Degenerate Fermi gas systems: white dwarf and neutron pulsar

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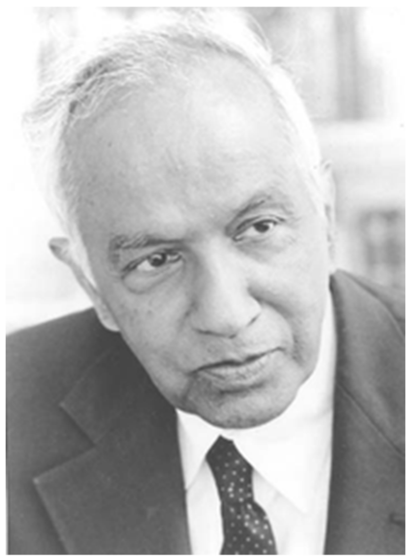
Subrahmanyan Chandrasekhar, FRS (October 19, 1910 – August 21, 1995) was an Indian origin American astrophysicist who, with William A. Fowler, won the 1983 Nobel Prize for Physics for key discoveries that led to the currently accepted theory on the later evolutionary stages of massive stars. Chandrasekhar was the nephew of Sir Chandrasekhara Venkata Raman, who won the Nobel Prize for Physics in 1930. Chandrasekhar served on the University of Chicago faculty from 1937 until his death in 1995 at the age of 84. He became a naturalized citizen of the United States in 1953.



http://en.wikipedia.org/wiki/Subrahmanyan_Chandrasekhar

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1. Introduction

Electron degeneracy is a stellar application of the Pauli Exclusion Principle, as is neutron degeneracy. No two electrons can occupy identical states, even under the pressure of a collapsing star of several solar masses. For stellar masses less than about 1.44 solar masses, the energy from the gravitational collapse is not sufficient to produce the neutrons of a neutron star, so the collapse is halted by electron degeneracy to form white dwarfs. This maximum mass for a white dwarf is called the Chandrasekhar limit. As the star contracts, all the lowest electron energy levels are filled and the electrons are forced into higher and higher energy levels, filling the lowest unoccupied energy levels. This creates an effective pressure which prevents further gravitational collapse.

(a) Earth
$M = 5.973610 \times 10^{24}$ kg
$R = 6.372 \times 10^{6}$ m

(b) Sun
$M = 1.988435 \times 10^{30}$ kg
$R = 6.9599 \times 10^{8}$ m

(c) Companion of Sirius: first white dwarf (Sirius B)
$M = 2.0 \times 10^{30}$ kg (≈ the mass of sun)
$R = 6.0 \times 10^{6}$ m (a little shorter than the Earth)

(d) Crab pulsar (neutron star)
$M = 1.4\ M_{sun} = 2.78 \times 10^{30}$ kg
$R = 1.2 \times 10^{3}$ m.

2. Kinetic energy of the ground state of fermion

The kinetic energy of the fermions in the ground state is given by

$$E_G = \frac{3}{5} N_f \varepsilon_F = \frac{3}{5} N_f \frac{\hbar^2}{2m_0} (3\pi^2 \frac{N_f}{V})^{2/3},$$

where N_f is the number of fermions, and m_0 is the mass of the fermion. Since $dE = TdS - PdV$, the pressure P is calculated as

$$\begin{aligned} P &= -\frac{\partial E_G}{\partial V} \\ &= -\frac{3}{5} N_f \frac{\hbar^2}{2m_0} \frac{2}{3} (3\pi^2 \frac{N_f}{V})^{-1/3} (-3\pi^2 \frac{N_f}{V^2}) \\ &= \frac{1}{5} (3\pi^2)^{2/3} \frac{\hbar^2}{m_0} (\frac{N_f}{V})^{5/3} \end{aligned}$$

The kinetic energy of fermions in the ground state can be rewritten as

$$E_G = \frac{3}{5} N \frac{\hbar^2}{2m_0} (3\pi^2 \frac{N_f}{\frac{4\pi}{3} R^3})^{2/3}$$

$$= \frac{3}{10} (\frac{9\pi}{4})^{2/3} \frac{\hbar^2}{m_0} \frac{N_f^{5/3}}{R^2} = 1.10495 \frac{\hbar^2}{m_0} \frac{N_f^{5/3}}{R^2} = \frac{B}{R^2}$$

The volume V is expressed by

$$V = \frac{4\pi}{3} R^3 .$$

where R is the radius of the system. We find that P becomes increases as the volume V decreases.

Here we note that the density of the system, ρ, is given by

$$\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3} R^3} .$$

and M is the total mass. The number density n_f for fermions is defined as

$$n_f = \frac{N_f}{V} = \frac{M}{V} \frac{N_f}{M} = \rho \frac{N_f}{m_f N_f} = \frac{\rho}{m_f} .$$

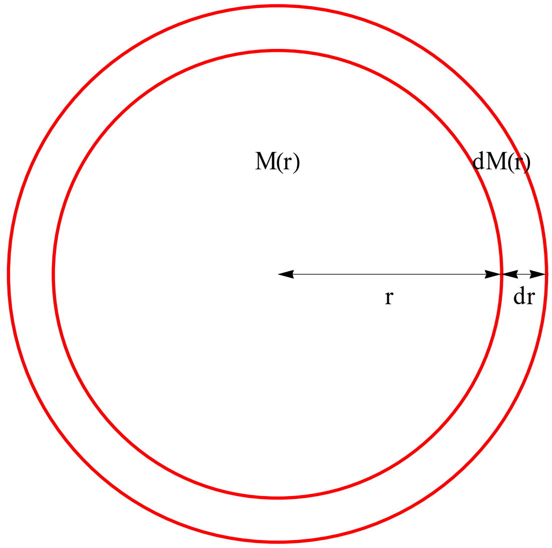
The average nearest neighbor distance between fermions can be evaluated

$$d = \left(\frac{1}{n_f} \right)^{1/3} = \left(\frac{m_f}{\rho} \right)^{1/3}$$

where m_f is the mass per fermion. Note that m_f is not always equal to the mass of each fermion (m_0). For the atoms with one proton and one neutron, there is one electron. In this case m_f should be equal to $m_f = m_p$ +mn, where m_p is the mass of proton and m_n is the mass of neutron.

3. Gravitational potential

We calculate the potential energy of the system.



Suppose that $M(r)$ is the mass of the system with radius r.

$$M(r) = \frac{4\pi}{3}\rho r^3, \qquad dM(r) = 4\pi r^2 \rho dr .$$

The potential energy is given by

$$U = -\int_0^R \frac{GM(r)dM(r)}{r} .$$

Noting that

$$\rho = \frac{3M}{4\pi R^3},$$

the potential energy is calculated as

$$U = -\int_0^R \frac{G(4\pi\rho)^2 r^5}{3r} dr = \frac{G}{3}(4\pi\rho)^2 \frac{1}{5} R^5 = -\frac{3GM^2}{5R} = -\frac{A}{R},$$

where

$$A = \frac{3GM^2}{5}$$

and G is the universal gravitational constant.

4. The total energy

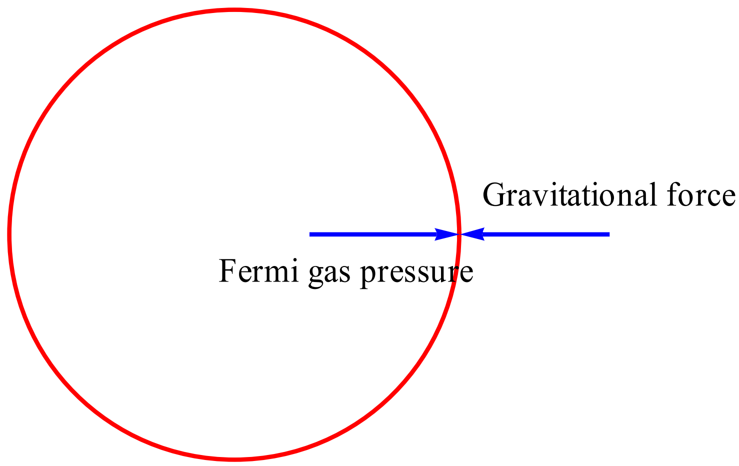
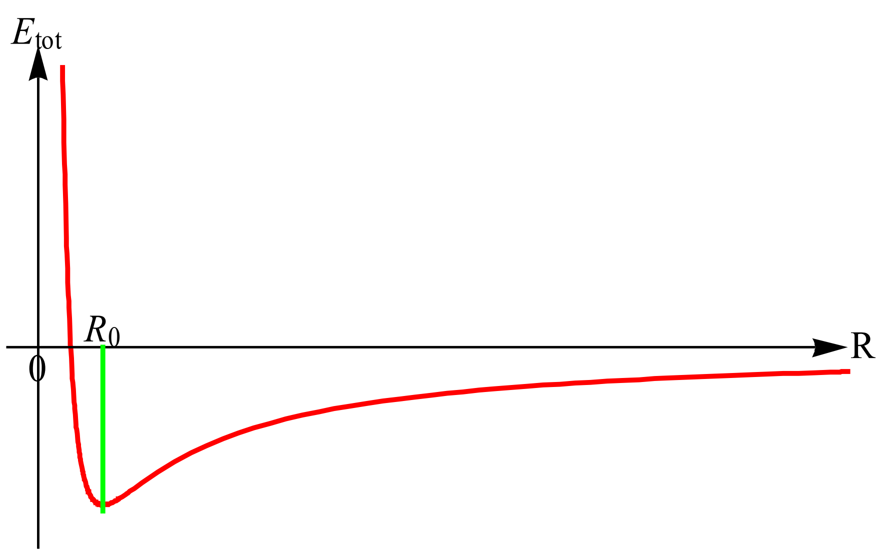


Fig. A balance between the gravitational force (inward) and the pressure of degenerate Fermi gas

The total energy is the sum of the gravitational and kinetic energies.

$$E_{tot} = f_{nonrel}(R) = -\frac{A}{R} + \frac{B}{R^2}$$



From the derivative of $f_{\mathrm{nonrel}}(R)$ with respect to R, we get the distance R in equilibrium.

$$f_{nonrel}{}'(R) = \frac{A}{R^2} - \frac{2B}{R^3} = 0,$$

or

$$R = R_0 = \frac{2B}{A}.$$

Thuis, for the nonrelativistic degenerate Fermi gas, there is a balance between the gravitational force (inward) and the force due to the degenerate Fermi gas pressure, leading to a stable radius R_0.

5. Relativistic degenerate Fermi gas

When N_f is the number of fermions, $V \to 0$, ε_F increases. When $\varepsilon_F \approx m_0c^2$, then the relativistic effect becomes important.

$$\frac{\varepsilon^2}{c^2} = p^2 + m_0{}^2c^2 \qquad \text{or} \qquad \varepsilon \approx cp = c\hbar k$$

where m_0 is the mass of fermion and m_f is the mass per fermion. Note that

$$dk = \frac{d\varepsilon}{c\hbar}, \qquad k = \frac{\varepsilon}{c\hbar}.$$

The density of states:

$$D(\varepsilon)d\varepsilon = \frac{2V}{(2\pi)^3}4\pi k^2 dk = \frac{2V}{8\pi^3}4\pi\left(\frac{\varepsilon}{c\hbar}\right)^2\frac{d\varepsilon}{c\hbar} = \frac{V}{\pi^2}\left(\frac{1}{c\hbar}\right)^3\varepsilon^2 d\varepsilon,$$

or

$$D(\varepsilon) = \frac{V}{\pi^2c^3\hbar^3}\varepsilon^2,$$

$$N_f = \int_0^{\varepsilon_F} D(\varepsilon)d\varepsilon = \frac{V}{\pi^2c^3\hbar^3}\int_0^{\varepsilon_F}\varepsilon^2 d\varepsilon = \frac{V}{\pi^2}\frac{1}{3c^3\hbar^3}\varepsilon_F{}^3,$$

or

$$\varepsilon_F = \pi c\hbar\left(\frac{3N_f}{\pi V}\right)^{1/3} = \pi c\hbar\left(\frac{3n_f}{\pi}\right)^{1/3}.$$

where

$$n_f = \frac{N_f}{V}.$$

The total energy in the ground state is obtained as

$$E_G = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon) d\varepsilon = \frac{V}{\pi^2 c^3 \hbar^3} \int_0^{\varepsilon_F} \varepsilon^3 d\varepsilon = \frac{V}{\pi^2} \frac{1}{4c^3\hbar^3} \varepsilon_F{}^4.$$

Using the expression of N_f, E_G can be rewritten as

$$E_G = \frac{3}{4} N_f \varepsilon_F = \frac{3}{4} N_f \varepsilon = \frac{3}{4} (3\pi^2)^{1/3} \hbar c N_f (\frac{N_f}{V})^{1/3}.$$

The pressure P is calculated as

$$P = \frac{E_G}{3V} = \frac{1}{4} (3\pi^2)^{1/3} \hbar c (\frac{N_f}{V})^{4/3} = \frac{1}{4} (3\pi^2)^{1/3} \hbar c (n_f)^{4/3}.$$

The total mass M is denoted as

$$M = N_f m_f,$$

where m_f is the mass per fermion. Note that m_f is not always equal to the mass of each fermion (m_0).

$$\begin{aligned} E_G &= \frac{3}{4} (3\pi^2)^{1/3} \hbar c \frac{M}{m_f} \left(\frac{M}{\frac{4\pi}{3} R^3 m_f}\right)^{1/3} \\ &= \frac{3}{4} (3\pi^2)^{1/3} \left(\frac{3}{4\pi}\right)^{1/3} \frac{\hbar c}{m_f{}^{4/3}} \frac{M^{4/3}}{R} \\ &= \frac{3}{4} \left(\frac{9\pi}{4}\right)^{1/3} \frac{\hbar c}{m_f{}^{4/3}} \frac{M^{4/3}}{R} = A \frac{M^{4/3}}{R} \end{aligned}$$

where

$$A = \frac{3}{4}(\frac{9\pi}{4})^{1/3}\frac{\hbar c}{m_f^{\ 4/3}} = 1.43937\frac{\hbar c}{m_f^{\ 4/3}}.$$

Since the gravitational energy is

$$-\frac{3GM^2}{5R},$$

the total energy (relativistic) is given by

$$E_{tot} = f_{rel}(R) = \frac{AM^{4/3}}{R} - \frac{3GM^2}{5R} = \frac{AM^{4/3} - \frac{3}{5}GM^2}{R}.$$

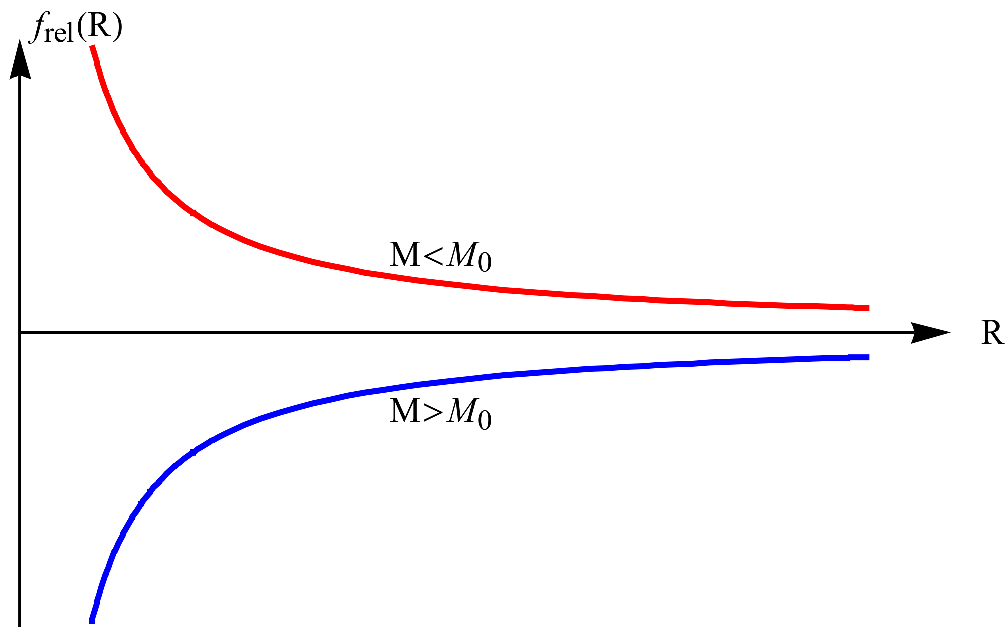


Fig. Schematic plot of $f_{rel}(R)$ vs R for $M>M_0$ and $M<M_0$. When $M = M_0$, $f_{rel}(R) = 0$. For $M>M_0$, the total energy decreases with decreasing R, leading to the stable state near $R = 0$. For For $M<M_0$, the total energy decreases with increasing R, leading to the stable state near $R = \infty$.

For $M>M_0$, R tends to zero, while for $M<M_0$, R tends to increase. The critical mass M_0 is evaluated from the condition,

$$AM_0^{4/3} = \frac{3}{5}GM_0^{\ 2}.$$

or

$$M_0 = \left(\frac{5\times 1.43937}{3G}\frac{\hbar c}{m_f^{\ 4/3}}\right)^{3/2} = \left(2.39895\frac{\hbar c}{Gm_f^{\ 4/3}}\right)^{3/2} = \frac{3.71562}{m_f^{\ 2}}\left(\frac{\hbar c}{G}\right)^{3/2}.$$

The interior of a white-dwarf star is composed of atoms like ^{12}C (6 electrons, 6 protons, and 6 neutrons) and ^{16}O (8 electrons, 8 protons, and 8 neutrons), which contain equal numbers of protons, neutrons, and electrons. Thus,

$$m_f = 2\ m_p$$

where m_p is the proton mass. Then we have

$$M_0 = 1.72438\ M_{sun}.$$

The currently accepted numerical value of the limit is about 1.4 M_{sun} (*Chandrasekhar limit*)

6. White dwarf with electron as ferimon

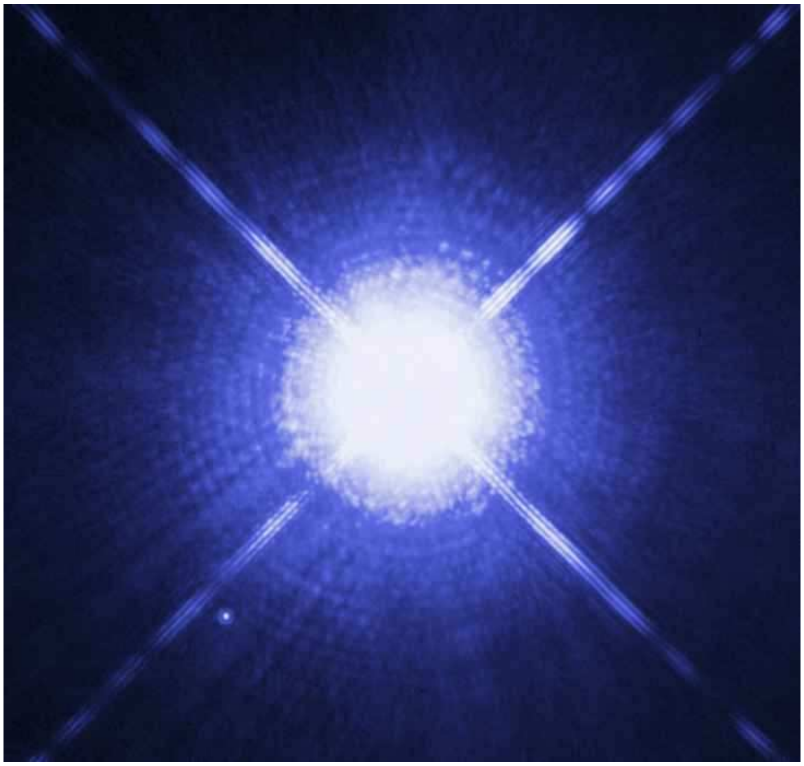


Fig. Image of Sirius A and Sirius B taken by the Hubble Space Telescope. Sirius B, which is a white dwarf, can be seen as a faint pinprick of light to the lower left of the much brighter Sirius A.

http://www.universetoday.com/wp-content/uploads/dog_star.jpg

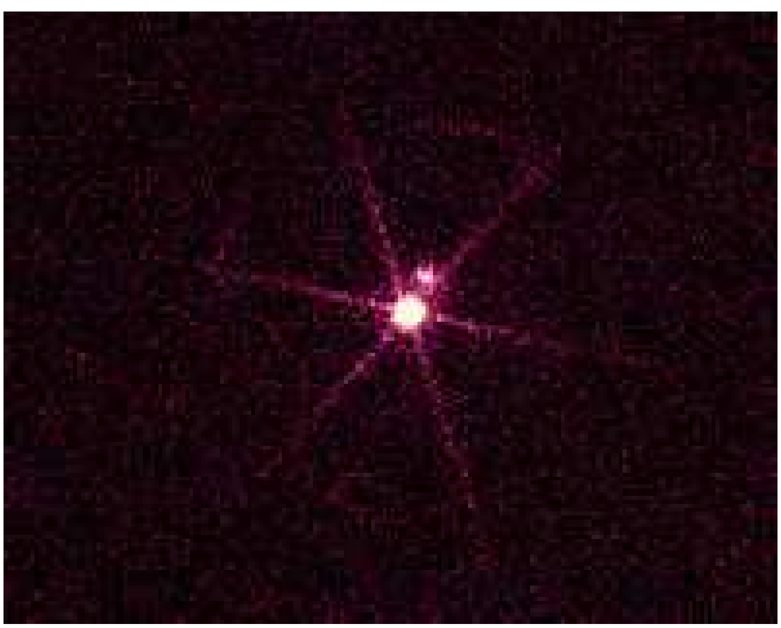


Fig. A Chandra X-ray Observatory image of the Sirius star system, where the spike-like pattern is due to the support structure for the transmission grating. The bright source is Sirius B. Credit: NASA/SAO/CXC.

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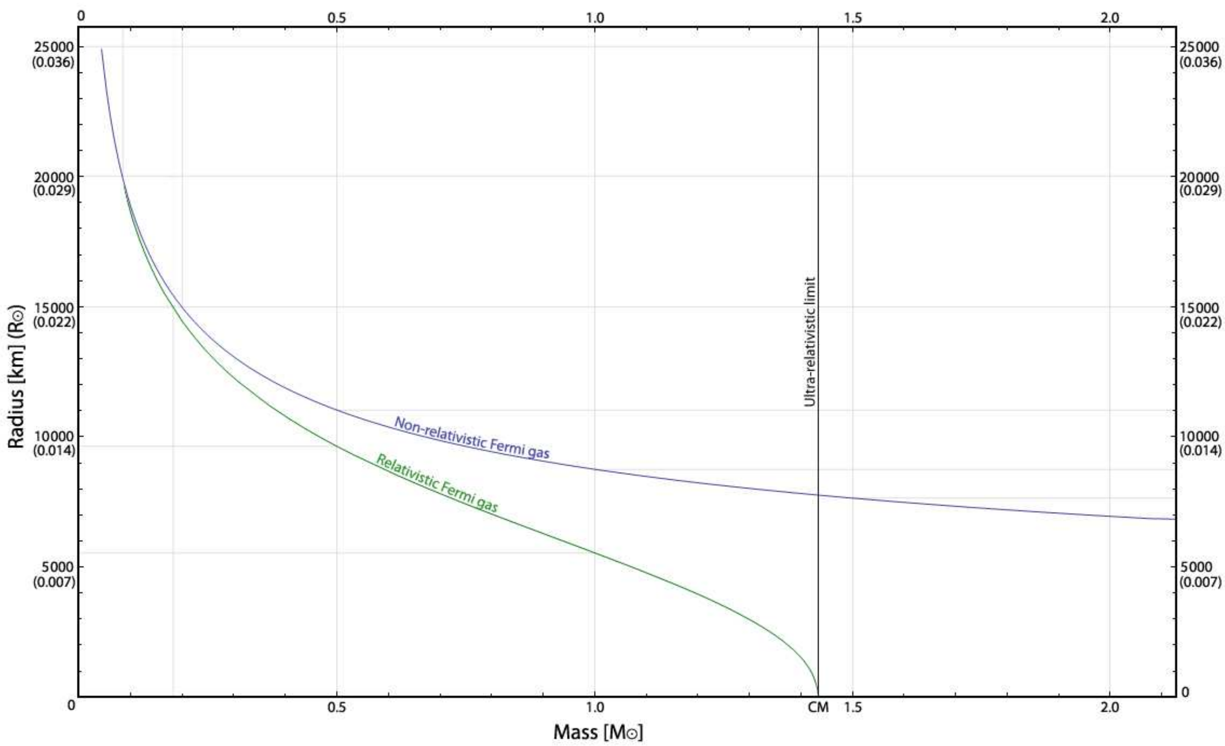

Fig. mass-radius relationship. mass and radius are in the unit of the mass and radius of sun.

http://upload.wikimedia.org/wikipedia/commons/8/81/WhiteDwarf_mass-radius.jpg

We consider the case of white dwarf. There is one electron per two protons (2 m_p mass for 1 electron, for example ^{12}C, six electrons, 6 protons, and 6 neutrons). These electrons are assumed to form a free electron Fermi gas, where

$$m_f = 2m_p \qquad m_0 = m_e.$$

Then the kinetic energy E_G can be given by

$$E_G \approx \frac{\hbar^2 M^{5/3}}{m_e R^2 m_f^{\ 5/3}} = \frac{B}{R^2},$$

and

$$B = \frac{\hbar^2 M^{5/3}}{m_e m_f^{\ 5/3}}.$$

The equilibrium distance R is given by

$$R = R_0 = \frac{2B}{A} = \frac{10\hbar^2}{3Gm_e m_f^{\ 5/3} M^{1/3}}.$$

When $m_f = 2\ m_p$,

$$C = M^{1/3} R_0 = \frac{2B}{A} = \frac{10\hbar^2}{3Gm_e m_f^{\ 5/3}} = 8.1488 \text{ x } 10^{16} \text{ kg}^{1/3} \text{ m}$$

The radius R is proportional to $M^{-1/3}$. When M is equal to the mass of sun, M_{sun}, then we have

$$R_0 = \frac{C}{M_{sun}^{\ 1/3}} = 6.48\ 02 \text{ x } 10^3 \text{ km}.$$

which is almost equal to the radius of Earth. The Fermi energy of the electrons is

$$\varepsilon_F = \frac{\hbar^2}{2m_e}(3\pi^2 n_f)^{2/3} = 2.362 \text{ x } 10^5 \text{ eV}.$$

where

$$\rho = \frac{M}{V} = \frac{m_f N_f}{V} = 1.744 \text{ x } 10^9 \text{ kg/m}^3.$$

or

$$n_f = \frac{N_f}{V} = \frac{\rho}{m_f} = \frac{1}{m_f}\frac{M}{\frac{4\pi}{3}R^3} = 5.21475 \text{ x } 10^{35} \text{ /m}^3.$$

The average distance between fermions is

$$d = \left(\frac{m_f}{\rho}\right)^{1/3} = 1.242 \times 10^{-12} \text{ m.}$$

((**Note**))

The more massive a white dwarf is, the smaller it is. The electrons must be squeezed closer together to provide the greater preasure needed to a more massive white dwarf.

7. Neutron star with neutron as fermion

We consider the case of neutron star. The system consists of only neutron (spin 1/2 fermion). We use

$$m_0 = m_n, \qquad m_f = m_n, \qquad M = M_{\text{sun}} \text{ (for convenience).}$$

Then E_G can be given by

$$E_G \approx \frac{\hbar^2 M^{5/3}}{m_n R^2 m_n^{5/3}} = \frac{\hbar^2 M^{5/3}}{R^2 m_n^{8/3}} = \frac{B}{R^2},$$

where

$$B = \frac{\hbar^2 M^{5/3}}{m_n^{8/3}}.$$

Then we have

$$M^{1/3} R_0 = C = \frac{2B}{A} = \frac{10\hbar^2}{3G m_n^{8/3}} = 1.40379 \times 10^{14} \text{ kg}^{1/3} \text{ m.}$$

If $M = M_{sun}$, then we get

$$R_0 = \frac{C}{M_{sun}^{1/3}} = 11.16 \text{ km.}$$

The Fermi energy is given by

$$\varepsilon_F = \frac{\hbar^2}{2m_n}(3\pi^2 n_f)^{2/3} = 4.3254 \text{ x } 10^7 \text{ eV}$$

where

$$n_f = \frac{N_f}{V} = \frac{\rho}{m_f} = \frac{1}{m_f}\frac{M}{\frac{4\pi}{3}R^3} = 2.03717 \text{ x } 10^{44} \text{ /m}^3.$$

with $m_f = m_n$. The average distance between fermions is

$$d = \left(\frac{1}{n_f}\right)^{1/3} = \left(\frac{m_n}{\rho}\right)^{1/3} = 1.6995 \text{ x } 10^{-15} \text{ m}.$$

The density ρ is

$$\rho = 3.412 \text{ x } 10^{17} \text{ kg/m}^3.$$

((**Note**))

The electrons are captured by nucleus. *N* decreases. However, *V* is also decreased. Then the number density *n* remains unchanged. Thus *P* does not change. When the system is further compressed, then all electrons are captured by nucleus.

$A_z + e^- = A_{z-1} + \nu$

$A_{z-1} + e^- = A_{z-2} + \nu$

....

$A_1 + e^- = A_0 + \nu$

where A_0 is a neutron and ν is a neutrino. z is the number of protons. Finally, nucleus is composed of only neutrons.

8. Crab pulsar (neutron star)

The Crab Pulsar (PSR B0531+21) is a relatively young neutron star. The star is the central star in the Crab Nebula, a remnant of the supernova SN 1054, which was widely observed on Earth in the year 1054. Discovered in 1968, the pulsar was the first to be connected with a supernova remnant. The optical pulsar is roughly 25 km in diameter and the pulsar "beams" rotate once every 33 ms, The outflowing relativistic wind from the neutron star generates synchrotron emission, which produces the bulk of the emission from the nebula, seen from radio waves through to gamma rays. The most dynamic feature in the inner part of the nebula is the point where the pulsar's equatorial wind slams into the surrounding nebula, forming a termination shock. The shape and position of this feature shifts rapidly, with the equatorial wind appearing as a series of wisp-like features that steepen, brighten, then fade as they move away from the pulsar into the main body of the nebula. The period of the pulsar's rotation is slowing by 36.4 ns per day due to the large amounts of energy carried away in the pulsar wind.

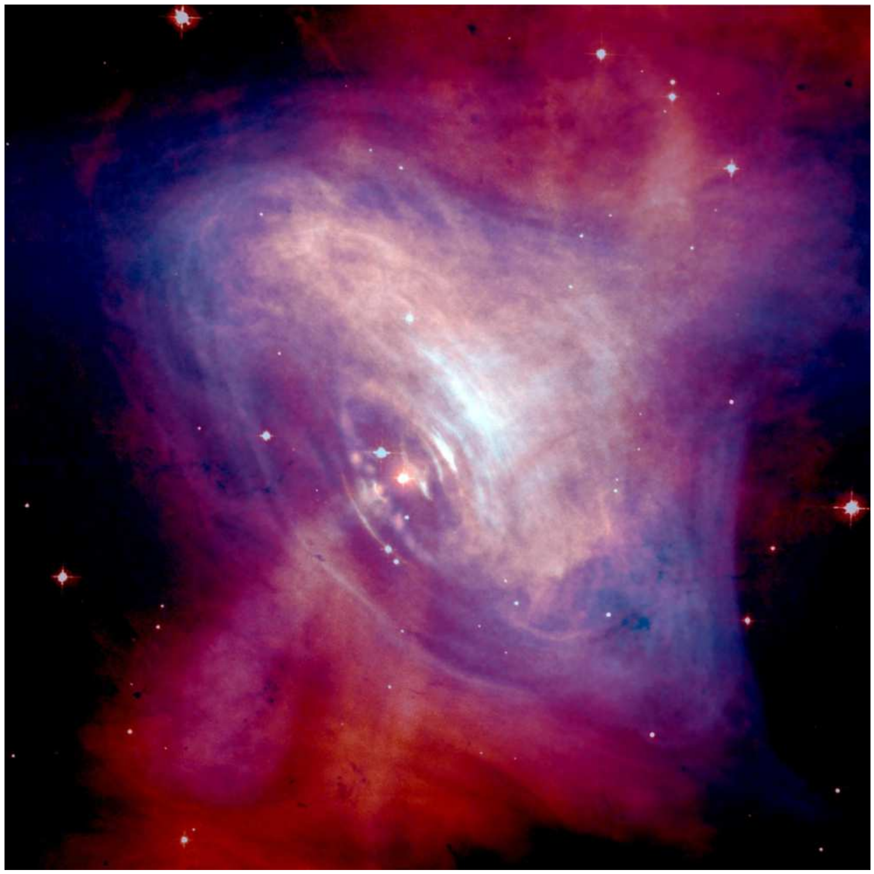

Fig. The Crab Nebula, which contains the Crab Pulsar. Image combines optical data from Hubble (in red) and X-ray images from Chandra (in blue). NASA/CXC/ASU/J. Hester

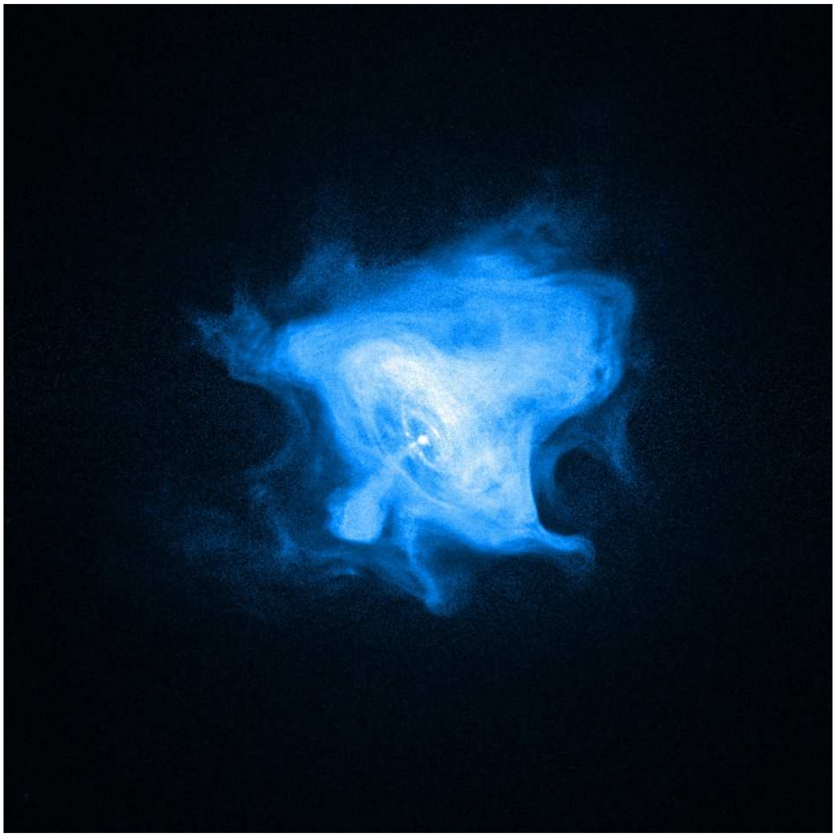

Fig. X-ray picture of Crab pulsar, taken by Chandra

http://en.wikipedia.org/wiki/Crab_Pulsar

Data of Crab pulsar:

ν = 30/s (T = 33 ms), M = 1.4 M_{sun}, R = 12 km. ΔT = 36.4 ns.

The density ρ is

$$\rho = \frac{M}{\frac{4\pi}{3}R^3} = 3.84598 \text{ x } 10^{17} \text{ kg/m}^3.$$

The moment of inertia I is calculated as

$$I = \frac{2}{5}MR^2 = 1.60347 \times 10^{38} \text{ kg m}^2.$$

The rotational kinetic energy is

$$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2}{5}MR^2\omega^2 = 2.91 \text{ x } 10^{42} \text{ J},$$

where the angular frequency ω is

$$\omega = \frac{2\pi}{T} = 190.4 \text{ rad/s}.$$

The loss of energy per day is

$$P = -\frac{4\pi^2}{T^3} I \frac{\Delta T}{\Delta t} = 7.42 \text{ x } 10^{31} \text{ W},$$

where ΔT = 36.4 ns per Δt = 1 day = 24 x 3600 s.

((**Note-1**)) **Density**

For a rotating object to remain bound, the gravitational force at the surface must exceed the centripetal acceleration:

$$m\frac{GM}{r^2} > mr\omega^2 \Rightarrow \frac{GM}{r^3} > \omega^2 = \frac{4\pi^2}{T^2} \Rightarrow \frac{G\rho}{r^3}\frac{4\pi}{3}r^3 > \frac{4\pi^2}{T^2} \Rightarrow \rho > \frac{3\pi}{T^2 G}.$$

For T = 33 ms, the density must be greater than 1.3×10^{11} g/cm^3 = 1.3x10^{14} kg/m^3. This exceeds the maximum possible density for a white dwarf.

((**Note-2**)) **Angular momentum conservation**

Suppose that the Sun (T = 25 days, radius 7×10^8 m, mass 1.988x10^{30} kg) were to collapse to a neutron star with a radius of 16 km. Using the angular momentum conservation law, we have

$$R_i^2\omega_i = R_f^2\omega_f,$$

or

$$\frac{\omega_f}{\omega_i} = \frac{R_i^2}{R_f^2} = \left(\frac{7\times10^8}{16\times10^3}\right)^2 = \frac{49\times10^{10}}{256} = 2\times10^9$$

In other words, the star is rotating 2×10^9 faster after the collapse than it was before.

$$\frac{T_f}{T_i} = \frac{1}{2 \times 10^9},$$

or

$$T_f = \frac{25 \times (24 \times 3600)}{2 \times 10^9} = 1ms.$$