Magnetoresistance of metals

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The electrical resistivity of metals can be measured along the direction of electric field. We define the magnetoresistance as the electrical resistivity when the magnetic field is applied along a direction perpendicular to the magnetic field (transverse magnetoresistance) or the magnetic field is applied along the direction of the electric field (longitudinal magnetoresistance). Here we discuss only the transverse magnetoresistance.

1. Introduction

We consider a free particle with mass m and a charge q in the presence of Electric field $\boldsymbol{E}$ and the magnetic field B along the z axis. We start with an equation of motion (classically),

$$m(\frac{d\boldsymbol{v}}{dt}+\frac{\boldsymbol{v}}{\tau})=q[\boldsymbol{E}+\frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B})]$$

where τ is the relaxation time. In the steady state ($\frac{d\boldsymbol{v}}{dt}=0$), we have

$$\boldsymbol{v}=\frac{q\tau}{m}[\boldsymbol{E}+\frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B})]$$

$$v_x=\frac{q\tau}{m}E_x+\frac{qB\tau}{mc}v_y$$

$$v_y=\frac{q\tau}{m}E_y-\frac{qB\tau}{mc}v_x$$

$$v_z=\frac{q\tau}{m}E_z$$

From this equation we get

$$v_x=\frac{cq\tau(cmE_x+Bq\tau E_y)}{c^2m^2+B^2q^2\tau^2}$$

$$v_y = \frac{cq\tau(cmE_y - Bq\tau E_x)}{c^2m^2 + B^2q^2\tau^2}$$

We use

$$\omega_c = \frac{qB}{mc}, \qquad \text{cyclotron angular frequency}$$

$$\mu = \frac{q\tau}{m} \qquad \text{(mobility)}$$

Thus we have

$$v_x = \frac{\mu(E_x + \omega_c\tau E_y)}{1+\omega_c{}^2\tau^2}, \qquad v_y = \frac{\mu(-\omega_c\tau E_x + E_y)}{1+\omega_c{}^2\tau^2}, \qquad v_z = \mu E_z$$

The current density:

$$J_x = nqv_x = \frac{nq\mu(E_x + \omega_c\tau E_y)}{1+\omega_c{}^2\tau^2} = \sigma_0 \frac{(E_x + \omega_c\tau E_y)}{1+\omega_c{}^2\tau^2}$$

$$J_y = nqv_y = \frac{nq\mu(-\omega_c\tau E_x + E_y)}{1+\omega_c{}^2\tau^2} = \sigma_0 \frac{(-\omega_c\tau E_x + E_y)}{1+\omega_c{}^2\tau^2}$$

$$J_z = nqv_z = nq\mu E_z = \sigma_0 E_z$$

where $\sigma_0 = nq\mu = \frac{nq^2\tau}{m}$ is the conductivity in the absence of fields.

The conductivity tensor

$$\boldsymbol{J} = \sigma\boldsymbol{E}$$

where

$$\sigma = \frac{\sigma_0}{1+\omega_c{}^2\tau^2}\begin{pmatrix} 1 & \omega_c\tau & 0 \\ -\omega_c\tau & 1 & 0 \\ 0 & 0 & 1+\omega_c{}^2\tau^2 \end{pmatrix}$$

The resistivity tensor:

$$\rho = \sigma^{-1} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & -\omega_c\tau & 0 \\ \omega_c\tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} & 0 \\ \rho_{yx} & \rho_{yy} & 0 \\ 0 & 0 & \rho_{zz} \end{pmatrix}$$

When $J_y = 0$,

$$E_x = \rho_{xx} J_x, \qquad E_y = \rho_{yx} J_x$$

wih

$$\rho_{xx} = \frac{1}{nq\mu} = \frac{m}{nq^2\tau} = \frac{1}{\sigma_0}, \qquad \rho_{yx} = \frac{\omega_c\tau}{nq\mu} = \frac{B}{nqc}$$

Hall coefficient:

$$R = \frac{E_y}{BJ_x} = \frac{\rho_{yx}}{B} = \frac{1}{nqc}$$

Note that the resistivity ρ_{xx} in the presence of magnetic fields is the as that in the absence of the field.

2. The case of two carriers (two-band model)

Next we consider two kinds of carriers (n_1, q_1, μ_1, and n_2, q_2, μ_2). Using the same procedure above discussed, we have

$$J_x = n_1 q_1 v_{1x} + n_2 q_2 v_{2x}, \qquad J_y = n_1 q_1 v_{1y} + n_2 q_2 v_{2y}$$

where

$$v_{1x} = \frac{\mu_1(E_x + \omega_{c1}\tau_1 E_y)}{1+\omega_{c1}{}^2\tau_1{}^2}, \qquad v_{1y} = \frac{\mu_1(-\omega_{c1}\tau_1 E_x + E_y)}{1+\omega_{c1}{}^2\tau_1{}^2}$$

$$v_{2x} = \frac{\mu_2(E_x + \omega_{c2}\tau_2 E_y)}{1+\omega_{c_2}{}^2\tau_2{}^2}, \qquad v_{2y} = \frac{\mu_2(-\omega_{c2}\tau_2 E_x + E_y)}{1+\omega_{c2}{}^2\tau_2{}^2}$$

with

$$\omega_{c1} = \frac{q_1 B}{mc}. \qquad \omega_{c2} = \frac{q_2 B}{mc}, \qquad \mu_1 = \frac{q_1 \tau_1}{m}, \qquad \mu_2 = \frac{q_2 \tau_2}{m}$$

Then we have

$$\begin{aligned} J_x &= n_1 q_1 \frac{\mu_1 (E_x + \omega_{c1}\tau_1 E_y)}{1+\omega_{c1}^2\tau_1^2} + n_2 q_2 \frac{\mu_2 (E_x + \omega_{c2}\tau_2 E_y)}{1+\omega_{c2}^2\tau_2^2} \\ &= (\frac{n_1 q_1 \mu_1}{1+\omega_{c1}^2\tau_1^2} + \frac{n_2 q_2 \mu_2}{1+\omega_{c2}^2\tau_2^2})E_x + (\frac{n_1 q_1 \mu_1 \omega_{c1}\tau_1}{1+\omega_{c1}^2\tau_1^2} + \frac{n_2 q_2 \mu_2 \omega_{c2}\tau_2}{1+\omega_{c2}^2\tau_2^2})E_y \end{aligned}$$

$$\begin{aligned} J_y &= n_1 q_1 \frac{\mu_1 (-\omega_{c1}\tau_1 E_x + E_y)}{1+\omega_{c1}^2\tau_1^2} + n_2 q_2 \frac{\mu_2 (-\omega_{c2}\tau_2 E_x + E_y)}{1+\omega_{c2}^2\tau_2^2} \\ &= -(\frac{n_1 q_1 \mu_1 \omega_{c1}\tau_1}{1+\omega_{c1}^2\tau_1^2} + \frac{n_2 q_2 \mu_2 \omega_{c2}\tau_2}{1+\omega_{c2}^2\tau_2^2})E_x \\ &\quad +(\frac{n_1 q_1 \mu_1}{1+\omega_{c1}^2\tau_1^2} + \frac{n_2 q_2 \mu_2}{1+\omega_{c2}^2\tau_2^2})E_y \end{aligned}$$

(a) Weak magnetic field ($\omega_c \tau << 1$)

$$J_x \approx (n_1 q_1 \mu_1 + n_2 q_2 \mu_2)E_x + (n_1 q_1 \mu_1 \omega_{c1}\tau_1 + n_2 q_2 \mu_2 \omega_{c2}\tau_2)E_y$$

$$J_y \approx -(n_1 q_1 \mu_1 \omega_{c1}\tau_1 + n_2 q_2 \mu_2 \omega_{c2}\tau_2)E_x + (n_1 q_1 \mu_1 + n_2 q_2 \mu_2)E_y$$

When $J_y = 0$

$$E_y = \frac{(n_1 q_1 \mu_1 \tau_1 \omega_{c1} + n_2 q_2 \mu_2 \omega_{c2}\tau_2)}{(n_1 q_1 \mu_1 + n_2 q_2 \mu_2)} E_x$$

$$\begin{aligned} J_x &\approx [(n_1 q_1 \mu_1 + n_2 q_2 \mu_2) + \frac{(n_1 q_1 \mu_1 \omega_{c1}\tau_1 + n_2 q_2 \mu_2 \omega_{c2}\tau_2)^2}{(n_1 q_1 \mu_1 + n_2 q_2 \mu_2)}]E_x \\ &= [(n_1 q_1 \mu_1 + n_2 q_2 \mu_2) + \frac{(n_1 q_1 \mu_1^2 + n_2 q_2 \mu_2^2)^2}{(n_1 q_1 \mu_1 + n_2 q_2 \mu_2)}\left(\frac{B}{c}\right)^2]E_x \end{aligned}$$

Hall coefficient

$$R = \frac{E_y}{BJ_x}$$
$$= \frac{(n_1 q_1 \mu_1 \tau_1 \omega_{c1} + n_2 q_2 \mu_2 \omega_{c2} \tau_2)}{B(n_1 q_1 \mu_1 + n_2 q_2 \mu_2)^2}$$
$$= \frac{(n_1 q_1 \mu_1^2 + n_2 q_2 \mu_2^2)}{c(n_1 q_1 \mu_1 + n_2 q_2 \mu_2)^2}$$

Note that

$$\omega_c \tau = \frac{B}{c} \mu$$

(b) The case of the strong magnetic field ($\omega_c \tau >> 1$)

$$J_x = (\frac{n_1 q_1 \mu_1}{\omega_{c1}^2 \tau_1^2} + \frac{n_2 q_2 \mu_2}{\omega_{c2}^2 \tau_2^2}) E_x + (\frac{n_1 q_1 \mu_1}{\omega_{c1} \tau_1} + \frac{n_2 q_2 \mu_2}{\omega_{c2} \tau_2}) E_y$$
$$= \frac{c^2}{B^2} (\frac{n_1 q_1}{\mu_1} + \frac{n_2 q_2}{\mu_2}) E_x + \frac{c}{B} (n_1 q_1 + n_2 q_2) E_y$$

$$J_y = -(\frac{n_1 q_1 \mu_1}{\omega_{c1} \tau_1} + \frac{n_2 q_2 \mu_2}{\omega_{c2} \tau_2}) E_x + (\frac{n_1 q_1 \mu_1}{\omega_{c1}^2 \tau_1^2} + \frac{n_2 q_2 \mu_2}{\omega_{c2}^2 \tau_2^2}) E_y$$
$$= -(n_1 q_1 + n_2 q_2) \frac{c}{B} E_x + \frac{c^2}{B^2} (\frac{n_1 q_1}{\mu_1} + \frac{n_2 q_2}{\mu_2}) E_y$$

When $J_y = 0$,

$$E_y = \frac{(n_1 q_1 + n_2 q_2)}{(\frac{n_1 q_1}{\mu_1} + \frac{n_2 q_2}{\mu_2})} \frac{B}{c} E_x .$$

$$J_x = [\frac{c^2}{B^2} (\frac{n_1 q_1}{\mu_1} + \frac{n_2 q_2}{\mu_2}) + \frac{(n_1 q_1 + n_2 q_2)^2}{(\frac{n_1 q_1}{\mu_1} + \frac{n_2 q_2}{\mu_2})}] E_x$$

The Hall coefficient:

$$R = \frac{E_y}{BJ_x}$$

$$= \frac{(n_1q_1 + n_2q_2)}{c[(n_1q_1 + n_2q_2)^2 + (\frac{n_1q_1}{\mu_1} + \frac{n_2q_2}{\mu_2})^2]}$$

3. Compensated metal (semi metals such as Bi)

In semimetal, there are both electron and holes. The number of holes is equal to that of electrons.

$$n_1 = n_2 = n, \qquad q_1 = -q_2 = e$$

Then the Hall coefficient is equal to zero. The current density (x-component) is

$$J_x = \frac{c^2}{B^2}(\frac{n_1q_1}{\mu_1} + \frac{n_2q_2}{\mu_2})E_x$$

So the magnetoresistance is proportional to B^2.

$$\frac{E_x}{J_x} = \frac{B^2}{c^2}\frac{1}{(\frac{n_1q_1}{\mu_1} + \frac{n_2q_2}{\mu_2})}$$

4. Origin of B^2 dependence in resistivity in semimetal

Inside the sample, electrons undergo a circular motion with the radius r_B in the presence of a magnetic field along the z axis. Note that

$$\frac{m{v_F}^2}{r_B} = \frac{e}{c}v_F B$$

or

$$\frac{mv_F}{r_B} = \frac{eB}{c}, \qquad \text{or} \qquad r_B = \frac{cp_F}{eB}$$

where

$$mv_F = p_F = \hbar k_F$$

When the electron is scattered by impurities and so on, electron shifts over the distance r_B to the direction of electric field. Correspondingly, the diffusion constant D is given by

$$D_B = \frac{r_B{}^2}{\tau}$$

i

Note that the conductivity σ is related to the diffusion constant as

$$\sigma = \frac{ne^2\tau}{m} \approx \frac{ne^2 D_0}{\varepsilon_F}$$

where n is the number density of electrons. This expression can be derived as follows in the free electron Fermi gas model. When D_0 is given by

$$D_0 = \frac{l^2}{\tau} = \frac{(v_F\tau)^2}{\tau} = v_F{}^2\tau = \frac{2\varepsilon_F}{m}\tau .$$

with $l = v_F\tau$ (mean free path). Using the expression of D_B, we have the conductivity in the presence of B as

$$\sigma = \frac{ne^2}{\varepsilon_F}\frac{r_B{}^2}{\tau} \propto r_B{}^2 \propto \frac{1}{B^2}$$

leading the B^2 dependence of the resistivity; $\rho \propto B^2$

5. Discussion

For the monovalent metal (Li, Na, K), there is one electron per unit cell. The valence band is half filled. For the odd number of electrons per unit cell, the Lorentz force is cancelled out by the Hall electric field built. The situation is similar to the case in the absence of magnetic field. The magnetoresistance becomes saturated in the limit of high magnetic field.

For divalent metals (Zn, Cd), there are two electrons per unit cell. The magnetoresistance is proportional to B^2. This implies that the number of holes is equal to that of electrons, which is predicted from the band structure model.

For trivalent metals (In, Al), there are three electrons per unit cell. The magnetoresistance becomes saturated in the high magnetic field. This implies that the number of holes is not equal to that of electrons. The Lorentz force is cancelled out by the Hall electric field built.

6. Magnetorestance for the open orbits

We now turn to an open section of Fermi surface about which an electron cannot perform closed orbits in the presence of a magnetic field. In this case the magnetorestance is proportional to B^2 in the limit of high magnetic field. This has been used to great deal in elucidating the Fermi surface of metals such as Cu, where changing the orientation of the magnetic field can produce open or closed orbits about the Fermi surface.

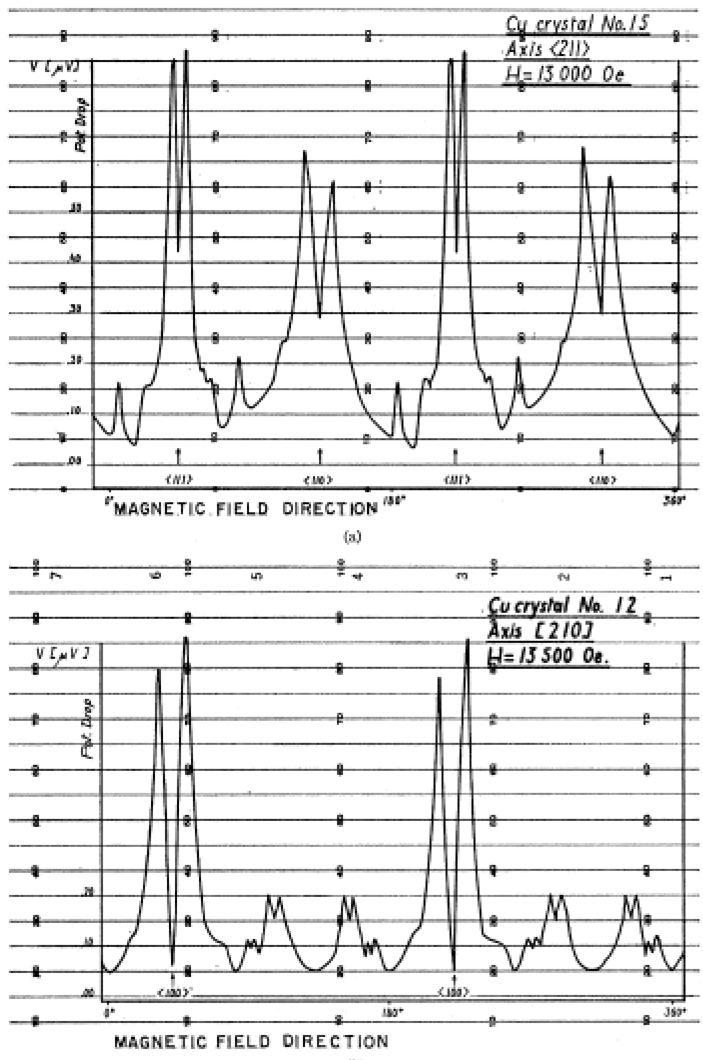

Fig. Angular dependence of transverse magneto-resistatnce of copper single crystal. B = 1.35 T. The current axis is $\langle 211 \rangle$. The magnetic field is rotated in the symmetric planes. (Funes and Coleman, Phys. Rev. 131, 2084 (1963).

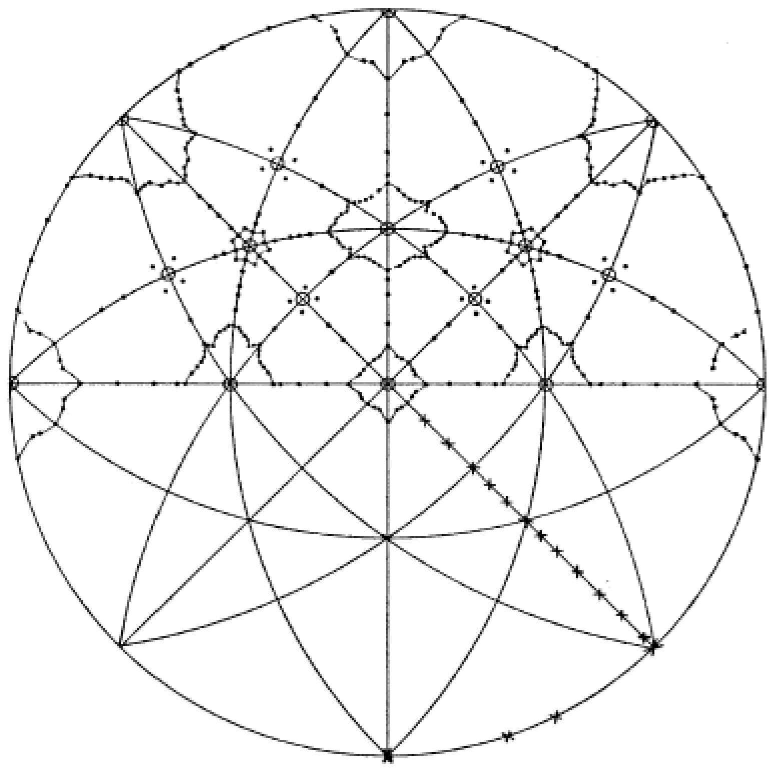

Fig. Stereographic plot of the field directions in which high maxima of resistance are observed. ○-deep minima; ●-High maxima; x-orientation of crystal axis used.

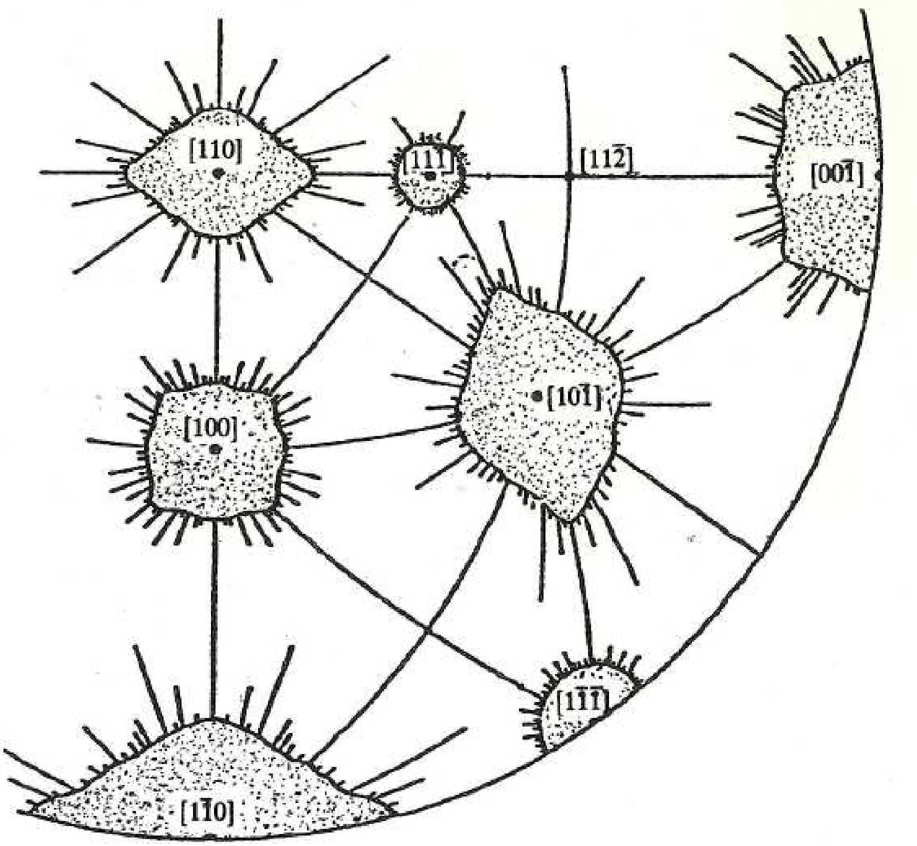

Fig. Klauder et al.

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A.J. Funes and R.V. Coleman, Phys. Rev. 131, 2084 (1963).
J.R. Klauder, W.A. Reed, G.F. Brennert and J.E. Kunzler, Phys. Rev. 141, 592 (1966).

APPENDIX
Kittel Chapter 9-10

10. *Open orbits and magnetoresistance.* We considered the transverse magnetoresistance of free electrons in Problem 6.9 and of electrons and holes in Problem 8.5. In some crystals the magnetoresistance saturates except in special crystal orientations. An open orbit carries current only in a single direction in the plane normal to the magnetic field; such carriers are not deflected by the field. In the arrangement of Fig. 6.14, let the open orbits be parallel to k_x; in real space these orbits carry current parallel to the y axis. Let $\sigma_{yy} = s\sigma_0$ be the conductivity of the open orbits; this defines the constant s. The magnetoconductivity tensor in the high field limit $\omega_c\tau \gg 1$ is

$$\sigma_0 \begin{pmatrix} Q^{-2} & -Q^{-1} & 0 \\ Q^{-1} & s & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $Q = \omega_c\tau$. (a) Show that the Hall field is $E_y = -E_x/sQ$. (b) Show that the effective resistivity in the x direction is $\rho = (Q^2/\sigma_0)(s/s+1)$, so that the resistivity does not saturate, but increases as B^2.

((**Solution**))

10a. $j_y = \sigma_0 (Q^{-1} E_x + sE_y) = 0$ in the Hall geometry, whence $E_y = -E_x/sQ$.

b. We have $j_x = \sigma_0 (Q^{-2} E_x - Q^{-1} E_y)$, and with our result for E_y it follows that

$$j_x = \sigma_0(Q^{-2} + s^{-1}Q^{-2})E_x,$$

whence $\rho = E_x/j_x = (Q^2/\sigma_0)\dfrac{s}{s+1}$.