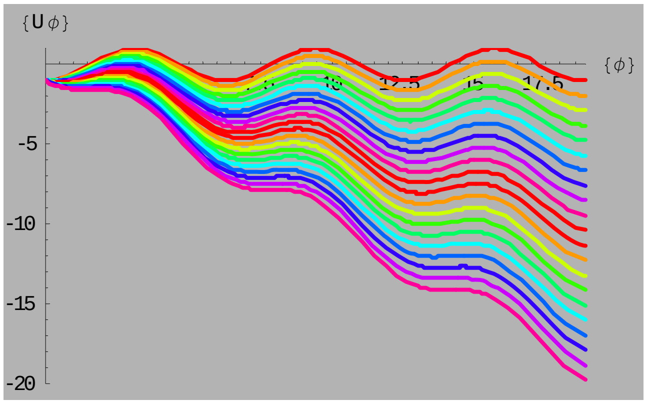
with $\kappa = I/I_c$, where Φ_0 ($=2\pi\hbar c/2e = 2.06783372\text{x}10^{-7}$ Gauss cm^2) is a magnetic quantum flux and $E_J = \Phi_0 I_c/(2\pi c)$ is the Josephson coupling constant. If ϕ is regarded as the coordinate x, the above equation corresponds to the equation of motion of a mass with $(\hbar/2e)^2 C$ in the presence of the potential $U(\phi)$. The second term of the left-hand side is a friction proportional to the velocity. When $I_N(t) = 0$ and the second term is neglected, the above equation can be rewritten as

$$\frac{1}{2}\left(\frac{\hbar}{2e}\right)^2 C\dot{\phi}^2 + U(\phi) = E = const\,, \tag{13}$$

where E is the total energy and is constant. For $\kappa>1$ (or $I>I_c$), $U(\phi)$ monotonically decreases with increasing ϕ: $d\phi/dt\neq 0$ (finite voltage-state). For $\kappa<1$ (or $I<I_c$), $U(\phi)$ has local minima. There are two solutions: $d\phi/dt = 0$ (no voltage state) or $d\phi/dt\neq 0$ (finite voltage-state) for large C. In this case, the *I-V* characteristic has a hysteresis.

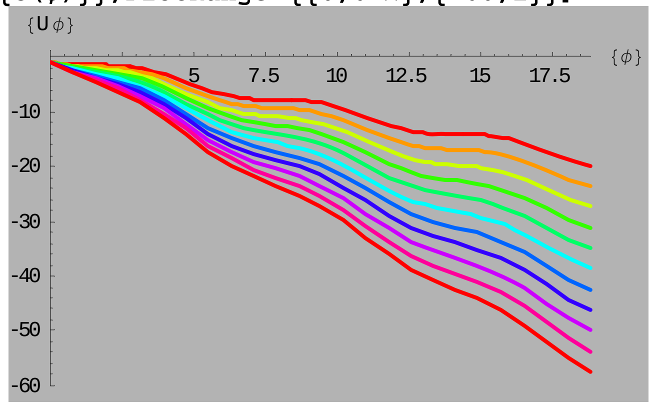
((Mathematica 5.2)) Program-2

```
U1=-( κ ϕ +
Cos[ϕ]);Plot[Evaluate[Table[U1,{κ,0,1,0.05}]],{ϕ,0,6 π},
PlotStyle→Table[Hue[0.1
i],{i,0,10}],Prolog→AbsoluteThickness[2],Background→GrayLev
el[0.7], AxesLabel→{{ϕ},{U(ϕ)}},PlotRange→{{0,6 π},{-20,1}}]
```



(-Graphics-)

```
  Plot[Evaluate[Table[U1,{κ,1,3,0.2}]],{ϕ,0,6 π},
PlotStyle→Table[Hue[0.1
i],{i,0,10}],Prolog→AbsoluteThickness[2],Background→GrayLevel[0.7],
AxesLabel→{{ϕ},{U(ϕ)}},PlotRange→{{0,6 π},{-60,1}}]
```



(-Graphics-)

Fig.6 Equivalent potential energy $U(\phi)$ with a parameter $\kappa = I/I_c$. (a) $0<\kappa<1$. $U(\phi)$ has local maxima and local minima. (b) $1<\kappa<3$. $U(\phi)$ increases with increasing ϕ.

Analogy: Simple rigid pendulum

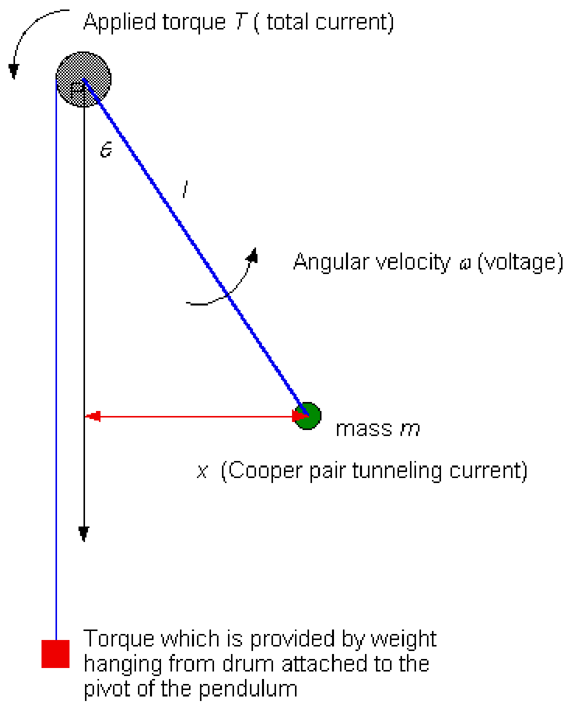


Fig.7 Simple pendulum with an applied torque.

We consider a simple rigid pendulum light stiff rod of length l with a bob of mass m.

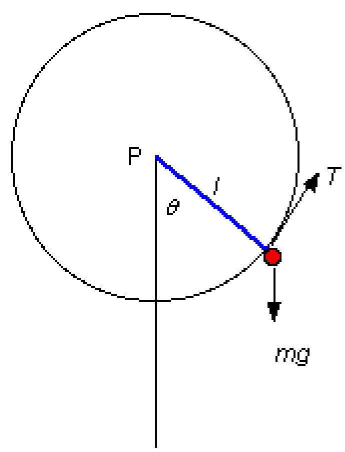


Fig.8 Free body diagram of the simple pendulum

The oendulum can rotate freely about the pivot P. The equation of motion is given by

$$I\ddot{\theta} = T - mgl\sin\theta - \eta\dot{\theta},$$

where T is an external torque, I is a moment of inertia around the pivot P and the third term of the right-hand side is a viscosity of air. There is an analogy between this pendulum and the Josephson junction,

$$T = I\ddot{\theta} + +\eta\dot{\theta} + mgl\sin\theta \text{ (pendulum).} \tag{14}$$

$$I = \frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c\sin\phi \text{ (Josepson).} \tag{15}$$

Table 1

Josephson junction	pendulum
phase difference ϕ	deflection θ
total current across junction I	applied torque T
Capacitance C	Moment of inertia
normal tunneling conductance $1/R$	viscous damping η
Josephson current $I_c \sin\phi$	horizontal displacement of bob $x = l\sin\theta$
voltage across junction V	angular velocity $\omega = \dot{\theta}$

(1) $T = mgl\sin\theta$, $\omega = \dfrac{d\theta}{dt} = 0$.

When a small torque is applied, the pendulum finally settles down at a constant angle of deflection θ. No angular velocity (ω = 0) corresponds to no voltage across a junction (V = 0). The junction is superconducting. $x = l \sin\theta$.

(2)

If the torque is gradually increased, the pendulum deflects to a greater but steady angle. We can pass more current through a junction without any voltage appearing.

(3)

Critical torque ($T_c = mgl$). This is the torque which deflects the pendulum through a right angle so that it is horizontal.

(4)

For $T>T_c$ the pendulum cannot remain at rest but rotates continuously. As the pendulum rotates, the horizontal deflection x oscillates. The angular velocity is always in the same direction. This corresponds to the case of $I>I_c$ and $V\neq 0$. A DC voltage will appear across the junction if the current passed through it exceeds a critical value.

On the half-cycle during which the bob is rising the rotation decelerates because gravity opposes the applied torque, but on the following half-cycle the bob is accelerated by gravity as it falls.

$$\nu = \frac{1}{2\pi}\left\langle \frac{d\theta}{dt} \right\rangle = \frac{2e}{2\pi\hbar} V_{dc} .$$

The phase is in one-to one correspondence with an angle of rotation of a damped pendulum, driven by a constant torque, in a constant gravitational field. The regime $I<I_c$ corresponds to a situation in which the applied torque is less than a critical torque necessary to raise the pendulum to an angle $\pi/2$. For $I>I_c$ the pendulum rotates in a manner such that the average energy dissipated per rotation is equal to the average work per rotation.

4.2 Differential equation for ϕ

For the sake of simplicity, we use the dimensionless quantities. Here we assume that

$$\eta = \frac{V}{I_c R}, \qquad \kappa = \frac{I}{I_c}, \qquad \tau = \omega_J t,$$

where ω_J is the Josephson plasma frequency and is defined by

$$\omega_J = \left(\frac{2eI_c}{\hbar C}\right)^{1/2},$$

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{d\phi}{d\tau}\omega_J .$$

Similarly we have

$$\frac{d^2\phi}{dt^2} = \omega_J^{\ 2}\frac{d^2\phi}{d\tau^2} .$$

Then we get

$$\frac{\hbar C}{2eI_c}\omega_J^{\ 2}\frac{d^2\phi}{d\tau^2} + \frac{\hbar}{2eRI_c}\omega_J\frac{d\phi}{d\tau} + \sin\phi = \kappa - \kappa_N(\tau),$$

or

$$\frac{d^2\phi}{d\tau^2} + \beta_J\frac{d\phi}{d\tau} + \sin\phi = \kappa - \kappa_N(\tau), \tag{16}$$

where

$$\beta_J = \frac{\hbar\omega_J}{2eRI_c} = \frac{\hbar\omega_J^{\ 2}}{2eRI_c\omega_J} = \frac{\hbar}{2eRI_c\omega_J}\frac{2eI_c}{\hbar C} = \frac{1}{\omega_J RC}.$$

The normalized voltage is described by

$$\eta(\tau) = \frac{V}{I_c R} = \frac{\hbar}{2e}\frac{1}{I_c R}\frac{d\phi}{dt} = \frac{\hbar\omega_J}{2eRI_c}\frac{d\phi}{d\tau} = \beta_J\frac{d\phi}{d\tau},$$

since

$$\dot{\phi} = \frac{2e}{\hbar}V .$$

We are interested in the DC current-voltage characteristic so we need to determine the time averaged voltage

$$\langle V\rangle = \left\langle\frac{\hbar}{2e}\frac{d\phi}{dt}\right\rangle = \left\langle I_c R\beta_J\frac{d\phi}{d\tau}\right\rangle = I_c R\beta_J\left\langle\frac{d\phi}{d\tau}\right\rangle,$$

or

$$<\eta> = \frac{\langle V\rangle}{I_c R} = \beta_J\left\langle\frac{d\phi}{d\tau}\right\rangle. \tag{17}$$

Note that the McCumber parametrer β_c is sometimes used in stead of β_J, where $\beta_c = 1/\beta_J^2$.

4.3 *I-V* characteristic for β_J » 1

For the special case β_J»1 (small capacitance limit), the above equation is reduced to

$$\beta_J\frac{d\phi}{d\tau} + \sin\phi = \kappa,$$

$$\langle V\rangle = \left\langle I_c R\beta_J\frac{d\phi}{d\tau}\right\rangle = I_0 R\beta_J\frac{1}{T}\int_0^T\frac{d\phi}{d\tau}d\tau = 2\pi I_0 R\frac{\beta_J}{T},$$

where T is a period;

$$T = \frac{2\pi}{\sqrt{\kappa^2 - 1}}\beta_J .$$

Here

$$T = \int_0^T d\tau = \int_0^{2\pi} \frac{d\phi}{\frac{d\phi}{d\tau}} = \beta_J \int_0^{2\pi} \frac{d\phi}{\kappa - \sin\phi},$$

or

$$\frac{T}{\beta_J} = \int_0^{2\pi} \frac{d\phi}{\kappa - \sin\phi} = \int_0^{\pi} \frac{d\phi}{\kappa - \sin\phi} + \int_0^{\pi} \frac{d\phi}{\kappa + \sin\phi} = \int_0^{\pi} [\frac{1}{\kappa - \sin\phi} + \frac{1}{\kappa + \sin\phi}] d\phi,$$

or

$$\frac{T}{\beta_J} = \frac{2\pi}{\sqrt{\kappa^2 - 1}} \quad \text{for } \kappa > 1,$$

$$\frac{T}{\beta_J} = 0 \qquad \text{for } \kappa < 1,$$

So we get

$$\langle V \rangle = 2\pi I_c R \frac{\beta_J}{T} = I_c R \sqrt{\kappa^2 - 1},$$

or

$$\frac{\langle V \rangle}{I_c R} = \sqrt{\kappa^2 - 1}, \tag{18}$$

or

$$\kappa = \sqrt{1 + \left(\frac{\langle V \rangle}{I_c R}\right)^2}. \tag{19}$$

((Mathematica 5.2)) Program-3

```
K1[x_] := ∫_0^π (1/(x - Sin[ϕ]) + 1/(x + Sin[ϕ])) dϕ; Y1 = K1[x]; Y2 = Simplify[Y1, x > 1]
```

$$\frac{2\pi}{\sqrt{-1 + x^2}}$$

```
eq1 = V1 == (2 π Ic R)/Y2 ; eq2 = Solve[eq1, x]
```

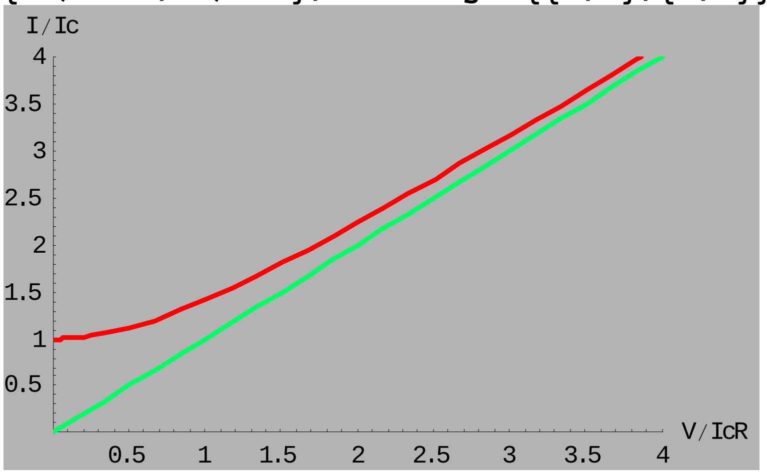
$$\left\{\left\{x \to -\sqrt{1 + \frac{V1^2}{Ic^2 R^2}}\right\}, \left\{x \to \sqrt{1 + \frac{V1^2}{Ic^2 R^2}}\right\}\right\}$$

```
I1 = Sqrt[1 + V1^2/(Ic^2 R^2)] /. {R → 1, Ic → 1}
```

$$\sqrt{1 + V1^2}$$

```
Plot[{I1,V1},{V1,0,4},PlotStyle→{Hue[0],Hue[0.4]},Prolog→Ab
```

```
soluteThickness[2],Background→GrayLevel[0.7],
AxesLabel→{"V/IcR","I/Ic"},PlotRange→{{0,4},{0,4}}]
```

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-Graphics-
```

Fig.9 I/I_c vs $V/(I_cR)$ curve for β_J>>1 (red curve). The green curve shows the Ohm's law: $I/I_c = V/(I_cR)$

5 Phase plane analysis[4]

We now consider a equation

$$\frac{d^2\phi}{d\tau^2} + \beta_J \frac{d\phi}{d\tau} + \sin\phi = \kappa\,, \tag{20}$$

or

$$\frac{d\Omega}{d\tau} = \kappa - \beta_J\Omega - \sin\phi\,, \tag{21}$$

where

$$\Omega(\tau) = \frac{d\phi}{d\tau}\,, \tag{22}$$

is proportional to voltage [$\eta = \beta_J \frac{d\phi}{d\tau}$]. Then we have

$$\frac{d^2\phi}{d\tau^2} = \frac{d\Omega(\tau)}{d\tau} = \frac{d\Omega}{d\phi}\frac{d\phi}{d\tau} = \frac{d\Omega}{d\phi}\Omega = \frac{d}{d\phi}\frac{\Omega^2}{2}\,.$$

The above equation can be rewritten as

$$\frac{d}{d\phi}\frac{\Omega^2}{2} + \beta_J\Omega + \sin\phi = \kappa\,. \tag{23}$$

The state of the system is represented at any time by a particular point in the (Ω,ϕ) plane. As the time τ varies, this point describes a trajectory. Each particular trajectory depends on the initial conditions. Thus for a fixed value of (Ω,ϕ) plane, the system is represented by a set of possible paths in the (Ω,ϕ) plane. Such a plot is often called a phase space diagram.

We begin by discussing the orbits when $\kappa = 0$ and $\beta_J = 0$.

$$\frac{d}{d\phi}\frac{\Omega^2}{2} + \sin\phi = 0 .$$

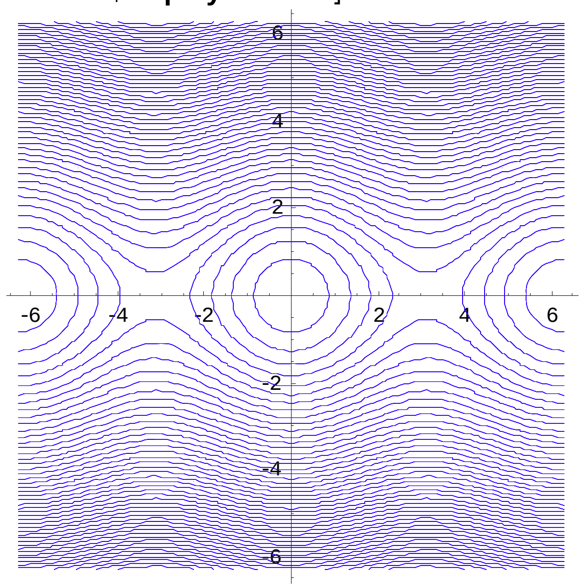
This equation can be integrated as

$$\frac{\Omega^2}{2} - \cos\phi = a = const . \tag{24}$$

Open orbits require that a always be larger than 2.

((Mathematica 5.2)) Program-4

```
<< Graphics`ImplicitPlot`; eq1 = 1/2 Ω^2 - Cos[ϕ];
pt1 =
 ImplicitPlot[eq1 == #, {ϕ, -2 π, 2 π}, {Ω, - 2 π, 2 π}, PlotPoints → 100,
    Contours → 50, PlotStyle → {Hue[0.7], Thickness[0.006]},
    DisplayFunction → Identity, PlotRange → All] & /@ Range[0.1, 1.2, 0.1];
Show[pt1, DisplayFunction → $DisplayFunction]
```




```
-Graphics-
```

Fig.10 The Ω vs ϕ plane trajectories. $\Omega^2/2 - \cos\phi = a$ where a is changed as a parameter. The closed orbits for a<1 ($\Omega^2(\phi = 0)/2 < 2$) and the open orbits for a>1 ($\Omega^2(\phi = 0)/2 > 2$).

6 Numerical calculation

In the plane of the parameters β_J and κ the situation can be summarized as follows.[6,7]

(a) For κ>1 and arbitrary β_J value no equilibrium point exists; there is only a periodic solution of the second kind. Therefore the junction will be in the finite voltage state.

(b) For κ<1, the situation is more complicated. The behavior depends on the particular value of β_J.

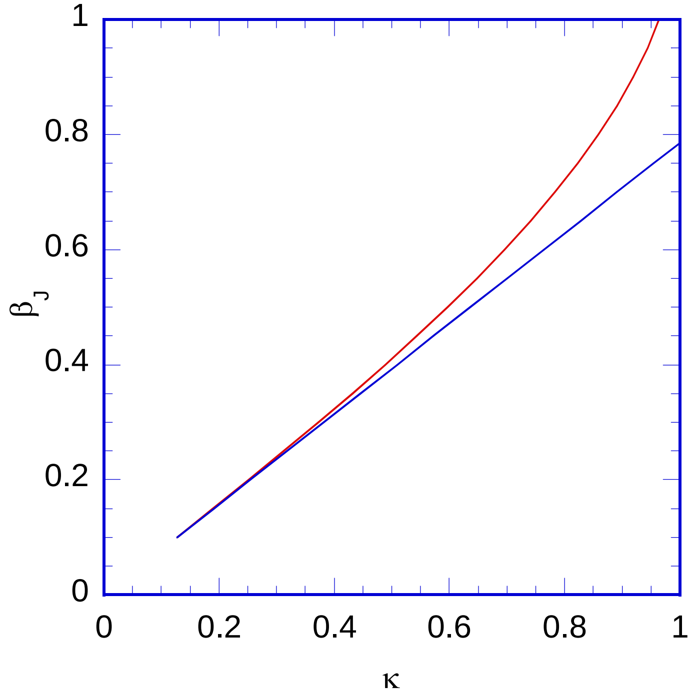


Fig.11 Critical line for $k_c(\beta_J)$ (denoted by red line) in the β_J vs κ plane. The blue line denotes the expression given by $\kappa_c(\beta_J) = \frac{4}{\pi}\beta_J$. The system undergoes stable oscillations when $\kappa > \kappa_c(\beta_J)$ for fixed β_J, in addition to the zero-voltage state.

For $\beta_J < 0.2$, the simple relation holds: $\kappa_c(\beta_J) = \frac{4}{\pi}\beta_J$.

A curve can be identified, denoted by $\kappa_c(\beta_J)$, which divides the plane into two regions corresponding to one or two stable state solutions, respectively.

We now solve the differential equation by using Mathematica 5.2

$$\Omega(\tau) = \frac{d\phi}{d\tau},$$

and

$$\frac{d\Omega(\tau)}{d\tau} + \beta_J \Omega(\tau) + \sin\phi(\tau) = \kappa .$$

Initial condition at $\tau = 0$ (or $t = 0$):

$$\Omega(\tau = 0) = v_0 \text{ and } \phi(\tau = 0) = \phi_0 .$$

We calculate the τ dependence of $\Omega(\tau)$ and $\phi(\tau)$ for $0 \le \tau \le \tau_{max}$ by using Mathematica 5.2 [NDSolve], where β_J and κ are changed as parameters.

(i) Curve of $\Omega(\tau)$ vs $\phi(\tau)$

(ii) The direction of the curves of $\Omega(\tau)$ vs $\phi(\tau)$, when τ increases [field vector]

(iii) The τ dependence of $\Omega(\tau)$ and $\phi(\tau)$.

(iv) The determination of the maximum and minimum values of $\Omega(\tau)$ in the long τ-region where $\Omega(\tau)$ is a well-defined oscillatory function of τ.

6.1 Simulation-1

((Mathematica 5.2)) Program-5

Phase space Ω vs ϕ with vector field

β_J = 0.6 is fixed. The current ratio κ is changed around the critical value κ_c = 0.6965. We show the phase diagram of the voltage $\phi(\tau)$ vs $\Omega(\tau)$ for various initial conditions. $\phi(0)$ = 0. $\Omega(0)$ = -10 – 10. We also show the vector field.

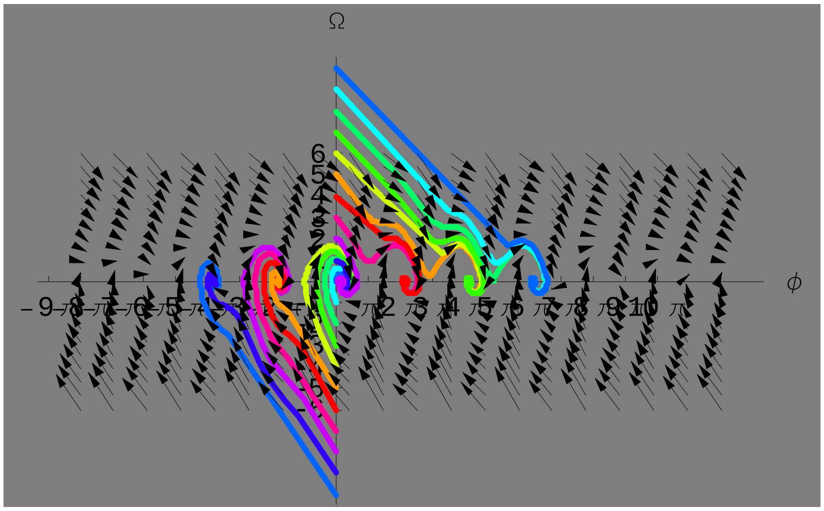
```
Clear["Global`*"]
 <<Graphics`Graphics`
 <<Graphics`PlotField`
 (*Subroutine, ParametricPlot in the phase space*)

phase[{ϕ0_,v0_},{βJ_,κ_},τmax_,opts__]:=Module[{numsol,numg
raph},numsol=NDSolve[{ Ω'[τ]+ βJ
Ω[τ]+Sin[ϕ[τ]]==κ,ϕ'[τ]==Ω[τ],ϕ[0]==ϕ0,Ω[0]==v0},{Ω[τ],ϕ[τ]},{
τ,0,τmax}]//Flatten;numgraph=ParametricPlot[{ϕ[τ],Ω[τ]}/.nu
msol,{τ,0,τmax},opts,
DisplayFunction→Identity]];field[{βJ_,κ_},{xmin_,xmax_},{ym
in_,ymax_},opts__]:=PlotVectorField[{y,-βJ y-
Sin[x]+κ},{x,xmin,xmax},{y,ymin,ymax},opts];

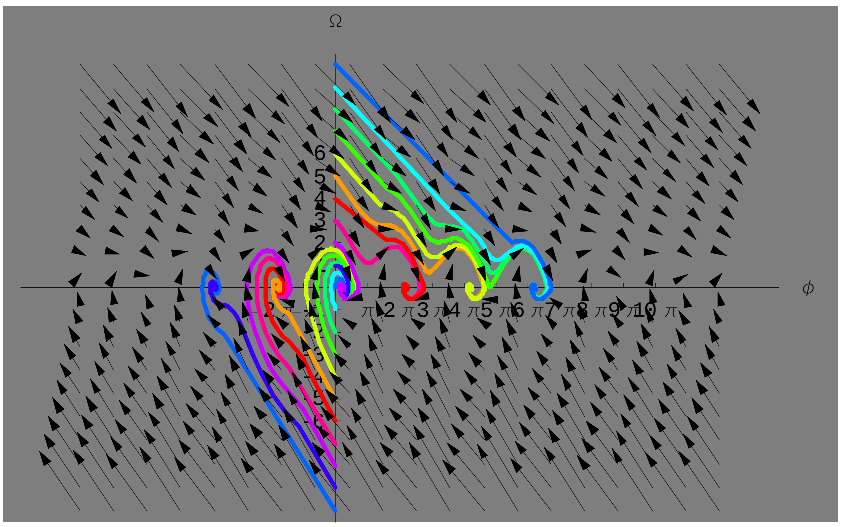
 phlist=phase[{0,#},{0.6,0.5},100,  PlotStyle→Hue[0.1
(#+6)], AxesLabel→{"ϕ","Ω"},Prolog→AbsoluteThickness[2],
Background→GrayLevel[0.5],PlotRange→All,Ticks→{  π Range[-
10,10], Range[-6,6]}, DisplayFunction→Identity]&/@Range[-
10,10,1];f1=field[{0.6,0.5},{-8 π,12 π},{-
6,6},PlotPoints→20,ScaleFunction→(0.4#&),ScaleFactor→None,D
isplayFunction→Identity];Show[phlist,f1,DisplayFunction→$Di
splayFunction]
```

Fig.12 The phase-plane trajectories in the Ω vs ϕ. β_J = 0.6. κ is changes as a parameter, κ =0.5 – 0.7.

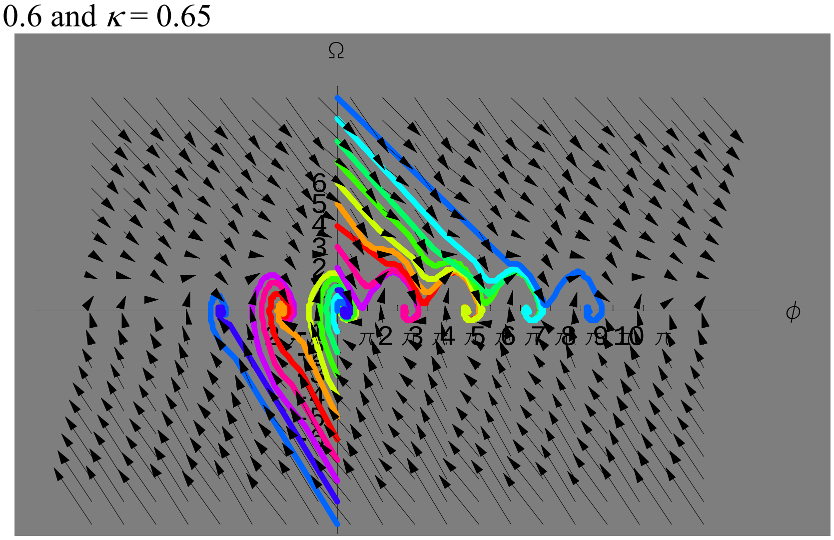
(1) β_J = 0.6 and κ = 0.5



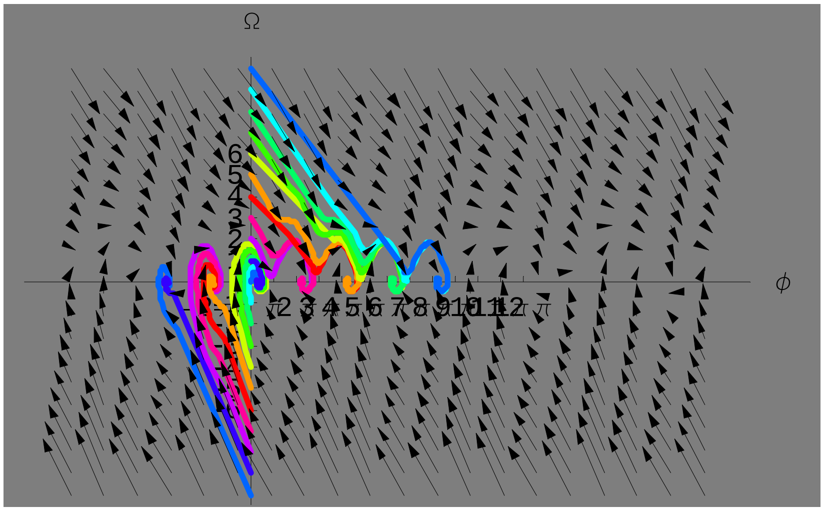
(2) $\beta_J = 0.6$ and $\kappa = 0.6$



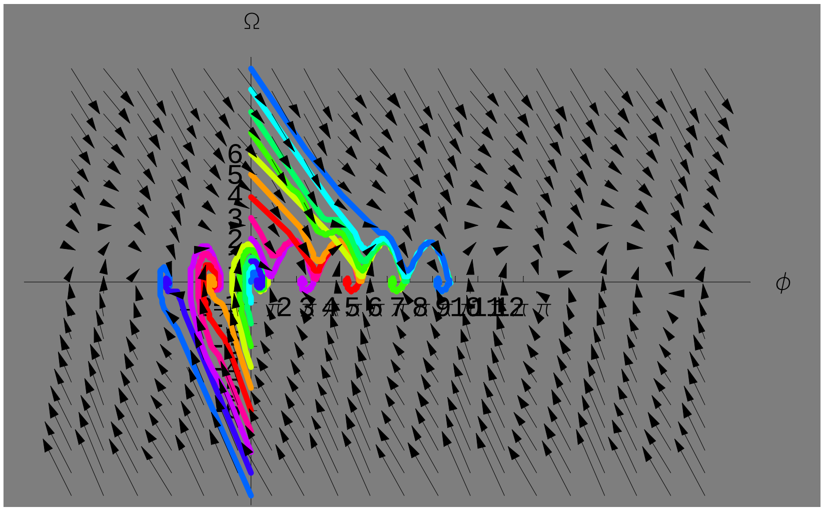
(3) $\beta_J = 0.6$ and $\kappa = 0.65$



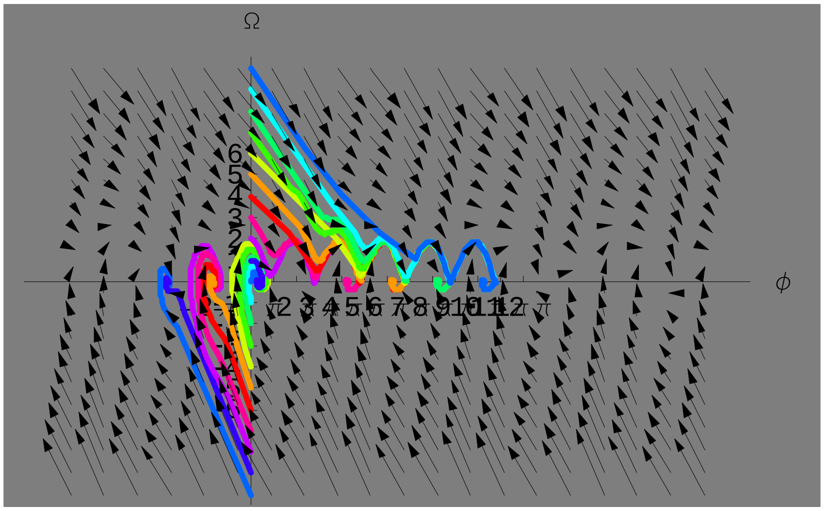
(4) $\beta_J = 0.6$ and $\kappa = 0.67$



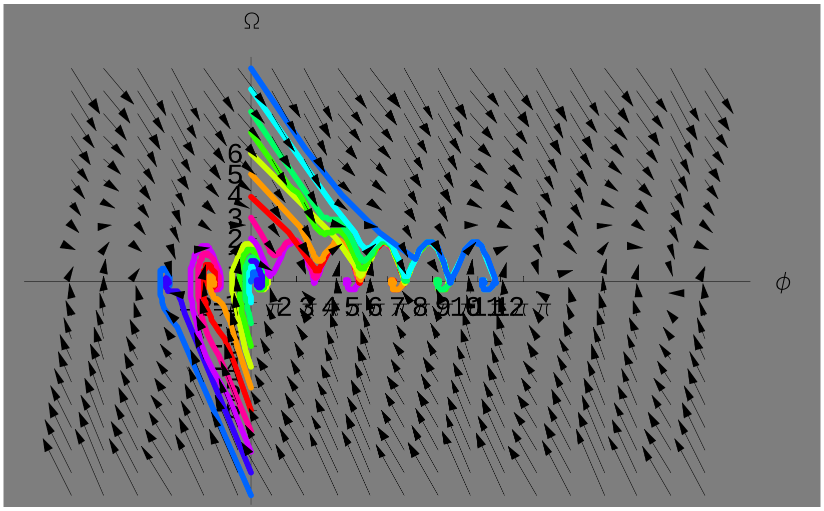
(5) $\beta_J = 0.6$ and $\kappa = 0.69$



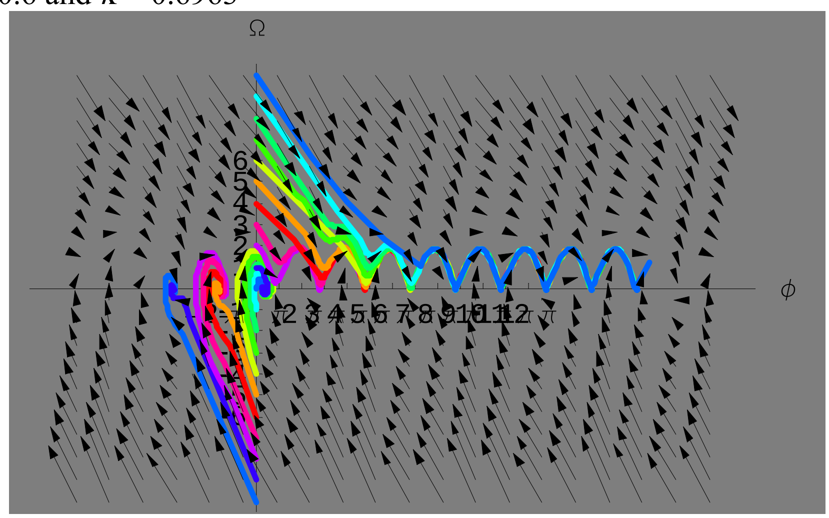
(6) $\beta_J = 0.6$ and $\kappa = 0.696$



(7) $\beta_J = 0.6$ and $\kappa = 0.6962$

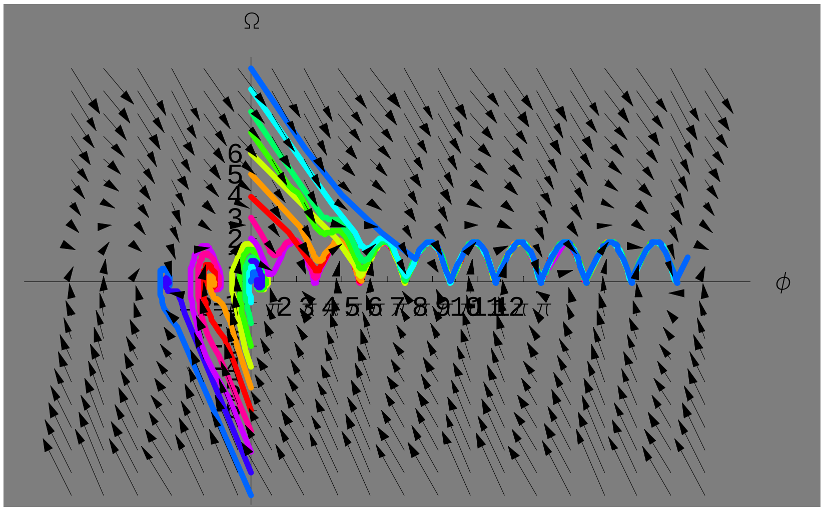


(8) $\beta_J = 0.6$ and $\kappa = 0.6963$

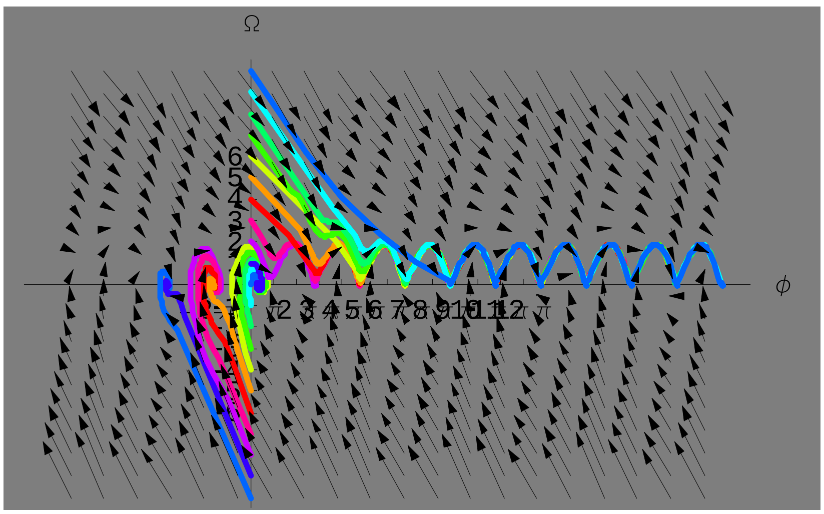


(9) $\beta_J = 0.6$ and $\kappa = 0.6965$

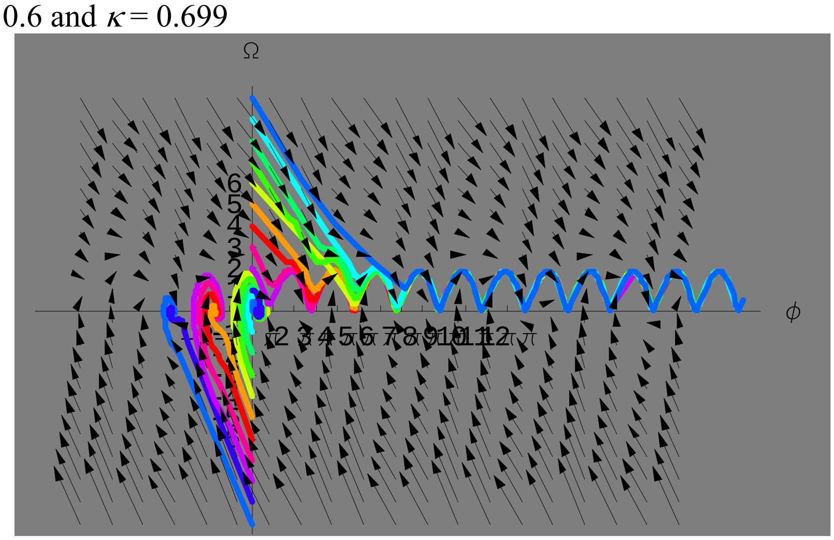
A stable periodic solution appears. The states of zero and finite voltage are both possible.



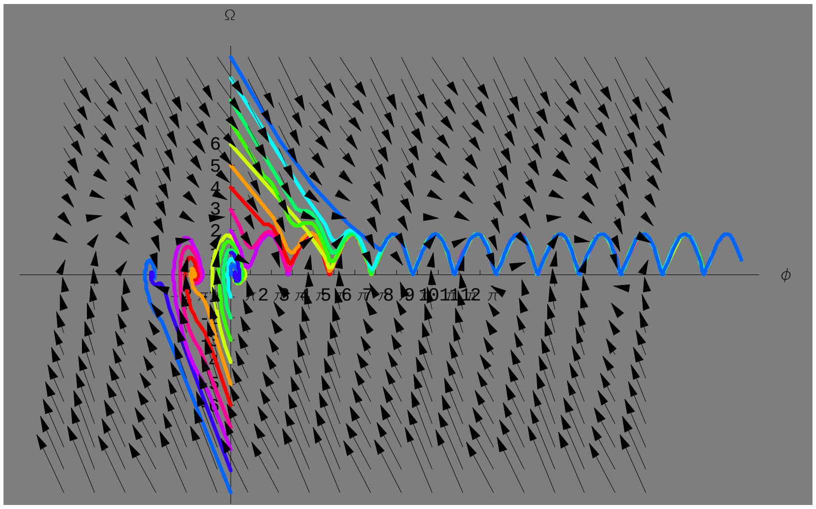
(10) $\beta_J = 0.6$ and $\kappa = 0.697$



(11) $\beta_J = 0.6$ and $\kappa = 0.699$



(12) $\beta_J = 0.6$ and $\kappa = 0.7$



6.2 Simulation-2

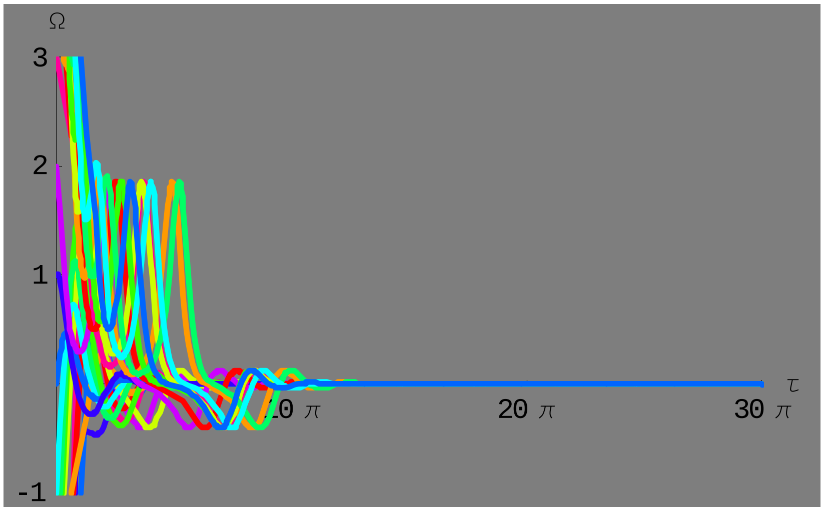
((Mathematica 5.2)) Program-6

$\beta_J = 0.6$ is fixed. The current ratio κ is changed around the critical value $\kappa_c = 0.6965$. We show the τ dependence of the voltage $\Omega(\tau)$ for various initial conditions. Note that the normalized DC voltage $\langle V\rangle / I_c R$ is defined by $<\eta>= \beta_J\langle\Omega\rangle$. For $\kappa>\kappa_c(\beta_J)$, $\Omega(\tau)$ is a sum of time-independent term $\langle\Omega)\rangle$ and a periodically oscillating function of τ. The average voltage corresponds to $<\eta>= \beta_J\langle\Omega\rangle$, where $\langle\Omega\rangle$ is the average of the maximum and minimum values of $\Omega(\tau)$ in the long τ- region, where $\Omega(\tau)$ is a well-defined periodically oscillating function of τ. The method to find the maximum and minimum values of $\Omega(\tau)$ will be shown in Sec.5.5 for convenience.

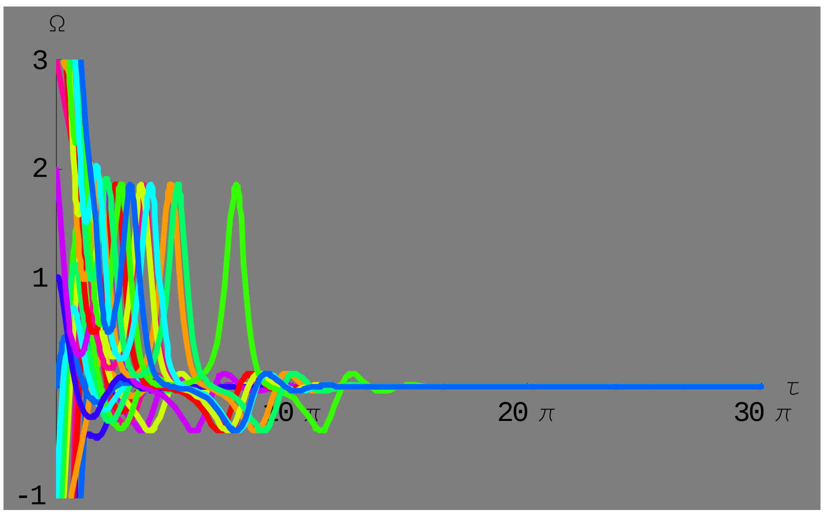
```
phlist=phase[{0,#},{0.6,0.693},100,  PlotStyle→Hue[0.1
(#+6)], AxesLabel→{"τ","Ω"},Prolog→AbsoluteThickness[2],
Background→GrayLevel[0.5],PlotRange→{{0,30 π},{-
1,3}},Ticks→{  π Range[0,50,10], Range[-6,6]},
DisplayFunction→Identity]&/@Range[-
10,10,1];Show[phlist,DisplayFunction→$DisplayFunction]
```

Fig.13 Ω vs τ for $\beta_J = 0.6$. The parameter κ is varied as a parameter, $\kappa = 0.693$ - .1.20.

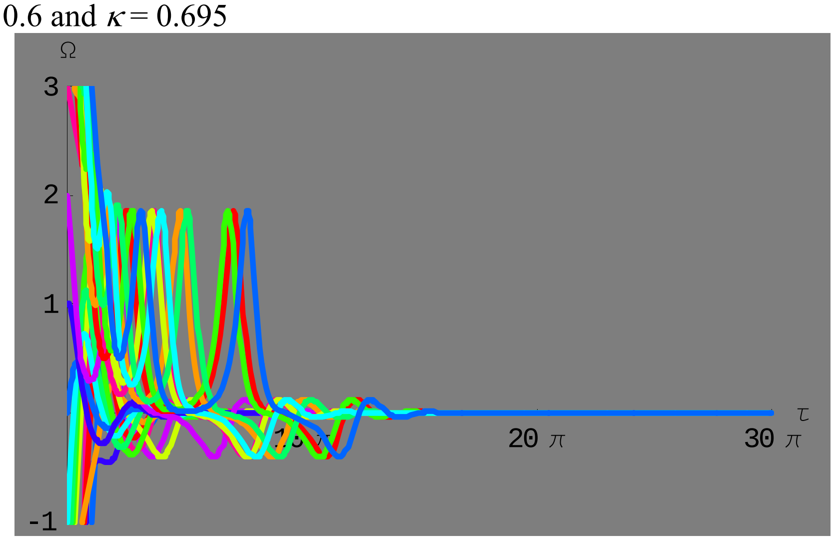
(1) $\beta_J = 0.6$ and $\kappa = 0.693$



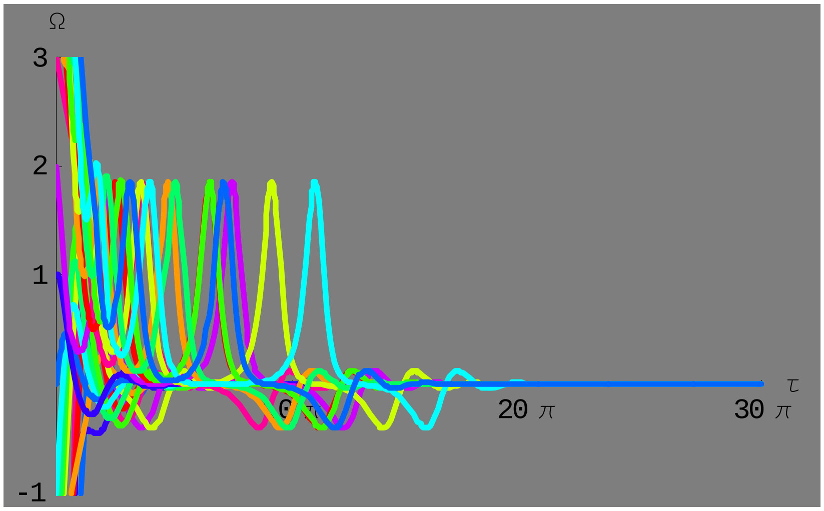
(2) $\beta_J = 0.6$ and $\kappa = 0.694$



(3) $\beta_J = 0.6$ and $\kappa = 0.695$

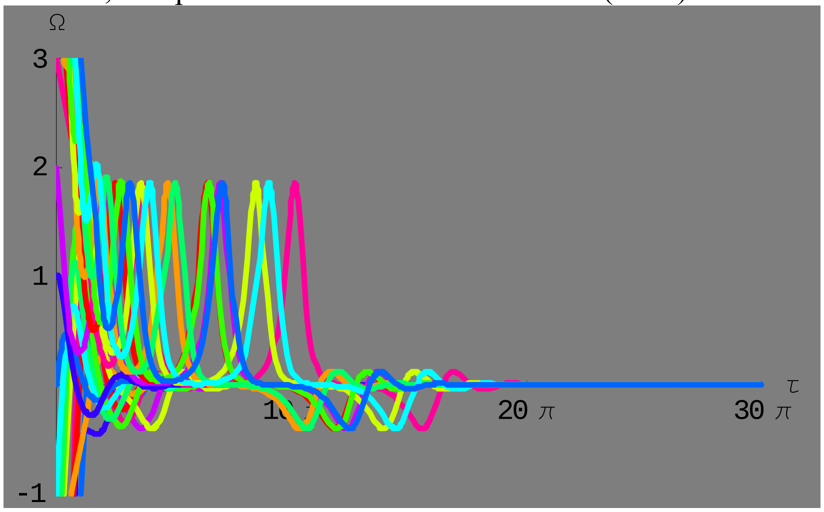


(4) $\beta_J = 0.6$ and $\kappa = 0.6960$



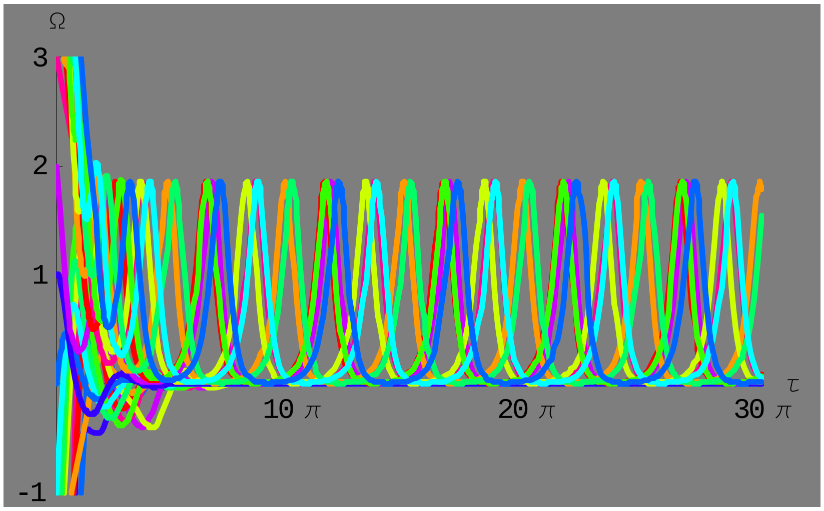
(5) $\beta_J = 0.6$ and $\kappa = 0.6962$

The average voltage η (= $\beta_J \langle \Omega(\tau) \rangle = V/RI_c$) is equal to 0, where $\beta_J = 0.6$ and κ (= I/I_c) = 0.6962, independent of the initial condition $\Omega(\tau = 0)$.

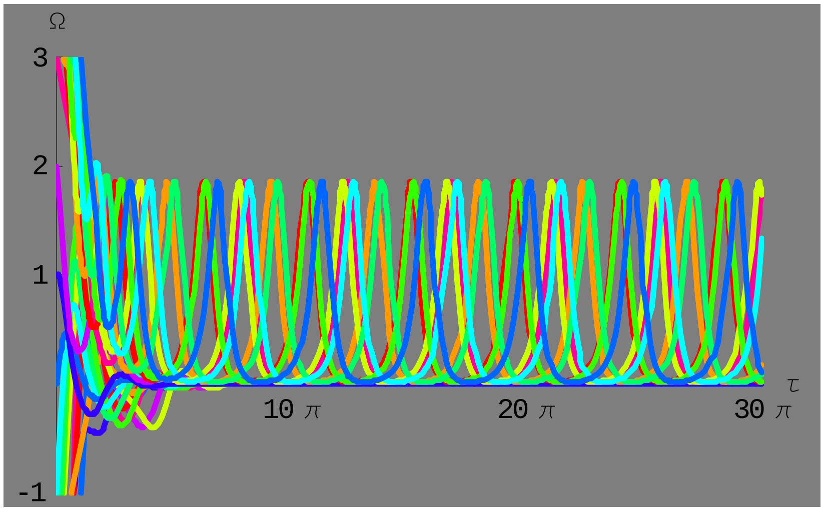


(6) $\beta_J = 0.6$ and $\kappa = 0.6965$

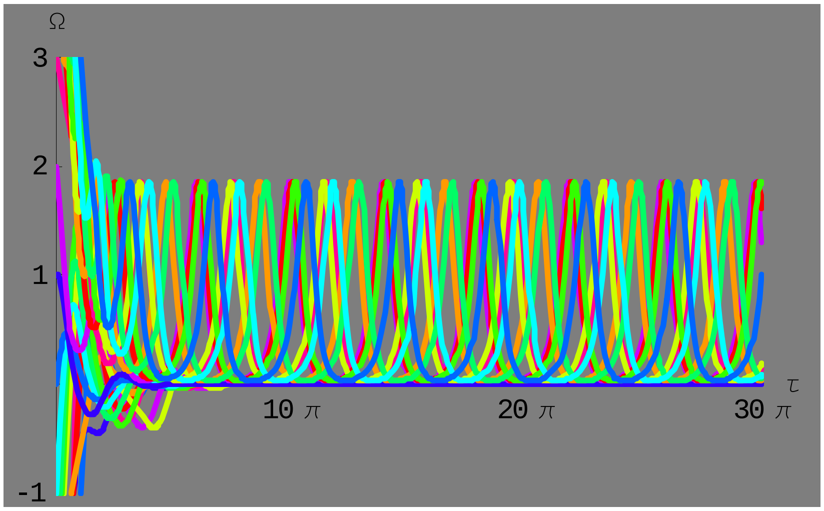
The average voltage η (= $\beta_J \langle \Omega(\tau) \rangle = V/RI_c$) is nearly equal to 0.6x0.95 = 0.57 where $\beta_J = 0.6$ and κ (= I/I_c) = 0.6965).



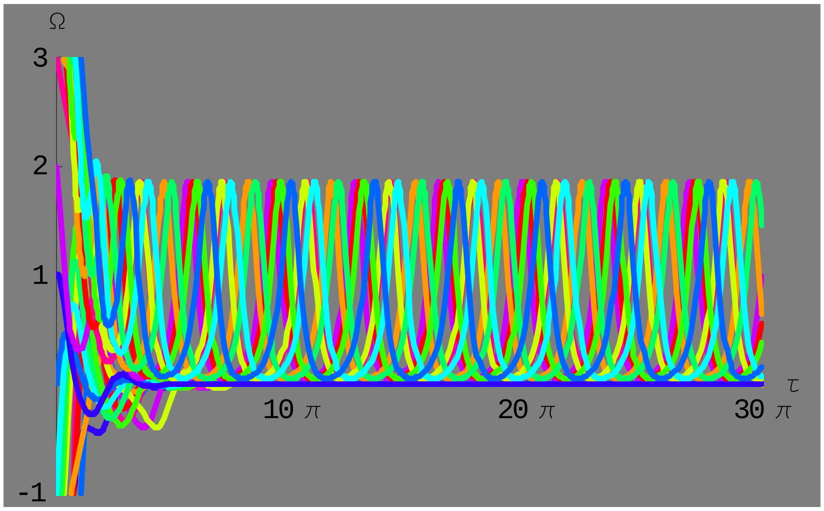
(7) $\beta_J = 0.6$ and $\kappa = 0.6970$



(8) $\beta_J = 0.6$ and $\kappa = 0.698$

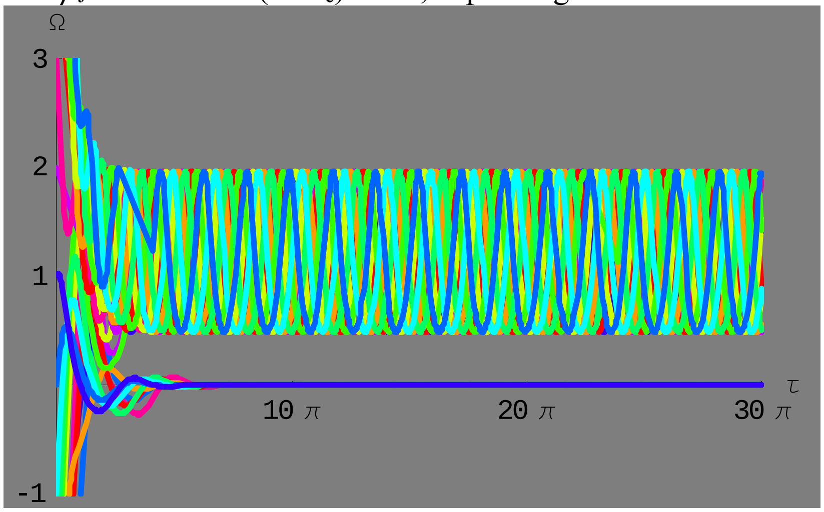


(9) $\beta_J = 0.6$ and $\kappa = 0.70$



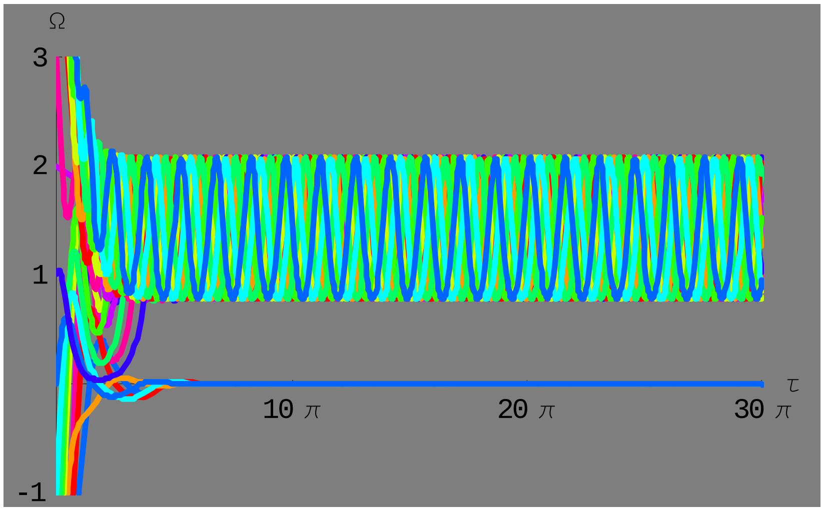
(10) $\beta_J = 0.6$ and $\kappa = 0.80$

The average voltage η $(= \beta_J \langle \Omega(\tau) \rangle)$ is equal to 0.6x1.22656 = 0.7359 and 0.6x0 =0 where $\beta_J = 0.6$ and κ $(= I/I_c) = 0.8$, depending on the initial condition $\Omega(\tau = 0)$.



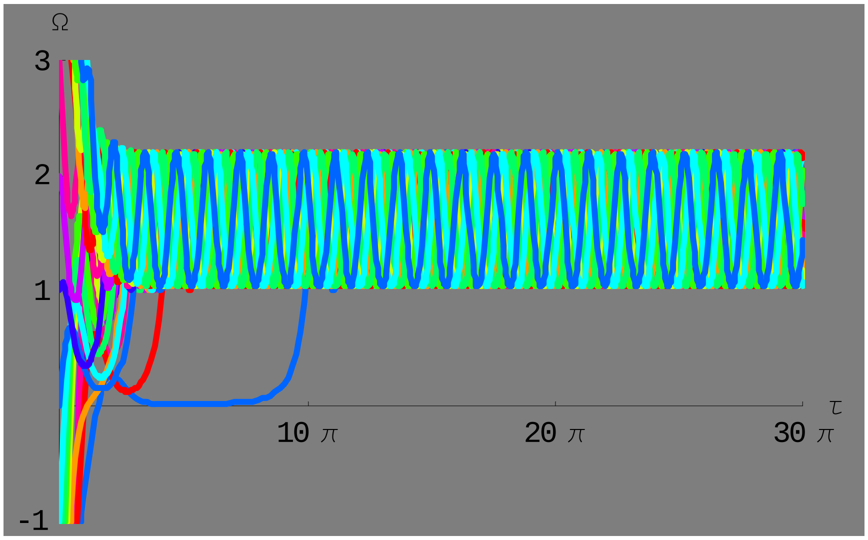
(11) $\beta_J = 0.6$ and $\kappa = 0.90$

The average voltage η $(= \beta_J \langle \Omega(\tau) \rangle)$ is nearly equal to 0.6x1.429 = 0.8574 and 0.6x0 =0 where $\beta_J = 0.6$ and κ $(= I/I_c) = 0.9$, depending on the initial condition $\Omega(\tau = 0)$. This implies the existence of the hysteresis behavior. The *I*-*V* curve with increasing *V* is different from that with decreasing *V*.



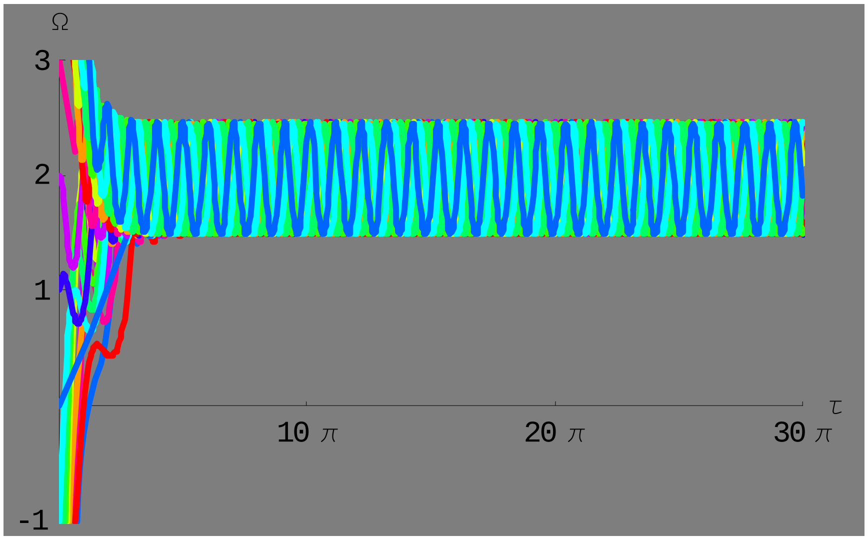
(12) $\beta_J = 0.6$ and $\kappa = 1.0$

The average voltage η ($= \beta_J \langle \Omega(\tau) \rangle$) is equal to 0.6x1.61579 = 0.9695, where β_J = 0.6 and κ ($= I/I_c$) = 1.0, independent of the initial condition $\Omega(\tau = 0)$. This implies no hysteresis behavior.



(13) $\beta_J = 0.6$ and $\kappa = 1.2$

The average voltage η ($= \beta_J \langle \Omega(\tau) \rangle$) is equal to 0.6x1.97065 = 1.1824, where β_J = 0.6 and κ ($= I/I_c$) = 1.2, independent of the initial condition $\Omega(\tau = 0)$. This implies no hysteresis behavior.



6.3 Simulation-3

((Mathematica 5.2)) Preogram-7

$\beta_J = 0.2$ is fixed. The current ratio κ is changed around the critical value $\kappa_c = 0.253$. We show the τ dependence of the voltage $\Omega(\tau)$ for various initial conditions.

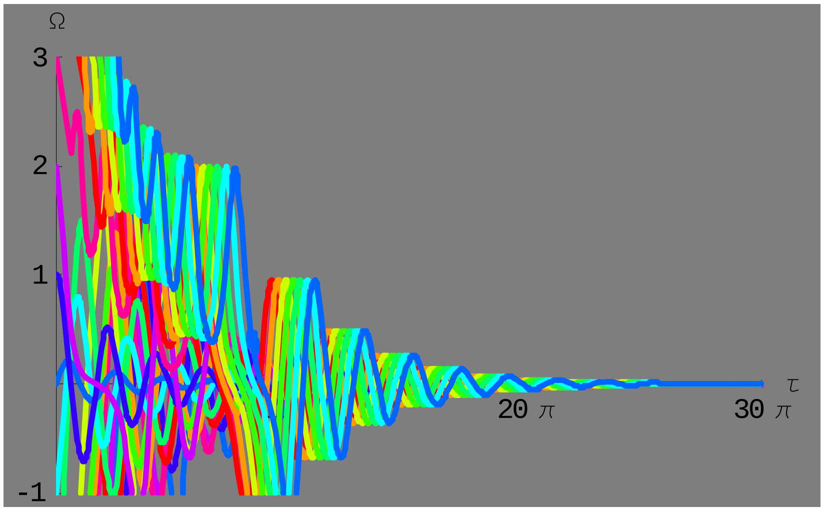
```
Clear["Global`*"]
 <<Graphics`Graphics`
 <<Graphics`PlotField`
 (*Subroutine, ParametricPlot in the phase space*)

phase[{ϕ0_,v0_},{βJ_,κ_},τmax_,opts__]:=Module[{numsol,numg
raph},numsol=NDSolve[{ Ω'[τ]+ βJ
Ω[τ]+Sin[ϕ[τ]]==κ,ϕ'[τ]==Ω[τ],ϕ[0]==ϕ0,Ω[0]==v0},{Ω[τ],ϕ[τ]},{
τ,0,τmax}]//Flatten;numgraph=Plot[Ω[τ]/.numsol,{τ,0,τmax},o
pts, DisplayFunction→Identity]]

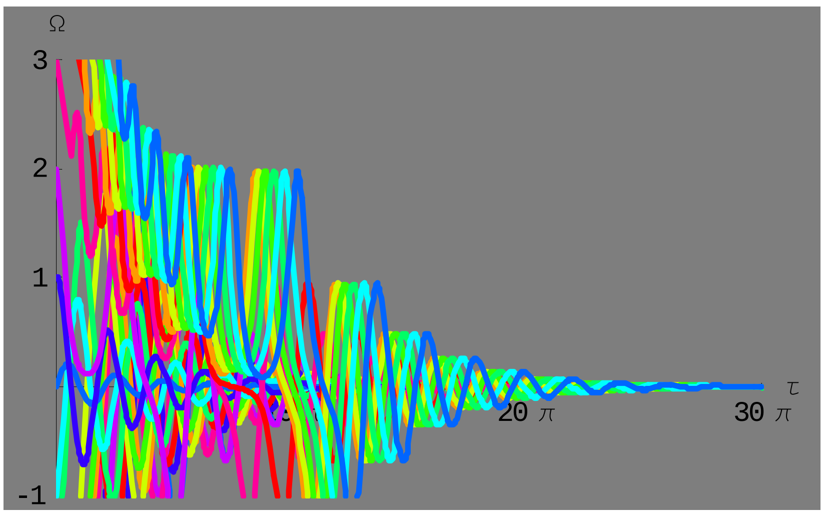
 phlist=phase[{0,#},{0.2,0.24},100,  PlotStyle→Hue[0.1
(#+6)], AxesLabel→{"τ","Ω"},Prolog→AbsoluteThickness[2],
Background→GrayLevel[0.5],PlotRange→{{0,30 π},{-
1,3}},Ticks→{  π Range[0,50,10], Range[-6,6]},
DisplayFunction→Identity]&/@Range[-
10,10,1];Show[phlist,DisplayFunction→$DisplayFunction]
```

Fig.14 Ω vs τ for $\beta_J = 0.2$. The parameter κ is varied as a parameter, $\kappa = 0.24$ - .0.50.

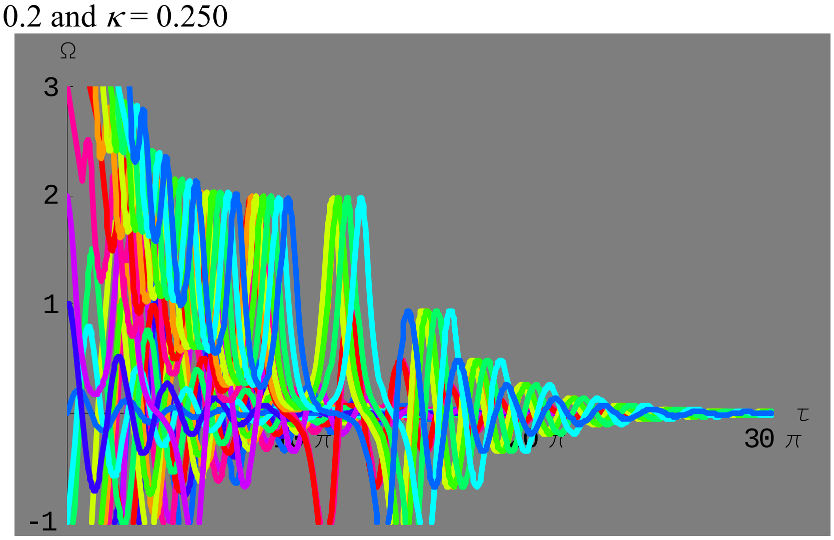
(1) $\beta_J = 0.2$ and $\kappa = 0.24$



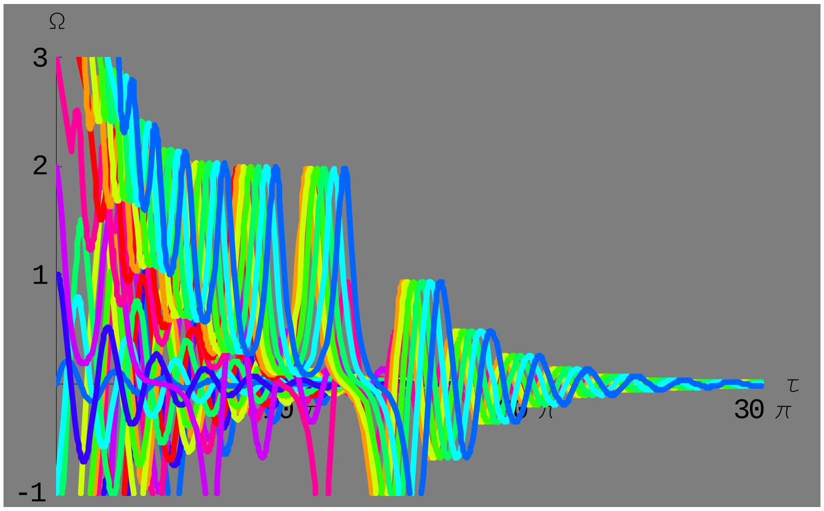
(2) $\beta_J = 0.2$ and $\kappa = 0.245$



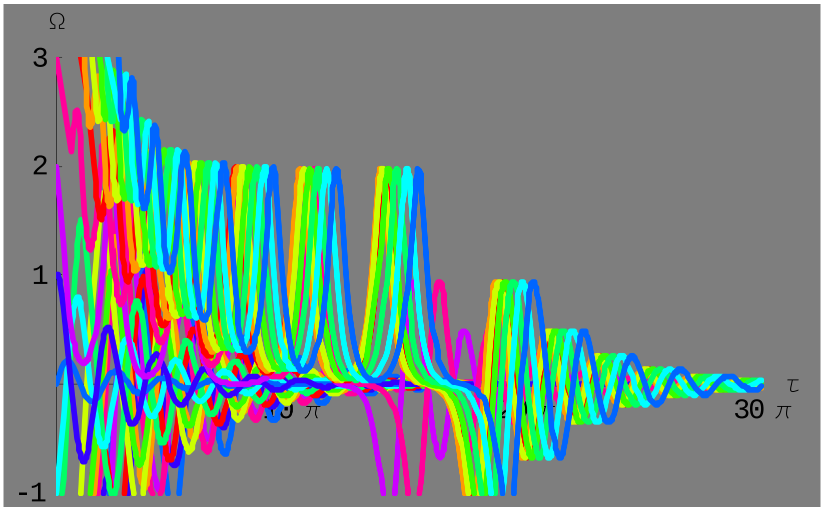
(3) $\beta_J = 0.2$ and $\kappa = 0.250$



(4) $\beta_J = 0.2$ and $\kappa = 0.251$

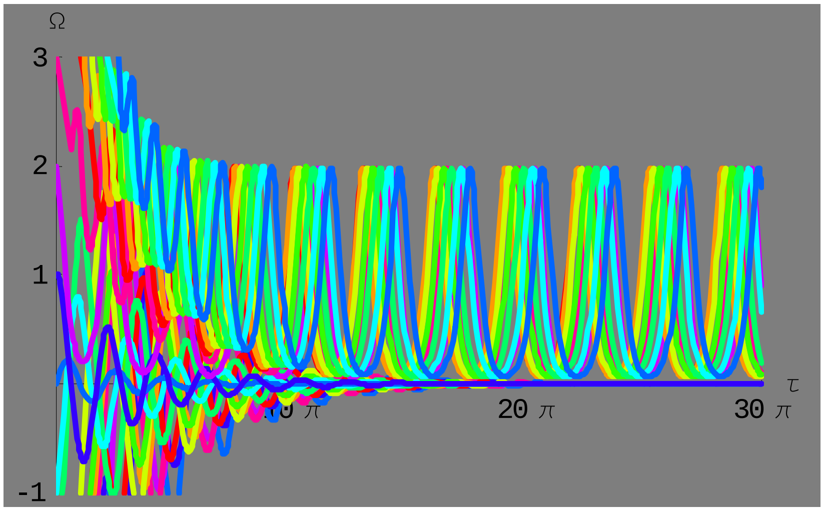


(5) $\beta_J = 0.2$ and $\kappa = 0.252$

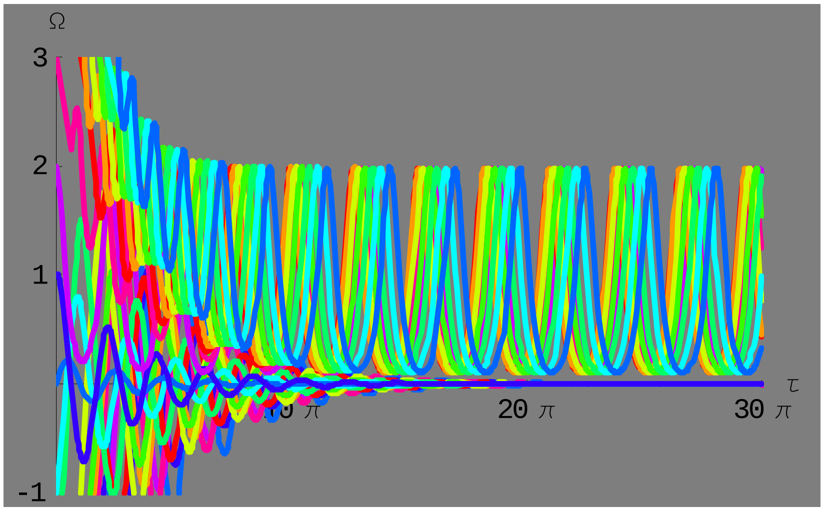


(6) $\beta_J = 0.2$ and $\kappa = 0.253$

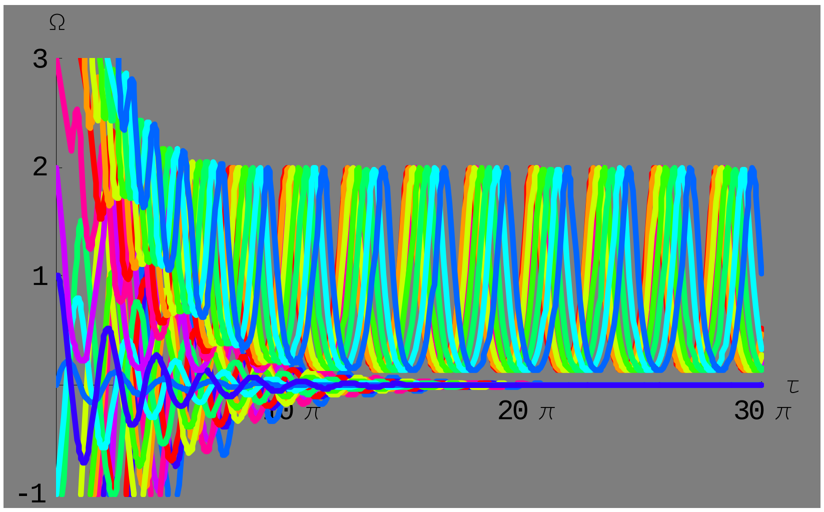
The average voltage η ($= \beta_J \langle \Omega(\tau) \rangle = V/RI_c$) is equal to 0.2x1.0269 = 0.20538 and 0.2x0 = 0, where $\beta_J = 0.2$ and κ ($= I/I_c$) = 0.253, depending on the initial condition $\Omega(\tau = 0)$.



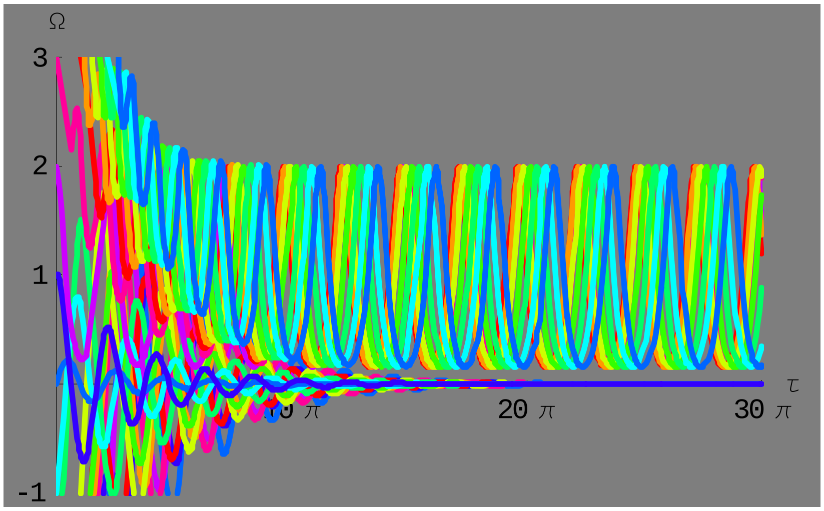
(7) $\beta_J = 0.2$ and $\kappa = 0.254$



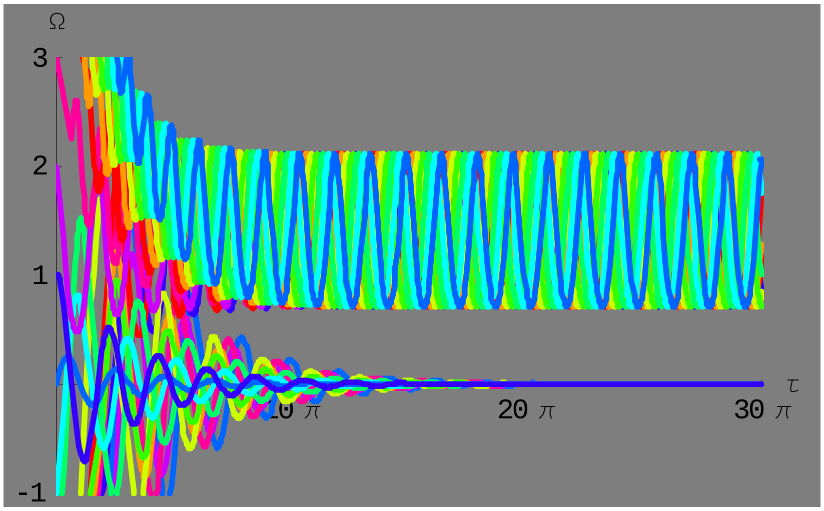
(8) $\beta_J = 0.2$ and $\kappa = 0.255$



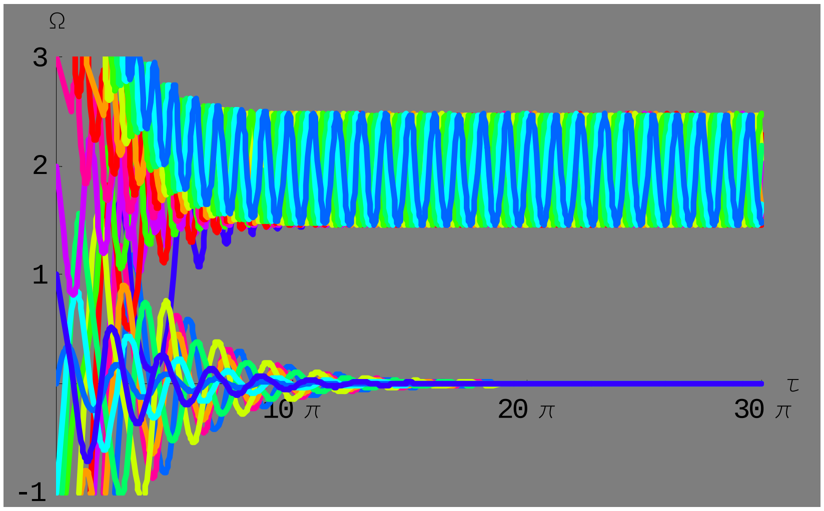
(9) $\beta_J = 0.2$ and $\kappa = 0.256$



(10) $\beta_J = 0.2$ and $\kappa = 0.30$

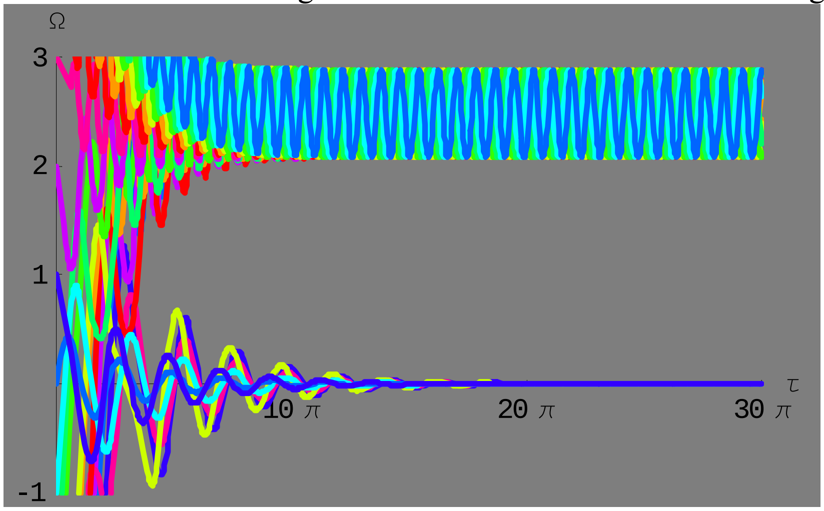


(11) $\beta_J = 0.2$ and $\kappa = 0.4$



(12) $\beta_J = 0.2$ and $\kappa = 0.5$

The average voltage η ($= \beta_J \langle \Omega(\tau) \rangle = V/RI_c$) is nearly equal to 0.2x2.48381 = 0.49676 and 0.2x0 = 0, where β_J = 0.2 and κ (= I/I_c) = 0.5, depending on the initial condition $\Omega(\tau = 0)$. This implies the existence of the hysteresis behavior. The *I-V* curve with increasing *V* is different from that with decreasing *V*.



6.4 Simulation-4

((Mathematica 5.2)) Program-8

β_J = 0.9 is fixed. The current ratio κ is changed around the critical value κ_c = 0.9197. We show the τ dependence of the voltage $\Omega(\tau)$ for various initial conditions: $\phi(0)$ = 0. $\Omega(0)$ = -10 – 10.

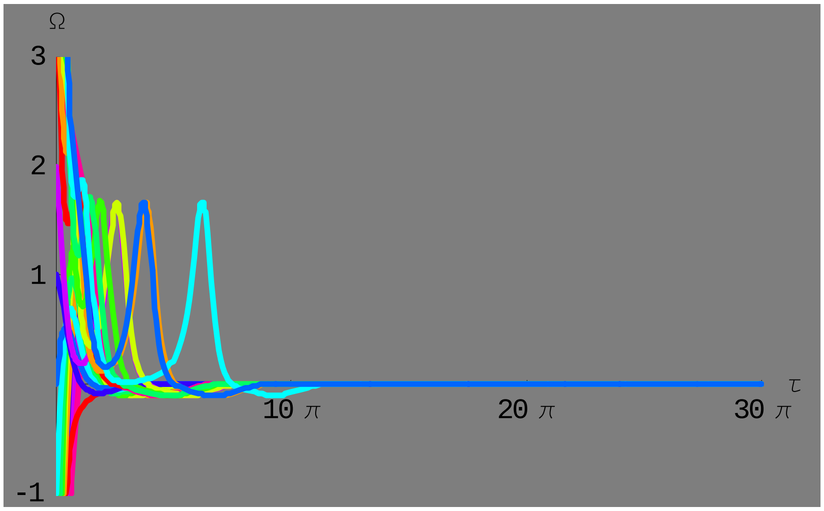
```
Clear["Global`*"]
 <<Graphics`Graphics`
 <<Graphics`PlotField`
 (*Subroutine, ParametricPlot in the phase space*)

phase[{ϕ0_,v0_},{βJ_,κ_},τmax_,opts__]:=Module[{numsol,numg
raph},numsol=NDSolve[{ Ω'[τ]+ βJ
Ω[τ]+Sin[ϕ[τ]]==κ,ϕ'[τ]==Ω[τ],ϕ[0]==ϕ0,Ω[0]==v0},{Ω[τ],ϕ[τ]},{
τ,0,τmax}]//Flatten;numgraph=Plot[Ω[τ]/.numsol,{τ,0,τmax},o
pts, DisplayFunction→Identity]]

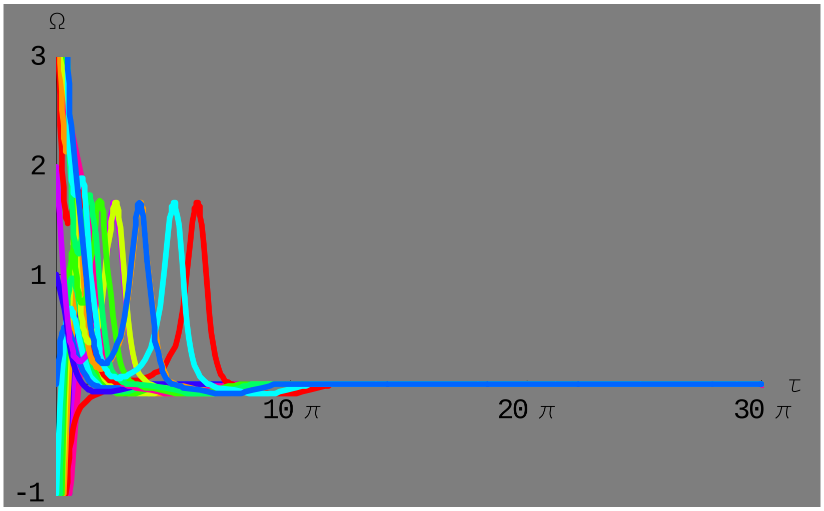
 phlist=phase[{0,#},{0.9,0.90},100,  PlotStyle→Hue[0.1
(#+6)], AxesLabel→{"τ","Ω"},Prolog→AbsoluteThickness[2],
Background→GrayLevel[0.5],PlotRange→{{0,30 π},{-
1,3}},Ticks→{  π Range[0,50,10], Range[-6,6]},
DisplayFunction→Identity]&/@Range[-
10,10,1];Show[phlist,DisplayFunction→$DisplayFunction]
```

Fig.15 Ω vs τ for β_J = 0.9. The parameter κ is varied as a parameter, κ = 0.90 - .2.0.

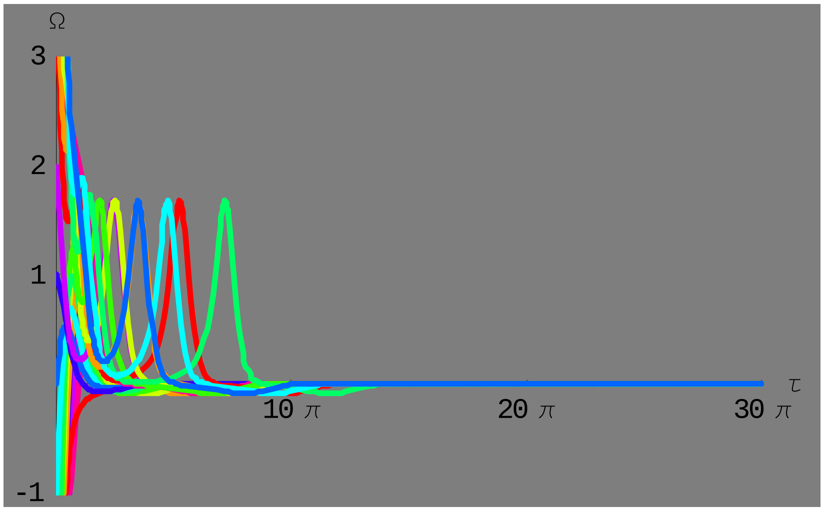
(1) $\beta_J = 0.90$ and $\kappa = 0.90$



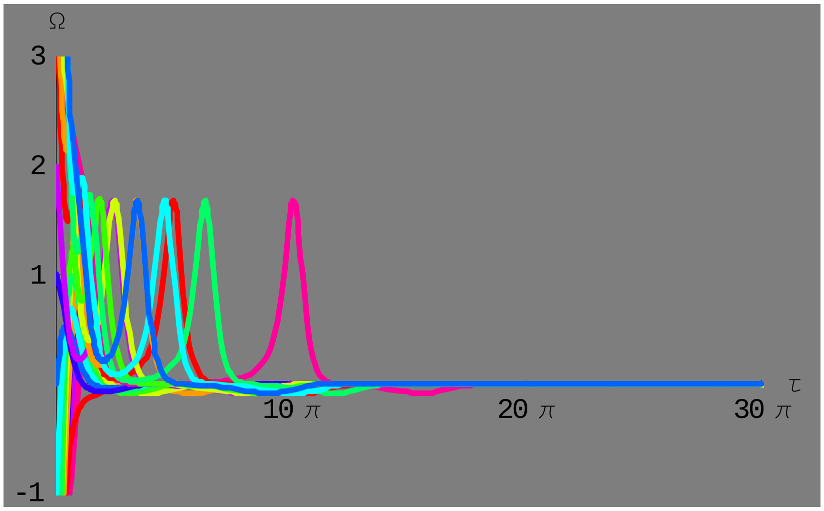
(2) $\beta_J = 0.90$ and $\kappa = 0.91$



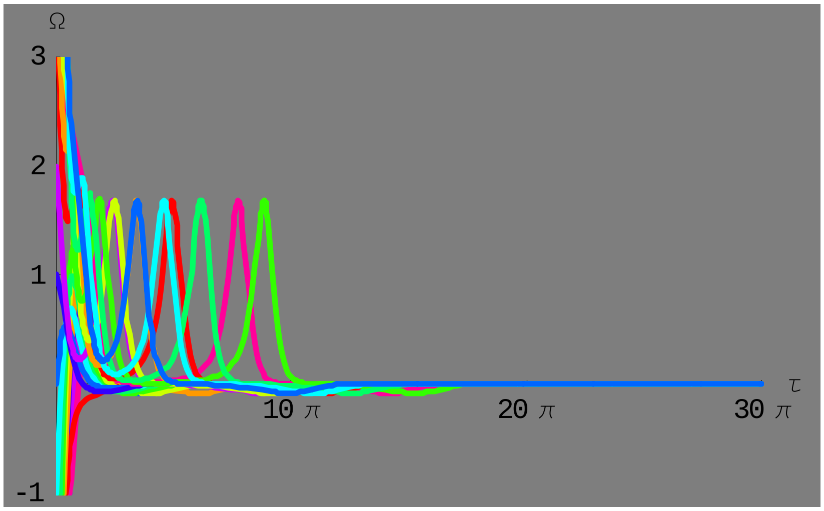
(3) $\beta_J = 0.90$ and $\kappa = 0.915$



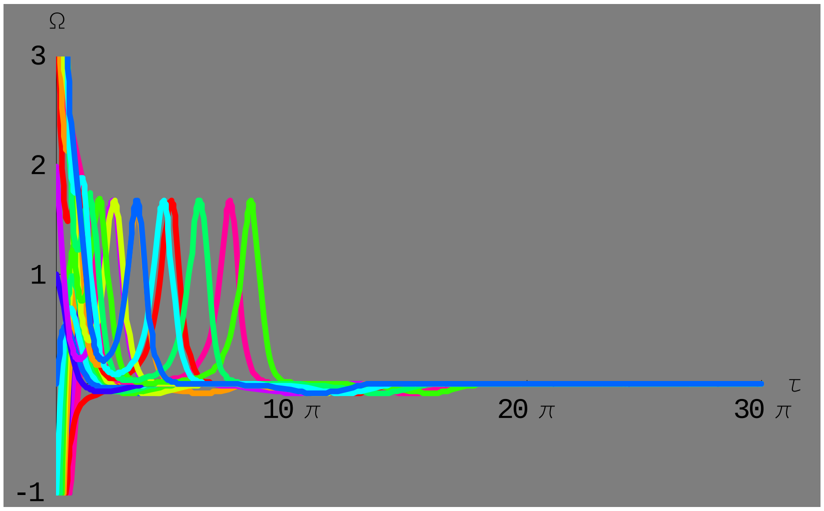
(4) $\beta_J = 0.90$ and $\kappa = 0.918$



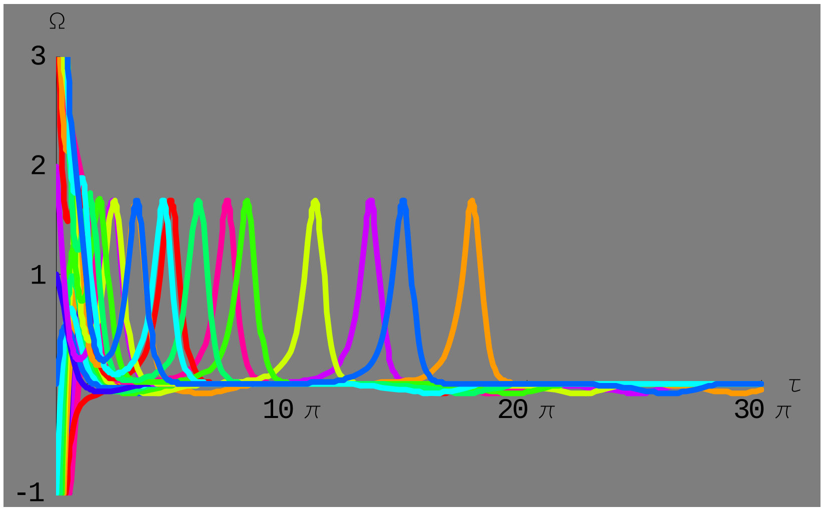
(5) $\beta_J = 0.90$ and $\kappa = 0.919$



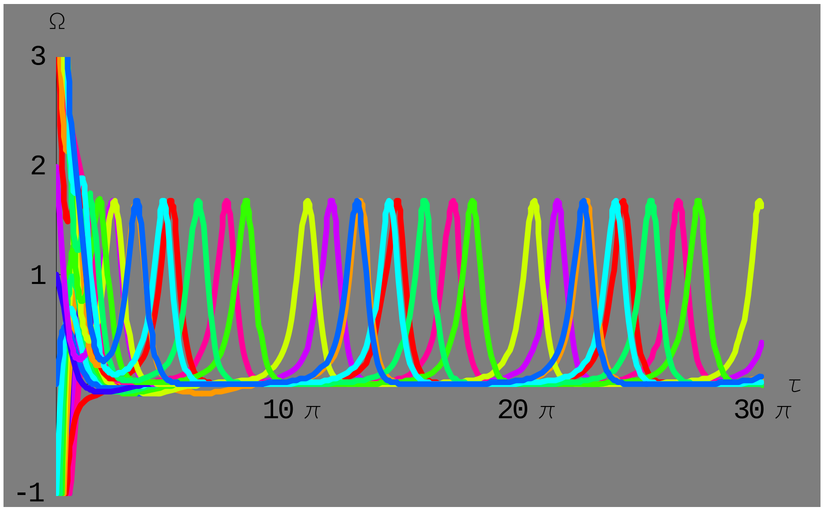
(6) $\beta_J = 0.90$ and $\kappa = 0.9195$



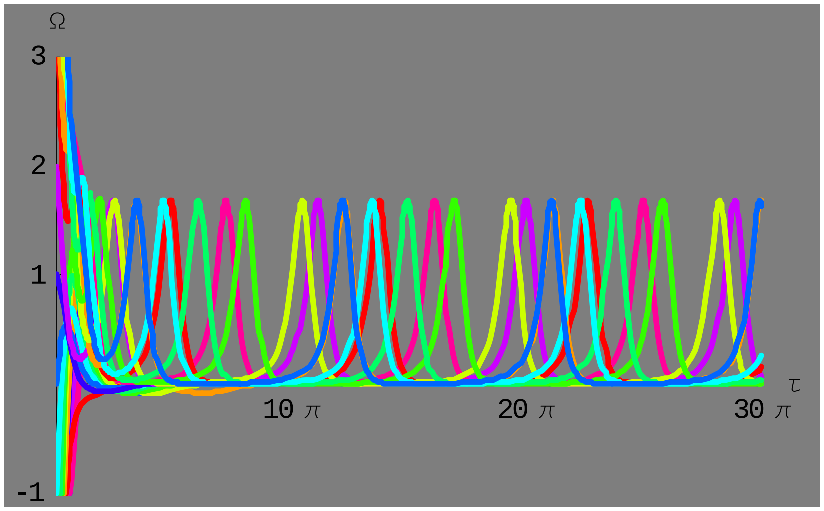
(7) $\beta_J = 0.90$ and $\kappa = 0.9197$



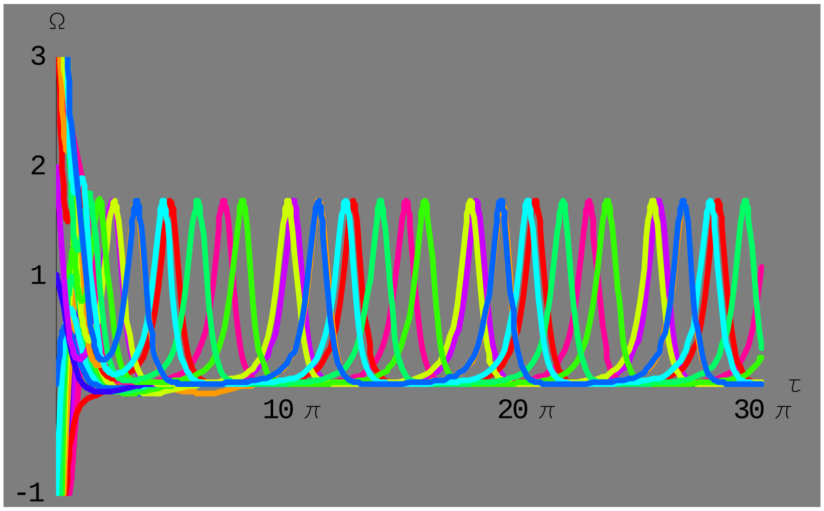
(8) $\beta_J = 0.90$ and $\kappa = 0.91975$



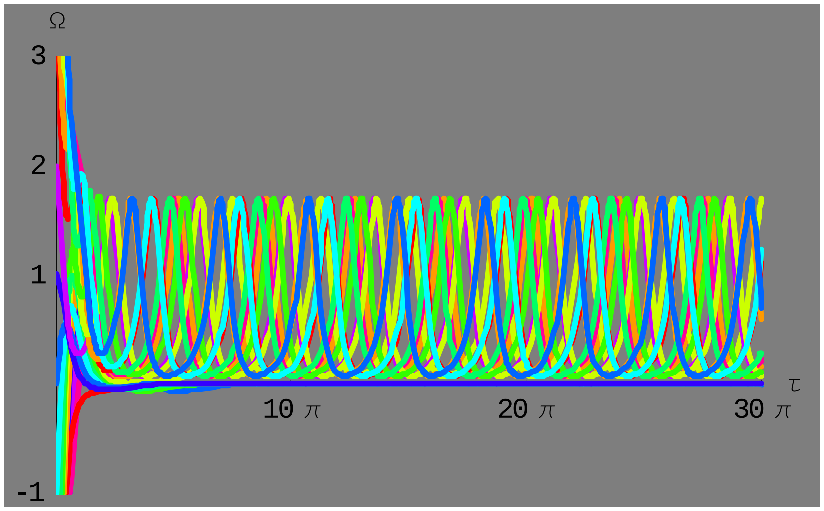
(9) $\beta_J = 0.90$ and κ = 0.9198



(10) $\beta_J = 0.90$ and κ = 0.92

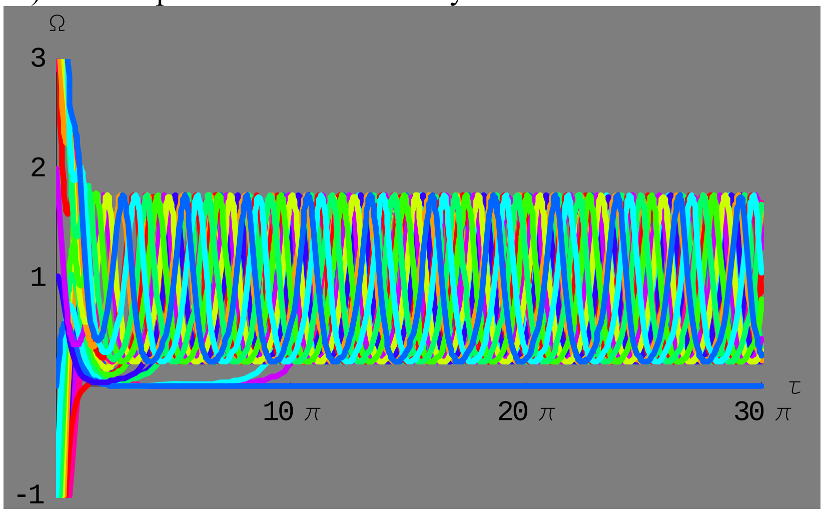


(11) $\beta_J = 0.90$ and κ = 0.94



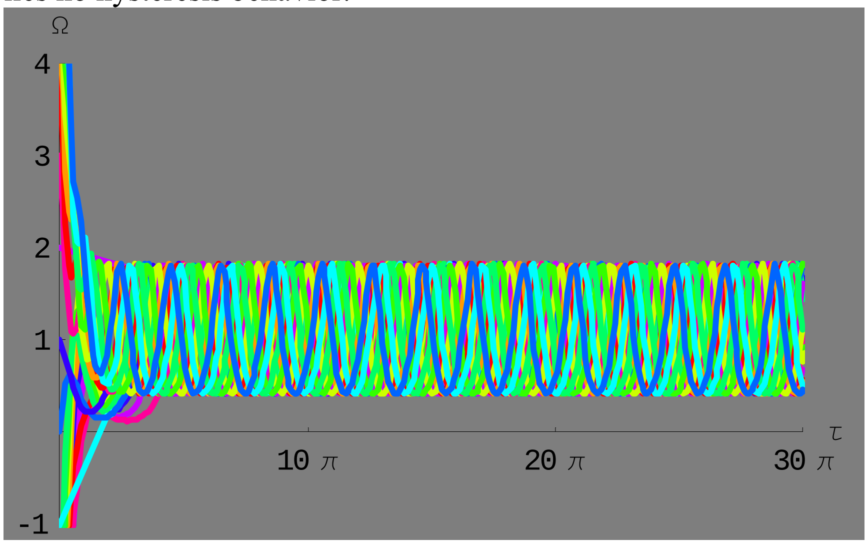
(12) $\beta_J = 0.90$ and $\kappa = 1$

The average voltage $<\eta>$ ($= \beta_J \langle \Omega(\tau) \rangle$) is equal to 0.9x0.98318 = 0.8849 and 0.9x0 =0 where $\beta_J = 0.9$ and κ ($= I/I_c$) = 1.0, depending on the initial condition $\Omega(\tau = 0)$. This implies the existence of hysteresis behavior.l



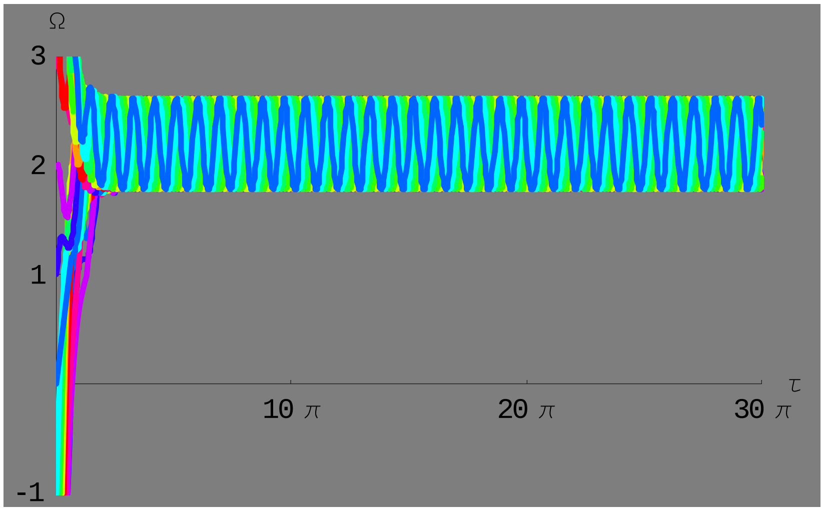
(13) $\beta_J = 0.90$ and $\kappa = 1.1$

The average voltage $<\eta>$ ($= \beta_J \langle \Omega(\tau) \rangle$) is equal to 0.9 x 1.12581 = 1.0132, where $\beta_J = 0.9$ and κ ($= I/I_c$) = 1.1, independent of the initial condition $\Omega(\tau = 0)$. This implies no hysteresis behavior.



(14) $\beta_J = 0.90$ and $\kappa = 2.0$

The average voltage $<\eta>$ ($= \beta_J \langle \Omega(\tau) \rangle$) is equal to 0.9 x 2.2029 = 1.9826, where $\beta_J = 0.9$ and κ ($= I/I_c$) = 2, which is independent of the initial condition $\Omega(\tau = 0)$.



6.5 Simulation: the relation of $<\eta>=\beta_J<\Omega>$ vs κ

Here we show how to determine the average voltage $<\eta>$ as a function of the current κ, where β_J is changed as a parameter.

(1) Using the following Mathematica 5.2 program, we find the maximum and minimum of $\Omega(\tau)$ in the long- time region where $\Omega(\tau)$ periodically oscillates with τ.

(2) The average $<\Omega>$ is calculated as (maximum+minimum)/2. The average voltage $<\eta>=\beta_J<\Omega>$ is plotted as a function of κ for each β_J (= 0.2 – 1.2).

((Mathematica 5.2)) Program-9

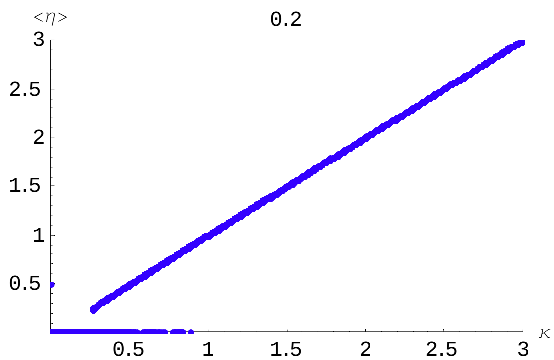
```
Clear["Global`*"]
 <<Graphics`Graphics`
 (*Subroutine, to find Maximum and minimum*)

phase1[{ϕ0_,v0_},{βJ_,κ_},τmax_,opts__]:=Module[{numsol1,num
graph},numsol1=NDSolve[{ Ω'[τ]+ βJ
Ω[τ]+Sin[ϕ[τ]]==κ,ϕ'[τ]==Ω[τ],ϕ[0]==ϕ0,Ω[0]==v0},{Ω[τ],ϕ[τ]},{
τ,0,τmax}]//Flatten;numgraph=Plot[ Ω[τ]/.numsol1,{τ,0,τmax},
opts,
DisplayFunction→Identity]];Max1[{ϕ0_,v0_},{βJ_,κ_},{τmin_,τ
max_}]:=Module[{numsol1},numsol1=NDSolve[{ Ω'[τ]+ βJ
Ω[τ]+Sin[ϕ[τ]]==κ,ϕ'[τ]==Ω[τ],ϕ[0]==ϕ0,Ω[0]==v0},{Ω[τ],ϕ[τ]},{
τ,0,τmax}]//Flatten;maximum=FindMaximum[βJ
Ω[τ]/.numsol1,{τ,τmin,τmax}]];Min1[{ϕ0_,v0_},{βJ_,κ_},{τmin_
,τmax_}]:=Module[{numsol1},numsol1=NDSolve[{ Ω'[τ]+ βJ
Ω[τ]+Sin[ϕ[τ]]==κ,ϕ'[τ]==Ω[τ],ϕ[0]==ϕ0,Ω[0]==v0},{Ω[τ],ϕ[τ]},{
τ,0,τmax}]//Flatten;minimum=FindMinimum[ βJ
Ω[τ]/.numsol1,{τ,τmin,τmax}]];Sei[βJ_,κ_]:=Module[{A1,B1,
ave1,list1},A1=Max1[{0,#},{βJ,κ},{20 π,30π}]&/@Range[-
10,10,1];  B1=Min1[{0,#},{βJ,κ},{20 π,30π}]&/@Range[-
10,10,1];ave1=(A1+B1)/2;list1=Table[{κ,ave1[[k,1]]},{k,1,21}
]];Nat1[βJ_]:=Flatten[Table[Sei[βJ,κ],{κ,0,3,0.01}],1];Saw1
```

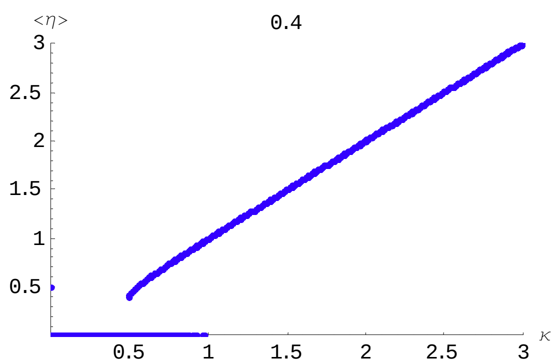
```
[βJ_]:=ListPlot[Nat1[βJ],PlotStyle→{Hue[0.7],PointSize[0.01
5]},AxesLabel→{"κ","<η>"},PlotLabel→NumberForm[βJ],
PlotRange→{{0,3},{0,3}}]
 Table[Saw1[βJ],{βJ,0.2,1.2,0.2}]
```

Fig.16 $<\eta>=\beta_J<\Omega>$ vs κ for $\beta_J = 0.2$, 0.4, 0,6, 0.8, 1.0, and 1.2. The number denoted in each curve is the value of β_J. For $\beta_J \le 1$, $<\eta>$ takes two values with $<\eta> = 0$ and finite value of $<\eta>$. Note that for $\beta_J = 1$ and 1.2, $<\eta>$ takes three values (multi-valued function) near $\kappa = 1$.

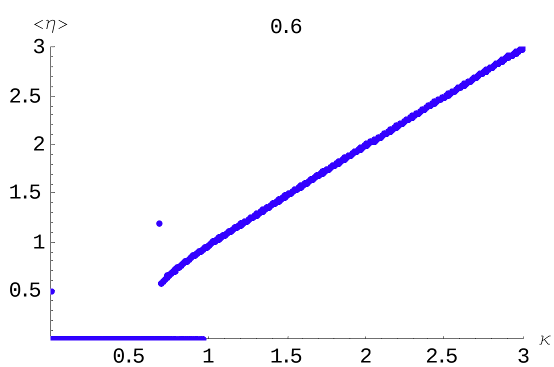
(1) $\beta_J = 0.2$



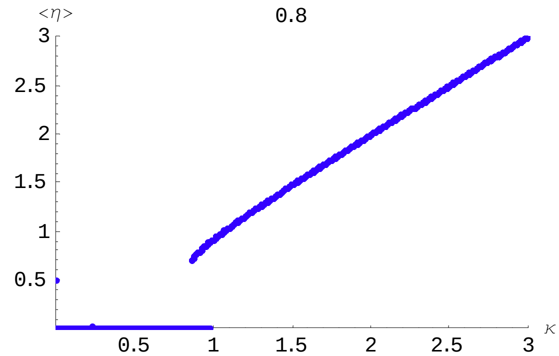
(2) $\beta_J = 0.4$



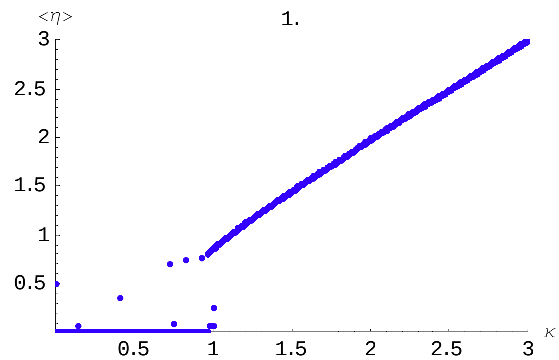
(3) $\beta_J = 0.6$



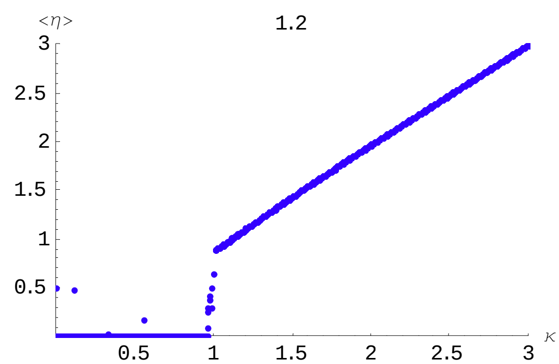
(4) $\beta_J = 0.8$



(5) $\beta_J = 1.0$



(6) $\beta_J = 1.2$



6.6 Result on κ vs $<\eta>$ from simulations

Figure 17 shows the $<\eta>$ vs κ curve for β_J = 0.1 – 1.2. For β_J = 0.6, no voltage drop develops until the value of κ reaches 1. At the point ($<\eta>$ = 0 and κ = 1) there occurs a transition from the zero-voltage state ($<\eta>$ = 0) to the finite-voltage state ($<\eta> \neq 0$). The $<\eta>$ vs κ curve approaches the straight line denoted by $<\eta> = \kappa$ with further increasing $<\eta>$. With decreasing $<\eta>$ from the high $<\eta>$-side, in turn, the $<\eta>$ vs κ curve starts to deviate from the straight line $<\eta> = \kappa$. The transition occurs from the finite-voltage state to the zero-voltage state at $\kappa = 0.6965$. Similar hysteresis behaviors are also seen for the cases of $\beta_J = 0.12$ - 0.9.

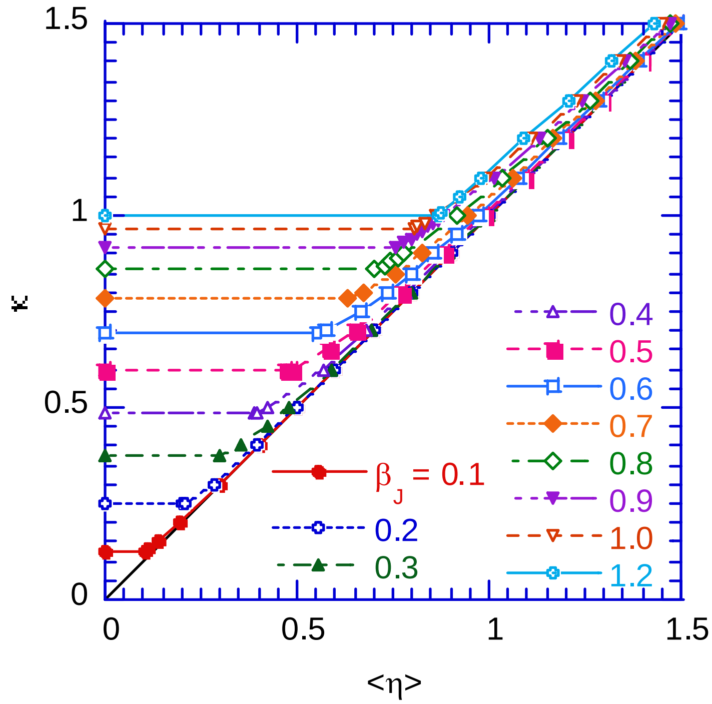


Fig.17 The *I-V* curve (κ vs $<\eta>$) for $\beta_J = 0.1 - 1.2$.

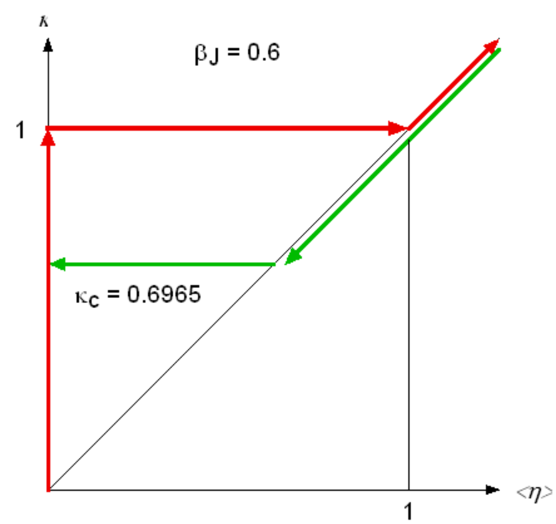


Fig.18 Schematic diagram of the *I-V* (κ vs $<\eta>$) trajectories as $<\eta$. changes. $\beta_J = 0.6$. At $<\eta> = 0$, κ changes from 0 to 1. At $\kappa=1$, $<\eta>$ changes from 0 to a value above 1. With further increasing $<\eta>$, the relation $\kappa = <\eta>$ holds valid (reversible). With decreasing $<\eta>$, in turn, the relation $\kappa = <\eta>$ still holds valid. There is a transition from this state to the zero-voltage state($<\eta> = 0$) at $\kappa = \kappa_c = 0.6965$.

7 DC SQUID (superconducting quantum interference device)[3]

7.1 Current density and flux quantization

In quantum mechanics, the current density is defined as

$$\mathbf{J} = \frac{q\hbar}{2mi}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^2|\psi|^2}{mc}\mathbf{A},$$

where q (=-2e, e>0) is a charge for electron pairs, m is a mass, $\boldsymbol{A}$ is a vector potential, and ψ is a wavefunction. When the wavefunction is given by the amplitude $|\psi(\boldsymbol{r})|$and the phase $\theta(\boldsymbol{r})$ as

$$\psi = |\psi(\mathbf{r})|e^{i\theta(\mathbf{r})},$$

then $\boldsymbol{J}$ can be rewritten as

$$\mathbf{J} = \frac{q\hbar}{m}|\psi|^2(\nabla\theta - \frac{q}{c\hbar}\mathbf{A}).$$

Note that this current density is invariant under the gauge transformation. $\mathbf{A}' = \mathbf{A} + \nabla\chi$ and $\theta' = \theta + q\chi/c\hbar$,

$$\mathbf{J}' = \frac{q\hbar}{m}|\psi|^2(\nabla\theta' - \frac{q}{c\hbar}\mathbf{A}') = \frac{q\hbar}{m}|\psi|^2(\nabla\theta' - \frac{q}{c\hbar}\mathbf{A}') = \frac{q\hbar}{m}|\psi|^2(\nabla\theta - \frac{q}{c\hbar}\mathbf{A}),$$

where $\psi'(\mathbf{r}) = e^{iq\chi/c\hbar}\psi(\mathbf{r}) = |\psi(\mathbf{r})|e^{i[\theta(\mathbf{r}) + q\chi/c\hbar]}$.

If we consider now a cylinder which may become superconductor in an external magnetic field and if we take a path from a surface at a distance which is larger than the penetration depth λ, then $\mathbf{J} = 0$. When $q = -2e$, we have

$$\mathbf{J} = -\frac{2e\hbar}{m}|\psi|^2(\nabla\theta + \frac{2e}{c\hbar}\mathbf{A}) = 0,$$

or

$$\nabla\theta = -\frac{2e}{c\hbar}\mathbf{A},$$

$$\oint\nabla\theta\cdot d\mathbf{l} = -\frac{2e}{c\hbar}\oint\mathbf{A}\cdot d\mathbf{l} = -\frac{2e}{c\hbar}\oint\nabla\times\mathbf{A}\cdot d\mathbf{a} = -\frac{2e}{c\hbar}\oint\mathbf{B}\cdot d\mathbf{a} = -\frac{2e}{c\hbar}\Phi = -2\pi\frac{\Phi}{\Phi_0},$$

where Φ is the magnetic flux inside the ring and $\Phi_0 = 2\pi\hbar c/(2e)$ (=2.06783372 x 10^{-7} Gauss cm^2) is a quantum fluxoid. In the last equation we apply the Stoke's theorem.

((Note))

The current flows along the ring. However, this current flows only on the surface boundary (region from the surface to the penetration depth λ). Inside of the system (region far from the surface boundary), there is no current since $\nabla\times\mathbf{H} = 4\pi\mathbf{J}/c$ and $\boldsymbol{H}$ = 0 (Meissner effect).

7.2 DC SQUID (double junctions): quantum mechanics

DC SQUID consists of two points contacts in parallel, forming a ring. Each contact forms a Josephson junctions of superconductor 1, insulating layer, and superconductor 2 (S_1-I-S_2). Suppose that a magnetic flux Φ passes through the interior of the loop.

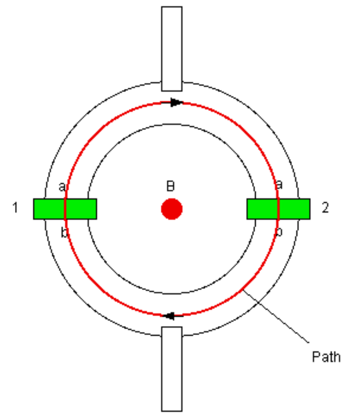


Fig.19 Schematic diagram of superconducting quantum interference device. δ_1 and δ_2 refer to two point-contact weak links. The rest of the circuit is strongly superconducting.

Here we have

$$\oint \nabla\theta \cdot d\mathbf{l} = \theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b}.$$

or

$$\theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b} = 2\pi\frac{\Phi}{\Phi_0}$$

or

$$\delta_1 - \delta_2 = 2\pi\frac{\Phi}{\Phi_0}$$

where $\delta_1 (= \theta_{1b} - \theta_{1a})$ is the phase difference between the superconductors a and b through the junction 1 and $\delta_2 = (\theta_{2b} - \theta_{2a})$ are is the phase difference between the superconductors a and b through the junction 2.

When $\boldsymbol{B} = 0$ (or $\Phi = 0$), we have $\delta_1 - \delta_2 = 0$. In general, we put the form

$$\delta_1 = \delta_0 + \frac{e}{\hbar c}\Phi, \qquad \delta_2 = \delta_0 - \frac{e}{\hbar c}\Phi.$$

The total current is given by

$$I = I_1 + I_2 = I_c[\sin(\delta_1) + \sin(\delta_2)]$$

$$= I_c[\sin(\delta_0 + \frac{e}{\hbar c}\Phi) + \sin(\delta_0 - \frac{e}{\hbar c}\Phi)]$$

$$= 2I_c \sin(\delta_0)\cos(\frac{e}{\hbar c}\Phi)$$

or

$$I = 2I_c \sin(\delta_0)\cos(\pi\frac{\Phi}{\Phi_0}). \tag{30}$$

The current varies with Φ and has a maximum of $2I_c$ when $\frac{e}{\hbar c}\Phi = s\pi$ (s: integers),

or

$$\Phi = \frac{\hbar c \pi}{e} s = \frac{hc}{2e} s = \Phi_0 s . \tag{31}$$

The simple two point contact device corresponds to a two-slit interference pattern, for which the physically interesting quantity is the modulus of the amplitude rather than the square modulus, as it is for optical interference patterns.

7.3 Analogy of the diffraction with double slits and single slit

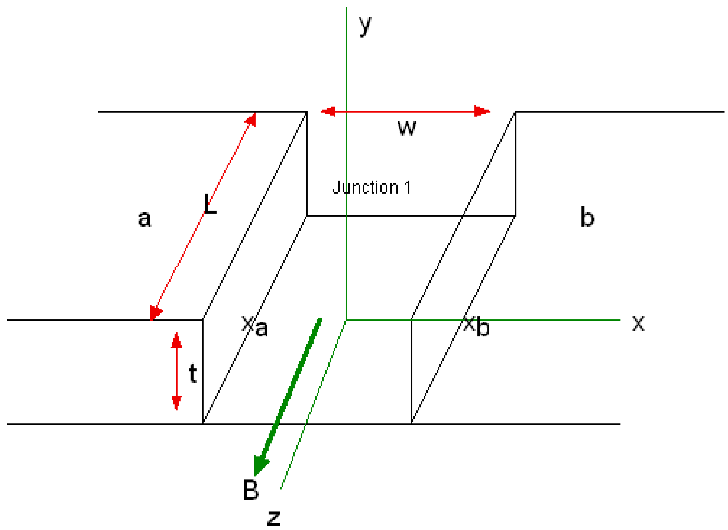


Fig.20 Diffraction effect of Josephson junction. A magnetic field ***B*** along the z direction, which is penetrated into the junction (in the normal phase).

We consider a junction (1) of rectangular cross section with magnetic field ***B*** applied in the plane of the junction, normal to an edge of width *w*.

$$J = J_0 \sin[\delta_1 + \frac{q}{\hbar c}\int_1^2 \mathbf{A}\cdot d\mathbf{l}] ,$$

with q = -2e. We use the vector potential ***A*** given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r}) = (-\frac{By}{2}, \frac{Bx}{2}, 0) ,$$

$$\mathbf{A}' = \mathbf{A} + \nabla\chi = (-By, 0, 0) ,$$

where

$$\chi = -\frac{Bxy}{2} .$$

Then we have

$$J = J_0 \sin[\delta_1 + \frac{q}{\hbar c}\int_{x_a}^{x_b}(-By)dx] = J_0 \sin[\delta_1 - \frac{qB}{\hbar c} yW] ,$$

$$dI_1 = JLdy = J_0 L \sin[\delta_1 - \frac{qB}{\hbar c} yW]dy ,$$

or

$$I_1 = J_0 L \int_{-t/2}^{t/2} \sin[\delta_1 - \frac{qB}{\hbar c} yW] dy .$$

Then we have

$$I_1 = J_0 L \frac{2\hbar c}{BWq} \sin(\delta_1) \sin(\frac{qB}{\hbar c} \frac{t}{2} W) .$$

Here we introduce the total magnetic flux passing through the area Wt ($\Phi_W = BWt$), $I_c = J_0 L t$, and

$$\frac{\Phi_W}{\Phi_0} = \frac{BWt}{\frac{2\pi\hbar c}{2e}} = \frac{eBWt}{\pi\hbar c}, \qquad \text{or} \qquad \pi \frac{\Phi_W}{\Phi_0} = \frac{eBWt}{\hbar c} .$$

Therefore we have

$$I_1 = I_c \sin(\delta_1) \frac{\sin(\frac{\pi\Phi_W}{\Phi_0})}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)} .$$

The total current is given by

$$I = I_1 + I_1 = I_c [\sin(\delta_1) + \sin(\delta_1)] \frac{\sin(\frac{\pi\Phi_W}{\Phi_0})}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)} ,$$

or

$$I = I_1 + I_1 = 2 I_c \sin(\delta_0) \cos(\pi \frac{\Phi}{\Phi_0}) \frac{\sin(\frac{\pi\Phi_W}{\Phi_0})}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)} . \qquad (32)$$

The short period variation is produced by interference from the two Josephson junctions, while the long period variation is a diffraction effect and arises from the finite dimensions of each junction. The interference pattern of $|I|^2$ is very similar to the intensity of the Young's double slits experiment. If the slits have finite width, the intensity must be multiplied by the diffraction pattern of a single slit, and for large angles the oscillations die out.

((Young's double slit experiment))

We consider the Young's double slits (the slits are separated by *d*). Each slit has a finite width *a*.

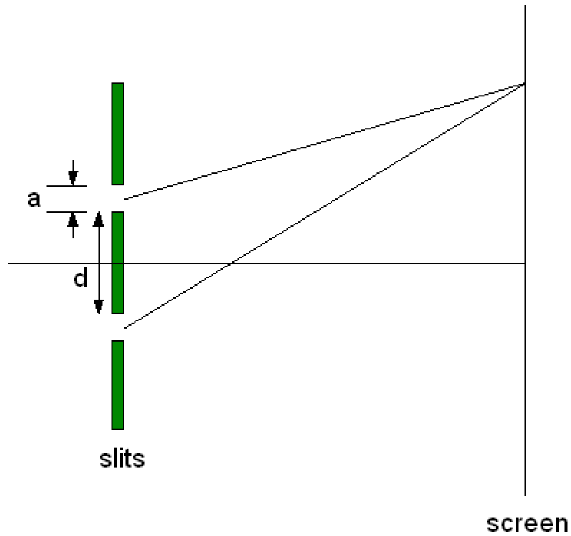


Fig.21 Geometric construction for describing the Young's double-slit experiment (not to scale).

((double slits))

E is the electric field of a light with the wavelength λ. d is the separation distance between the centers of the slits.

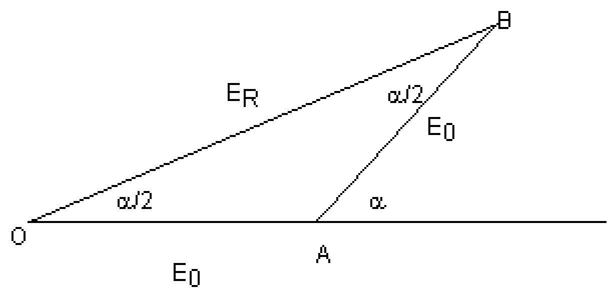


Fig.22 A reconstruction of the resultant phasor E_R which is the combination of two phasors (E_0).

$$E_R = 2E_0 \cos\frac{\alpha}{2}.$$

The intensity $I \propto E_R^{\ 2} = 4E_0^{\ 2}\cos^2\frac{\alpha}{2} = 2E_0^{\ 2}(1+\cos\alpha)$.],

where the phase difference α is given by

$$\alpha = \frac{2\pi d}{\lambda}\sin\theta .$$

((single slit))

We assume that each slit has a finite width a.

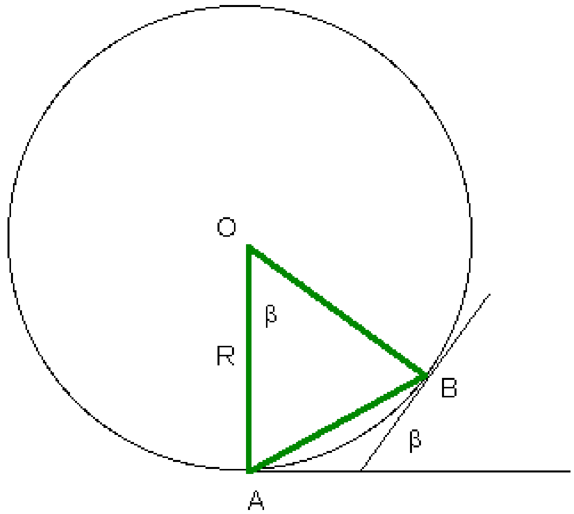


Fig.23 Phasor diagram for a large number of coherent sources. All the ends of phasors lie on the circular arc of radius R. The resultant electric field magnitude E_R equals the length of the chord.

$$E_0 = R\beta,$$

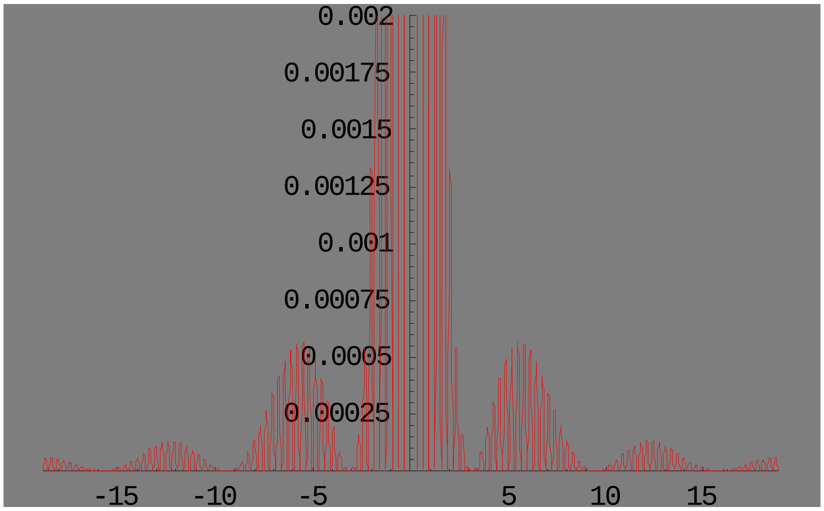
$$E_R = 2R\sin\frac{\beta}{2} = 2\frac{E_0}{\beta}\sin\frac{\beta}{2} = E_0\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}.$$

where the phase difference β is given by $\beta = \frac{2\pi a}{\lambda}\sin\theta$. Then the resultant intensity I for the double slits (the distance d) (each slit has a finite width a) is given by

$$I = I_m\cos^2\frac{\alpha}{2}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2 = \frac{I_m}{2}(1+\cos\alpha)\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2.$$

((Mathematica 5.2)) Program-10

```
f[α_, β_] := (1 + Cos[α]) Sin[β/2]^2/(β/2)^2
 Plot[Evaluate[Table[f[α,N α],{N,20,20}],{α,-15 π,15 π}],
PlotPoints→200, PlotStyle→Table[Hue[0.3
i],{i,0,10}],PlotRange→{{- 6 π,6 π},{0,0.002}},
Prolog→AbsoluteThickness[1.2],Background→GrayLevel[0.5]]
```



-Graphics-

Fig.24 The combined effects of two-slit and single-slit interference. The pattern consists of a diffraction envelope and interference fringes.

7.4 DC SQUID Juntion based on th RSJ model

7.4.1 Formulation

The DC SQUID consists of two Josephson junctions connected in parallel on a superconducting loop of inductance *L*.

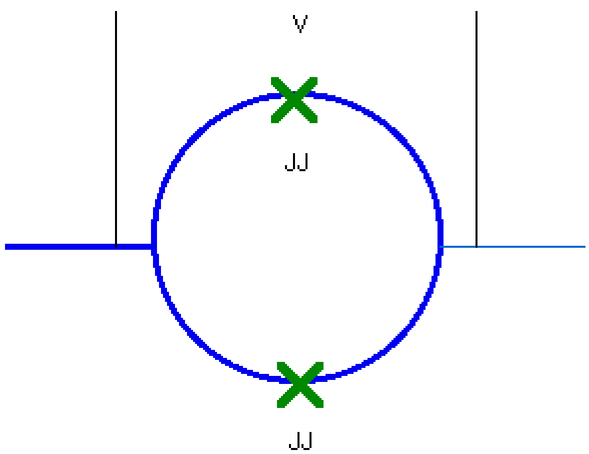


Fig.25 Simple notation for the DC SQUID consisting of two Josephson junctions (J.J.) in parallel.

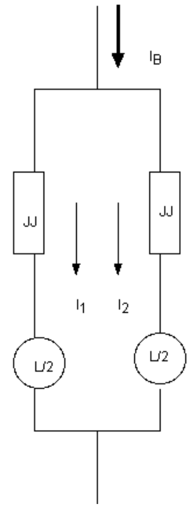


Fig.26 Equivalent circuit of the DC SQUID.

As shown in Fig.26, the total current is given by

$$I_B = I_1 + I_2 .$$

The total magnetic flux is given by $\Phi = \Phi_{ext} + LI_s$, where L is the total self-inductance ($L = L_1 + L_2$ and $L_1 = L_2 = L/2$ in this case)

$$I_s = \frac{I_2 - I_1}{2},$$

where I_s is the loop (circulating) current.

$$I_c \sin\phi_1 + \frac{V_1}{R} + C\dot{V}_1 = I_1,$$

$$I_c \sin\phi_2 + \frac{V_2}{R} + C\dot{V}_2 = I_2,$$

$$\dot{\phi}_1 = \frac{2e}{\hbar}V_1, \qquad \dot{\phi}_2 = \frac{2e}{\hbar}V_2.$$

Since

$$V = V_1 + \frac{L}{2}\frac{dI_1}{dt}, \quad V = V_2 + \frac{L}{2}\frac{dI_2}{dt}, \text{ and } I_B = I_1 + I_2,$$

and the total voltage V is given by the simple form,

$$V = \frac{1}{2}(V_1 + V_2) + \frac{L}{4}\frac{dI_B}{dt} = \frac{1}{2}(V_1 + V_2) = \frac{\hbar}{4e}\left(\frac{d\phi_1}{dt} + \frac{d\phi_2}{dt}\right)$$

since $dI_B / dt = 0$ (or I_B is independent of t).

For the sake of simplicity, we use the dimensionless quantities. Here we assume that

$$\eta = \frac{V}{I_c R}, \qquad \kappa = \frac{I}{I_c}, \qquad \tau = \omega_J t .$$

Then we have

$$\frac{\hbar C}{2eI_c}\omega_J{}^2 \frac{d^2\phi_1}{d\tau^2} + \frac{\hbar}{2eRI_c}\omega_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \kappa_1,$$

or

$$\frac{d^2\phi_1}{d^2\tau} + \beta_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \kappa_1 ,$$

where

$$\beta_J = \frac{1}{\omega_c CR} .$$

Similarly,

$$\frac{d^2\phi_2}{d^2\tau} + \beta_J \frac{d\phi_2}{d\tau} + \sin\phi_2 = \kappa_2 .$$

We also have

$$\kappa_1 + \kappa_2 = \kappa_B ,$$
$$\kappa_2 - \kappa_1 = 2\kappa_s ,$$

or

$$2\kappa_1 = \kappa_B - 2\kappa_s ,$$

where

$$\kappa_B = \frac{I_B}{I_c} , \qquad \kappa_1 = \frac{I_1}{I_c} , \qquad \kappa_2 = \frac{I_2}{I_c} , \qquad \kappa_s = \frac{I_s}{I_c} = \frac{\kappa_2 - \kappa_1}{2}$$

Then we have

$$\frac{d^2\phi_1}{d^2\tau} + \beta_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \frac{\kappa_B}{2} - \kappa_s . \tag{33}$$

Similarly we have

$$\frac{d^2\phi_2}{d^2\tau} + \beta_J \frac{d\phi_2}{d\tau} + \sin\phi_2 = \frac{\kappa_B}{2} + \kappa_s . \tag{34}$$

The phases are related to the external magnetic flux by

$$\phi_1 - \phi_2 = 2\pi \frac{\Phi}{\Phi_0} = 2\pi(\frac{LI_s}{\Phi_0} + \frac{\Phi_{ext}}{\Phi_0}) = 2\pi(\frac{\beta}{2}\kappa_s + \frac{\Phi_{ext}}{\Phi_0}) ,$$

or

$$\kappa_s = \frac{2}{\beta}(\frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{ext}}{\Phi_0}) , \tag{35}$$

where

$$\beta = \frac{2I_c L}{\Phi_0} .$$

The normalized voltage η is given by

$$\eta = \frac{1}{2}\beta_J (\frac{d\phi_1}{d\tau} + \frac{d\phi_2}{d\tau}) . \tag{36}$$

7.4.2 Two-dimensional (2D) SQUID potential

Equations (33) and (34) describing the DC SQUID dynamics can be regarded as an equation of motion of a point mass in a field of force with a 2D SQUID potential $U(\phi_1, \phi_2)$.

$$\frac{d^2\phi_1}{d^2\tau} + \beta_J \frac{d\phi_1}{d\tau} = -\sin\phi_1 + \frac{\kappa_B}{2} - \kappa_s = -\frac{2e}{\hbar I_c} \frac{\partial U(\phi_1, \phi_2)}{\partial \phi_1}$$

and

$$\frac{d^2\phi_2}{d^2\tau}+\beta_J\frac{d\phi_2}{d\tau}=-\sin\phi_2+\frac{\kappa_B}{2}+\kappa_s=-\frac{2e}{\hbar I_c}\frac{\partial U(\phi_1,\phi_2)}{\partial\phi_2}$$

or

$$\frac{\partial U(\phi_1,\phi_2)}{\partial\phi_1}=\frac{\hbar I_c}{2e}(\sin\phi_1-\frac{\kappa_B}{2}+\kappa_s)=\frac{\Phi_0}{2\pi c}I_c[\sin\phi_1-\frac{\kappa_B}{2}+\frac{2}{\pi\beta}(\frac{\phi_1-\phi_2}{2}-\pi\frac{\Phi_{ext}}{\Phi_0})], \qquad (37)$$

$$\frac{\partial U(\phi_1,\phi_2)}{\partial\phi_2}=\frac{\hbar I_c}{2e}(\sin\phi_2-\frac{\kappa_B}{2}-\kappa_s)=\frac{\Phi_0}{2\pi c}I_c[\sin\phi_2-\frac{\kappa_B}{2}-\frac{2}{\pi\beta}(\frac{\phi_1-\phi_2}{2}-\pi\frac{\Phi_{ext}}{\Phi_0})]. \qquad (38)$$

Thus the normalized 2D SQUID potential $\widetilde{U}(\phi_1,\phi_2)$ $[=U(\phi_1,\phi_2)/(2E_J)]$ is obtained as

$$\widetilde{U}(\phi_1,\phi_2)=\frac{1}{\pi\beta}(\frac{\phi_1-\phi_2}{2}-\pi\frac{\Phi_{ext}}{\Phi_0})^2-\cos(\frac{\phi_1-\phi_2}{2})\cos(\frac{\phi_1+\phi_2}{2})-\frac{\kappa_B}{2}(\frac{\phi_1+\phi_2}{2}), \qquad (39)$$

where $E_J=\Phi_0 I_c/(2\pi c)$ is the Josephson coupling constant. It is convenient to introduce new variable

$$x=\frac{\phi_1+\phi_2}{2\pi}, \qquad y=\frac{\phi_1-\phi_2}{2\pi}.$$

The loop current κ_s is related to y as

$$\kappa_s=\frac{2}{\beta}(y-\frac{\Phi_{ext}}{\Phi_0}).$$

Then the 2D SQUID potential $\widetilde{U}(\phi_1,\phi_2)$ is rewritten as

$$\widetilde{U}(x,y)=\frac{\pi}{\beta}(y-\frac{\Phi_{ext}}{\Phi_0})^2-\frac{\kappa_B}{2}\pi x-\cos(\pi x)\cos(\pi y).$$

Here we make a contour plot of $U(\phi_1,\phi_2)$ in the ϕ_1-ϕ_2 plane, as β, Φ_{ext}/Φ_0, and κ_B are changed as parameters. The parameters β (= 1) and Φ_{ext}/Φ_0 (= 0.25) are fixed. The current κ_B is changed as a parameter. As will be shown in Fig.32, the critical current $(\kappa_B)_c$ is equal to 1.628 for β = 1 and Φ_{ext}/Φ_0 = 0.25. In Fig.27 we show the contour plot of $U(\phi_1,\phi_2)$ in the ϕ_1-ϕ_2 plane, where β = 1 and Φ_{ext}/Φ_0 = 0.25. For κ_B = 0.4, $U(\phi_1,\phi_2)$ has multiple metasatable state separated by saddle points on the $\phi_2 = \phi_1$ line. With increasing κ_B, these saddle points gradually disappear. At $\kappa>(\kappa_B)_c$ it seems that all the saddle points disappear, suggesting no stable state corresponding to local minima of the potential energy.

((**Mathematica 5.2**)) Program 11

```
(*2D SQUID potential,β = 1 - 2,κB = 1 - 3;n0 =Ξ/Φ0 ( 0 - 0.5), pp= points*)
 F[ϕ1_, ϕ1_, β_, κB_, n0_] :=
  π/β ((ϕ1 - ϕ2)/(2 π) - n0)^2 - κB/2 π ((ϕ1 + ϕ2)/(2 π)) - Cos[π ((ϕ1 + ϕ2)/(2 π))] Cos[π ((ϕ1 - ϕ2)/(2 π))]

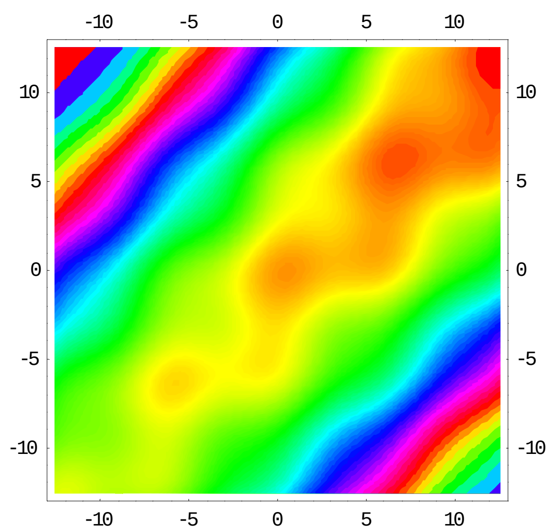
mp[pp_,β_,κB_,n0_]:=Module[{ss1,ss2},ss1=ContourPlot[F[ϕ1,ϕ1,β,κB,n0], {ϕ1,-4 π, 4 π},{ϕ2,-4 π, 4 π},PlotPoints -> pp, ContourLines ->True,PlotRange -> All, ColorFunction -> Hue,
```

```
AspectRatio -> Automatic,
Compiled→False];ss2=ListContourPlot[ss1[[1]],

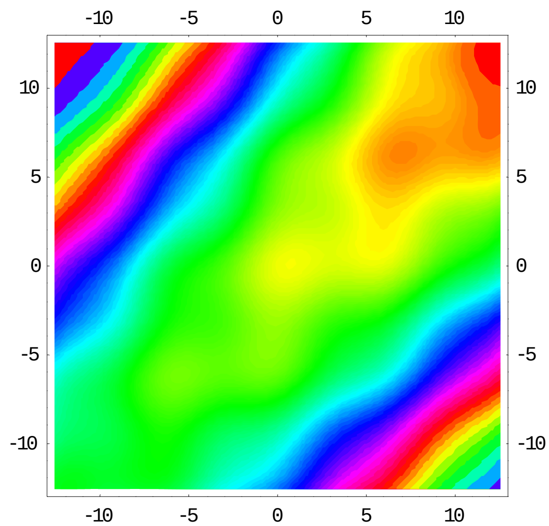
Contours→(#[[pp/2]]&/@Partition[Sort[Flatten[ss1[[1]]]],pp]
),
  ColorFunction→(Hue[2 #]&),ContourLines→False,
MeshRange→{{-4 π,4 π},{-4 π,4 π}},
  DisplayFunction→Identity];Show[ss2,
  DisplayFunction→$DisplayFunction,FrameTicks→True,
  AspectRatio→Automatic]]
 Table[mp[100,1.0,κB,0.25],{κB,0,2.0,0.2}]
```

Fig. 27 The contour plot of $U(\phi_1,\phi_2)$ in the in the ϕ_1-ϕ_2 plane: ϕ_1 is x-axis and ϕ_2 is the y axis. κ_B = 0.4, 0.8, 1.2, 1.6, and 1.8. β = 1. Φ_{ext}/Φ_0 = 0.25. $(\kappa_B)_c$ = 1.628.

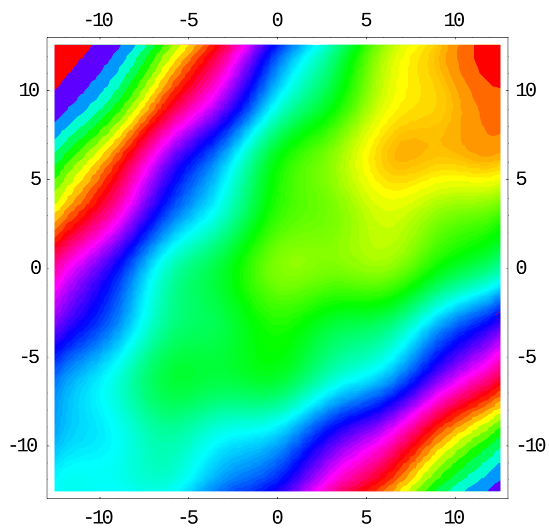
(1) κ_B =0.4, β = 1.0 and Φ/Φ_0 = 0.25



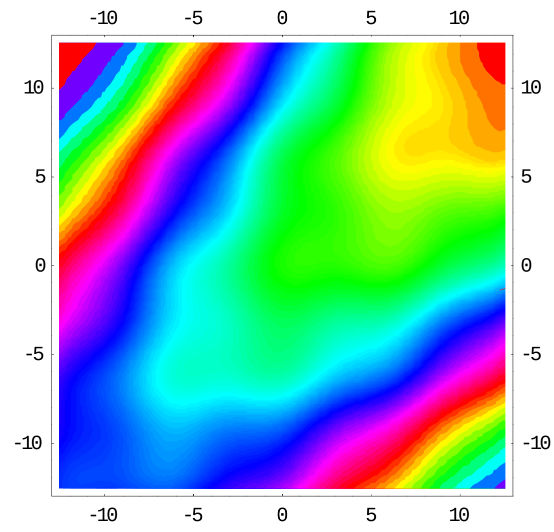
(2) κ_B =0.8, β = 1.0 and Φ/Φ_0 = 0.25



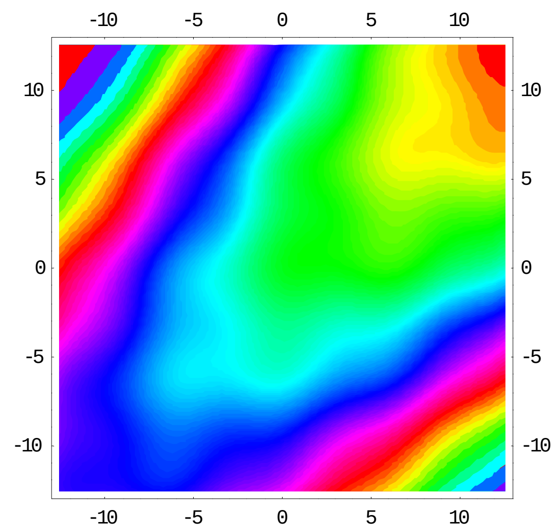
(3) κ_B =1.2, β = 1.0 and Φ/Φ_0 = 0.25



(4) κ_B =1.6, β = 1.0 and Φ/Φ_0 = 0.25



(5) κ_B =1.8, β = 1.0 and Φ/Φ_0 = 0.25



7.5 Simple case: β_J »1 and β = 0

For simplicity we assume that β_J »1. This assumption is appropriate for the operation of DC SQUID.

First we consider the critical current at $V = 0$. We also assume that $\beta = 0$.

$$I_B = I_1 + I_2 = I_c(\sin\phi_1 + \sin\phi_2) = I_c[\sin\phi_1 + \sin(\phi_1 - 2\pi\frac{\Phi_{ext}}{\Phi_0})],$$

since

$$\phi_2 = \phi_1 - 2\pi\frac{\Phi_{ext}}{\Phi_0}.$$

Then we have

$$I_B = 2I_c \sin(\phi_1 - \pi\frac{\Phi_{ext}}{\Phi_0})\cos(\pi\frac{\Phi_{ext}}{\Phi_0}).$$

The maximum of I_B is

$$I_{\max} = 2I_c \left| \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \right|. \tag{40}$$

The critical current is a periodic function of the external magnetic flux.

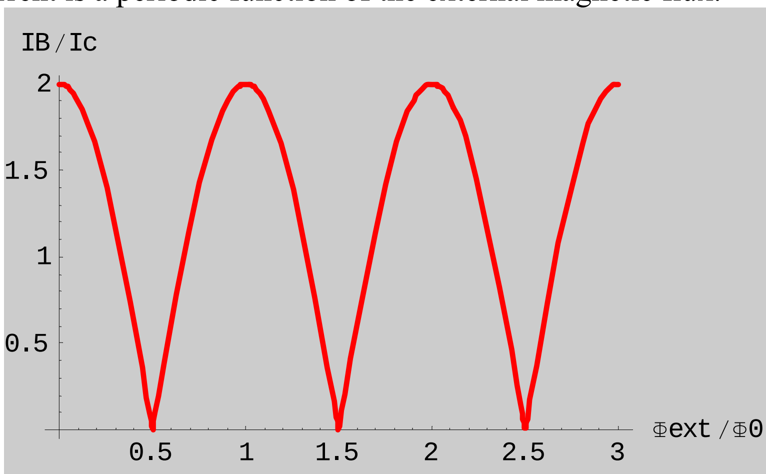


Fig.28 Ideal case for the I_B/I_c vs Φ_{ext}/Φ_0 curve in the DC SQUID, where I_B is the maximum supercurrent. $I_B = 2\ I_c$ when $\Phi_{ext}/\Phi_0 = n$ (integer) and $I_B = 0$ for $\Phi_{ext}/\Phi_0 = n + 1/2$.

We consider the general case (but still $L = 0$ and β_J »1)

$$\beta_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \frac{\kappa_B}{2} - \kappa_s, \tag{41}$$

$$\beta_J \frac{d\phi_2}{d\tau} + \sin\phi_2 = \frac{\kappa_B}{2} + \kappa_s, \tag{42}$$

$$\phi_2 = \phi_1 - 2\pi \frac{\Phi_{ext}}{\Phi_0}. \tag{43}$$

From the addition of Eqs.(41) and (42) with the help of the relation Eq.(43), we have

$$\beta_J \frac{d}{d\tau}(\phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0}) + \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \sin(\phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0}) = \frac{\kappa_B}{2}.$$

When we introduce a new parameter

$$\varphi_1 = \phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0},$$

we have the final form

$$\beta_J \frac{d\varphi_1}{d\tau} + \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \sin\varphi_1 = \frac{\kappa_B}{2}. \tag{44}$$

We are interested in the DC current-voltage characteristic so we need to determine the time averaged voltage

$$V = \left\langle \frac{\hbar}{2e} \frac{d\phi_1}{dt} \right\rangle = \left\langle I_c R \beta_J \frac{d\phi_1}{d\tau} \right\rangle = \frac{I_c R \beta_J}{T} \int_0^T \frac{d\phi_1}{d\tau} d\tau = I_c R \beta_J \frac{2\pi}{\tau}.$$

Here

$$\tau = \int_0^{2\pi} \frac{d\phi_1}{\frac{d\phi_1}{d\tau}} = \int_0^{2\pi} \frac{\beta_J d\phi_1}{\frac{\kappa_B}{2} - \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \sin\varphi_1} = \frac{\beta_J}{\cos(\pi \frac{\Phi_{ext}}{\Phi_0})} \int_0^{2\pi} \frac{d\phi_1}{\kappa' - \sin\varphi_1} = \frac{1}{\cos(\pi \frac{\Phi_{ext}}{\Phi_0})} \tau',$$

$$\kappa' = \frac{\kappa_B}{2\cos(\pi \frac{\Phi_{ext}}{\Phi_0})},$$

where

$$\tau' = \int_0^{2\pi} \frac{d\phi}{\kappa' - \sin\phi} = \int_0^{\pi} \frac{d\phi}{\kappa' - \sin\phi} + \int_0^{\pi} \frac{d\phi}{\kappa' + \sin\phi} = \int_0^{\pi} [\frac{1}{\kappa' - \sin\phi} + \frac{1}{\kappa' + \sin\phi}] d\phi,$$

or

$$\tau' = \frac{2\pi}{\sqrt{\kappa'^2 - 1}} \quad \text{for } \kappa > 1,$$

$$\tau' = 0 \qquad \text{for } \kappa < 1.$$

Then we have

$$V = 2\pi I_c R \beta_J \frac{1}{\tau} = \frac{2\pi I_c R}{\tau'} \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) = I_c R \sqrt{\kappa'^2 - 1} \cos(\pi \frac{\Phi_{ext}}{\Phi_0}),$$

or

$$\frac{V}{I_c R} = \left| \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \right| \sqrt{\kappa'^2 - 1},$$

or

$$|\kappa'| = \sqrt{1 + \left(\frac{V}{I_c R \cos(\pi \frac{\Phi_{ext}}{\Phi_0})} \right)^2} = \frac{1}{2\left|\cos(\pi \frac{\Phi_{ext}}{\Phi_0})\right|} \frac{I_B}{I_c},$$

or

$$\frac{I_B}{2I_c} = \sqrt{\cos^2(\pi \frac{\Phi_{ext}}{\Phi_0}) + (\frac{V}{RI_c})^2},$$

or

$$\frac{< V >}{RI_c} = \sqrt{(\frac{I_B}{2I_c})^2 - \cos^2(\pi \frac{\Phi_{ext}}{\Phi_0})},$$

or

$$< V > = \frac{R}{2} \sqrt{I_B{}^2 - 4 I_c{}^2 \cos^2(\pi \frac{\Phi_{ext}}{\Phi_0})}. \qquad (45)$$

When this equation for the voltage is compared with that for one Josephson junction with $\beta_J \gg 1$

$$\langle V \rangle = R\sqrt{I^2 - I_c{}^2}.$$

We find that the critical current is $2I_c|\cos(\pi\Phi_{ext}/\Phi_0)|$. This means that the critical current is $2I_c$ for $\Phi_{ext}/\Phi_0 = n$ (integer) and zero for $\Phi_{ext}/\Phi_0 = n+1/2$. In other words, the critical current is a periodic function of Φ with a period of Φ_0. However, the actual critical current does not oscillate between 0 and $2I_c$ because of the finite self-inductance L. In the above model, L (or β = 0) is assumed to be zero. The critical current varies between $2I_c$ and finite value depending on the value of β (see the detail in Sec.8).

When the total current I_B is constant, the voltage across the DC SQUID periodically changes with the external magnetic flux. This is the phenomenon one exploit to create the most sensitive magnetic field detection.

((Note)) Figure 29 is obtained from the Instruction manual of Mr. SQUID.[9]

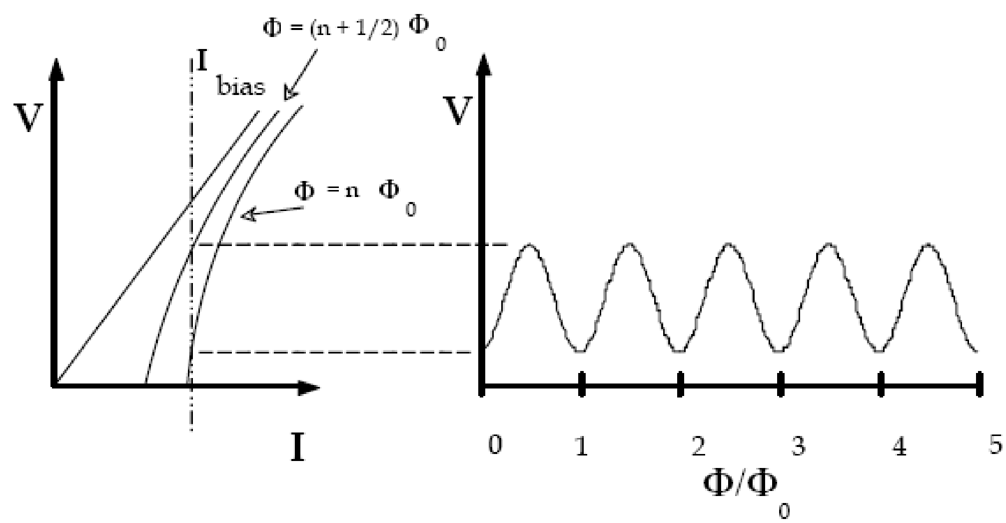


Fig.29 Detected voltage vs the magnetic flux Φ/Φ_0. The current I_B is kept at fixed value which is a little larger than $2I_c$. The detected voltage shows a maximum for $\Phi = (n+1/2)\Phi_0$, and a minimum for $\Phi = n\Phi_0$. The detected voltage is a periodic function of Φ with a period of Φ_0. (This figure is copied from the User Guide of Mr SQUID[9]).

7.6 More general case: β_J »1 and finite β

In order to avoid hysteresis in the *I-V* curve one usually choose over-damped junction. Here we start with the differential equations given by

$$\beta_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \frac{\kappa_B}{2} - \kappa_s, \quad (46)$$

$$\beta_J \frac{d\phi_2}{d\tau} + \sin\phi_2 = \frac{\kappa_B}{2} + \kappa_s, \quad (47)$$

$$\phi_1 - \phi_2 = 2\pi(\frac{\beta}{2}\kappa_s + \frac{\Phi_{ext}}{\Phi_0}). \quad (48)$$

or

$$\kappa_s = \frac{2}{\beta}(\frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{ext}}{\Phi_0}).$$

Thus we have two differential equations

$$\beta_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \frac{\kappa_B}{2} - \frac{2}{\beta}(\frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{ext}}{\Phi_0}),$$

$$\beta_J \frac{d\phi_2}{d\tau} + \sin\phi_2 = \frac{\kappa_B}{2} + \frac{2}{\beta}(\frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{ext}}{\Phi_0}),$$

$$\eta = \frac{1}{2}\beta_J(\frac{d\phi_1}{d\tau} + \frac{d\phi_2}{d\tau}) = \frac{1}{2}(\kappa_B - \sin\phi_1 - \sin\phi_2),$$

where the initial conditions [$\phi_1(0)$ and $\phi_2(0)$] are chosen appropriately

8 Simulation

The differential equations for $\phi_1(\tau)$ and $\phi_2(\tau)$ are numerically solved by using the Mathematica 5.2. We show our calculation on the τ dependence of η. The parameters κ_B, β_J, β and Φ_{ext}/Φ_0 are appropriately changed for our calculations.

8.1 Relation of <η> vs κ_B with Φ_{ext}/Φ_0 as a parameter

We calculate the relation <η> vs κ_B where Φ_{ext}/Φ_0 = 0, 0.05,0.1, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, and 0.5. We choose β_J = 10 for the over-damped case. So that no hysteresis is seen in the *I-V* curve. The parameter β is changed as a parameter: β = 0.02 - 3. The voltage <η> suddenly increases from zero to a finite value at the critical current which is dependent on the magnetic flux Φ_{ext}/Φ_0 and β.

(i) Using the following Mathematica 5.2 program, we find the maximum and minimum of $\Omega(\tau)$ in the long time region where $\Omega(\tau)$ periodically oscillates with τ.

(ii) The average <Ω> is calculated as (maximum+minimum)/2. The average voltage $<\eta>=\beta_J<\Omega>$ is plotted as a function of κ for the fixed β_J (= 10) and β, where the magnetic flux Φ_{ext}/Φ_0 is changed as a parameter.

(iii) The voltage is equal to zero for κ<(κ_B)$_c$. It suddenly increase with increasing κ above (κ_B)$_c$. We determine the critical current (κ_B)$_c$ as a function of the magnetic flux where β is changed as a parameter.

((Mathematica 5.2)) Program-12

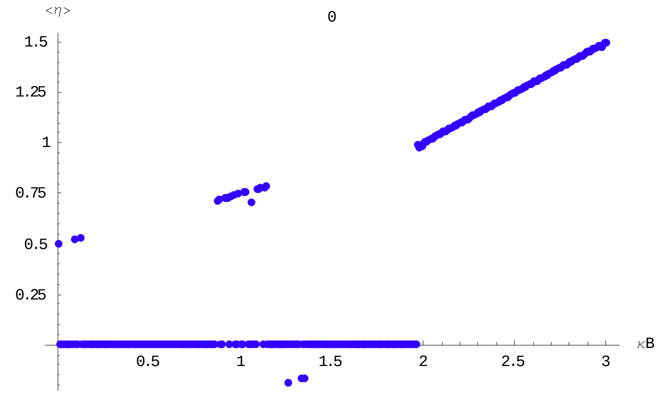
```
Clear["Global`*"]
 <<Graphics`Graphics`
 (*Subroutine, DC SQUID beta=1 betaJ=10 voltage vs magnetic
flux*)
```

```
DCSQ[{ϕ01_, ϕ02_}, {βJ_, κB_, β_, N0_}, τmax_, opts__] :=
 Module[{numso1, numgraph},
  numso1 =
   NDSolve[{ βJ ϕ1'[τ] + Sin[ϕ1[τ]] == κB/2 - 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
     βJ ϕ2'[τ] + Sin[ϕ2[τ]] == κB/2 + 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0), ϕ1[0] == ϕ01,
     ϕ2[0] == ϕ02}, {ϕ1[τ], ϕ2[τ]}, {τ, 0, τmax}] // Flatten;
  numgraph = Plot[((κB - Sin[ϕ1[τ]] - Sin[ϕ2[τ]])/2) /. numso1, {τ, 0, τmax},
    opts, DisplayFunction → Identity]];
Max1[{ϕ01_, ϕ02_}, {βJ_, κB_, β_, N0_}, {τmin_, τmax__}] :=
 Module[{numso1},
  numso1 =
   NDSolve[{ βJ ϕ1'[τ] + Sin[ϕ1[τ]] == κB/2 - 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
     βJ ϕ2'[τ] + Sin[ϕ2[τ]] == κB/2 + 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0), ϕ1[0] == ϕ01,
     ϕ2[0] == ϕ02}, {ϕ1[τ], ϕ2[τ]}, {τ, 0, τmax}] // Flatten;
  maximum = FindMaximum[((κB - Sin[ϕ1[τ]] - Sin[ϕ2[τ]])/2) /. numso1,
    {τ, τmin, τmax}]];
Min1[{ϕ01_, ϕ02_}, {βJ_, κB_, β_, N0_}, {τmin_, τmax__}] :=
 Module[{numso1},
  numso1 =
   NDSolve[{ βJ ϕ1'[τ] + Sin[ϕ1[τ]] == κB/2 - 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
     βJ ϕ2'[τ] + Sin[ϕ2[τ]] == κB/2 + 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0), ϕ1[0] == ϕ01,
     ϕ2[0] == ϕ02}, {ϕ1[τ], ϕ2[τ]}, {τ, 0, τmax}] // Flatten;
  maximum = FindMinimum[((κB - Sin[ϕ1[τ]] - Sin[ϕ2[τ]])/2) /. numso1, {τ, τmin, τmax}]]
```

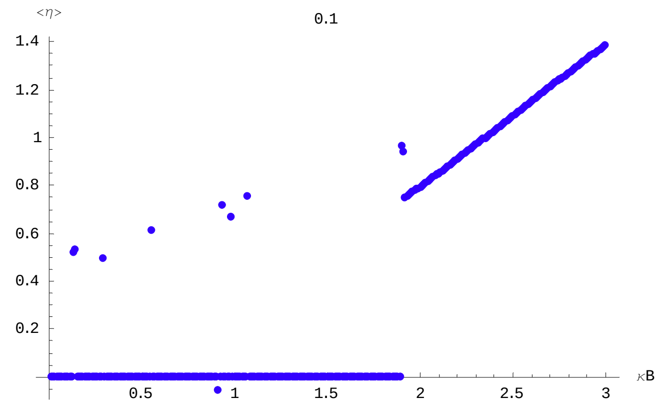
```
Sei[βJ_,β_,N0_]:=Module[{A1,B1,list1,ave1},A1=Max1[{0,0},{β
J,#,β,N0},{400 π,800π}]&/@Range[0,3,0.01];
B1=Min1[{0,0},{βJ,#,β,N0},{400
π,800π}]&/@Range[0,3,0.01] ;ave1=(A1+B1)/2;
list1=Table[{0.01 (k-1),ave1[[k,1]]},{k,1,301}]]
 h[n0_]:=ListPlot[Evaluate[Sei[10,1,n0]],PlotStyle→{Hue[2
n0],PointSize[0.01]},AxesLabel→{"κB","η"},PlotLabel→NumberF
orm[n0]]
 f1=Table[h[p],{p,0,0.5,0.05}]
```

Fig.30 $<\eta>=\beta_J<\Omega>$ vs κ (β_J = 10 and β = 1) for Φ_{ext}/Φ_0 = 0 – 0.5. The number denoted in each curve is the value of Φ_{ext}/Φ_0. Note that $<\eta>$ takes several values at the same κ around $\Phi_{ext}/\Phi_0 = 0.4$. All solutions are plotted in the figures. Some values are unphysical. We do not understand why so many values appear. Some solutions may correspond to metastable states.

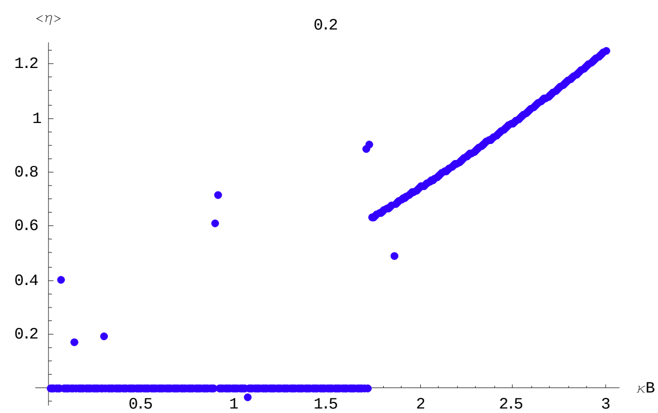
(1) $\Phi_{ext}/\Phi_0 = 0$. $\beta = 1$ and $\beta_J = 10$



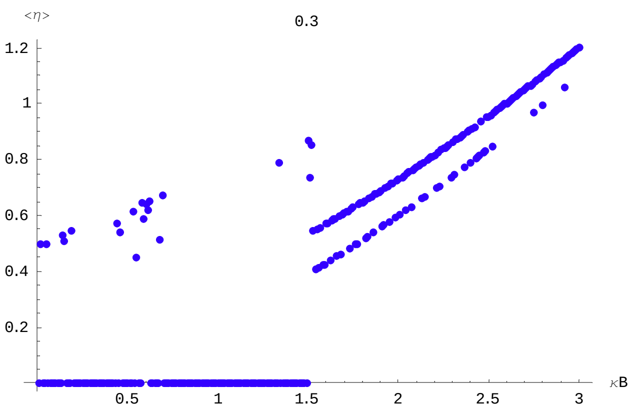
(2) $\Phi_{ext}/\Phi_0 = 0.1$. $\beta = 1$ and $\beta_J = 10$



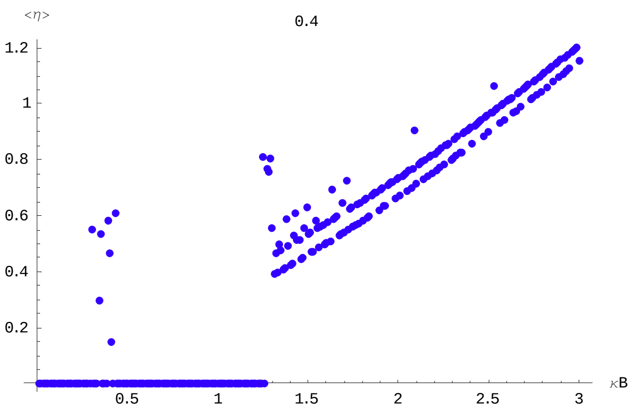
(3) $\Phi_{ext}/\Phi_0 = 0.2$. $\beta = 1$ and $\beta_J = 10$



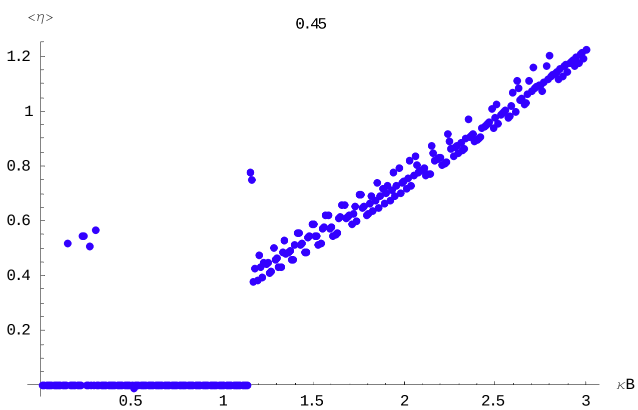
(4) $\Phi_{ext}/\Phi_0 = 0.3$. $\beta = 1$ and $\beta_J = 10$



(5) $\Phi_{ext}/\Phi_0 = 0.4$. $\beta = 1$ and $\beta_J = 10$



(6) $\Phi_{ext}/\Phi_0 = 0.45$. $\beta = 1$ and $\beta_J = 10$



(7) $\Phi_{ext}/\Phi_0 = 0.5$. $\beta = 1$ and $\beta_J = 10$

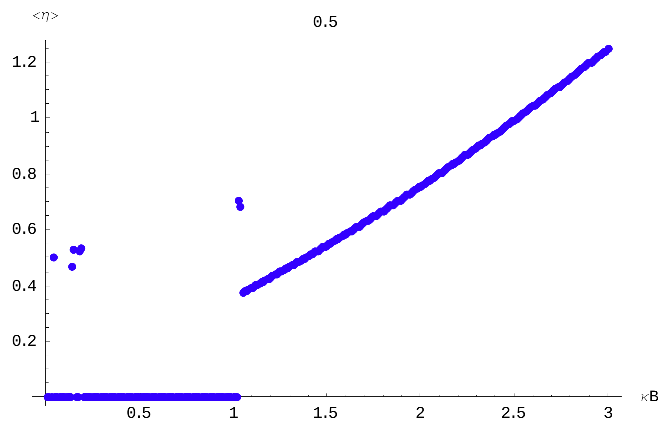
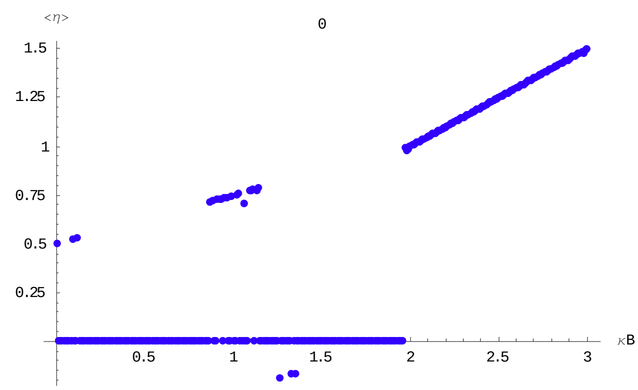
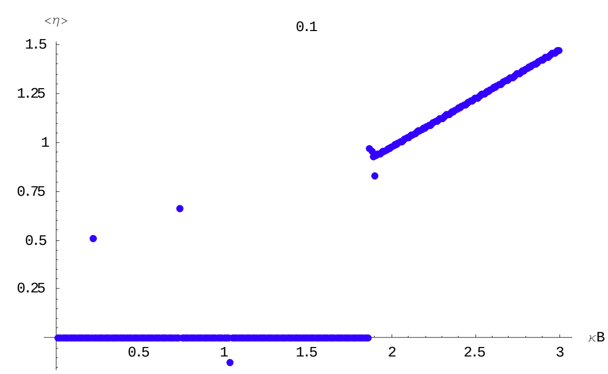


Fig.31 $<\eta>=\beta_J<\Omega>$ vs κ (β_J = 10) and (β = 0.3) for Φ_{ext}/Φ_0=0 – 0.5. The number denoted in each curve is the value of Φ_{ext}/Φ_0. Note that $<\eta>$ takes several values at the same κ around Φ_{ext}/Φ_0 = 0.4 (multi-valued function). All solutions are plotted in the figures. Some values are unphysical. We do not understand why so many values appear. Some solutions correspond to metastable states.

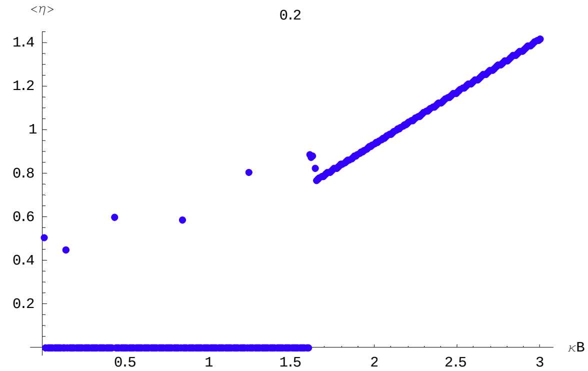
(1) Φ_{ext}/Φ_0 = 0. β = 0.3 and β_J = 10



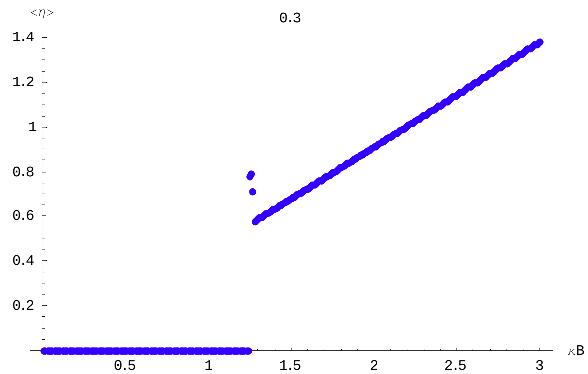
(2) Φ_{ext}/Φ_0 = 0.1. β = 0.3 and β_J = 10



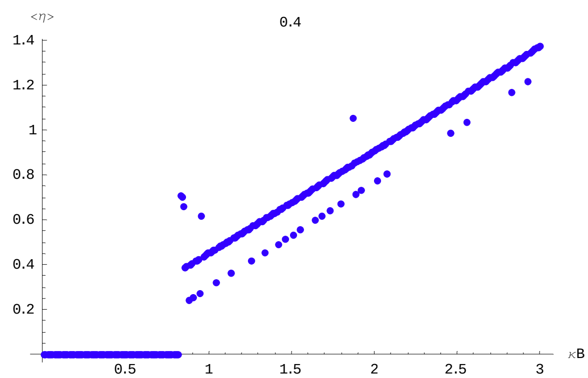
(3) $\Phi_{ext}/\Phi_0 = 0.2$. $\beta = 0.3$ and $\beta_J = 10$



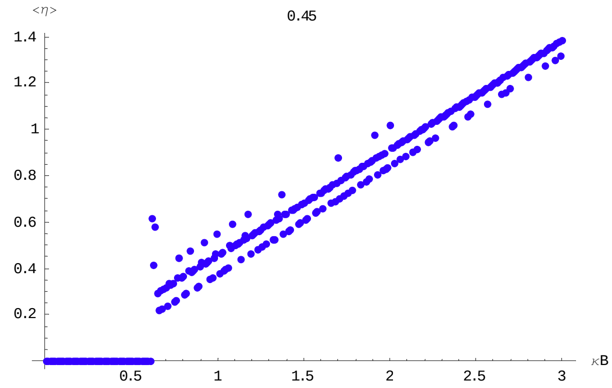
(4) $\Phi_{ext}/\Phi_0 = 0.3$. $\beta = 0.3$ and $\beta_J = 10$



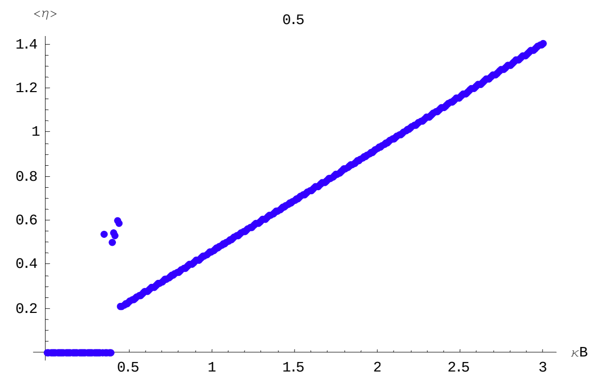
(4) $\Phi_{ext}/\Phi_0 = 0.4$. $\beta = 0.3$ and $\beta_J = 10$



(5) $\Phi_{ext}/\Phi_0 = 0.45$. $\beta = 0.3$ and $\beta_J = 10$



(6) Φ_{ext}/Φ_0 = 0.5. β = 0.3 and β_J = 10



8.2 Critical current vs Φ_{ext}/Φ_0 with β as a parameter

From the above simulation we find that the critical current $(\kappa_B)_c$ decreases with increasing the magnetic flux from 2 at Φ_{ext}/Φ_0 = 0 to some finite value (but not zero) at Φ_{ext}/Φ_0 = 0.5 because of the finite value of β (finite inductance). First we estimate the critical current analytically based on an approximation $\sin\phi \approx 2\phi/\pi$ for $|\phi|<\pi/2$ (one can easily prove this using the Mathematica 5.2). We use the following approximations,

$$\kappa_1 = \sin\phi_1 = \frac{2}{\pi}\phi_1, \qquad \kappa_2 = \sin\phi_2 = \frac{2}{\pi}\phi_2, \qquad \kappa_s = \frac{\kappa_2 - \kappa_1}{2}, \quad \kappa_B = \kappa_1 + \kappa_2,$$

$$\phi_1 - \phi_2 = 2\pi\left(\frac{\beta}{2}\kappa_s + \frac{\Phi_{ext}}{\Phi_0}\right) = \frac{\pi}{2}(\kappa_1 - \kappa_2) = -\pi\kappa_s \text{ or } \beta\kappa_s + \frac{2\Phi_{ext}}{\Phi_0} = -\kappa_s .$$

From this relation we have

$$\kappa_s = -\frac{\Phi_{ext}}{\Phi_0}\frac{2}{\beta+1} = \frac{\kappa_2 - \kappa_1}{2},$$

$$\kappa_1 + \kappa_2 = \kappa_B, \quad \kappa_1 - \kappa_2 = \frac{\Phi_{ext}}{\Phi_0}\frac{4\kappa_B}{(\beta+1)},$$

or

$$\kappa_1 = \frac{\kappa_B}{2} + \frac{2}{\beta+1}\frac{\Phi_{ext}}{\Phi_0}, \tag{49}$$

$$\kappa_2 = \frac{\kappa_B}{2} - \frac{2}{\beta+1}\frac{\Phi_{ext}}{\Phi_0}.$$

The value of κ_1 has a maximum at Φ_{ext}/Φ_0 = 1/2. The condition for the critical current is that the maximum of κ_1 should be equal to 1. As Φ_{ext}/Φ_0 changes from zero to 1/2, the value of κ_1 changes from $\kappa_1 = \kappa_B/2$ to

$$\kappa_1 = \frac{\kappa_B}{2} + \frac{1}{\beta+1},$$

Since the critical current of κ_1 is equal to 1, the critical current of κ_B should be equal to

$$(\kappa_1)_c = \frac{(\kappa_B)_c}{2} + \frac{1}{\beta+1} = 1$$

or

$$(\kappa_B)_c = 2(1-\frac{1}{\beta+1}) = \frac{2\beta}{\beta+1}. \qquad (50)$$

Note that the change in the SQUID voltage is approximated by

$$\Delta V = R_N I_c \Delta\kappa_1 = R_N I_c \frac{1}{\beta+1} = \frac{R_N I_c}{\beta+1}, \qquad (51)$$

where $\Delta\kappa_1 = 1/(\beta+1)$: $\kappa_1 = \kappa_B/2$ at Φ_{ext}/Φ_0 = 0 and $\kappa_1 = (\kappa_B/2) + 1/(\beta+1)$. We assume that the normal-state resistance of the DC SQUID is $R_N/2$: the slope of *I-V* curve is given by $R_N/2$, but not by R_N. So the resistance of each Josephson junction is R_N since the parallel configuration. In our Mr. SQUID, we have $R_N/2$ = 1.44 Ω and $2I_c$ = 66mA.

In Figs.30 and 31, we show the plot of $<\eta>$ vs κ_B. The zero-voltage state ($<\eta>$ = 0) is stable for $\kappa_B \le (\kappa_B)_c$, where $(\kappa_B)_c$ is the critical current. Figure 32 shows the critical current $(\kappa_B)_c$ as a function of Φ_{ext}/Φ_0, where β is changed as a parameter and β_J = 10. We find that $(\kappa_B)_c$ decreases with increasing the magnetic flux Φ_{ext}/Φ_0. There occurs a transition at $\kappa_B = (\kappa_B)_c$ from the zero-voltage state ($<\eta>$ = 0) to a finite-voltage state ($<\eta> \approx \kappa_B$). In Fig.33 we show the plot of $(\kappa_B)_c$ as a function of β at Φ_{ext}/Φ_0 = 1/2, where β_J = 10. The data point fall well on the solid line denoted by $(\kappa_B)_c = 2\beta/(\beta+1)$.

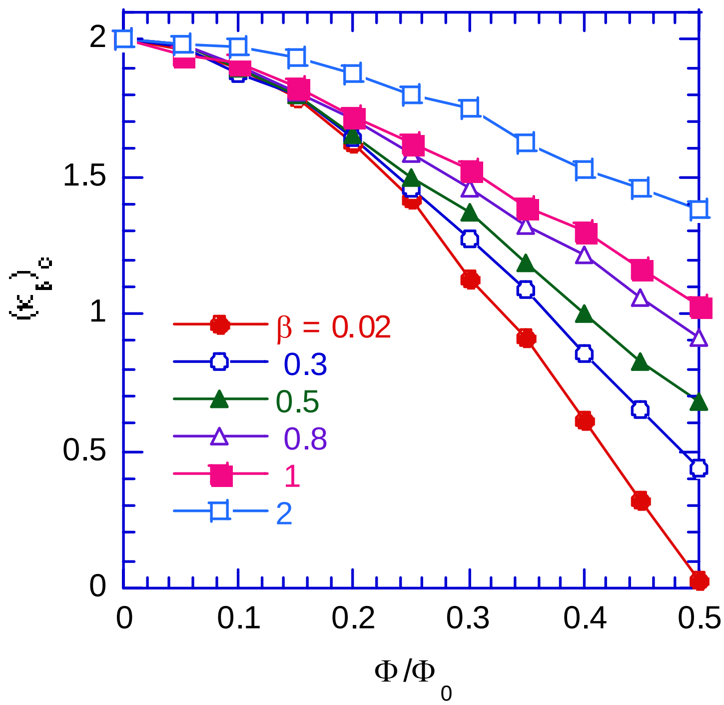


Fig.32 The critical current $(\kappa_B)_c$ as a function of Φ_{ext}/Φ_0, where β is changed as a parameter. $\beta_J = 10$.

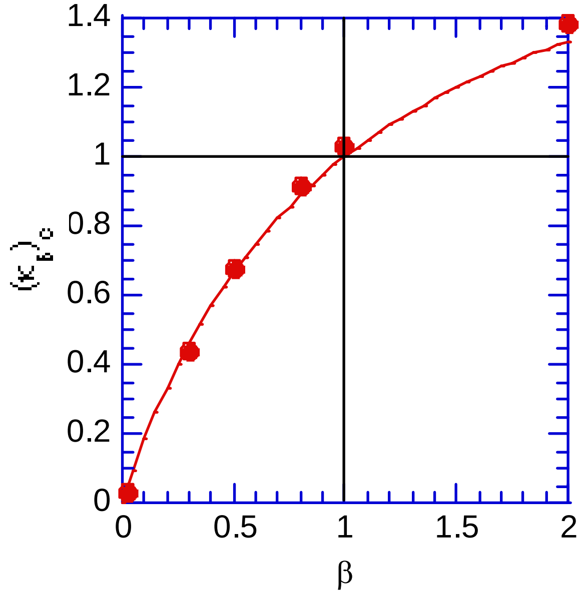


Fig.33 The critical current $(\kappa_B)_c$ at $\Phi_{ext}/\Phi_0 = 1/2$ as a function of β. $\beta_J = 10$. The solid line denotes the expression given by $(\kappa_B)_c = 2\beta/(1+\beta)$.

8.3 Relation of $<\eta>$ vs Φ_{ext}/Φ_0 with κ_B as a parameter

We calculate the magnetic flux (Φ_{ext}/Φ_0) dependence on the average voltage, where κ_B is changed as a parameter. When κ_B is fixed, the average voltage $<\eta>$ periodically changes with Φ_{ext}/Φ_0 [the periodicity $\Delta(\Phi_{ext}/\Phi_0) = 1$]. In Fig.34, we show our calculation for κ_B = 1.5 – 2.9. We find that $<\eta>$ is a multivalued function of Φ_{ext}/Φ_0. We think that the lowest curve may be a stable solution. This curve has a maximum at Φ_{ext}/Φ_0 = 1/2 and 0 near Φ_{ext}/Φ_0 = 0 and 1. Note that we do not take into account of the effect of Johnson noise. This is a principle of the DC SQUID. The element plays a role of the transformation between the voltage and the magnetic flux.

((Mathematica 5.2)) Program-13

```
Clear["Global`*"]
 <<Graphics`Graphics`
 (*Subroutine, DC SQUID beta=1 betaJ=10 voltage vs magnetic
flux*)
```

```
DCSQ[{ϕ01_, ϕ02_}, {βJ_, κB_, β_, N0_}, τmax_, opts__] :=
 Module[{numso1, numgraph},
  numso1 =
   NDSolve[
     {βJ ϕ1'[τ] + Sin[ϕ1[τ]] == κB/2 - 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
      βJ ϕ2'[τ] + Sin[ϕ2[τ]] == κB/2 + 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
      ϕ1[0] == ϕ01, ϕ2[0] == ϕ02}, {ϕ1[τ], ϕ2[τ]}, {τ, 0, τmax}] //
    Flatten;
  numgraph = Plot[((κB - Sin[ϕ1[τ]] - Sin[ϕ2[τ]])/2) /. numso1,
    {τ, 0, τmax}, opts, DisplayFunction → Identity]];
Max1[{ϕ01_, ϕ02_}, {βJ_, κB_, β_, N0_}, {τmin_, τmax__}] :=
 Module[{numso1},
  numso1 =
   NDSolve[
     {βJ ϕ1'[τ] + Sin[ϕ1[τ]] == κB/2 - 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
      βJ ϕ2'[τ] + Sin[ϕ2[τ]] == κB/2 + 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
      ϕ1[0] == ϕ01, ϕ2[0] == ϕ02}, {ϕ1[τ], ϕ2[τ]}, {τ, 0, τmax}] //
    Flatten;
  maximum =
   FindMaximum[((κB - Sin[ϕ1[τ]] - Sin[ϕ2[τ]])/2) /. numso1,
    {τ, τmin, τmax}]];
Min1[{ϕ01_, ϕ02_}, {βJ_, κB_, β_, N0_}, {τmin_, τmax__}] :=
 Module[{numso1},
  numso1 =
   NDSolve[
     {βJ ϕ1'[τ] + Sin[ϕ1[τ]] == κB/2 - 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
      βJ ϕ2'[τ] + Sin[ϕ2[τ]] == κB/2 + 2/β ((ϕ1[τ] - ϕ2[τ])/(2 π) - N0),
      ϕ1[0] == ϕ01, ϕ2[0] == ϕ02}, {ϕ1[τ], ϕ2[τ]}, {τ, 0, τmax}] //
    Flatten;
  maximum =
   FindMinimum[((κB - Sin[ϕ1[τ]] - Sin[ϕ2[τ]])/2) /. numso1,
    {τ, τmin, τmax}]]

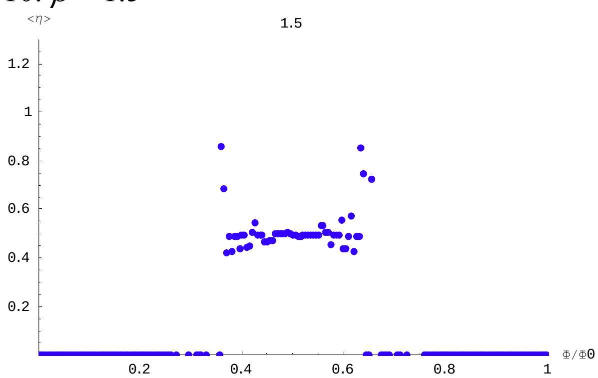
Sawako[βJ_,β_,κB_]:=Module[{A1,B1,list1,ave1},A1=Max1[{0,0}
,{βJ,κB,β,#},{400 π,800π}]&/@Range[0,1,0.005];
B1=Min1[{0,0},{βJ,κB,β,#},{400
```

```
π,800π}]&/@Range[0,1,0.005] ;ave1=(A1+B1)/2;
list1=Table[{0.005 (k-1),ave1[[k,1]]},{k,1,201}]]

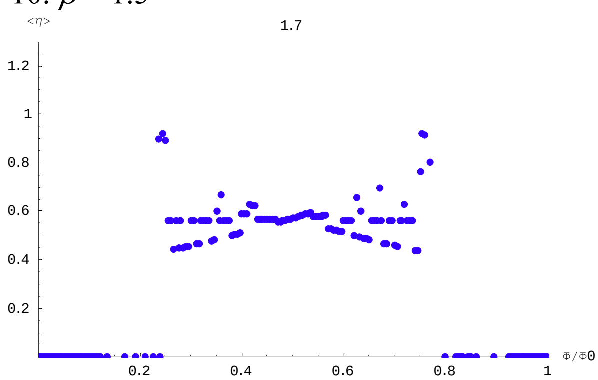
g[κB_]:=ListPlot[Evaluate[Sawako[10,1.5,κB]],PlotStyle→{Hue
[0.7],PointSize[0.015]},AxesLabel→{"Φ/Φ0","<η>"},PlotLabel→
NumberForm[κB], PlotRange→{{0,1},{0,1.3}}]
 f1=Table[g[κB],{κB,0.5,3,0.1}]
```

Fig.34 The average voltage $<\eta>$ vs Φ_B/Φ_0, where κ_B is changed as a parameter. $\beta_J = 10$. $\beta = 1.5$.

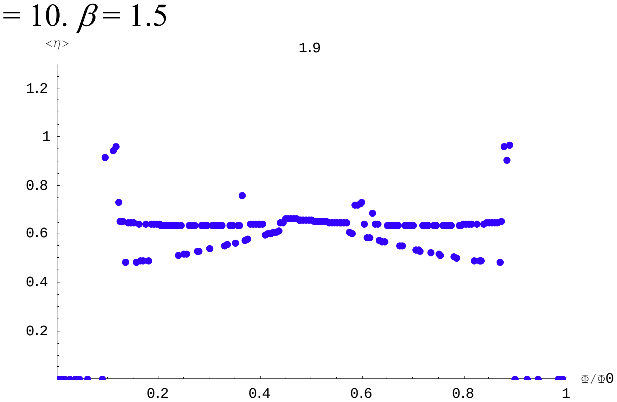
(1) $\kappa_B = 1.5$. $\beta_J = 10$. $\beta = 1.5$



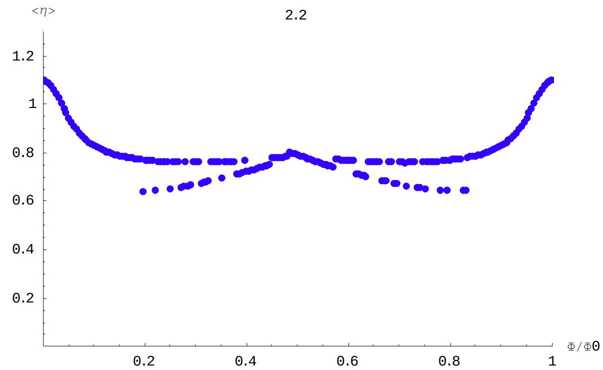
(2) $\kappa_B = 1.7$. $\beta_J = 10$. $\beta = 1.5$



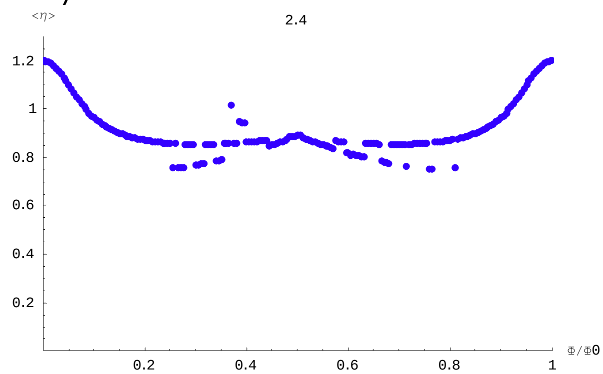
(3) $\kappa_B = 1.9$. $\beta_J = 10$. $\beta = 1.5$



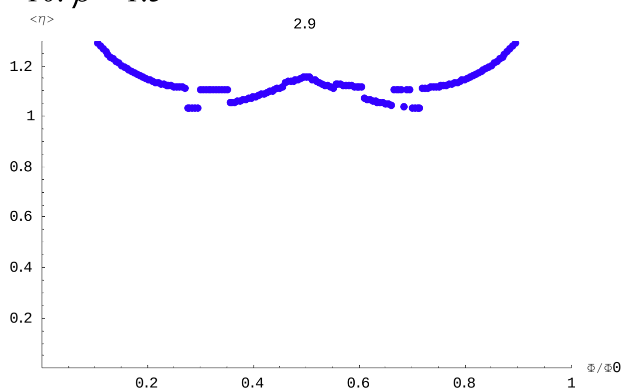
(4) $\kappa_B = 2.2$. $\beta_J = 10$. $\beta = 1.5$



(5) $\kappa_B = 2.4.$ $\beta_J = 10.$ $\beta = 1.5$



(6) $\kappa_B = 2.9.$ $\beta_J = 10.$ $\beta = 1.5$



8.4 Loop current

Here we discuss how the loop current changes with the time τ, depending on the total current κ_B and the external magnetic flux Φ_{ext}. The loop current is given by

$$\kappa_s = \frac{2}{\beta}\left(\frac{\phi_1 - \phi_2}{2\pi} - \frac{\Phi_{ext}}{\Phi_0}\right). \qquad (52)$$

Here we consider one typical case: $\beta_J = 10$, $\beta = 1$, and Φ_{ext}/Φ_0 being changed as a parameter. We choose the initial condition that $\phi_1(0) = \phi_2(0) = 0$.

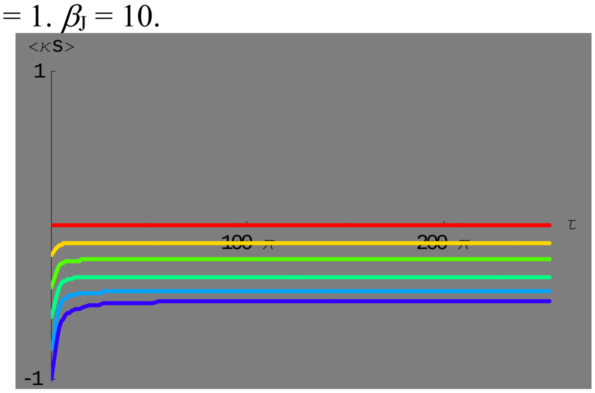
((Mathematica 5.2)) Program-14

```
Clear["Global`*"]
```

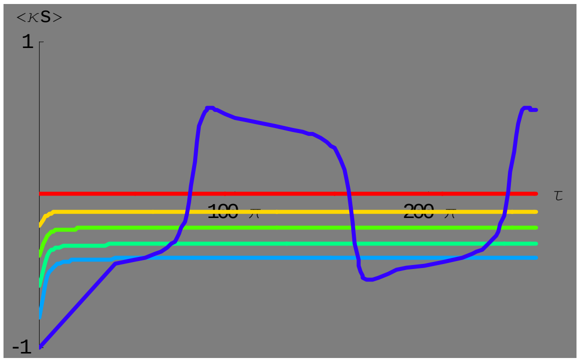
```
<<Graphics`Graphics`
 (*Subroutine, DC SQUID Magnetic flux dependence of loop
current*)
DCSQ[{φ01_, φ02_}, {βJ_, κB_, β_, N0_}, τmax_, opts__] :=
  Module[{numsol, numgraph},
   numsol =
    NDSolve[{ βJ φ1'[τ] + Sin[φ1[τ]] == κB/2 - 2/β ((φ1[τ] - φ2[τ])/(2 π) - N0),
      βJ φ2'[τ] + Sin[φ2[τ]] == κB/2 + 2/β ((φ1[τ] - φ2[τ])/(2 π) - N0), φ1[0] == φ01,
      φ2[0] == φ02}, {φ1[τ], φ2[τ]}, {τ, 0, τmax}] // Flatten;
   numgraph = Plot[2/β ((φ1[τ] - φ2[τ])/(2 π) - N0) /. numsol, {τ, 0, τmax},
     opts, DisplayFunction → Identity]];
 phlist=DCSQ[{0,0},{10,1,1,#},3000,  PlotStyle→Hue[1.4
(#+5)], AxesLabel→{"τ","<κs>"},Prolog→AbsoluteThickness[2],
Background→GrayLevel[0.5],PlotRange→{{0,800},{-
1,1}},Ticks→{  π Range[0,200,100], Range[-1,2]},
DisplayFunction→Identity]&/@Range[0,0.5,0.1 ];Show[phlist,D
isplayFunction→$DisplayFunction]
```

Fig.35 κ_S vs τ where Φ_{ext}/Φ_0 = 0.1, 0.2, 0.3, 0.4, and 0.5. κ_B is changed as a parameter. β = 1. β_J = 10. Note that in the figures the y axis should be κ_S, but not <κ_S>.

(1) κ_B = 1.0 and Φ_{ext}/Φ_0 = 0, (red), 0.1, 0.2, 0.3, 0.4, and 0.5 (purple) from the top to the bottom. β = 1. β_J = 10.

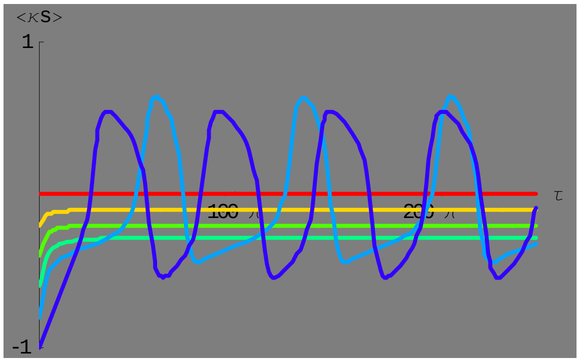


(2) κ_B = 1.1 and Φ_{ext}/Φ_0 = 0.1, 0.2, 0.3, 0.4, and 0.5. β = 1. β_J = 10.



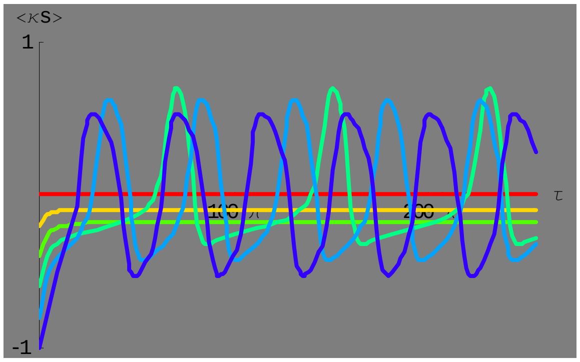
(3) $\kappa_B = 1.4$ and $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4$, and 0.5. $\beta = 1$. $\beta_J = 10$.

The loop current starts to oscillate with time only for $\Phi_{ext}/\Phi_0 = 0.4$ and 0.5.

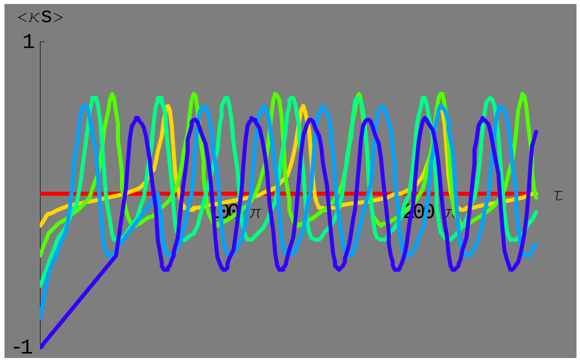


(4) $\kappa_B = 1.6$ and $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4$, and 0.5. $\beta = 1$. $\beta_J = 10$.

The loop current starts to oscillate with time only for $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4$ and 0.5.



(5) $\kappa_B = 2.0$ and $\Phi_{ext}/\Phi_0 = 0.1, 0.2, 0.3, 0.4$, and 0.5. $\beta = 1$. $\beta_J = 10$.



9 CONCLUSION

We have discussed the physics of the Josephson junction and the principle of the DC SQUID. We do not discuss the *rf* SQUID (consisting of only one Josephson junction) the principle of the SQUID magnetometer. The SQUID magnetometer is the most sensitive measurement device. It can measure magnetic flux on the order of one flux quantum. The magnetic properties of magnetic systems including spin glass, superspin glass, and superparamagnet are studied using the SQUID magnetometer (MPMS XT-5, Quantum Design) in our Laboratory. Finally we want to say that a book (in Japanese) written by Ohtsuka[12] is very useful for our understanding of the principle of the RSI model and DC SQUID model.

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This book was very useful for us in writing this lecture note. However,unfortunately this book was written in Japanese.

APPENDIX

For convenience, the following programs of Mathematica 5.2 are presented.

(1) Program-5 (Fig.12): Phase space Ω vs ϕ with vector field. $\beta_J = 0.60$. κ is changed as a parameter.

(2) Program-6 (Fig.13): Ω vs τ for $\beta_J = 0.60$. κ is changed as a parameter.

(3) Program-9 (Fig.16): average voltage vs current

(4) Program-13 (Fig.34) average voltage vs magnetic flux