Fermi liquid theory Masatsugu Sei Suzuki Department of Physics, SUNY at Binghamton (Date: January 13, 2019)

1. Introduction

Here we consider the normal state of interacting Fermi particles without any long range order. The normal state at low temperatures is called the Fermi liquid and is considered to be the system of free quasiparticles that is continuously connected with free Fermi gas. The concept of the Fermi liquid was introduced and developed by L. D. Landau. Landau's Fermi liquid theory, which concentrates rich contents.

The electrons are fermion with spin 1/2. The system of many electrons is a collection of Fermi particles. The behavior of electrons in solids can be well explained in terms of quantum mechanics; the nature of duality of wave and particle. The electron has a negative charger. There is a repulsive Coulomb interaction between electrons. Nevertheless, the electron in metal is well described by a free fermion without any interactions. Why is a simple free electron model so useful in spite of complicated nature of electrons in metals? What is the role of interactions between electrons? Here we discuss the concept of the Fermi liquid theory which is essential to understanding the effectiveness of free electron model.

Here we discuss the relaxation time in the Fermi liquid theory. Several methods for the derivation of the relaxation time will be introduced below.

2. Relaxation time due to the collisions of Fermi particles

The interaction between electron in metal is relatively large. It seems that there are many collisions between electrons repeatedly. This is not the case. Because of the Pauli's exclusion principle, the probability for collisions is suppressed as low. The reciprocal of the relaxation time for the scattering of electrons is given by

$$\frac{\hbar}{\tau} \simeq \frac{\left(k_B T\right)^2}{\varepsilon_F}$$

liquid ³He.

When $k_BT \ll \varepsilon_F$, $\frac{\hbar}{\tau}$ becomes sufficiently small. The scattering effect is neglected. The wave vector **k** of electron can be regarded as good quantum number. This is the fundamental of the Landau Fermi liquid theory. Typical examples of Fermi liquids are (i) electrons in metals and (ii)

According to the Fermi golden rule, the relaxation time due to the scattering of electrons is given by

$$\frac{1}{\tau(\mathbf{p}_{1})} = \frac{2\pi}{\hbar} \frac{1}{N_{A}^{2}} \sum_{\mathbf{p}_{2}, \mathbf{p}_{1}', \mathbf{p}_{2}'} U^{2} \delta(\varepsilon_{1} + \varepsilon_{2} - \varepsilon_{1}' - \varepsilon_{2}')$$
$$\delta_{\mathbf{p}_{1} + \mathbf{p}_{2}, \mathbf{p}_{1}' + \mathbf{p}_{2}'} f_{\mathbf{p}_{2}} (1 - f_{\mathbf{p}_{1}'}) (1 - f_{\mathbf{p}_{2}'})$$

where N_A is the number of particles.



Fig. Four states of electrons in the vicinity of the Fermi surface. See the detail in the text.

3. Derivation of the relaxation time by Anderson

It is no doubt well known that the resulting mean free time varies as the square of the energy increment,

$$\Delta E = \frac{\hbar^2}{2m} [(k_F + a)^2 - k_F]^2 = \frac{\hbar^2}{m} k_F a$$
$$\frac{\hbar}{\tau} = \text{consts.} \times \frac{|V|^2 (\Delta E)^2}{\varepsilon_F^3}$$

This may be seen by observing that for any final state with the hole in state k' energy conservation requires:

$$k^{2} - (k - q)^{2} = (k' + q)^{2} - k'^{2}$$

or

$$k^{2} - (k^{2} + q^{2} - 2k \cdot q) = (k^{2} + q^{2} - 2k \cdot q) - k^{2}$$

or

$$(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{q}=\boldsymbol{q}^2$$

or

$$(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{q}) \cdot \boldsymbol{q} = 0$$



Fig. Four states: $|\mathbf{k}\rangle$ (A, outside the Fermi sphere), $|\mathbf{k} - q\rangle$ (B, outside the Fermi surface), $|\mathbf{k}'\rangle$ (D, inside the Fermi sphere), $|\mathbf{k}' + q\rangle$ (C, outside the Fermi sphere). H' is the center of the circle (denoted by green line) where the points A, B, C, and D lie on. The diameter of the circle is $|\mathbf{k} - \mathbf{k}'|$.

This is the equation of a sphere whose diameter is $|\mathbf{k} - \mathbf{k}'|$: thus we know that $\mathbf{k}' + \mathbf{q}$ must lie on the sphere whose diameter is the vector from \mathbf{k} to \mathbf{k}' . When a is small, this sphere can be approximated by planes through \mathbf{k} and \mathbf{k}' , since $\mathbf{k}', \mathbf{k}, \mathbf{k} - \mathbf{q}$, and $\mathbf{k}' + \mathbf{q}$ must all lie near the Fermi surface (this is true except for $\mathbf{k}' \approx -\mathbf{k}$, which is a very small fraction of the available states). Now clearly, if \mathbf{k}' lies within $(a \cos \theta)$ of the surface, two types of scattering process are possible; if we pick a $\mathbf{q} \perp (\mathbf{k} - \mathbf{k}')$. and small $(\mathbf{q} > |\mathbf{k}_F - \mathbf{k}'| \cos \theta, \mathbf{q} < a \cos \theta)$, we may have

$$|k\rangle \rightarrow |k-q\rangle, \qquad |k'\rangle \rightarrow |k'+q\rangle$$

or

$$|k\rangle \rightarrow |k'+q\rangle, \qquad |k'\rangle \rightarrow |k-q\rangle$$

In either case, the possible ranges both of k' and of q for a given θ are proportional to $a \propto \Delta E$: thus the total scattering probability at this θ is $(\Delta E)^2$. I leave it as an exercise to perform the integration over θ and get the full expression for the \hbar/τ of the electron of a given ΔE .

$$x = \frac{a}{k_F}$$

$$\overrightarrow{OA} = k_F[(-(1+x)\cos\theta, (1+x)\sin\theta] \quad \text{outside the Fermi surface}$$

$$\overrightarrow{OB} = k_F[(-(1+x)\cos\theta, (1-x)\sin\theta] \quad \text{outside the Fermi surface}$$

$$q = k_F(0, 2x\sin\theta)$$

$$\overrightarrow{OH} = k_F[0, \sin\theta]$$

$$\overrightarrow{OH} = k_F[\cos\theta, \sin\theta]$$

$$\overrightarrow{OD} = \overrightarrow{OG} + \frac{q}{2}$$

$$= k_F[\cos\theta, \sin\theta] + k_F(0, x\sin\theta) \quad \text{on the Fermi surface}$$

$$= k_F[\cos\theta, (1+x)\sin\theta]$$

$$\overrightarrow{OC} = \overrightarrow{OG} - \frac{q}{2}$$

$$= k_F[\cos\theta, (1-x)\sin\theta]$$
inside the Fermi surface

$$= k_F[\cos\theta, (1-x)\sin\theta]$$

(a) We note that

$$\left| \overrightarrow{OB} \right| = k_F \sqrt{(1+x)^2 \cos^2 \theta + (1-x)^2 \sin^2 \theta}$$
$$= k_F \sqrt{1+x^2 + 2x \cos(2\theta)}$$

When the point B is on the Fermi sphere with $\left|\overrightarrow{OB}\right| = 1$, we have

$$x + 2\cos(2\theta)$$
,

or

$$x = -2\cos(2\theta)$$
 $\theta = \frac{1}{2}\arccos(-x/2)$

When x = 0, $\theta = \frac{\pi}{4}$

(b)

We note that

$$\left| \overrightarrow{OC} \right| = k_F \sqrt{\cos^2 \theta + (1+x)^2 \sin^2 \theta}$$
$$= k_F \sqrt{1 + x(2+x) \sin^2 \theta}$$
$$\left| \overrightarrow{OD} \right| = k_F \sqrt{\cos^2 \theta + (1-x)^2 \sin^2 \theta}$$
$$= k_F \sqrt{1 - x(2-x) \sin^2 \theta}$$

indicating that

$$\left|\overrightarrow{OC}\right| > k_F, \qquad \left|\overrightarrow{OD}\right| < k_F$$





2. Approach by Abrikosov and Khalatnikov



Fig.

Let us define the probability for a weakly interacting gas. If there is a particle 1 outside the Fermi sphere, then the process of first order in the interaction will be as follows. Particle 1 interacts with particle 2 inside the Fermi sphere, following which the two particles pass over to states 1' and 2' outside the Fermi sphere. Because of the Pauli principle, this is the only possibility. The law of momentum conservation requires that

$$p_1 + p_2 = p_1' + p_2'$$

and, in accordance with what has been seen said above

$$p_1 > p_F$$
, $p_2 < p_F$, $p_1' > p_F$, $p_2' > p_F$,

The planes (p_1, p_2) and (p_1', p_2') do not coincide, generally speaking, and in Fig.1 they are simply superposed by rotation. The scattering probability is given to within a constant by

$$W \propto \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') d^3 \boldsymbol{p}_2 d^3 \boldsymbol{p}_1'$$

The integration is carried out only over p_2 and p_1 ', since p_2 ' is determined by the law of momentum conservation. The angle between the vectors p_1 ' and p_2 ' is actually specified by the law of energy conservation. The integration over this angle eliminates the δ -function. It now remains only to integrate the absolute values of the vectors.

Suppose that p_1 is close to p_F . Then, all the remaining momenta will also be close to p_F in absolute value, and, consequently, in **Fig.1** they will make nearly equal angles with the horizontal line (with the sum $p_1 + p_2$). Hence, from the relationship between the projections on this axis we can write the relation between the absolute values:

$$p_1' \approx p_1 + p_2 - p_2'$$
.

Since $p_2' > p_F$, it follows that we have

$$p_1' < p_1 + p_2 - p_F$$

But at the same time $p_1 > p_F$, from which it follows that

$$p_F < p_1' < p_1 + p_2 - p_F$$
 or $2p_F < p_1 + p_2$

The upper limit for p_2 is p_F . Thus, we have

$$p_F - p_1 < p_2 - p_F < 0$$

 $0 < p_1' - p_F < (p_1 - p_F) + (p_2 - p_F)$

On integration we obtain

$$\iint dp_2 dp_1' = \frac{1}{2} (p_1 - p_F)^2$$

Hence

$$\gamma \propto (p_1 - p_F)^2$$

The complete formula for γ can be obtained from dimensionality considerations. It must be proportional to the square of the interaction constant and, according to the above calculation, to the quantity $(p_1 - p_F)^2$. Following this, we have to introduce a further factor made up of p_F , *m* and \hbar in such a manner that the result has the dimension of energy.



Fig. Area shaded in green. $A = \frac{1}{2} x_0^2$.



Fig.



Fig. The conservation of momentum and energy. $\overrightarrow{OA} = p_1 \cdot \overrightarrow{OC} = p_2 \cdot \overrightarrow{OB} = p_1' \cdot \overrightarrow{OC} = p_2$. $\overrightarrow{OB} = p_1' \cdot \overrightarrow{OD} = p_2' \cdot \overrightarrow{AP} = p_2 \cdot \overrightarrow{BP} = p_2 \cdot \overrightarrow{OP} = p_1 + p_2 = p_1' + p_2'$. H is the center of mass.



Fig. Point B and point D can rotate around the axis OP by the same angle. The point H is the center of mass between \overrightarrow{OA} and \overrightarrow{OB} . Triangle OAP is fixed. In other words, the vector \overrightarrow{OP} is fixed. The vector \overrightarrow{AB} is always parallel to the vector \overrightarrow{CD} from the momentum and energy conservation laws: $\overrightarrow{AB} = -\overrightarrow{CD}$. $p_1 + p_2 = p_1' + p_2' = \overrightarrow{OP}$ or $p_1 - p_1' = -(p_2 - p_2')$



Note that θ is the angle between p_1 and p_2 ;

$$p_1 = p_0(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}),$$
 $p_2 = p_0(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}),$
 $(p_1')_r \approx p_0 \sin\frac{\theta}{2}$

where $\angle GBJ = \theta$, and $\angle BOJ \approx \angle AJO \approx \frac{\theta}{2}$. The vector **f** is in the plane where p_1 and p_2 lie;

$$\boldsymbol{f} = (f_z, f_r)$$

where f_z is the component of the vector f along the axis parallel to $p_1 + p_2$ and f_r is the perpendicular component. We have the relation

$$p_1' = p_1 + f$$

or

$$p_{1}'^{2} = (p_{1})^{2} + f^{2} + 2p_{1} \cdot f$$

= $(p_{1})^{2} + f^{2} + 2p_{1}(f_{z}\cos\frac{\theta}{2} + f_{r}\sin\frac{\theta}{2})$
 $\approx (p_{1})^{2} [1 + 2\frac{1}{p_{1}}(f_{z}\cos\frac{\theta}{2} + f_{r}\sin\frac{\theta}{2})]$

or

$$p_1' = p_1 + f_z \cos\frac{\theta}{2} + f_r \sin\frac{\theta}{2}$$

We also have the relation

$$p_{2}' = p_{2} - f$$

or

$$(p_2')^2 = (p_2)^2 + f^2 - 2p_2 \cdot f$$

 $\approx (p_2)^2 [1 - 2\frac{p_2 \cdot f}{(p_2)^2}]$

or

$$p_2' = p_2(1 - f_z \cos\frac{\theta}{2} + f_r \sin\frac{\theta}{2})$$

Jacobian:

$$df_z df_z = \frac{\partial(f_z, f_r)}{\partial(p_1', p_2')} dp_1' dp_2' = \frac{dp_1' dp_2'}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

with

$$\frac{\partial(p_1', p_2')}{\partial(f_z, f_r)} = \begin{vmatrix} \frac{\partial p_1'}{\partial f_z} & \frac{\partial p_1'}{\partial f_r} \\ \frac{\partial p_2'}{\partial f_z} & \frac{\partial p_2'}{\partial f_r} \end{vmatrix} = \sin \theta$$

For p_2 , we use the polar co-ordinate (p_2, θ, ϕ_2) around the vector p_1 .

$$d^{3}\boldsymbol{p}_{2} = p_{2}^{2}dp_{2}\sin\theta d\theta d\phi_{2}$$
$$d^{3}\boldsymbol{p}_{1}' = (\boldsymbol{p}_{1}')_{r}df_{z}df_{r}d\phi$$
$$= p_{0}\sin\frac{\theta}{2}\frac{dp_{1}'dp_{2}'}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}d\phi$$
$$= p_{0}\frac{dp_{1}'dp_{2}'}{2\cos\frac{\theta}{2}}d\phi$$

and

$$d^{3}\boldsymbol{p}_{1}'d^{3}\boldsymbol{p}_{2} = p_{0}\frac{dp_{1}'dp_{2}'}{2\cos\frac{\theta}{2}}d\phi p_{0}^{2}dp_{2}\sin\theta d\theta d\phi_{2}$$
$$= p_{0}^{3}d\phi d\phi_{2}\sin\frac{\theta}{2}d\theta dp_{1}'dp_{2}'dp_{2}$$

We put

$$x = \beta(\varepsilon_p - \mu)$$

$$\approx \beta(\frac{p^2 - p_0^2}{2m})$$

$$= \frac{\beta}{2m}(p + p_0)(p - p_0)$$

$$\approx \frac{\beta p_0}{m}(p - p_0)$$

$$dx = \frac{\beta p_0}{m} dp$$

$$p_2^2 dp_2 \approx p_0^2 \frac{m}{\beta p_0} dx = \frac{mp_0}{\beta} dx$$

$$dp_1' = \frac{m}{\beta p_0} dx_1', \qquad dp_2' = \frac{m}{\beta p_0} dx_2'$$

$$d^3 p_1' d^3 p_2 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') = p_0^3 d\phi d\phi_2 \sin \frac{\theta}{2} d\theta dp_1' dp_2' dp_2$$

$$\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')$$

$$= p_0^3 d\phi d\phi_2 \sin \frac{\theta}{2} d\theta$$

$$\left(\frac{m}{\beta p_0}\right)^3 dx_1' dx_2' dx_2 \delta(\frac{x_1 + x_2 - x_1' - x_2'}{\beta})$$

$$= m^3 \left(\frac{1}{\beta}\right)^2 d\phi d\phi_2 \sin \frac{\theta}{2} d\theta$$

$$dx_1' dx_2' dx_2 \delta(x_1 + x_2 - x_1' - x_2')$$

The relaxation time is obtained as

$$\frac{1}{\tau} = m^{3}(k_{B}T)^{2} \int \sin \frac{\theta}{2} d\theta d\phi d\phi_{2}$$

$$\int \delta(x_{1} + x_{2} - x_{1}' - x_{2}') dx_{2} dx_{1}' dx_{2}' f(x_{2}) [1 - f(x_{1}')] [1 - f(x_{2}')]$$

$$= 2\pi m^{3} (k_{B}T)^{2} \int \sin \frac{\theta}{2} d\theta d\phi$$

$$\int \delta(x_{1} + x_{2} - x_{1}' - x_{2}') dx_{2} dx_{1}' dx_{2}' f(x_{2}) [1 - f(x_{1}')] [1 - f(x_{2}')]$$

with

$$f(\varepsilon_2)[1 - f(\varepsilon_1')][1 - f(\varepsilon_2')] = \frac{1}{e^{x_2} + 1} \frac{1}{e^{-x_1'} + 1} \frac{1}{e^{-x_2'} + 1}$$

where

$$\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') = \delta[\frac{1}{\beta}(x_1 + x_2 - x_1' - x_2')]$$
$$= \frac{1}{\beta}\delta(x_1 + x_2 - x_1' - x_2')$$

Note that the integral given by

$$F(a) = \int_{0}^{\infty} dx \int_{0}^{\infty} dy \left(\frac{1}{e^{x+y-a}+1}\right) \left(\frac{1}{1+e^{-x}}\right) \left(\frac{1}{1+e^{-y}}\right)$$

has a finite value when 0 < a < 1. When a = 0,

$$F(0) = \frac{\pi^2}{24} = 0.41123$$

((See APPENDIX))

Thus the relaxation time is

$$\frac{1}{\tau} = A \frac{\left(k_B T\right)^2}{\hbar \varepsilon_F}$$

where A is a dimensionless number and is of the order of unity.

((Heisenberg's principle of uncertainty))

This equation can be explained using the Heisenberg's principle of uncertainty,

$$\frac{1}{\tau} \approx \frac{1}{\hbar} (k_B T) \left(\frac{k_B T}{\varepsilon_F} \right) = \frac{1}{\hbar} \frac{(k_B T)^2}{\varepsilon_F}$$

where $\left(\frac{k_B T}{\varepsilon_F}\right)$ is the fraction of electrons in the energy interval $k_B T$ (thermal energy) in the vicinity of c

vicinity of ε_F .

((Note))

The density of states for free electrons

$$D(\varepsilon) = a\varepsilon^{1/2}$$
 (*a*, constant)

The total number of electrons;

$$N = \int_{0}^{\infty} d\varepsilon D(\varepsilon) f(\varepsilon) \approx \int_{0}^{\varepsilon_{F}} a\varepsilon^{1/2} d\varepsilon = \frac{2a}{3} \varepsilon_{F}^{3/2}$$

The fraction of number of electrons in the vicinity ε_F nis

$$\Delta N \approx D(\varepsilon_F) k_B T = a \varepsilon_F^{-1/2} (k_B T)$$

Thus we have the fraction

$$\frac{\Delta N}{N} = \frac{a\varepsilon_F^{1/2}(k_B T)}{\frac{2a}{3}\varepsilon_F^{3/2}} = \frac{3}{2} \left(\frac{k_B T}{\varepsilon_F}\right) \approx \frac{k_B T}{\varepsilon_F}$$

3. Approach by Kittel



Fig. The point H is the wave vector of the center of mass of 1 and 2. All pairs of states 1' and 2' conserve momentum and energy if they lie at opposite ends of a diameter of the small sphere. The small sphere is drawn from the center of mass of 1 and 2. But not all pairs of points 1' and 2' are allowed by the exclusion principle, for both 1', 2' must lie outside the Fermi sphere; the fraction allowed is $\Delta \varepsilon / \varepsilon_F$. (Kittel, ISSP 8-th edition)

It is an astonishing property of metals that conduction electron, although crowded together only 2 Å apart, travel long distances between collisions with each other. The mean free paths for electron-electron collisions are longer than 10^4 Å at room temperature and longer than 10 cm at 1K.

Two factors are responsible for these long mean free paths, without the free-electron model of metal would have little value. The most powerful factor is the exclusion principle, and the second factor is the screening of Coulomb interaction between electrons.

We show how the Pauli's exclusion principle reduces the collision frequency of an electron that has a low excitation energy ε_1 outside a filled Fermi sphere. We estimate the effect of the exclusion principle on the two collision, $1+2 \rightarrow 1'+2'$ between an electron in the excited state (1) and an electron in the filled state (2) in the Fermi surface.

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We show how the Pauli's exclusion principle reduces the collision frequency of an electron that the energy ε_1 is outside a filled Fermi sphere. We estimate the effect of the exclusion principle on the two collisions, $1+2 \rightarrow 1+2'$ between an electron in the excited state (1') and an electron in the filled state (2') in the Fermi surface. Because of the exclusion principle the states 1' and 2' of the electrons after collision must lie outside the Fermi sphere, all states within the sphere being already occupied; thus both states are outside the Fermi sphere.

The conservation of energy $(\varepsilon_1 + \varepsilon_2 = \varepsilon_1 + \varepsilon_2)$ requires that $|\varepsilon_2 - \varepsilon_F| < \varepsilon_1 - \varepsilon_F$. This means that collisions are possible only if the state 2 lies within a shell of thickness $a = \Delta \varepsilon$ within the Fermi surface. Thus the fraction $\Delta \varepsilon / \varepsilon_F$ of the electrons in filled states provides a suitable target for electron 1. But even if the target electron 2 is in the suitable energy shell, only a small fraction of the final states compatible with conservation of energy and momentum are allowed by the exclusion principle. This gives a second factor of $\Delta \varepsilon / \varepsilon_F$. In Fig. we show a small sphere on which all pairs of states 1', 2' at opposite ends of a diameter satisfy the conservation laws, but collisions can occur only if both 1' and 2' lie outside the Fermi sphere. The product of the two fractions is $(\Delta \varepsilon / \varepsilon_F)^2$.



Fig.

 $p_1 = k$, $p_2 = k'$, $p_1' = k - q$, $p_2' = k' + q$

4. Heat capacity and entropy for Fermi gas The internal energy U_{FG} of the Fermi gas

$$\frac{U_{FG}}{N\varepsilon_F} \approx \frac{3}{5} \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F}\right)^2\right]^{5/2} \left[1 + \frac{5\pi^2}{8} \left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$
$$\approx \frac{3}{5} \left[1 - \frac{5\pi^2}{24} \left(\frac{k_B T}{\varepsilon_F}\right)^2\right] \left[1 + \frac{5\pi^2}{8} \left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$
$$= \frac{3}{5} \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

or

$$U_{FG} = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F} \right)^2 \right]$$

The heat capacity:

$$C_{FG} = \frac{dU_{FG}}{dT} = \frac{3}{5} N \varepsilon_F \frac{5\pi^2}{12} (\frac{k_B}{\varepsilon_F})^2 2T = \frac{N\pi^2 k_B^2}{2\varepsilon_F} T$$

or

The heat capacity C_{FG} :

$$\frac{C_{FG}}{Nk_B} = \frac{\pi^2 k_B T}{2\varepsilon_F}.$$

The entropy S_{FG} :

$$dS_{FG} = \frac{C_{FG}}{T}dT = \frac{N\pi^2 k_B^2}{2\varepsilon_F}dT$$

or

$$S_{FG} = \frac{N\pi^2 k_B^2}{2\varepsilon_F} T$$
$$\frac{S_{FG}}{Nk_B} = \frac{\pi^2 k_B}{2\varepsilon_F} T = \frac{\pi^2}{2} \frac{T}{T_E}$$

((Note))

In general case,

$$C_{FG} = \frac{\pi^2}{3} D(\varepsilon_F) k_B^2 T$$

where $D(\mathcal{E}_F)$ is the density of states at the Fermi energy \mathcal{E}_F Using the relation $D(\mathcal{E}_F) = \frac{3N}{2\mathcal{E}_F}$, we have the specific heat for free electron Fermi gas

$$C_{FG} = \frac{\pi^2}{3} \frac{3N}{2\varepsilon_F} k_B^2 T = \frac{\pi^2 N k_B}{2} \frac{k_B T}{\varepsilon_F} = \frac{\pi^2 N k_B}{2} \frac{T}{T_F}$$

or

$$\frac{C_{FG}}{Nk_{B}} = \frac{\pi^{2}}{2} \frac{T}{T_{F}} = 0.998948 \ T(K)$$

where $T_F = 4.94 \text{ K}$ for liquid ³He.

5. Pauli paramagnetism for Fermi gas model

The Pauli paramagnetism is given by

$$\chi_{FG} = \mu^2 D(\varepsilon_F) = \frac{3}{2} \frac{N\mu^2}{\varepsilon_F} = \frac{3}{2} \frac{N\mu^2}{k_B T_F}$$

with

$$D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}$$

The magnetic moment of liquid ³He is

$$\mu = \gamma \hbar I = \frac{1}{2} \gamma \hbar$$

where $\mu = -2.1227 \mu_N$ and $I = \frac{1}{2}$. The nuclear magneton is defined by

$$\mu_N = \frac{e\hbar}{2m_p c}$$

 $(m_{\rm p}$ is the mass of proton). The gyromagnetic ratio (the ratio of the magnetic moment to the angular momentum) is given by

$$\gamma = \frac{\mu}{\hbar I} = \frac{-2.1227\,\mu_N}{\hbar/2} = -4.2454\,\frac{\mu_N}{\hbar} = -4.2454\,\mathrm{x}\,4789.43 = -2.033\,\mathrm{x}\,10^4\,\mathrm{(rad/s\,G)}$$

with

$$\frac{\mu_N}{\hbar} = \frac{e}{2m_pc} = 4789.43$$

where

$$f = \frac{\omega}{2\pi} = \frac{\gamma B}{2\pi} = 32.361 \text{ (MHz/T)}$$

The Pauli susceptibility can be rewritten as

$$\chi_{FG} = n \frac{3\mu^2}{2\varepsilon_F} = n \frac{3\left(\frac{\gamma}{2}\hbar\right)^2}{2k_B T_F} = n \frac{3\gamma^2\hbar^2}{8k_B T_F}$$

6. Fermi energy of liquid ³He Liquid ³He as a Fermi gas

spin
$$I = \frac{1}{2}$$
 (fermion)
Density $\rho = 0.081$ g/cm³

The Fermi energy:

$$\varepsilon_F = \frac{\hbar^2}{2m_0} (3\pi^2 n)^{2/3}$$

where m_0 is the mass of ³He atom,

$$m_0 = \frac{3.016 \,g}{6.022 \times 10^{23}} = 0.5 \times 10^{-23} = 5.0 \times 10^{-24} \,g$$

The number density:

$$n = \frac{N}{V} = \frac{N}{M} \frac{M}{V} = \frac{\rho}{m_0} = \frac{0.081}{5.0 \times 10^{-24}} = 1.62 \times 10^{22} / cm^3 = 1.62 \times 10^{28} / m^3$$

Then the Fermi energy is

$$\varepsilon_F = 6.815 \times 10^{-16} \text{ erg} = 4.254 \times 10^{-4} \text{ eV}.$$

The Fermi temperature

$$T_F = \frac{\varepsilon_F}{k_B} = 4.94 \text{ K}.$$

That is only a little higher than the boiling point, 3.2 K.

7. Heat capacity of fermi liquid

As predicted for the fermi gas mode, the heat capacity of liquid ³He should be

$$\frac{C_{FG}}{Nk_BT} = \frac{\pi^2}{2T_F} = 0.9989 \,\mathrm{K}^{-1}$$

So although the linear temperature dependence agrees with experiment, the predicted coefficient is too small by almost a factor of 3. According to the Fermi liquid theory, the heat capacity of the liquid ³He is predicted as

$$\frac{C_{FL}}{Nk_BT} = \frac{\pi^2}{2T_F} \left(1 + \frac{F_1^{(s)}}{3}\right) \approx \frac{\pi^2}{2T_F} 2.80$$

where $F_1^{(s)} = 5.39$ for P (pressure) = 0. Similarly, the entropy S is predicted as

$$\frac{S_{FL}}{Nk_BT} = \frac{\pi^2}{2T_F} \left(1 + \frac{F_1^{(s)}}{3}\right) \approx \frac{\pi^2}{2T_F} 2.80$$

The entropy of the solid, meanwhile, should be

$$k_B \ln 2^N = k_B N \ln 2$$

since each nucleus has two possible spin orientations. This constant value should apply down to very low temperatures, when the nuclear spins finally align and the entropy freezes out. Here is a sketch of both entropy functions. We make a plot of the entropy of the fermi gas and Fermi liquid as a function of T as well as the entropy of solid ³He.



For the Fermi gas, we have

$$T_{FG} = \frac{2T_F}{\pi^2} \ln 2 = 0.69 \text{ K}$$

where

$$\frac{S_{FG}}{Nk_B} = \frac{\pi^2}{2T_F} T_{FG} = \ln 2 \; .$$

For the Fermi liquid, we have

$$T_{FL} = \frac{2T_F}{\pi^2} \frac{\ln 2}{2.80} = 0.25$$
 K.

where

$$\frac{S_{FL}}{Nk_B} = \frac{\pi^2}{2T_F} 2.80T_{FL} = \ln 2$$

The temperature T_{FL} (=0.25 K) is much lower than the temperature T_{FG} (=0.69 K).

According to the Clausius-Clapeyron relation, the slope of the solid-liquid phase boundary on a graph of *P* vs *T* should be proportional to the entropy difference, $S_{liquid} - S_{solid}$. The above analysis predicts that the slope should be positive for *T*>0.25 K, and negative at lower temperatures. The experimental phase diagram shows just this behavior, with the transition from positive to negative slope at about 0.3 K, just slightly higher than the prediction 0.25 K. This discrepancy could be because of lattice vibrations giving the solid some additional entropy, and/or the entropy of the liquid no longer being quite linear at relatively high temperatures. At very low temperatures, where the entropy of the solid also goes to zero, the phase boundary becomes horizontal.

8. Entropy

The entropy is mainly due to the nuclear spin of the solid ³He around T = 0.3 K. The ³He atoms form a lattice. Below 10 mK the nuclear spins starts to order because of the antiferromagnetic exchange interaction. The nuclear spins antiferromagnetically ordered below the Neel temperature $T_N = 2$ mK. The antiferromagnetic exchange interaction is -0.85 mK.

 $V_{liq} > V_{Solid}$

For example, at T = 0.3 K, it is determined experimentally that

 $\frac{V_{liq} - V_{Solid}}{N} = 2.1 \times 10^{-24} cm^3 / atom$



9. Pomeranchuk cooling

In the *P*-*V* phase diagram, the solid-liquid boundary below 0.3 K has a negative slope. This is very unusual. Note that the solid-liquid boundary for most materials (except water) has a positive slope. What are the entropies for the liquid phase and solid phase? The ³He atom is a fermion. The heat capacity is proportional to *T* at low temperatures

$$C_{liquid} = N_A \frac{\pi^2}{2} k_B \left(\frac{T}{T_F}\right) 2.80$$

The entropy is evaluated as

$$S_{liquid} = \int \frac{C_{liquid}}{T} dT = N_A \frac{\pi^2}{2} k_B \left(\frac{T}{T_F}\right) 2.80$$

where T_F is the Fermi temperature. The entropy is proportional to T.

The entropy of solid ³He is dominated by the much larger contribution of the disordered nuclear spins (spin 1/2). Each the nuclear spin (spin 1/2) of ³He atom has a magnetic moment, just like a paramagnetic salts. In the paramagnetic state where the directions of spins are random, the solid entropy is given by

$$S_{soilid} = N_A k_B \ln 2$$
.

Note that these spins are antiferromagnetically ordered with a Neel temperature $T_N \approx 1$ mK and the entropy drops rapidly to zero. The liquid entropy coincides with the solid entropy at a characteristic temperature T_1 (=0.32 K). Below T_1 , the solid entropy is higher (more disordered) than the liquid entropy. Above T_1 the liquid entropy (more disordered) is higher than the solid entropy. Such a cross-over of the liquid and solid entropies of ³He at about 0.32 K produces a pronounced minimum in the melting curve at about 2.93 MPa, followed by a rise in the melting pressure to 3.45 MPa at T = 0 K. This negative slope is used to produce adiabatic compressional cooling along the melting curve, a technique called Pomeranchuk cooling after its proposer, and used by Osheroff *et al.* (1972a) in their discovery of the superfluid phases of ³He.



Fig. Entropy per atom in the coexisting solid and liquid phases of ³He. The entropy of the liquid phase is less than that of the solid phase below a characteristic temperature $T_1 = \frac{2 \ln 2}{\pi^2} T_F$. The dotted line from *a* to *b* corresponds to slow adiabatic compression from pure liquid to pure solid.



The phase boundary is expressed by the Clausius-Clapeyron equation

$$\frac{dP}{dT} = \frac{S_{liq} - S_{solid}}{V_{liq} - V_{solid}}$$

In the phase diagram of ³He around 0.3 K,

$$\frac{dP}{dT} > 0 \text{ for } T > 0.3 \text{ K} \quad \text{corresponding to } S_{liq} > S_{solid}$$

$$\frac{dP}{dT} < 0 \text{ for } T < 0.3 \text{ K} \quad \text{corresponding to } S_{liq} < S_{solid}$$

since $V_{liq} > V_{Solid}$ in the vicinity of 0.3 K.

Consider the liquid-solid boundary (denoted by red open circle) below 0.32 K, where $S_{liq} < S_{solid}$, When temperature is increased, the entropy increases. The only way this can happen here is to solid which has higher entropy. So the liquid freezes when the temperature rises. This explains the negative slope below 0.32 K.

In order for entropy to increase, heat must be absorbed from the surrounding (like when a normal solid melts). This gives rise to a cooling effect. Higher pressure forces the liquid to become

solid. This gives a cooling effect. This compressional cooling" method was proposed by Issak Pomeranchuk in 1950.

Pomenranchuk predicted that adiabatic compression of coexisting solid and liquid would cool ³He. If the initial temperature is below 0.32 K, then an increase in the external pressure moves the helium along the melting curve up and to the left: toward higher pressure and lower temperatures. Typically, the temperature decreases from 0.3 K to 1 mK. The solid-liquid boundary (arrow) below 0.3 K has a negative slope.

APPENDIX-I Fermi liquid theory

The life time a quasi-particle excitation dressed by interactions is rather long owing to the surrounding degenerate Fermi sea. Thus the physical properties at $T \ll T_F$ may be well described by a theory based on quasi-particle excitations, the so-called Landau Fermi liquid theory. Since the critical temperature T_c of this superfluid is of the order of $T_F/1000$, similar to that for superconductivity in a metal, one can naturally study the system on the basis of this theory.

Table 6.1. The molar volume, the Landau parameters at $T < T_c$ and the critical temperature of liquid ³He

P (bar)	V (cm ³)	$n \times 10^{21}$ (cm ⁻³)	$\frac{k_{\rm F} \times 10^7}{(\rm cm^{-1})}$	m* / m	$F_{1}^{(s)}$	$F_{0}^{(s)}$	$F_{0}^{(a)}$	T _c (mK)	T _{AB} (mK)
0	36.84	16.3	7.84	2.80	5.39	9.30	-0.6951	0.929	_
15	28.89	20.8	8.49	4.28	9.85	41.73	-0.753	2.067	_
34.4	25.50	23.6	8.87	5.85	14.56	88.47	-0.753	2.491	1.933

Fermi liquid relations between the Landau parameters and experimentally measured quantities

(a) Effective mass:

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(s)}$$

(b) Specific heat:

$$\frac{C_N}{C_g} = \frac{m^*}{m} = 1 + \frac{1}{3}F_1^{(s)}$$

where

$$C_g = \frac{1}{2}\pi^2 N k_B \frac{T}{T_F}$$

(c) Spin suceptibility:

$$\frac{\chi_N}{\chi_g} = \frac{m^*}{m} \frac{1}{1 + F_0^{(a)}} = \frac{1 + \frac{1}{3}F_1^{(s)}}{1 + F_0^{(a)}}$$

where

$$\chi_g = \mu_B^2 \frac{3N}{2\varepsilon_F} = \frac{3\mu_B^2 N}{2k_B} \frac{1}{T_F}$$

(d) Compressibility:

$$\frac{\kappa_N}{\kappa_g} = \frac{m^*}{m} \frac{1}{1 + F_0^{(s)}} = \frac{1 + \frac{1}{3}F_1^{(s)}}{1 + F_0^{(s)}}$$

where

$$\kappa_g = \frac{3}{2} \frac{1}{n\varepsilon_F}$$

(e) The sound velocity:

$$\frac{c^2}{c_g} = \frac{1 + F_0^{(s)}}{1 + \frac{1}{3}F_1^{(s)}}$$

where

$$T_F = \frac{\hbar^2}{2mk_B} (3\pi^2 n), \qquad T_F^* = \frac{\hbar^2}{2m^* k_B} (3\pi^2 n)$$
$$p_F = \hbar (3\pi^2 n)^{2/3}$$

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APPENDIX

The integral

$$F(a) = \int_{0}^{\infty} dx \int_{0}^{\infty} dy \left(\frac{1}{e^{x+y-a}+1}\right) \left(\frac{1}{1+e^{-x}}\right) \left(\frac{1}{1+e^{-y}}\right)$$

When a = 0,

$$F(0) = \frac{\pi^2}{24} = 0.41123$$

When a increases, the function F(a) increases as shown in Figure below.



Fig. Plot of F(a) as a function of a.