**Ginzburg-Landau Theory for Superconductivity**
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton, Binghamton
(Date   April 29, 2012)

Vitaly Lazarevich Ginzburg ForMemRS (Russian: Вита́лий Ла́заревич Ги́нзбург; October 4, 1916 – November 8, 2009) was a Soviet theoretical physicist, astrophysicist, Nobel laureate, a member of the Russian Academy of Sciences and one of the fathers of Soviet hydrogen bomb. He was the successor to Igor Tamm as head of the Department of Theoretical Physics of the Academy's physics institute (FIAN), and an outspoken atheist.
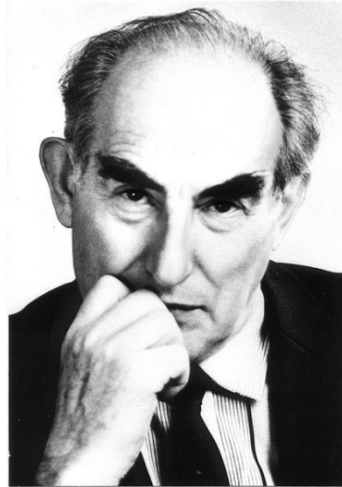
http://en.wikipedia.org/wiki/Vitaly_Ginzburg

Alexei Alexeyevich Abrikosov (Russian: Алексе́й Алексе́евич Абрико́сов; born June 25, 1928) is a Soviet and Russian theoretical physicist whose main contributions are in the field of condensed matter physics. He was awarded the Nobel Prize in Physics in 2003.

Abstract

The original article (2006) were put in my web site
http://www2.binghamton.edu/physics/docs/ginzburg-landau.pdf

After six years, I realize that there are some mistakes in this article. In Spring 2012, I have an opportunity to teach Solid State Physics (Phys.472/572). The present article is the revised version of the original article. The Mathematica programs are also revised, since they become old.

Here we discuss the phenomenological approach (Ginzburg-Landau (GL) theory) of the superconductivity, first proposed by Ginzburg and Landau long before the development of the microscopic theory [the Bardeen-Cooper-Schrieffer (BCS) theory]. The complicated mathematical approach of the BCS theory is replaced by a relatively simple second-order differential equation with the boundary conditions. In principle, the GL equation allows the order parameter, the field and the currents to be calculated. However, these equations are still non-linear and the calculations are rather complicated and in general purely numerical. The time dependent GL theory is not included in this article.

So far there have been many excellent textbooks on the superconductivity.[1-7] Among them, the books by de Gennes,[1] Tinkham,[6] and Nakajima[5] (the most popular textbook in Japan, unfortunately in Japanese) were very useful for our understanding the phenomena of superconductivity. There have been also very nice reviews,[8,9] on the superconducting phenomena. We also note that one will find many useful mathematical techniques in the Mathematica book written by Trott.[10]

Content

1.   Introduction

It is surprising that the rich phenomenology of the superconducting state could be quantitatively described by the GL theory,[11] without knowledge of the underlying microscopic mechanism based on the BCS theory. It is based on the idea that the superconducting transition is one of the second order phase transition.[12] In fact, the universality class of the critical behavior belongs to the three-dimensional *XY* system such as liquid ^{4}He. The general theory of the critical behavior can be applied to the superconducting phenomena. The order parameter is described by two components (complex number $\psi = |\psi|e^{i\theta}$). The amplitude $|\psi|$ is zero in the normal phase above a superconducting transition temperature T_c and is finite in the superconducting phase below T_c. In the presence of an external magnetic field, the order parameter has a spatial variation. When the spatial variation of the order parameter is taken into account, the free energy of the system can be expressed in terms of the order parameter ψ and its spatial derivative of ψ . In general this is valid in the vicinity of T_c below T_c, where the amplitude $|\psi|$ is small and the length scale for spatial variation is long.

The order parameter ψ is considered as a kind of a wave function for a particle of charge q^* and mass m^*. The two approaches, the BCS theory and the GL theory, remained completely separate until Gorkov[13] showed that, in some limiting cases, the order parameter $\psi(\mathbf{r})$ of the GL theory is proportional to the pair potential $\Delta(\mathbf{r})$. At the same time this also shows that $q^* = 2e$ (<0) and $m^* = 2m$. Consequently, the Ginzburg-Landau theory acquired their definitive status.

The GL theory is a triumph of physical intuition, in which a wave function $\psi(\mathbf{r})$ is introduced as a complex order parameter. The parameter $|\psi(\mathbf{r})|^2$ represents the local density of superconducting electrons, $n_s(\mathbf{r})$. The macroscopic behavior of superconductors (in particular the type II superconductors) can be explained well by this

GL theory. This theory also provides the qualitative framework for understanding the dramatic supercurrent behavior as a consequence of quantum properties on a macroscopic scale.

The superconductors are classed into two types of superconductor: type-I and type-II superconductors. The Ginzburg-Landau parameter κ is the ratio of λ to ξ, where λ is the magnetic-field penetration depth and ξ is the coherence length of the superconducting phase. The limiting value $\kappa = 1/\sqrt{2}$ separating superconductors with positive surface energy $(\kappa < 1/\sqrt{2})$ (type-I) from those with negative surface energy $(\kappa > 1/\sqrt{2})$ (type-II), is properly identified. For the type-II superconductor, the superconducting and normal regions coexist. The normal regions appear in the cores (of size ξ) of vortices binding individual magnetic flux quanta $\Phi_0 = 2\pi\hbar c/\left|q^*\right|$ on the scale λ, with the charge $\left|q^*\right| = 2\left|e\right|$ appearing in Φ_0 a consequence of the pairing mechanism. Since $\lambda > \xi$, the vortices repel and arrange in a so-called Abrikosov lattice. In his 1957 paper, Abrikosov[14] derived the periodic vortex structure near the upper critical field H_{c2}, where the superconductivity is totally suppressed, determined the magnetization $M(H)$, calculated the field H_{c1} of first penetration, analyzed the structure of individual vortex lines, found the structure of the vortex lattice at low fields.

2.   Background

In this chapter we briefly discuss the Maxwell's equation, Lagrangian, gauge transformation, and so on, which is necessary for the formulation of the GL equation.

2.1   Maxwell's equation

The Maxwell's equations (cgs units) are expressed in the form

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \times \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \dfrac{4\pi}{c}\mathbf{j} + \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} \end{cases}, \qquad (2.1)$$

B: magnetic induction (the microscopic magnetic field)
E: electric field
ρ: charge density
J: current density
c: the velocity of light
H: the applied external magnetic field

The Lorentz force is given by

$$\mathbf{F} = q[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})]. \qquad (2.2)$$

The Lorentz force is expressed in terms of fields **E** and **B** (gauge independent, see the gauge transformation below)).

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0, \qquad \text{(equation of continuity)} \qquad (2.3)$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad (2.4)$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \qquad (2.5)$$

where A is a vector potential and ϕ is scalar potential.

2.2 Lagrangian of particles with mass *m** and charge *q** in the presence of electric and magnetic field

The Lagrangian L is given by

$$L = \frac{1}{2}m^{*}\mathbf{v}^{2} - q^{*}(\phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}), \qquad (2.6)$$

where m^{*} and q^{*} are the mass and charge of the particle. Canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m^{*}\mathbf{v} + \frac{q^{*}}{c}\mathbf{A}. \qquad (2.7)$$

Mechanical momentum (the measurable quantity)

$$\boldsymbol{\pi} = m^{*}\mathbf{v} = \mathbf{p} - \frac{q^{*}}{c}\mathbf{A}. \qquad (2.8)$$

The Hamiltonian is given by

$$H = \mathbf{p}\cdot\mathbf{v} - L = (m^{*}\mathbf{v} + \frac{q^{*}}{c}\mathbf{A})\cdot\mathbf{v} - L = \frac{1}{2}m^{*}\mathbf{v}^{2} + q^{*}\phi = \frac{1}{2m^{*}}(\mathbf{p} - \frac{q^{*}}{c}\mathbf{A})^{2} + q^{*}\phi. \qquad (2.9)$$

The Hamiltonian formalism uses A and ϕ, and not **E** and **B**, directly. The result is that the description of the particle depends on the gauge chosen.

2.3 Gauge transformation: Analogy from classical mechanics[15,16]

When **E** and **B** are given, ϕ and A are not uniquely determined.
If we have a set of possible values for the vector potential A and the scalar potential ϕ, we obtain other potentials A' and ϕ' which describes the same electromagnetic field by the gauge transformation,

$$\mathbf{A'} = \mathbf{A} + \nabla \chi, \tag{2.10}$$

$$\phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t}, \tag{2.11}$$

where χ is an arbitrary function of r.

The Newton's second law indicates that the position and the velocity take on, at every point, values independent of the gauge. Consequently,

$$\mathbf{r'} = \mathbf{r} \quad \text{and} \quad \mathbf{v'} = \mathbf{v},$$

or

$$\boldsymbol{\pi'} = \boldsymbol{\pi}, \tag{2.12}$$

Since $\boldsymbol{\pi} = m^* \mathbf{v} = \mathbf{p} - \dfrac{q^*}{c} \mathbf{A}$, we have

$$\mathbf{p'} - \frac{q^*}{c} \mathbf{A'} = \mathbf{p} - \frac{q^*}{c} \mathbf{A}, \tag{2.13}$$

or

$$\mathbf{p'} = \mathbf{p} + \frac{q^*}{c}(\mathbf{A'} - \mathbf{A}) = \mathbf{p} + \frac{q^*}{c} \nabla \chi. \tag{2.14}$$

In the Hamilton formalism, the value at each instant of the dynamical variables describing a given motion depends on the gauge chosen.

2.4 Gauge invariance in quantum mechanics

In quantum mechanics, we describe the states in the old gauge and the new gauge as We denote $|\psi\rangle$ and $|\psi'\rangle$ the state vectors relative to these gauges. The analogue of the relation in the classical mechanics is thus given by the relations between average values.

$$\langle \psi' | \hat{\mathbf{r}} | \psi' \rangle = \langle \psi | \hat{\mathbf{r}} | \psi \rangle \text{ (gauge invariant)}, \tag{2.15}$$

$$\langle \psi' | \hat{\boldsymbol{\pi}} | \psi' \rangle = \langle \psi | \hat{\boldsymbol{\pi}} | \psi \rangle \text{ (gauge invariant)}, \tag{2.16}$$

$$\langle \psi' | \hat{\mathbf{p}} | \psi' \rangle = \langle \psi | \hat{\mathbf{p}} + \frac{q^*}{c} \nabla \chi | \psi \rangle. \tag{2.17}$$

We now seek a unitary operator $\hat{U}$ which enables one to go from $|\psi\rangle$ to $|\psi'\rangle$:

$$|\psi'\rangle = \hat{U}|\psi\rangle. \tag{2.18}$$

From the condition, $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$, we have

$$\hat{U}^+\hat{U} = \hat{U}\hat{U}^+ = \hat{1}. \tag{2.19}$$

From the condition, $\langle\psi'|\hat{\mathbf{r}}|\psi'\rangle = \langle\psi|\hat{\mathbf{r}}|\psi\rangle$,

$$\hat{U}^+\hat{\mathbf{r}}\hat{U} = \hat{\mathbf{r}}, \tag{2.20}$$

or

$$[\hat{\mathbf{r}}, \hat{U}] = 0 = i\hbar\frac{\partial\hat{U}}{\partial\hat{\mathbf{p}}}. \tag{2.21}$$

$\hat{U}$ is independent of $\hat{\mathbf{p}}$.

From the condition, $\langle\psi'|\hat{\mathbf{p}}|\psi'\rangle = \langle\psi|\hat{\mathbf{p}} + \frac{q}{c}\nabla\chi|\psi\rangle$,

$$\hat{U}^+\hat{\mathbf{p}}\hat{U} = \hat{\mathbf{p}} + \frac{q}{c}\nabla\chi, \tag{2.22}$$

or

$$[\hat{\mathbf{p}}, \hat{U}] = +\frac{q}{c}\hat{U}\nabla\chi = \frac{\hbar}{i}\frac{\partial}{\partial\hat{\mathbf{r}}}\hat{U}. \tag{2.23}$$

Now we consider the form of $\hat{U}$. The unitary operator $\hat{U}$ commutes with $\hat{\mathbf{r}}$ and χ:

$$\hat{U} = \exp(\frac{iq^*\chi}{c\hbar}). \tag{2.24}$$

The wave function is given by

$$\langle\mathbf{r}|\psi'\rangle = \langle\mathbf{r}|\hat{U}|\psi\rangle = \exp(\frac{iq^*\chi}{c\hbar})\langle\mathbf{r}|\psi\rangle. \tag{2.25}$$

For the wave function, the gauge transformation corresponds to a phase change which varies from one point to another, and is not, therefore, a global phase factor. Here we show that

$$\langle \psi' | \hat{\mathbf{p}} - \frac{q^*}{c} \mathbf{A}' | \psi' \rangle = \langle \psi | \hat{\mathbf{p}} - \frac{q^*}{c} \mathbf{A} | \psi \rangle, \tag{2.26}$$

$$\langle \psi' | \hat{\mathbf{p}} - \frac{q^*}{c} \mathbf{A}' | \psi' \rangle = \langle \psi | \hat{\mathbf{p}} + \frac{q^*}{c} \nabla \chi - \frac{q^*}{c} (\mathbf{A} + \nabla \chi) | \psi \rangle = \langle \psi | \hat{\mathbf{p}} - \frac{q^*}{c} \mathbf{A} | \psi \rangle, \tag{2.27}$$

since

$$\langle \psi' | \hat{\mathbf{p}} | \psi' \rangle = \langle \psi | \hat{\mathbf{p}} + \frac{q^*}{c} \nabla \chi | \psi \rangle, \tag{2.28}$$

and

$$\langle \psi' | \frac{q^*}{c} \mathbf{A}' | \psi' \rangle = \langle \psi | \hat{U}^+ \frac{q^*}{c} \mathbf{A}' \hat{U} | \psi \rangle = \langle \psi | \frac{q^*}{c} (\mathbf{A} + \nabla \chi) | \psi \rangle. \tag{2.29}$$

2.5   Hamiltonian under the gauge transformation

$$i\hbar \frac{\partial}{\partial t} | \psi \rangle = \hat{H} | \psi \rangle, \tag{2.30}$$

$$i\hbar \frac{\partial}{\partial t} | \psi' \rangle = \hat{H}' | \psi' \rangle, \tag{2.31}$$

$$i\hbar \frac{\partial}{\partial t} \hat{U} | \psi \rangle = \hat{H}' \hat{U} | \psi \rangle, \tag{2.32}$$

or

$$i\hbar [\frac{\partial \hat{U}}{\partial t} | \psi \rangle + \hat{U} \frac{\partial}{\partial t} | \psi \rangle] = \hat{H}' \hat{U} | \psi \rangle. \tag{2.33}$$

Since $\dfrac{\partial \hat{U}}{\partial t} = \dfrac{iq^*}{\hbar c} \dfrac{\partial \chi}{\partial t} \hat{U}$,

$$-\frac{q^*}{c} \frac{\partial \chi}{\partial t} \hat{U} | \psi \rangle + \hat{U} i\hbar \frac{\partial}{\partial t} | \psi \rangle = \hat{H}' \hat{U} | \psi \rangle, \tag{2.35}$$

or

$$-\frac{q^*}{c} \frac{\partial \chi}{\partial t} \hat{U} | \psi \rangle + \hat{U} \hat{H} | \psi \rangle = \hat{H}' \hat{U} | \psi \rangle, \tag{2.36}$$

or

$$\hat{H}'\hat{U} = -\frac{q^*}{c}\frac{\partial \chi}{\partial t}\hat{U} + \hat{U}\hat{H} .$$ (2.37)

Thus we have

$$\hat{H}' = -\frac{q^*}{c}\frac{\partial \chi}{\partial t} + \hat{U}\hat{H}\hat{U}^+ .$$ (2.38)

Note that

$$\hat{U}^+\hat{\mathbf{p}}\hat{U} = \hat{\mathbf{p}} + \frac{q}{c}\nabla\chi ,$$ (2.39)

or

$$\hat{\mathbf{p}}\hat{U} = \hat{U}(\hat{\mathbf{p}} + \frac{q}{c}\nabla\chi) ,$$ (2.40)

or

$$\hat{\mathbf{p}}\hat{U} = \hat{U}\hat{\mathbf{p}} + \frac{q}{c}\hat{U}\nabla\chi ,$$ (2.41)

or

$$\hat{U}\hat{\mathbf{p}}\hat{U}^+ = (\hat{\mathbf{p}} - \frac{q}{c}\nabla\chi) .$$ (2.42)

From this relation, we also have

$$\hat{U}(\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A})\hat{U}^+ = (\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A} - \frac{q^*}{c}\nabla\chi) = (\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}') ,$$ (2.43)

$$\hat{U}(\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A})^2\hat{U}^+ = (\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}')^2 .$$ (2.44)

Then

$$\hat{H}' = -\frac{q^*}{c}\frac{\partial \chi}{\partial t} + \hat{U}[\frac{1}{2m^*}(\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A})^2 + q^*\phi]\hat{U}^+ ,$$ (2.45)

or

$$\hat{H}'= -\frac{q^*}{c}\frac{\partial \chi}{\partial t} + \frac{1}{2m^*}(\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}')^2 + q^*\phi = \frac{1}{2m^*}(\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}')^2 + q^*\phi'. \quad (2.46)$$

Therefore the new Hamiltonian can be written in the same way in any gauge chosen.

2.6   Invariance of physical predictions under a gauge transformation
The current density is invariant under the gauge transformation.

$$\mathbf{J} = \frac{1}{m^*}\mathrm{Re}[\langle \psi | \hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A} | \psi \rangle], \quad (2.47)$$

$$\langle \psi' | \hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}' | \psi' \rangle = \langle \psi | \hat{U}^+ (\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}')\hat{U} | \psi \rangle, \quad (2.48)$$

$$\hat{U}^+ (\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}')\hat{U} = \hat{\mathbf{p}} + \frac{q^*}{c}\nabla\chi - \frac{q^*}{c}\mathbf{A}' = \hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}, \quad (2.49)$$

$$\mathbf{J}' = \frac{1}{m^*}\mathrm{Re}[\langle \psi' | \hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}' | \psi' \rangle] = \frac{1}{m^*}\mathrm{Re}[\langle \psi | \hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A} | \psi \rangle]. \quad (2.50)$$

The density is also gauge independent.

$$\rho' = |\langle \mathbf{r} | \psi' \rangle|^2 = \rho = |\langle \mathbf{r} | \psi \rangle|^2. \quad (2.51)$$

3.   Basic concepts
3.1   London's equation
We consider an equation given by

$$\langle \mathbf{p} \rangle = m^* \langle \mathbf{v} \rangle + \frac{q^*}{c}\mathbf{A}. \quad (3.1)$$

We assume that $\langle \mathbf{p} \rangle = 0$ or

$$m^* \langle \mathbf{v} \rangle + \frac{q^*}{c}\mathbf{A} = 0. \quad (3.2)$$

The current density is given by

$$\mathbf{J} = q^* |\psi|^2 \langle \mathbf{v} \rangle = -\frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A}. \text{ (London equation)} \quad (3.3)$$

This equation corresponds to a London equation. From this equation, we have

11

$$\nabla \times \mathbf{J} = -\frac{q^{*2}|\psi|^2}{m^*c}\nabla \times \mathbf{A} = -\frac{q^{*2}|\psi|^2}{m^*c}\mathbf{B} . \tag{3.4}$$

Using the Maxwell's equation

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} , \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0 ,$$

we ge

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{4\pi}{c}\nabla \times \mathbf{J} = -\frac{4\pi n^* q^{*2}}{m^*c^2}\mathbf{B} , \tag{3.5}$$

where

$$n^* = |\psi|^2 = \text{constant} \qquad \text{(independent of } r\text{)}$$

$$\lambda_L^2 = \frac{m^*c^2}{4\pi n^* q^{*2}} : \qquad (\lambda_L; \text{ penetration depth})$$

Then

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\frac{1}{\lambda_L^2}\mathbf{B} , \tag{3.6}$$

or

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B} . \tag{3.7}$$

In side the system, *B* become s zero, corresponding to the Meissner effect.

3.2   The quantum mechanical current density *J*

The current density is given by

$$\mathbf{J} = \frac{q^*\hbar}{2m^*i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A} . \tag{3.8}$$

Now we assume that

$$\psi = |\psi|e^{i\theta} . \tag{3.9}$$

Since

$$\psi^* \nabla \psi - \psi \nabla \psi^* = 2i |\psi|^2 \nabla \theta , \qquad (3.10)$$

we have

$$\mathbf{J} = \frac{q^* \hbar}{m^*} |\psi|^2 (\nabla \theta - \frac{q^*}{c\hbar} \mathbf{A}) = q^* |\psi|^2 \mathbf{v}_s \qquad (3.11)$$

or

$$\hbar \nabla \theta = \frac{q^*}{c} \mathbf{A} + m^* \mathbf{v}_s . \qquad (3.12)$$

This equation is generally valid. Note that J is gauge-invariant. Under the gauge transformation, the wave function is transformed as

$$\psi'(\mathbf{r}) = \exp(\frac{iq^* \chi}{\hbar c}) \psi(\mathbf{r}) . \qquad (3.13)$$

This implies that

$$\theta \to \theta' = \theta + \frac{q^* \chi}{\hbar c} , \qquad (3.14)$$

Since $\mathbf{A}' = \mathbf{A} + \nabla \chi$, we have

$$\mathbf{J}' = \hbar (\nabla \theta' - \frac{q^*}{c\hbar} \mathbf{A}') = \hbar [\nabla(\theta + \frac{q^* \chi}{\hbar c}) - \frac{q^*}{c\hbar} (\mathbf{A} + \nabla \chi)] = \hbar (\nabla \theta - \frac{q^*}{c\hbar} \mathbf{A}) . (3.15)$$

So the current density is invariant under the gauge transformation.

Here we note that

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp[i\theta(\mathbf{r})] . \qquad (3.16)$$

and

$$\mathbf{p} \psi(\mathbf{r}) = \frac{\hbar}{i} \nabla \{ \exp[i\theta(\mathbf{r})] |\psi(\mathbf{r})| \} = \frac{\hbar}{i} [i\psi(\mathbf{r}) \nabla \theta(\mathbf{r}) + \exp[i\theta(\mathbf{r})] \nabla |\psi(\mathbf{r})|] . \qquad (3.17)$$

If $|\psi(\mathbf{r})|$ is independent of r, we have

$$\mathbf{p} \psi(\mathbf{r}) = [\hbar \nabla \theta(\mathbf{r})] \psi(\mathbf{r}) \qquad (3.18)$$

13

or

$$\mathbf{p} = \hbar \nabla \theta(\mathbf{r}).$$ (3.19)

Then we have the following relation

$$\mathbf{p} = \hbar \nabla \theta = \frac{q^*}{c} \mathbf{A} + m^* \mathbf{v}_s$$ (3.20)

when $|\psi(\mathbf{r})|$ is independent of r.

3.3. London gauge

Here we assume that $|\psi|^2 = n_s^* = $ constant. Then we have

$$\mathbf{p} = \hbar \nabla \theta = \frac{q^*}{c} \mathbf{A} + m^* \mathbf{v}_s,$$ (3.21)

$$\mathbf{J}_s = q^* |\psi|^2 \mathbf{v}_s = q^* n_s^* \mathbf{v}_s.$$ (3.22)

Then

$$\mathbf{p} = \hbar \nabla \theta = \frac{q^*}{c} \mathbf{A} + \frac{m^*}{q^* n_s^*} \mathbf{J}_s.$$ (3.23)

We define

$$\Lambda = \frac{m^*}{n_s^* q^{*2}}.$$ (3.24)

Now the above equation is rewritten as

$$\mathbf{p} = \hbar \nabla \theta = \frac{q^*}{c} \mathbf{A} + q^* \Lambda \mathbf{J}_s,$$ (3.25)

$$\nabla \times \mathbf{p} = \frac{q^*}{c} (\nabla \times \mathbf{A}) + q^* \Lambda (\nabla \times \mathbf{J}_s) = 0,$$ (3.26)

or

$$\nabla \times (\frac{1}{c\Lambda} \mathbf{A} + \mathbf{J}_s) = 0,$$ (3.27)

or

$$\frac{1}{c\Lambda}\mathbf{B} + \nabla \times \mathbf{J}_s = 0 . \tag{3.28}$$

From the expression

$$\mathbf{p} = \frac{q^*}{c}\mathbf{A} + q^*\Lambda\mathbf{J}_s \tag{3.29}$$

and $p = 0$, we have a London equation

$$\mathbf{J}_s = -\frac{1}{c\Lambda}\mathbf{A} . \tag{3.30}$$

For the supercurrent to be conserved, it is required that

$$\nabla \cdot \mathbf{J}_s = -\frac{1}{c\Lambda}\nabla \cdot \mathbf{A} = 0$$

or

$$\nabla \cdot \mathbf{A} = 0 . \tag{3.31}$$

From the condition that no current can pass through the boundary of a superconductor, it is required that

$$\mathbf{J}_s \cdot \mathbf{n} = 0 ,$$

or

$$\mathbf{A} \cdot \mathbf{n} = 0 . \tag{3.32}$$

The conditions ($\nabla \cdot \mathbf{A} = 0$ and $\mathbf{A} \cdot \mathbf{n} = 0$) are called London gauge.
We now consider the gauge transformation:

$$\mathbf{A'} = \mathbf{A} + \nabla\chi$$

$$\mathbf{J}_s' = -\frac{1}{c\Lambda}\mathbf{A'}, \tag{3.33}$$

$$\nabla \cdot \mathbf{J}_s' = -\frac{1}{c\Lambda}\nabla \cdot \mathbf{A'} = -\frac{1}{c\Lambda}(\nabla \cdot \mathbf{A} + \nabla^2\chi) = 0 , \tag{3.34}$$

$$\mathbf{J}_s{}'\cdot\mathbf{n} = -\frac{1}{c\Lambda}\mathbf{A}'\cdot\mathbf{n} = -\frac{1}{c\Lambda}(\mathbf{A}\cdot\mathbf{n} + \nabla\chi\cdot\mathbf{n}). \tag{3.35}$$

The London equation is gauge-invariant if we throw away any part of A which does not satisfy the London gauge.

$$\mathbf{A}' = \mathbf{A} + \nabla\chi ,$$

where $\nabla^2\chi = 0$ and $\nabla\chi\cdot\mathbf{n} = 0$ are satisfied,

3.4 Flux quantization
We start with the current density

$$\mathbf{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\mathbf{A}) = q^*|\psi|^2\mathbf{v}_s . \tag{3.36}$$

Suppose that $n_s^* = |\psi|^2 =$ constant, then we have

$$\nabla\theta = \frac{m^*}{q^*\hbar n_s^*}\mathbf{J}_s + \frac{q^*}{c\hbar}\mathbf{A} , \tag{3.37}$$

or

$$\oint\nabla\theta\cdot d\mathbf{l} = \frac{m^*}{q^*\hbar n_s^*}\oint\mathbf{J}_s\cdot d\mathbf{l} + \frac{q^*}{c\hbar}\oint\mathbf{A}\cdot d\mathbf{l} . \tag{3.38}$$

The path of integration can be taken inside the penetration depth where $\mathbf{J}_s = 0$.

$$\oint\nabla\theta\cdot d\mathbf{l} = \frac{q^*}{c\hbar}\oint\mathbf{A}\cdot d\mathbf{l} = \frac{q^*}{c\hbar}\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \frac{q^*}{c\hbar}\int\mathbf{B}\cdot d\mathbf{a} = \frac{q^*}{c\hbar}\Phi , \tag{3.40}$$

where Φ is the magnetic flux. Then we find that

$$\Delta\theta = \theta_2 - \theta_1 = 2\pi n = \frac{q^*}{c\hbar}\Phi \tag{3.41}$$

where n is an integer. The phase θ of the wave function must be unique, or differ by a multiple of 2π at each point,

$$\Phi = \frac{2\pi c\hbar}{|q^*|}n . \tag{3.42}$$

The flux is quantized. When $|q^*| = 2|e|$, we have a magnetic quantum fluxoid;

$$\Phi_0 = \frac{2\pi c\hbar}{2|e|} = \frac{ch}{2|e|} = 2.06783372 \times 10^{-7} \text{ Gauss cm}^2 \tag{3.43}$$

4. Ginzburg-Landau theory-phenomenological approach
4.1 The postulated GL equation

We introduce the order parameter $\psi(\mathbf{r})$ with the property that

$$\psi^*(\mathbf{r})\psi(\mathbf{r}) = n_s(\mathbf{r}), \tag{4.1}$$

which is the local concentration of superconducting electrons. We first set up a form of the free energy density $F_s(\mathbf{r})$,

$$F_s(\mathbf{r}) = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{\mathbf{B}^2}{8\pi}, \tag{4.2}$$

where β is positive and the sign of α is dependent on temperature.

4.2 The derivation of GL equation and the current density by variational method

We must minimize the free energy with respect to the order parameter $\psi(r)$ and the vector potential $A(r)$. We set

$$\Im = \int F_s(\mathbf{r})dr, \tag{4.3}$$

where the integral is extending over the volume of the system. If we vary

$$\psi(\mathbf{r}) \to \psi(\mathbf{r}) + \delta\psi(\mathbf{r}) \text{ and } \mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + \delta\mathbf{A}(\mathbf{r}), \tag{4.4}$$

we obtain the variation in the free energy such that

$$\Im + \delta\Im.$$

By setting $\delta\Im = 0$, we obtain the GL equation

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi = 0, \tag{4.5}$$

and the current density

$$\mathbf{J}_s = \frac{q^*\hbar}{2m^*i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A} \tag{4.6}$$

17

or

$$\mathbf{J}_s = \frac{q^*}{2m^*}[\psi^*(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi + \psi(-\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi^*].\tag{4.7}$$

At a free surface of the system we must choose the gauge to satisfy the boundary condition that no current flows out of the superconductor into the vacuum.

$$\mathbf{n}\cdot\mathbf{J}_s = 0.\tag{4.8}$$

4.2.1   Derivation of GL equation by variational method

((**Mathematica program-1**))
Variational method using "VarialtionalD" of the Mathematica.
Note that ψ_c is the complex conjugate of ψ.

Derivation of Ginzburg Landau equation

```
Clear["Global`*"]; << "VariationalMethods`"; Needs["VectorAnalysis`"];
SetCoordinates[Cartesian[x, y, z]];
A = {A1[x, y, z], A2[x, y, z], A3[x, y, z]};

eq1 = α ( ψ[x, y, z] ψc[x, y, z] ) + 1/2 β (ψ[x, y, z]² ψc[x, y, z]²) +

    1/(2 m) (((ℏ/i Grad[ψ[x, y, z]] - q/c A ψ[x, y, z]) .

    (- ℏ/i Grad[ψc[x, y, z]] - q/c A ψc[x, y, z])) // Expand;
eq2 = VariationalD[eq1, ψc[x, y, z], {x, y, z}] // Expand;
```

We need to calculate the following

```
OP1 := (ℏ/i D[#, x] - q/c A1[x, y, z] # &); OP2 := (ℏ/i D[#, y] - q/c A2[x, y, z] # &);

OP3 := (ℏ/i D[#, z] - q/c A3[x, y, z] # &);

eq3 =
  α ψ[x, y, z] + β ψ[x, y, z]² ψc[x, y, z] +
    1/(2 m) (OP1[OP1[ψ[x, y, z]]] + OP2[OP2[ψ[x, y, z]]] + OP3[OP3[ψ[x, y, z]]]) //
  Expand;
eq4 = eq2 - eq3

0
```

18

4.2.2. Derivation of current density by variational method ((Mathematica Program-2))

```
Clear["Global`*"];

<< "VariationalMethods`"; Needs["VectorAnalysis`"];
SetCoordinates[Cartesian[x, y, z]];

A = {A1[x, y, z], A2[x, y, z], A3[x, y, z]}; H = {H1, H2, H3};

eq1 = 1/(2 m) (ℏ/i Grad[ψ[x, y, z]] - q/c A ψ[x, y, z]) . (- ℏ/i Grad[ψc[x, y, z]] - q/c A ψc[x, y, z]) +
      1/(8 π) (Curl[A]).(Curl[A]) - 1/(4 π) H.Curl[A]  // Expand;
eq2 = VariationalD[eq1, A1[x, y, z], {x, y, z}] // Expand;
```

```
eq3 =
1/c (c/(4 π) Curl[Curl[A]] + q^2/(m c) ψ[x, y, z] ψc[x, y, z] A -
     (q ℏ)/(i 2 m) (ψc[x, y, z] Grad[ψ[x, y, z]] - ψ[x, y, z] Grad[ψc[x, y, z]]));

eq4 = eq3[[1]] // Expand;

eq2 - eq4 // Simplify
```

```
0
```

5   Basic properties
5.1   GL free energy and Thermodynamic critical field H_c

$A = 0$ and $\psi = \psi_\infty$ (real) has no space dependence. Why ψ is real?
We have a gauge transformation;

$$\mathbf{A'} = \mathbf{A} + \nabla\chi \text{ and } \psi'(\mathbf{r}) = \exp(\frac{iq^*\chi}{c\hbar})\psi(\mathbf{r}). \tag{5.1}$$

We choose

$$\mathbf{A'} = \mathbf{A} + \nabla\chi = \mathbf{A}$$

with $\chi = \chi_0 = $ constant. Then we have

$$\psi'(\mathbf{r}) = \exp(\frac{iq^*\chi_0}{c\hbar})\psi(\mathbf{r}),$$

or

$$\psi(\mathbf{r}) = \exp(-\frac{iq^*\chi_0}{c\hbar})\psi'(\mathbf{r}).$$

Even if ψ' is complex number, ψ can be real number.

$$F_s = F_N + \alpha\psi_\infty^2 + \frac{1}{2}\beta\psi_\infty^4.$$ (5.2)

When $\dfrac{\partial F}{\partial\psi_\infty} = 0$, F_s has a local minimum at

$$\psi_\infty = (-\alpha/\beta)^{1/2} = (|\alpha|/\beta)^{1/2}.$$ (5.3)

Then we have

$$F_s - F_N = -\frac{\alpha^2}{2\beta} = -\frac{H_c^2}{8\pi},$$ (5.4)

from the definition of the thermodynamic critical field:

$$H_c = \left(\frac{4\pi\alpha^2}{\beta}\right)^{1/2} = H_c(0)(1-\frac{T^2}{T_c^2}).$$ (5.5)

(parabolic law). Suppose that β is independent of T, then

$$\alpha = -\sqrt{\frac{\beta}{4\pi}}H_c(0)(1-\frac{T^2}{T_c^2}) \approx 2\sqrt{\frac{\beta}{4\pi}}H_c(0)(\frac{T}{T_c}-1) = \alpha_0(\frac{T}{T_c}-1),$$ (5.6)

where

$$\alpha_0 = 2\sqrt{\frac{\beta}{4\pi}}H_c(0).$$

The parameter α is positive above T_c and is negative below T_c. Note that $\beta>0$. For $T<T_c$, the sign of α is negative:

$$F_s - F_N = \alpha_0(t-1)\psi_\infty^2 + \frac{1}{2}\beta\psi_\infty^4,$$ (5.7)

where $\alpha_0 > 0$ and $t = T/T_c$ is a reduced temperature.
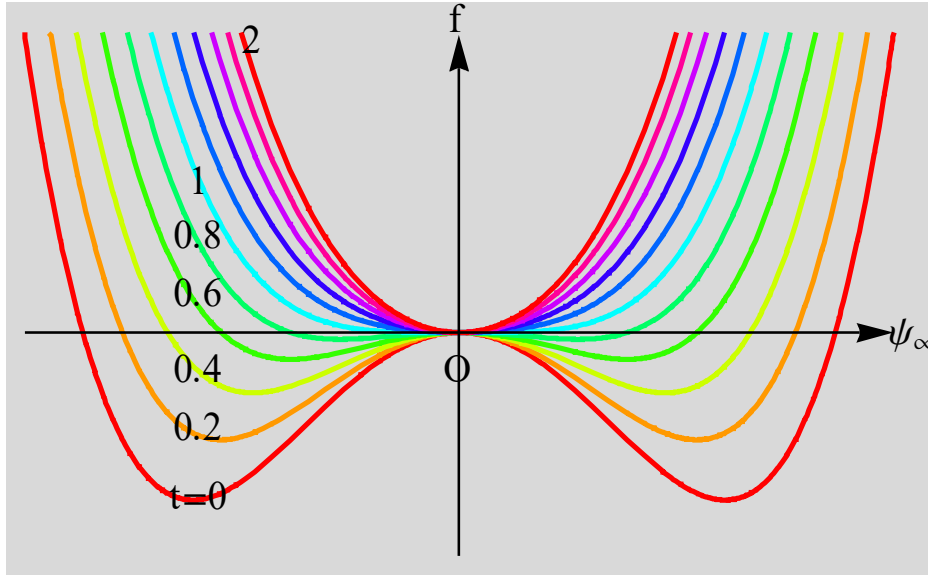
Fig.1   The GL free energy functions expressed by Eq.(5.7), as a function of ψ_∞. $\alpha_0 = 3$. $\beta = 1$. t is changed as a parameter. $t = T/T_c$. $(t = 0 - 2)$ around $t = 1$.

Fig.2   The order parameter ψ_∞ as a function of a reduced temperature $t = T/T_c$. We use $\alpha_0 = 3$ and $\beta = 1$ for this calculation.

5.2   Coherence length ξ

We assume that $A = 0$. We choose the gauge in which ψ is real.

$$\alpha\psi + \beta\psi^3 - \frac{\hbar^2}{2m^*}\frac{d^2}{dx^2}\psi = 0.$$ (5.8)

We put $\psi = \psi_\infty f$.

$$-\frac{\hbar^2}{2m^*\alpha}\frac{d^2f}{dx^2} + f - f^3 = 0.$$ (5.9)

We introduce the coherence length

$$\xi^2 = -\frac{\hbar^2}{2m^*\alpha} = \frac{\hbar^2}{2m^*|\alpha|} = \frac{2\pi n_s^* \hbar^2}{m^* H_c^2},$$ (5.10)

where

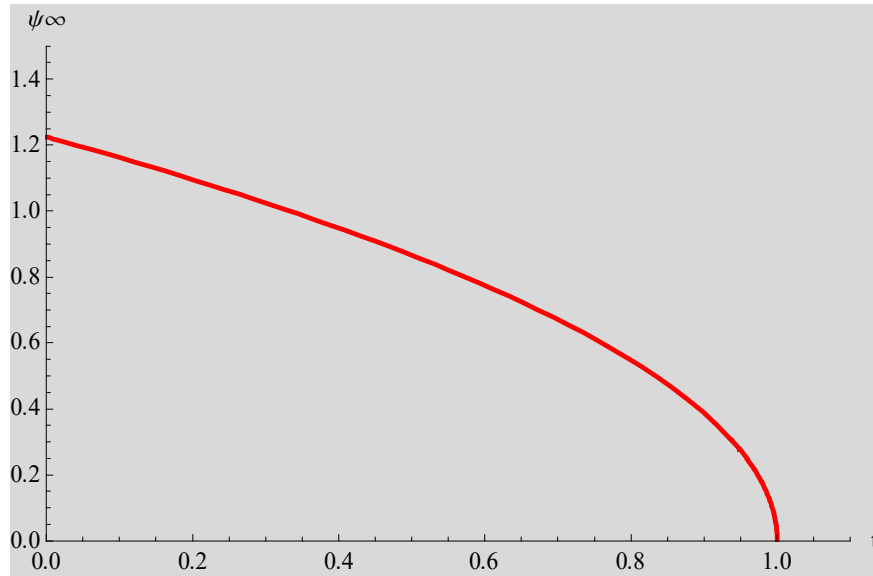$$\frac{\alpha^2}{2\beta} = \frac{H_c^2}{8\pi}, \qquad \psi_\infty^2 = \frac{|\alpha|}{\beta} = n_s,$$ (5.11)

or

$$\xi = \sqrt{\frac{\hbar^2}{2m^*\alpha_0}}\,|1 - \frac{T}{T_c}|^{-1/2}.$$ (5.12)

Then we have

$$\xi^2\frac{d^2f}{dx^2} + f - f^3 = 0,$$ (5.13)

with the boundary condition $f = 1$, $df/dx = 0$ at $x = \infty$ and $f = 0$ at $x = 0$.

$$\xi^2\frac{df}{dx}\frac{d^2f}{dx^2} + (f - f^3)\frac{df}{dx} = 0,$$ (5.14)

or

$$\frac{d}{dx}\frac{\xi^2}{2}\left(\frac{df}{dx}\right)^2 = \frac{d}{dx}(\frac{f^4}{4} - \frac{f^2}{2}),$$ (5.15)

or

22

$$\frac{\xi^2}{2}\left(\frac{df}{dx}\right)^2 = \frac{1}{4}(1-f^2)^2 , \tag{5.16}$$

or

$$\frac{df}{dx} = \frac{1}{\sqrt{2}\xi}(1-f^2). \tag{5.17}$$

The solution of this equation is given by

$$f = \tanh\left(\frac{x}{\sqrt{2}\xi}\right). \tag{5.18}$$

((Mathematica))

Ginzburg-Landau equation; coherence length

```
Clear["Global`*"]; eq1 =  f'[x] ==  1/(√2 ξ) (1 - f[x]^2);

eq2 = DSolve[{eq1, f[0] == 0}, f[x], x];
f[x_] = f[x] /. eq2[[1]] // ExpToTrig // Simplify
```

$\mathrm{Tanh}\left[\frac{x}{\sqrt{2}\,\xi}\right]$
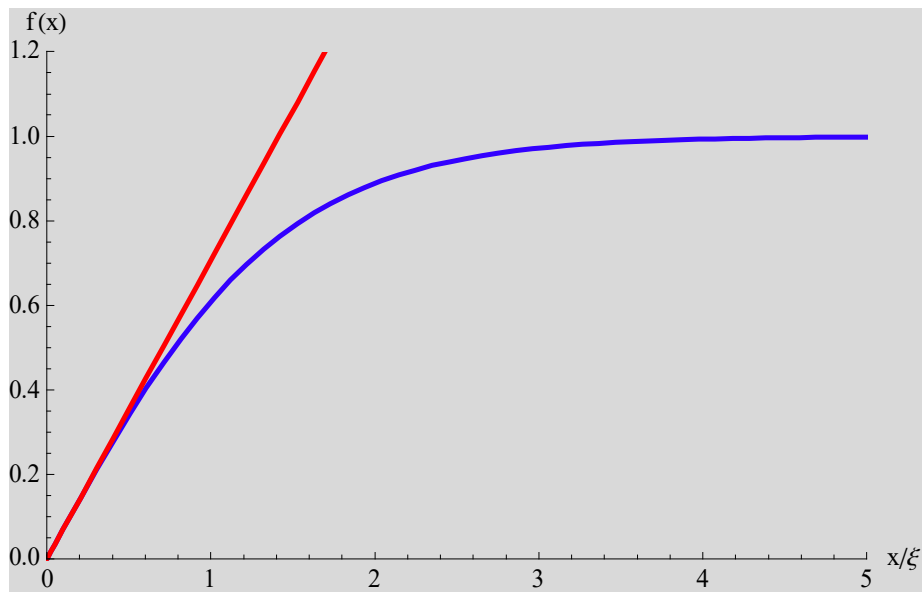
Fig.3 Normalized order parameter $f(x)$ expressed by Eq.(5.18) as a function of x/ξ.
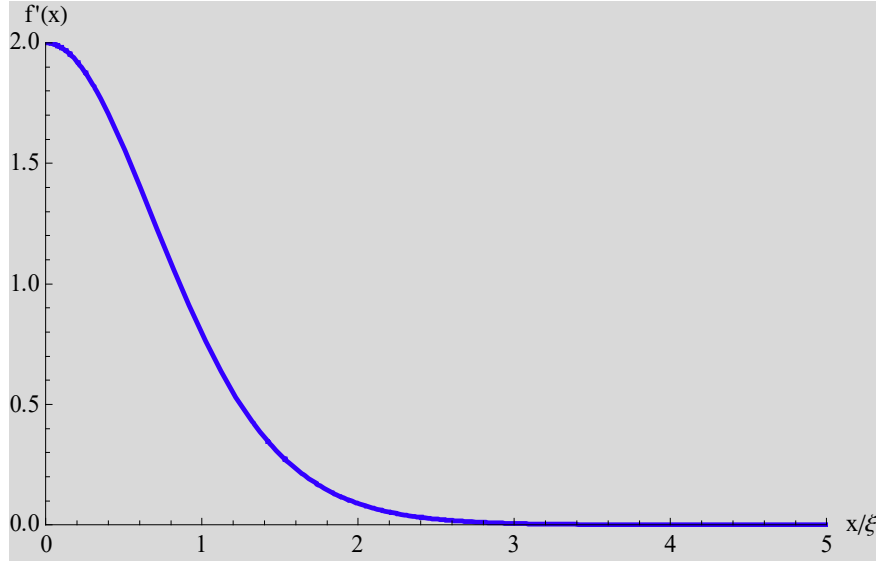
Fig.4 Derivative $f'(x)$ as a function of x/ξ.

5.3 Magnetic field penetration depth λ

$$\mathbf{J}_s = \frac{q^*\hbar}{2m^*i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A}\,.\qquad(5.19)$$

We assume that $\psi = \psi_\infty$ (real).

$$\mathbf{J}_s = -\frac{q^{*2}\psi_\infty^2}{m^*c}\mathbf{A}\ \ \text{(London's equation)},\qquad(5.20)$$

$$\nabla\times\mathbf{J}_s = -\frac{q^{*2}\psi_\infty^2}{m^*c}\nabla\times\mathbf{A} = -\frac{q^{*2}\psi_\infty^2}{m^*c}\mathbf{B}\,,\qquad(5.21)$$

where

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}_s\,,\qquad \mathbf{B} = \nabla\times\mathbf{A}\,,\qquad \nabla\cdot\mathbf{B} = 0\,.\qquad(5.22)$$

Then we have

$$\nabla\times(\frac{c}{4\pi}\nabla\times\mathbf{B}) = -\frac{q^{*2}\psi_\infty^2}{m^*c}\mathbf{B}\,,\qquad(5.23)$$

or

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\frac{4\pi q^{*2} \psi_\infty^2}{m^* c^2} \mathbf{B}. \tag{5.24}$$

Then we have London's equation

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda^2} \mathbf{B}, \tag{5.25}$$

where

$$\lambda^{-2} = \frac{4\pi q^{*2} \psi_\infty^2}{m^* c^2}. \tag{5.26}$$

λ is the penetration depth

$$\lambda = \sqrt{\frac{m^* c^2}{4\pi q^{*2} \psi_\infty^2}} = \sqrt{\frac{m^* c^2 \beta}{4\pi q^{*2} |\alpha|}}. \tag{5.27}$$

The solution of the above differential equation is given by

$$B_Z(x) = B_Z(x = 0) \exp(-x/\lambda), \tag{5.28}$$

where the magnetic field is directed along the z axis.

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} = \frac{c}{4\pi} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & B_z(x) \end{vmatrix} = \frac{c}{4\pi} \left(0, -\frac{\partial}{\partial x} B_z(x), 0\right). \tag{5.29}$$

The current $\mathbf{J}$ flows along the y direction.

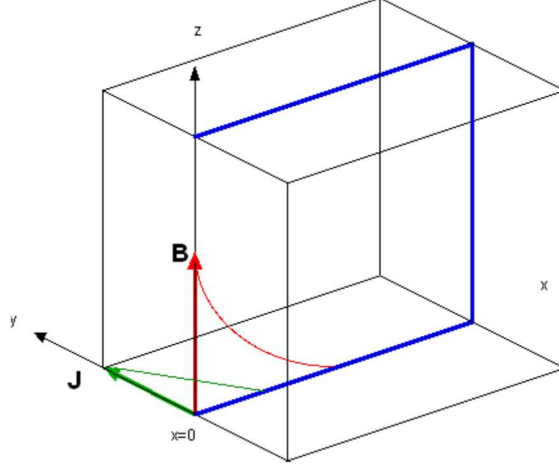$$J_y = \frac{cB_z(x = 0)}{4\pi\lambda} \exp(-x/\lambda). \tag{5.30}$$

Fig.5   The distribution of the magnetic induction $B(x)$ (along the z axis) and the current density (along the *y* axis) near the boundary between the normal phase and the superconducting phase. The plane with $x = 0$ is the boundary.

5.4 Parameters related to the superconductivity

Here the superconducting parameters are listed for convenience.

$$n_s^* = \psi_\infty^2 = \frac{|\alpha|}{\beta}, \qquad H_c^2 = 4\pi n_s^* |\alpha| = 4\pi |\alpha| \psi_\infty^2 = 4\pi \frac{|\alpha|^2}{\beta}, \qquad (5.31)$$

$$H_c^2 = 4\pi \psi_\infty^4 \beta \qquad (5.32)$$

$$|\alpha| = \frac{H_c^2}{4\pi \psi_\infty^2}, \qquad \beta = \frac{H_c^2}{4\pi \psi_\infty^4}, \qquad \Phi_0 = \frac{2\pi \hbar c}{|q^*|}, \qquad (5.33)$$

$$\lambda = \sqrt{\frac{m^* c^2 \beta}{4\pi q^{*2} |\alpha|}}, \qquad \xi = \frac{\hbar}{\sqrt{2m^* |\alpha|}}, \qquad \kappa = \frac{\lambda}{\xi} = \frac{cm^* \sqrt{\beta}}{\sqrt{2\pi \hbar} |q^*|}. \qquad (5.34)$$

Then we have

$$\lambda = \sqrt{\frac{m^* c^2}{4\pi n_s^* q^{*2}}}, \qquad \xi = \frac{\sqrt{2\pi n_s^*} \, \hbar}{H_c \sqrt{m^*}}, \qquad \kappa = \frac{cm^* H_c}{2\sqrt{2}\pi \hbar n_s^* |q^*|}, \qquad (5.35)$$

$$\frac{\lambda \xi}{\Phi_0} = \frac{1}{2\sqrt{2}\pi H_c}, \qquad \frac{\kappa}{\lambda^2} = \frac{\sqrt{2} |q^*| H_c}{c\hbar}, \qquad \frac{\kappa}{\lambda^2} \Phi_0 = 2\sqrt{2}\pi H_c, \qquad (5.36)$$

26

$$H_c = \frac{c\hbar}{\sqrt{2}|q^*|\lambda\xi}. \tag{5.37}$$

We also have

$$\kappa = \frac{2\sqrt{2}\pi\lambda^2 H_c}{\Phi_0}, \qquad \sqrt{2}H_c = \frac{\Phi_0}{2\pi\lambda\xi}. \tag{5.38}$$

Note:

$$|q^*| = 2e \, (>0) \quad m^* = 2m, \qquad n_s^* = n_s/2.$$

((**Mathematica Program-5**))

n1 = ns*, q1 = q*, α1 = Abs[α], m1 = m* *

```
rule1 = {α1 → Hc²/(4 π n1) , β → Hc²/(4 π n1²) , Φ0 → (2 π ℏ c)/q1}; λ = √(m1 c² β)/(4 π q1² α1) ;
```

```
ξ = ℏ/√(2 m1 α1) ;
```

```
κ = λ/ξ // PowerExpand
```

$$\frac{c\ m1\ \sqrt{\beta}}{\sqrt{2\ \pi}\ q1\ \hbar}$$

```
λ1 = λ /. rule1 // PowerExpand
```

$$\frac{c\ \sqrt{m1}}{2\ \sqrt{n1}\ \sqrt{\pi}\ q1}$$

```
ξ1 = ξ /. rule1 // PowerExpand
```

$$\frac{\sqrt{n1}\ \sqrt{2\ \pi}\ \hbar}{Hc\ \sqrt{m1}}$$

```
(λ ξ)/Φ0 /. rule1 // PowerExpand
```

$$\frac{1}{2\ \sqrt{2}\ Hc\ \pi}$$

```
κ1 = κ /. rule1 // PowerExpand
```

$$\frac{c\ Hc\ m1}{2\ \sqrt{2}\ n1\ \pi\ q1\ \hbar}$$

```
κ/λ² /. rule1 // PowerExpand
```

$$\frac{\sqrt{2}\ Hc\ q1}{c\ \hbar}$$

```
κ/λ² Φ0 /. rule1 // PowerExpand
```

$$2\ \sqrt{2}\ Hc\ \pi$$

6. Formulation of GL equation
6.1 General theory[2]

In summary we have the GL equation;

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi = 0,\tag{6.1}$$

and

$$\mathbf{J}_s = \frac{c}{4\pi}\nabla\times(\nabla\times\mathbf{A}) = \frac{c}{4\pi}[\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}] = \frac{q^*\hbar}{2m^*i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A},$$

$$\tag{6.2}$$

$$\mathbf{B} = \nabla\times\mathbf{A}.\tag{6.3}$$

The GL equations are very often written in a form which introduces only the following dimensionless quantities.

$$\psi = \psi_\infty f,\qquad \mathbf{r} = \lambda\boldsymbol{\rho},\qquad \mathbf{h} = \frac{\mathbf{B}}{\sqrt{2}H_c},\tag{6.4}$$

where

$$\lambda^{-2} = \frac{4\pi q^{*2}\psi_\infty^2}{m^*c^2},\ \xi^2 = -\frac{\hbar^2}{2m^*\alpha} = \frac{\hbar^2}{2m^*|\alpha|},H_c^2 = \frac{4\pi\alpha^2}{\beta},\ \psi_\infty^2 = \frac{|\alpha|}{\beta} = -\frac{\alpha}{\beta},$$

$$\tag{6.5}$$

$$\Phi_0 = \frac{2\pi\hbar c}{|q^*|} = -\frac{2\pi\hbar c}{q^*}\ \text{(since }q^* = 2e<0),\tag{6.6}$$

$$\frac{1}{\sqrt{2}H_c\lambda^2} = \frac{2\pi}{\Phi_0\kappa},\qquad \sqrt{2}H_c\lambda^2 = \frac{\Phi_0}{2\pi}\kappa,\qquad \kappa = \frac{\lambda}{\xi}.\tag{6.7}$$

We have

$$\left(\frac{1}{i\kappa}\nabla_\rho + \tilde{\mathbf{A}}\right)^2 f = f - |f|^2 f,\tag{6.8}$$

where

$$\widetilde{\mathbf{A}} = \frac{2\pi\xi}{\Phi_0}\mathbf{A}\,.$$ (6.9)

Here we use the relation

$$\mathbf{h} = \frac{\mathbf{B}}{\sqrt{2}H_c} = \frac{1}{\sqrt{2}H_c}\nabla\times\mathbf{A} = \frac{1}{\sqrt{2}H_c}\frac{\Phi_0}{2\pi\lambda\xi}\nabla_\rho\times\widetilde{\mathbf{A}} = \nabla_\rho\times\widetilde{\mathbf{A}}\,,$$ (6.10)

or

$$\mathbf{h} = \nabla_\rho\times\widetilde{\mathbf{A}}\,,$$ (6.11)

$$\begin{aligned}\mathbf{J}_s &= \frac{c}{4\pi}\nabla\times(\nabla\times\mathbf{A}) = \frac{c}{4\pi}[\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}]\\ &= \frac{q^*\hbar}{2m^*i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A}\end{aligned}$$ (6.12)

$$\frac{c}{4\pi\lambda^2}\frac{\Phi_0}{2\pi\xi}\nabla_\rho\times(\nabla_\rho\times\widetilde{\mathbf{A}}) = \frac{q^*\hbar}{2m^*i}\psi_\infty^2\frac{1}{\lambda}[f^*\nabla_\rho f - f\nabla f^*] - \frac{q^{*2}\psi_\infty^2}{m^*c}\frac{\Phi_0}{2\pi\xi}|f|^2\widetilde{\mathbf{A}}\,,$$ (6.13)

$$\nabla_\rho\times(\nabla_\rho\times\widetilde{\mathbf{A}}) = \frac{i}{2\kappa}[f^*\nabla_\rho f - f\nabla f^*] - |f|^2\widetilde{\mathbf{A}}\,,$$ (6.14)

since

$$\lambda^{-2} = \frac{4\pi q^{*2}\psi_\infty^2}{m^*c^2}$$ (6.15)

and

$$\begin{aligned}\frac{4\pi\lambda^2}{c}\frac{2\pi\xi}{\Phi_0}\frac{q^*\hbar}{2m^*}\psi_\infty^2\frac{1}{\lambda} &= \frac{4\pi\lambda^2}{c}\frac{2\pi\xi}{\Phi_0}\frac{q^*\hbar}{2m^*}\frac{1}{\lambda}\frac{m^*c^2}{4\pi q^{*2}\lambda^2} = \frac{2\pi\xi}{\Phi_0}\frac{\hbar c}{2q^*}\frac{1}{\lambda}\\ &= -\frac{2\pi}{\Phi_0}\frac{\hbar c}{2|q^*|}\frac{1}{\kappa} = -\frac{1}{\kappa}\end{aligned}$$ (6.16)

$$\frac{q^{*2}}{m^*c}\psi_\infty^2 = \frac{c}{4\pi\lambda^2}\,.$$ (6.17)

30

In summary we have the following equations:

$$\left(\frac{1}{i\kappa}\nabla_\rho + \tilde{\mathbf{A}}\right)^2 f = f - |f|^2 f, \tag{6.18}$$

$$\nabla_\rho \times (\nabla_\rho \times \tilde{\mathbf{A}}) = \frac{i}{2\kappa}[f^*\nabla_\rho f - f\nabla f^*] - |f|^2 \tilde{\mathbf{A}}, \tag{6.19}$$

$$\mathbf{h} = \nabla_\rho \times \tilde{\mathbf{A}}. \tag{6.20}$$

Gauge transformation:

$$\mathbf{A} = \mathbf{A}_0 + \nabla\chi, \tag{6.21}$$

$$\psi(r) = \exp(\frac{iq^*\chi}{c\hbar})\psi_0(r), \tag{6.22}$$

$$\tilde{\mathbf{A}} = \frac{2\pi\xi}{\Phi_0}\mathbf{A}, \qquad \psi = \psi_\infty f, \tag{6.23}$$

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_0 + \frac{2\pi\xi}{\Phi_0}\nabla\chi = \tilde{\mathbf{A}}_0 + \frac{2\pi\xi}{\Phi_0}\frac{1}{\lambda}\nabla_\rho\chi = \tilde{\mathbf{A}}_0 + \frac{1}{\kappa}\nabla_\rho(\frac{2\pi}{\Phi_0}\chi), \tag{6.24}$$

$$f = \exp(\frac{iq^*\chi}{c\hbar})f_0 = \exp(-\frac{i2\pi}{\Phi_0}\chi)f. \tag{6.25}$$

Here we assume that $-\dfrac{2\pi}{\Phi_0}\chi = \varphi_0$. Then we have

$$f = \exp(i\varphi_0)f_0, \qquad \tilde{\mathbf{A}} = \tilde{\mathbf{A}}_0 - \frac{1}{\kappa}\nabla_\rho\varphi_0. \tag{6.26}$$

We can choose the order parameter f as a real number f_0 such that f_0 is real, f_0 is a constant.
$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_0$.

$$\left(\frac{1}{i\kappa}\nabla_\rho + \tilde{\mathbf{A}}_0\right)^2 f_0 = f_0 - f_0^3, \tag{6.27}$$

$$\nabla_\rho \times (\nabla_\rho \times \tilde{\mathbf{A}}_0) = \nabla_\rho \times \mathbf{h} = -f_0^2\tilde{\mathbf{A}}_0, \tag{6.28}$$

$$\mathbf{h} = \nabla_\rho \times \tilde{\mathbf{A}}_0, \tag{6.29}$$

$$\left(\frac{1}{i\kappa}\nabla_\rho + \widetilde{\mathbf{A}}_0\right)^2 f_0 = f_0 - f_0^3$$

$$= -\frac{1}{\kappa^2}\nabla_\rho^2 f_0 + \left(\widetilde{\mathbf{A}}_0\right)^2 f_0 + \frac{1}{i\kappa}[\nabla_\rho \cdot (\widetilde{\mathbf{A}}_0 f_0) + (\widetilde{\mathbf{A}}_0 \cdot \nabla_\rho f_0)]$$

(6.30)

Using the formula of the vector analysis

$$\nabla_\rho \cdot [\nabla_\rho \times (\nabla_\rho \times \widetilde{\mathbf{A}}_0)] = 0 = -\nabla_\rho \cdot (f_0^2 \widetilde{\mathbf{A}}_0)$$

$$= -\nabla_\rho f_0^2 \cdot \widetilde{\mathbf{A}}_0 - f_0^2 \nabla_\rho \cdot \widetilde{\mathbf{A}}_0 = -2f_0 \nabla_\rho f_0 \cdot \widetilde{\mathbf{A}}_0 - f_0^2 \nabla_\rho \cdot \widetilde{\mathbf{A}}_0$$

(6.31)

or

$$-2\widetilde{\mathbf{A}}_0 \cdot \nabla_\rho f_0 \cdot = f_0 \nabla_\rho \cdot \widetilde{\mathbf{A}}_0,$$

(6.32)

We also have

$$\nabla_\rho \cdot (\widetilde{\mathbf{A}}_0 f_0) = \widetilde{\mathbf{A}}_0 \cdot \nabla_\rho f_0 + f_0 \nabla_\rho \cdot \widetilde{\mathbf{A}}_0.$$

(6.33)

We have

$$f_0 - f_0^3 = -\frac{1}{\kappa^2}\nabla_\rho^2 f_0 + \left(\widetilde{\mathbf{A}}_0\right)^2 f_0 + \frac{1}{i\kappa}[f_0 \nabla_\rho \cdot \widetilde{\mathbf{A}}_0 + 2\widetilde{\mathbf{A}}_0 \cdot \nabla_\rho f_0]$$

$$= -\frac{1}{\kappa^2}\nabla_\rho^2 f_0 + \left(\widetilde{\mathbf{A}}_0\right)^2 f_0$$

(6.34)

or

$$-\frac{1}{\kappa^2}\nabla_\rho^2 f_0 + \left(\widetilde{\mathbf{A}}_0\right)^2 f_0 = f_0 - f_0^3.$$

(6.35)

We also have

$$\nabla_\rho \times \mathbf{h} = -f_0^2 \widetilde{\mathbf{A}}_0,$$

(6.36)

$$\widetilde{\mathbf{A}}_0 = -\frac{1}{f_0^2}\nabla_\rho \times \mathbf{h},$$

(6.37)

$$\mathbf{h} = \nabla_\rho \times \widetilde{\mathbf{A}}_0 = -\nabla_\rho \times [\frac{1}{f_0^2}\nabla_\rho \times \mathbf{h}]$$

$$= -\nabla_\rho \left(\frac{1}{f_0^2}\right) \times (\nabla_\rho \times \mathbf{h}) - \frac{1}{f_0^2}\nabla_\rho \times (\nabla_\rho \times \mathbf{h})$$

(6.38)

or

$$\mathbf{h} = \frac{2}{f_0^3} \nabla_\rho f_0 \times (\nabla_\rho \times \mathbf{h}) - \frac{1}{f_0^2} \nabla_\rho \times (\nabla_\rho \times \mathbf{h}), \tag{6.39}$$

or

$$f_0^2 \mathbf{h} = \frac{2}{f_0} \nabla_\rho f_0 \times (\nabla_\rho \times \mathbf{h}) - \nabla_\rho \times (\nabla_\rho \times \mathbf{h}). \tag{6.40}$$

In summary,

$$-\frac{1}{\kappa^2} \nabla_\rho^2 f_0 + \frac{1}{f_0^3} (\nabla_\rho \times \mathbf{h})^2 = f_0 - f_0^3, \tag{6.41}$$

$$f_0^2 \mathbf{h} = \frac{2}{f_0} \nabla_\rho f_0 \times (\nabla_\rho \times \mathbf{h}) - \nabla_\rho \times (\nabla_\rho \times \mathbf{h}), \tag{6.42}$$

$$\widetilde{\mathbf{A}}_0 = -\frac{1}{f_0^2} \nabla_\rho \times \mathbf{h}. \tag{6.43}$$

6.2   Special case

We now consider the one dimensional case. For convenience we remove the subscript ρ and $\boldsymbol{r} = (x, y, z)$ instead of $\boldsymbol{\rho}$. We also use A for $\widetilde{\mathbf{A}}_0$ and f for f_0. f is real.

$$-\frac{1}{\kappa^2} \nabla^2 f + \frac{1}{f^3} (\nabla \times \mathbf{h})^2 = f - f^3, \tag{6.44}$$

$$f^2 \mathbf{h} = \frac{2}{f} \nabla f \times (\nabla \times \mathbf{h}) - \nabla \times (\nabla \times \mathbf{h}), \tag{6.45}$$

$$\mathbf{A} = -\frac{1}{f^2} \nabla \times \mathbf{h}, \tag{6.46}$$

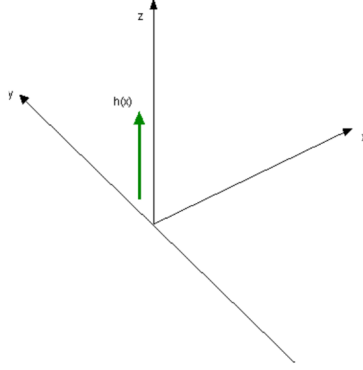$$\mathbf{h} = \nabla \times \mathbf{A}. \tag{6.47}$$

Fig.6   The direction of the magnetic induction $h(x)$.

x, y and z are dimensionless.

$$\nabla \times \mathbf{h} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & h(x) \end{vmatrix} = \left(0, -\frac{dh}{dx}, 0\right), \tag{6.48}$$

$$\nabla \times (\nabla \times \mathbf{h}) = \nabla(\nabla \cdot \mathbf{h}) - \nabla^2 \mathbf{h} = -\nabla^2 \mathbf{h} = -\frac{d^2 h}{dx^2} \mathbf{e}_z, \tag{6.49}$$

$$-\frac{1}{\kappa^2} \frac{d^2 f}{dx^2} + \frac{1}{f^3} \left(\frac{dh}{dx}\right)^2 = f - f^3, \tag{6.50}$$

$$f^2 h = -\frac{2}{f} \frac{df}{dx} \frac{dh}{dx} + \frac{d^2 h}{dx^2} \qquad \text{for the } z \text{ component,} \tag{6.51}$$

or

$$h = \frac{\partial}{\partial x} \left(\frac{1}{f^2} \frac{\partial h}{\partial x}\right), \tag{6.52}$$

$$\mathbf{A} = \frac{1}{f^2} \frac{dh}{dx} \mathbf{e}_y. \tag{6.53}$$

(a) $\kappa \ll 1$ (type I superconductor)

$$f - f^3 + \frac{1}{\kappa^2} \frac{d^2 f}{dx^2} = 0. \tag{6.54}$$

The solution of this equation is already given above.

(b)　　$\kappa \gg 1$ (type-II superconductor)

$$f - f^3 - \frac{1}{f^3}\left(\frac{\partial h}{\partial x}\right)^2 = 0. \tag{6.55}$$

7　　Free energy
7.1　　Helmholtz and Gibbs free energies

The analysis leading to the GL equation can be done either in terms of the Helmholtz free energy (GL) or in terms of the Gibbs free energy.

(a)　　The Helmholtz free energy
　　　It is appropriate for situations in which $B =<H>$; macroscopic average is held constant rather than H, because if B is constant, there is no induced emf and no energy input from the current generator.
(b)　　The Gibbs free energy
　　　It is appropriate for the case of constant H.

The analysis leading to the GL equations can be done either in terms of the Helmholtz free energy or in terms of the Gibbs energy. The Gibbs free energy is appropriate for the case of constant H.

$$g = f - \frac{1}{4\pi}\mathbf{B}\cdot\mathbf{H} \quad \text{(Legendre transformation).} \tag{7.1}$$

B is a magnetic induction (microscopic magnetic field) at a given point of the superconductors.

The Helmholtz free energy is given by

$$f_s = f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{\mathbf{B}^2}{8\pi}, \tag{7.2}$$

The Gibbs free energy is expressed by

$$g_s = f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{\mathbf{B}^2}{8\pi} - \frac{1}{4\pi}\mathbf{B}\cdot\mathbf{H}. \tag{7.3}$$

(i)　　At $x = -\infty$ (normal phase)

$$f(-\infty) = f_n + \frac{H_c^2}{8\pi}, \tag{7.4}$$

$$g(-\infty) = g_n = f_n + \frac{H_c^2}{8\pi} - \frac{H_c^2}{4\pi} = f_n - \frac{H_c^2}{8\pi}. \tag{7.5}$$

(ii)   At $x = \infty$ (superconducting phase)

$$g_s = g_s(\infty) = f_s = f_n + \alpha \psi_\infty{}^2 + \frac{1}{2}\beta \psi_\infty{}^4 = f_n - \frac{\alpha^2}{2\beta} = f_n - \frac{H_c{}^2}{8\pi}. \tag{7.6}$$

The interior Gibbs free energy

$$E_g = \int d\mathbf{r}[g_s(\mathbf{r}) - g_n]$$

$$= \int d\mathbf{r}[f_n - g_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{B^2}{8\pi} - \frac{1}{4\pi}BH_c] \tag{7.7}$$

or

$$E_g = \int d\mathbf{r}[\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{1}{8\pi}(B - H_c)^2] \tag{7.8}$$

7.2   Derivation of surface energy
We have a GL equation;

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi = 0. \tag{7.9}$$

If one multiplies the GL equation by ψ^* and integrates over all $\mathbf{r}$ by parts, one obtain the identity

$$\int d\mathbf{r}[\alpha|\psi|^2 + \beta|\psi|^4 + \frac{1}{2m^*}\psi^*\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi] = 0, \tag{7.10}$$

which is equal to

$$\int d\mathbf{r}[\alpha|\psi|^2 + \beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2] = 0, \tag{7.11}$$

Here we show that the quantity I defined by

$$I = \int d\mathbf{r}[\psi^*\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi - \left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2]. \tag{7.12}$$

is equal to 0, which is independent of the choice of ψ and ψ^*. The variation of I is calculated as

$$\delta I = \int d\mathbf{r}[\Lambda \delta\psi + \Gamma \delta\psi^*] \tag{7.13}$$

using the Mathematica (VariationalD). We find that $\Lambda = \Gamma = 0$ for any ψ and ψ^*. I is independent of the choice of ψ and ψ^*. When we choose $\psi = \psi^* = 0$. Then we have $I = 0$.

((**Mathematica Program-6**))

```
Clear["Global`*"]; << "VariationalMethods`"; Needs["VectorAnalysis`"];
SetCoordinates[Cartesian[x, y, z]];

A = {A1[x, y, z], A2[x, y, z], A3[x, y, z]};
eq1 =
  1
 ---- ((( ℏ Grad[ψ[x, y, z]] - q A ψ[x, y, z]) .
 2 m     ( i                    c

    ( ℏ                         q
    (- - Grad[ψc[x, y, z]] - - A ψc[x, y, z])) // Expand;
    ( i                       c

OP1 := ( ℏ D[#, x] - q A1[x, y, z] # &);
        ( i          c

OP2 := ( ℏ D[#, y] - q A2[x, y, z] # &);
        ( i          c

OP3 := ( ℏ D[#, z] - q A3[x, y, z] # &);
        ( i          c

eq2 =
    1
  ---- ψc[x, y, z]
  2 m
     (OP1[OP1[ψ[x, y, z]]] + OP2[OP2[ψ[x, y, z]]] + OP3[OP3[ψ[x, y, z]]]) //
   Expand;

eq3 = eq1 - eq2 // Expand;

VariationalD[eq3, ψc[x, y, z], {x, y, z}]

0

VariationalD[eq3, ψ[x, y, z], {x, y, z}]

0
```

Subtracting Eq.(7.11) from Eq.(7.8), we obtain the concise form

$$E_g = \int d\mathbf{r}[-\frac{1}{2}\beta|\psi|^4 + \frac{1}{8\pi}(B-H_c)^2]. \tag{7.14}$$

Suppose that the integrand depends only on the x axis. Then we can define the surface energy per unit area as γ:

$$\gamma = \int_{-\infty}^{\infty} dx[-\frac{1}{2}\beta|\psi|^4 + \frac{1}{8\pi}(B-H_c)^2], \tag{7.15}$$

which is to be equal to

$$\gamma = \delta\frac{1}{8\pi}H_c^2. \tag{7.16}$$

Then we obtain a simple expression

$$\delta = \int_{-\infty}^{\infty} dx[-\frac{|\psi|^4}{\psi_\infty^4} + (1-\frac{B}{H_c})^2]. \tag{7.17}$$

The second term is a positive diamagnetic energy and the first term is a negative condensation energy due to the superconductivity. When f and B/H_c are defined by

$$f = \frac{\psi}{\psi_\infty}, \qquad h = \frac{B}{\sqrt{2}H_c}. \tag{7.18}$$

Here we use $x = \lambda\tilde{x}$ ($\tilde{x}$ is dimensionless). For simplicity, furthermore we use x instead of $\tilde{x}$. Then we have

$$\delta = \lambda\int_{-\infty}^{\infty} dx[-f^4 + (1-\sqrt{2}h)^2] = 2\lambda\int_{-\infty}^{\infty} dx[\frac{1}{2}(1-f^4) + h^2 - \sqrt{2}h]. \tag{7.19}$$

7.3 Surface energy calculation for the two cases

(a) $\kappa \ll 1$ (type-I superconductor)

When $h = 0$ for $x>0$ and $\frac{|\psi|^4}{\psi_\infty^4} = f^4$ with $f = \tanh\left(\frac{x}{\sqrt{2}\kappa}\right)$, we have

$$\delta \approx \lambda\int_{-\infty}^{\infty} dx(1-f^4) = \frac{4\sqrt{2}}{3}\kappa\lambda = \frac{4\sqrt{2}}{3}\xi = 1.8856\xi > 0, \tag{7.20}$$

a result first obtained by Ginzburg and Landau. So the surface energy is positive.

((Mathematica Program-7))

```
Clear["Global`*"];
```

$$I1 = \int_0^\infty \left(1 - \text{Tanh}\left[\frac{x}{\sqrt{2}\,\kappa}\right]\right)^4 dx \,//\, \text{Simplify}[\#, \kappa > 0] \,\&$$

$$\frac{4\sqrt{2}\,\kappa}{3}$$

(b) $\kappa \gg 1$ (type-II superconductor)

Here we assume that λ is much larger than ξ.

$$f^4(1 - f^2) = \left(\frac{\partial h}{\partial x}\right)^2.$$

As h must decrease with increasing x, we have

$$\frac{1}{f^2}\frac{\partial h}{\partial x} = -(1 - f^2)^{1/2}, \qquad (7.21)$$

and

$$h = \frac{\partial}{\partial x}\left(\frac{1}{f^2}\frac{\partial h}{\partial x}\right) = -\frac{\partial}{\partial x}(1 - f^2)^{1/2}. \qquad (7.22)$$

Here f obeys the following differential equation. From Eq.(7.22) we have

$$\frac{\partial h}{\partial x} = -\frac{\partial^2}{\partial x^2}(1 - f^2)^{1/2}. \qquad (7.23)$$

Using Eq.(7.21),

$$\frac{\partial h}{\partial x} = -\frac{\partial^2}{\partial x^2}(1 - f^2)^{1/2} = -(1 - f^2)^{1/2} f^2, \qquad (7.24)$$

or

$$\frac{\partial^2}{\partial x^2}(1 - f^2)^{1/2} = (1 - f^2)^{1/2}(f^2 - 1 + 1) = (1 - f^2)^{1/2} - (1 - f^2)^{3/2}. \qquad (7.25)$$

When $u = (1 - f^2)^{1/2}$, we have

$$\frac{\partial^2 u}{\partial z^2} = u - u^3,$$ (7.26)

$$\left(\frac{\partial u}{\partial z}\right)^2 = u^2 - \frac{1}{2}u^4 + const.$$ (7.27)

When $u = 0$,

$$\frac{du}{dx} = 0 \ (\text{at } f = 1 \ (x = \infty) \qquad \text{at } \frac{du}{dx} = 0).$$

So we have

$$\left(\frac{du}{dx}\right)^2 = u^2 - \frac{1}{2}u^4 = u^2(1 - \frac{1}{2}u^2),$$ (7.28)

or

$$\frac{du}{dx} = -u(1 - \frac{1}{2}u^2)^{1/2},$$ (7.29)

since du/dx must be negative. We solve the problem with the boundary condition; $u = 1$ at $x = 0$. The solution for u is

$$u = \frac{2(-2 + \sqrt{2})e^x}{-3 + 2\sqrt{2} - e^{2x}}.$$

which is obtained from the Mathematica Program-9 (see below).

$$\delta = \lambda \int_{-\infty}^{\infty} dx[\frac{1}{2}(1 - f^4) + h^2 - \sqrt{2}h],$$ (7.30)

or

$$\delta = \lambda \int_{-\infty}^{\infty} dx[2u^2(1 - \frac{u^2}{2}) - \sqrt{2}u(1 - \frac{u^2}{2})^{1/2}],$$ (7.31)

or

$$\delta = \lambda \int_{-\infty}^{\infty} du[2u^2(1 - \frac{u^2}{2}) - \sqrt{2}u(1 - \frac{u^2}{2})^{1/2}]\frac{dx}{du}du,$$ (7.32)

$$\delta = \lambda \int_1^0 du [2u^2(1-\frac{u^2}{2}) - \sqrt{2}u(1-\frac{u^2}{2})^{1/2}]\frac{1}{-u(1-\frac{1}{2}u^2)^{1/2}}du \qquad (7.33)$$

or

$$\delta = \lambda \int_0^1 du [2u(1-\frac{u^2}{2})^{1/2} - \sqrt{2}]du = -\lambda \frac{4}{3}(\sqrt{2}-1) < 0, \qquad (7.34)$$

Thus, for $\kappa \gg 1$ the surface energy is negative.

((**Mathematica Program-8**))

```
Clear["Global`*"];
```

$$I1 = \int_0^1 \left(2\,u\,\sqrt{1-\frac{u^2}{2}} \;-\; \sqrt{2} \right) d\!\!u$$

$$-\frac{4}{3}\left(-1 + \sqrt{2}\right)$$

((**Mathematica Program-9**)) Negative surface energy

41

```
Clear["Global`*"]; eq1 = u'[x] == -u[x] (1 - u[x]^2/2)^(1/2) ;
```

```
eq2 = DSolve[{eq1, u[0] == 1}, u[x], x] // Simplify;
u[x_] = u[x] /. eq2[[1]] // Simplify
```

$$\frac{2 \left(-2+\sqrt{2}\right) e^x}{-3+2\sqrt{2} - e^{2 x}}$$

```
h1 = Plot[u[x], {x, 0, 5}, PlotStyle → {Thick, Hue[0]}, PlotPoints → 100,
   Background → LightGray, AxesLabel → {"x", "u"}];
```

```
h2 = Plot[√(1 - u[x]^2) , {x, 0, 5}, PlotStyle → {Thick, Hue[0]}, PlotPoints → 100,
   PlotRange → {{0, 5}, {0, 1}}, Background → LightGray, AxesLabel → {"x", "f"}];
```

```
F = 2 u[x]^2 (1 - u[x]^2/2) - √2 u[x] (1 - u[x]^2/2)^(1/2) // Simplify;
```

```
h3 = Plot[F, {x, 0, 10}, PlotStyle → {Thick, Hue[0]}, PlotPoints → 100,
   Background → LightGray, AxesLabel → {"x", "F"}];
Integrate[F, {x, 0, ∞}]
```

$$-\frac{4}{3} \left(-1+\sqrt{2}\right)$$

```
% // N
```

```
-0.552285
```

```
F1 = D[F, x];
h4 = Plot[F1, {x, 0, 10}, PlotStyle → {Thick, Hue[0]}, PlotPoints → 100,
   Background → LightGray, AxesLabel → {"x", "dFdx"}, PlotRange → All];
FindRoot[F1 == 0, {x, 1, 2}]
```

{x → 1.14622}

```
pt1 = {h1, h2, h3, h4}; Show[GraphicsGrid[Partition[pt1, 2]]]
```
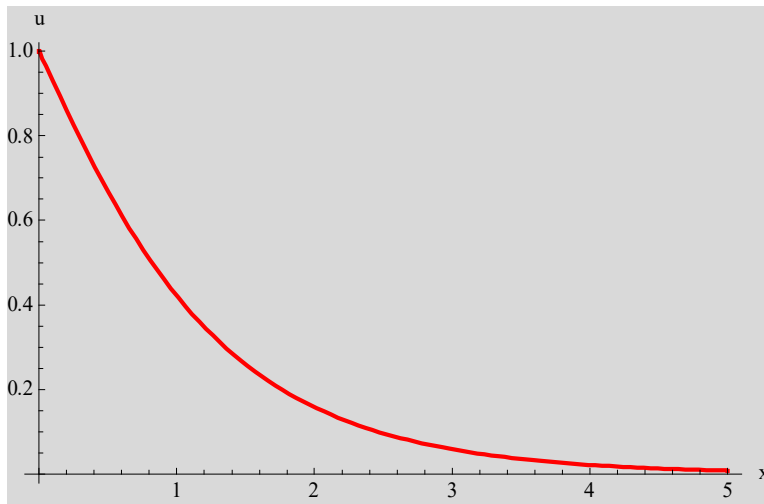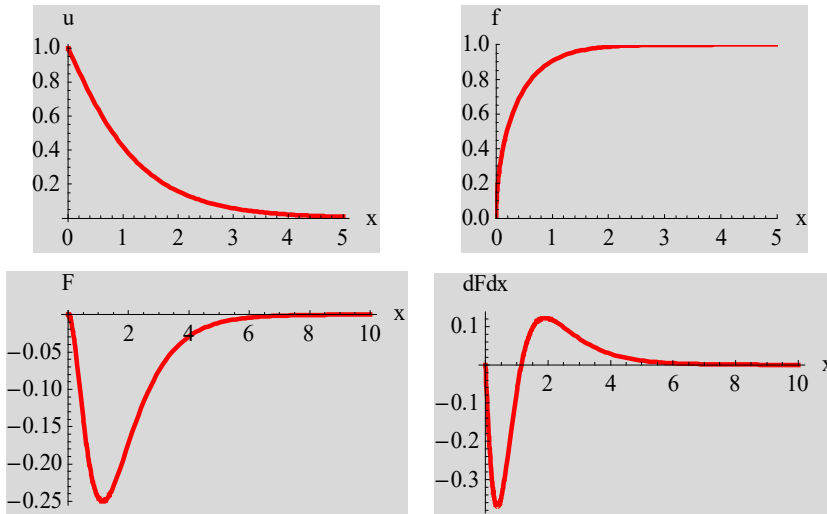

Fig.7   Plot of $u = (1 - f^2)^{1/2}$ vs x. x (dimensionless parameter) is the ratio of the distance to λ.
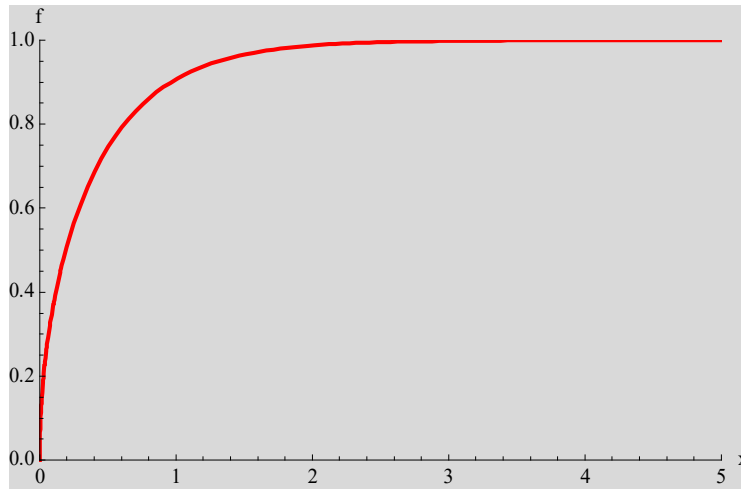
Fig.8   Plot of $f = \sqrt{1 - u^2}$ vs x.
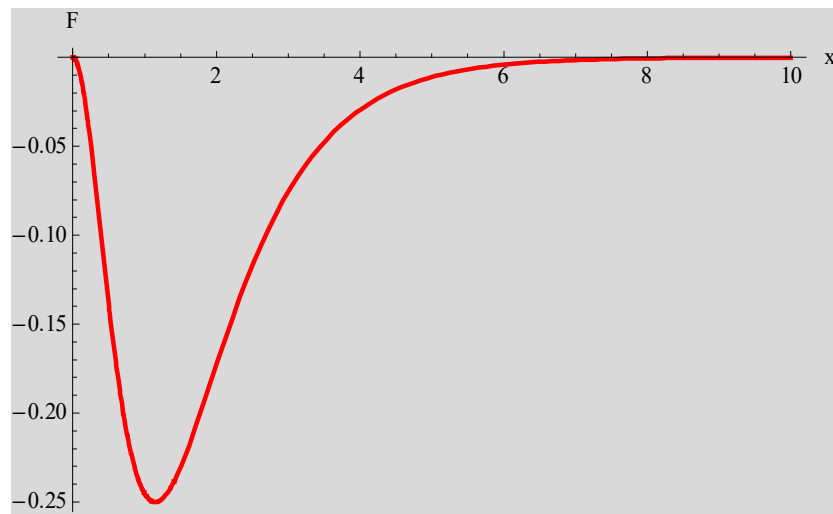
Fig.9   Plot of the surface energy $\sigma_s = 2u^2(1 - \frac{u^2}{2}) - \sqrt{2}u(1 - \frac{u^2}{2})^{1/2}$ as a function of x. $\sigma_s = F(x)$.
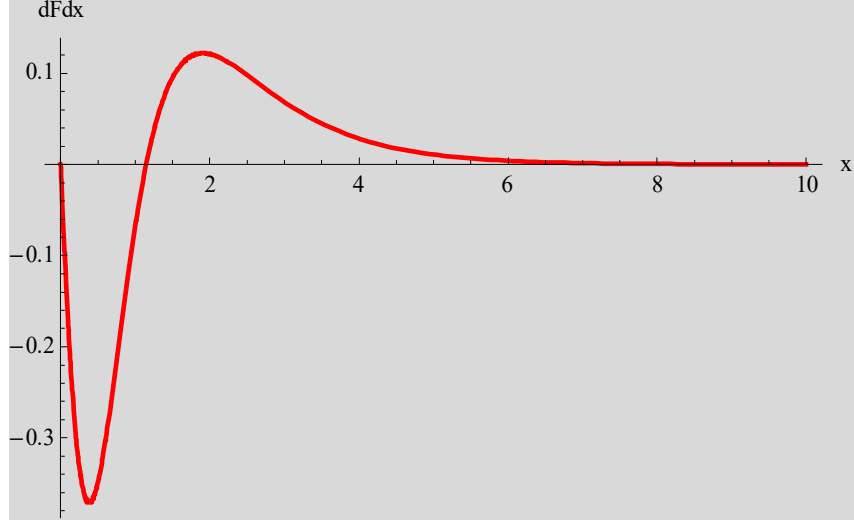
Fig.10　　　　Plot of the derivative $d\sigma_s/dx = dF/dx$ with respect to x.

7.4　Criterion between type-I and type-II superconductor: $\kappa = 1/\sqrt{2}$

The boundary between the type-II and type-I superconductivity can be defined by finding the value of κ which corresponds to a surface energy equal to zero.

$$\frac{\delta}{2} = \int_{-\infty}^{\infty} dx[\frac{1}{2}(1-f^4) + h^2 - \sqrt{2}h].\tag{7.35}$$

It is clear that this value is zero if

$$\frac{1}{2}(1-f^4) + h^2 - \sqrt{2}h,\tag{7.36}$$

or

$$[h - \frac{1}{\sqrt{2}}(1-f^2)][h - \frac{1}{\sqrt{2}}(1+f^2)] = 0\tag{7.37}$$

or

$$h = \frac{1}{\sqrt{2}}(1-f^2)\tag{7.38}$$

Substituting this into the two equations

$$f - f^3 + \frac{1}{\kappa^2}\frac{d^2f}{dx^2} - \frac{1}{f^3}\left(\frac{\partial h}{\partial x}\right)^2 = 0\tag{7.39}$$

45

$$f^2 h = \frac{d^2 h}{dx^2} - \frac{2}{f}\frac{df}{dx}\frac{dh}{dx} \qquad\qquad (7.40)$$

From the calculation by Mathematica, we conclude that $\kappa = 1/\sqrt{2}$.

((Mathematica Program-10))

Determination of $\kappa = 1\Big/\sqrt{2}$

```
Clear["Global`*"];
```

$$\text{eq1} = f[x] - f[x]^3 + \frac{1}{\kappa^2}\, f''[x] - \frac{1}{f[x]^3}\, h'[x]^2 == 0;$$

$$\text{eq2} = f[x]^2\, h[x] == h''[x] - \frac{2}{f[x]}\, f'[x]\, h'[x];$$

$$\text{rule1} = \left\{ h \to \left(\frac{1}{\sqrt{2}}\, (1 - f[\#]^2)\, \& \right) \right\};$$

```
eq3 = eq1 /. rule1 // Simplify;
eq31 = Solve[eq3, f''[x]] // Expand;
eq32 = f''[x] /. eq31[[1]]
```

$$-\kappa^2\, f[x] + \kappa^2\, f[x]^3 + \frac{2\,\kappa^2\, f'[x]^2}{f[x]}$$

```
eq4 = eq2 /. rule1 // Simplify;
eq41 = Solve[eq4, f''[x]] // Expand;
eq42 = f''[x] /. eq41[[1]]
```

$$-\frac{f[x]}{2} + \frac{f[x]^3}{2} + \frac{f'[x]^2}{f[x]}$$

```
Solve[eq32 == eq42, κ]
```

$$\left\{ \left\{ \kappa \to -\frac{1}{\sqrt{2}} \right\},\ \left\{ \kappa \to \frac{1}{\sqrt{2}} \right\} \right\}$$

---

8. Application of the GL equation
8.1 Critical current of a thin wire or film[1,4]
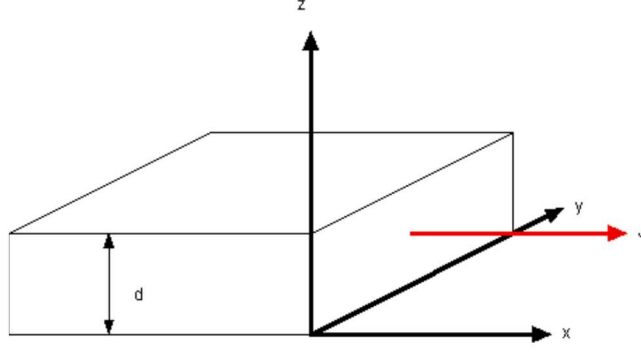
Fig.11   The direction of the current density.

The direction of current is the x axis. The direction of the thickness is the z axis. The following three equations are valid in general.

$$\mathbf{J}_s = \frac{q^* \hbar}{m^*} |\psi|^2 (\nabla \theta - \frac{q^*}{c\hbar} \mathbf{A}) = q^* |\psi|^2 \mathbf{v}_s ,$$ (8.1)

$$f_s - f_n = \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4 + \frac{1}{2m^*} \left| (\frac{\hbar}{i} \nabla - \frac{q^*}{c} \mathbf{A}) \psi \right|^2 + \frac{\mathbf{B}^2}{8\pi} ,$$ (8.2)

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\theta(\mathbf{r})} .$$ (8.3)

We consider the case when

$$|\psi(\mathbf{r})|^2 = n_s^* = \text{constant}$$ (8.4)

Then we have

$$(\frac{\hbar}{i} \nabla - \frac{q^*}{c} \mathbf{A}) \psi = (\hbar \nabla \theta - \frac{q^*}{c} \mathbf{A}) \psi = (m^* \mathbf{v}_s) \psi ,$$ (8.5)

and

$$f_s - f_n = \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4 + \frac{m^*}{2} |\psi|^2 v_s^2 + \frac{\mathbf{B}^2}{8\pi} .$$ (8.6)

We now consider a thin film. We assume that d$<<\xi(T)$ in order to have $|\psi|^2 = n_s^* =$ constant. $|\psi|$ has the same value everywhere. The minimum of f_s with respect to $|\psi|$ is obtained for

47

$$\alpha + \beta|\psi|^2 + \frac{m^*}{2}v_s^2 = 0, \tag{8.7}$$

for a given v_s, where $\mathbf{B}^2/(8\pi)$ is neglected. Here $\alpha = -|\alpha|$ and $|\psi| = \psi_\infty f$ with $\psi_\infty^2 = -\alpha/\beta = |\alpha|/\beta$. Then we have

$$\frac{m^*}{2}v_s^2 = -\alpha - \beta\psi_\infty^2 f^2 = |\alpha|(1 - f^2). \tag{8.8}$$

The corresponding current is a function of f,

$$J_s = q^*|\psi|^2 v_s = q^*\psi_\infty^2 \sqrt{\frac{2|\alpha|}{m^*}} f^2 \sqrt{1 - f^2}. \tag{8.9}$$

We assume that $J_1 = f^2\sqrt{1 - f^2}$. This has a maximum value when $\partial J_1/\partial f = 0$;

$$J_{1\max} = \frac{2}{3\sqrt{3}} \text{ at } f = \sqrt{\frac{2}{3}},$$

$$J_s = \frac{2}{3\sqrt{3}} q^*\psi_\infty^2 \sqrt{\frac{2|\alpha|}{m^*}} = \frac{2}{3\sqrt{3}} q^*\psi_\infty^2 \frac{\hbar}{m^*\xi} \tag{8.10}$$

((Mathematica Program-11))
Critical current of a thin wire

```
Clear["Global`*"]; J1 = f² (1 - f²)^1/2;
k1 = Plot[J1, {f, 0, 1}, PlotStyle → {Red, Thick},
   Background → LightGray,  AxesLabel → {"f", "J1"}];

H1 = D[J1, f] // Simplify; H2 = Solve[H1 == 0, f]

{{f → 0}, {f → -√(2/3)}, {f → √(2/3)}}


max1 = J1 /. H2[[3]]
2
---
3 √3
```

48
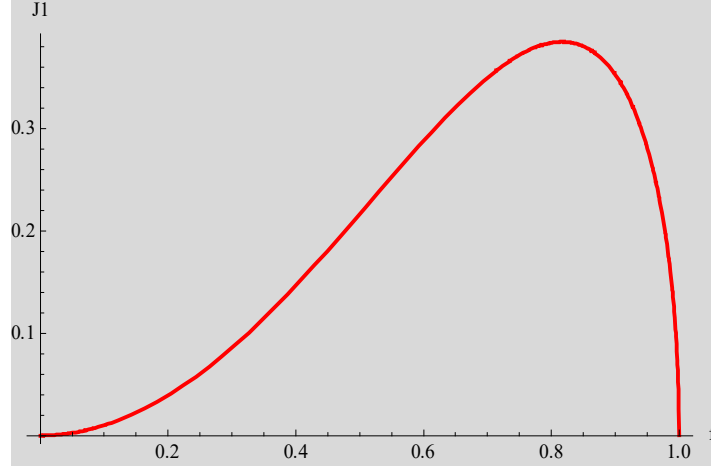
Fig.12   The normalized current density J_1 vs f. $J_1 = f^2 \sqrt{1 - f^2}$.

8.2.   Parallel critical field of thin film
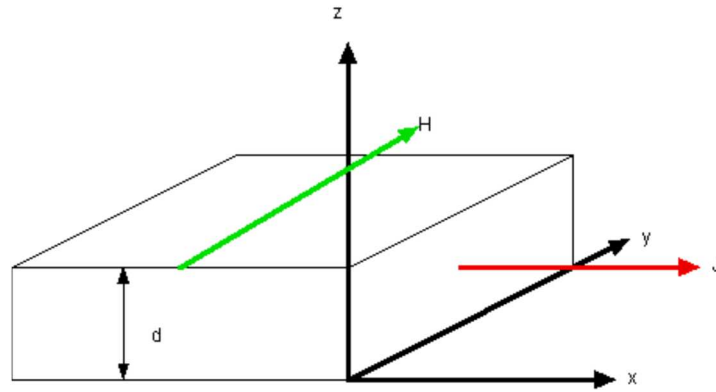
Fig.13   Configuration of the external magnetic field H and the current density J.

Helmholtz free energy:

$$f_s - f_n = \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^2 + \frac{\mathbf{B}^2}{8\pi} .$$ (8.11)

Gibbs free energy:

$$g_s = f_n + \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^2 + \frac{\mathbf{B}^2}{8\pi} - \frac{1}{4\pi}\mathbf{B}\cdot\mathbf{H} ,$$ (8.12)

or

$$g_s = f_n + \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^2 + \frac{(\mathbf{B}-\mathbf{H})^2}{8\pi} - \frac{H^2}{8\pi}. \qquad (8.13)$$

So the Gibbs free energy per unit area of film is

$$G_s = \int\limits_{-d/2}^{d/2} g_s(z)dz = \int\limits_{-d/2}^{d/2} [f_n + \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^2 + \frac{(\mathbf{B}-\mathbf{H})^2}{8\pi} - \frac{H^2}{8\pi}]dz,$$

$$(8.14)$$

$$J_x = \frac{q^*\hbar}{m^*}|\psi|^2 (\frac{\partial\theta}{\partial x} - \frac{q^*}{c\hbar}A_x) = q^*|\psi|^2 v_x. \qquad (8.15)$$

When θ is constant,

$$v_x = -\frac{q^*}{m^* c} A_x(z). \qquad (8.16)$$

$$\mathbf{B} = (0, B_y(z), 0) = \nabla \times \mathbf{A} = (0, \frac{\partial A_x(z)}{\partial z}, 0). \qquad (8.17)$$

or

$$B_y(z) = \frac{\partial A_x(z)}{\partial z}, \qquad (8.18)$$

or

$$B_y(z) = \frac{\partial A_x(z)}{\partial z} \approx H, \qquad (8.19)$$

or

$$A_x(z) = Hz, \quad v_x = -\frac{q^*}{m^* c} Hz, \qquad (8.20)$$

$$G_s = \int\limits_{-d/2}^{d/2} g_s(z)dz$$

$$= \int\limits_{-d/2}^{d/2} [f_n + \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 - \frac{H^2}{8\pi} + \frac{m^*}{2}|\psi|^2 \left(\frac{q^* H}{m^* c}\right)^2 z^2 + \frac{(B-H)^2}{8\pi}]dz \qquad ^{,}$$

$$(8.21)$$

When the last term is negligibly small, we obtain

$$G_s \approx (f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 - \frac{H^2}{8\pi})d + \frac{m^*}{2}|\psi|^2\left(\frac{q^*H}{m^*c}\right)^2\frac{2}{3}\left(\frac{d}{2}\right)^3, \qquad (8.22)$$

or

$$G_s \approx F_n + d[(\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 - \frac{H^2}{8\pi}) + \frac{q^{*2}H^2d^2}{24m^*c^2}|\psi|^2]. \qquad (8.24)$$

Similarly,

$$F_s = F_n + \int_{-d/2}^{d/2}[\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^2 + \frac{B^2}{8\pi}]dz, \qquad (8.25)$$

$$F_s = F_n + d(\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{B^2}{8\pi}) + \frac{q^{*2}H^2d^3}{24m^*c^2}|\psi|^2. \qquad (8.26)$$

Minimizing the expression of F_s with respect to $|\psi|^2$, we find

$$\alpha + \beta|\psi|^2 + \frac{q^{*2}H^2d^2}{24m^*c^2} = 0, \qquad (8.27)$$

$$\frac{|\psi|^2}{\psi_\infty^2} = \left(1 - \frac{q^{*2}H^2d^2}{24m^*c^2|\alpha|}\right) = 1 - \frac{H^2}{H_c^2}\frac{d^2}{24\lambda^2} = 1 - \frac{H^2}{H_c^2}\frac{\varepsilon^2}{24}, \qquad (8.28)$$

or

$$\left(\frac{H}{H_c}\right)^2 = \frac{24}{\varepsilon^2}(1 - f^2), \qquad (8.29)$$

where

$$\varepsilon = \frac{d}{\lambda}. \qquad (8.30)$$

This is rewritten as

$$f^2 = 1 - \frac{H^2}{H_{c//}^2}, \qquad (8.31)$$

where $H_{c//}$ is defined as

$$H_{c//} = 2\sqrt{6}\frac{H_c}{\varepsilon}.$$
(8.32)

This parallel critical field $H_{c//}$ can exceed the thermodynamic critical field H_c when ε is lower than $2\sqrt{6} = 4.899$.

We now consider the difference between G_s and G_n

$$G_n = F_n - \frac{H^2}{8\pi}d$$
(8.33)

where

$$F_n = df_n, \qquad \text{and} \qquad G_n = dg_n$$

The difference is given by

$$G_n - G_s = (-\frac{H^2}{8\pi}) - [(\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 - \frac{H^2}{8\pi}) + \frac{q^{*2}H^2d^2}{24m^*c^2}|\psi|^2].$$
(8.34)

or

$$G_n - G_s = -[(\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4) + \frac{q^{*2}H^2d^2}{24m^*c^2}|\psi|^2].$$
(8.35)

Noting that

$$\alpha + \beta|\psi|^2 + \frac{q^{*2}H^2d^2}{24m^*c^2} = 0,$$
(8.36)

we obtain

$$G_n - G_s = -\frac{1}{2}\beta|\psi|^4 + \beta|\psi|^4 = \frac{1}{2}\beta|\psi|^4 > 0.$$
(8.37)

The superconducting state is energetically favorable.

8.3. Parallel critical fields of thick films[1]

We now consider the case when an external magnetic field is applied to a thick film with a thickness d. The field is applied along the plane. We solve the GL equation with the appropriate boundary condition. The magnetic induction and the current density are given by

$$B_y(z) = H \frac{\cosh(z\frac{f}{\lambda})}{\cosh(\varepsilon\frac{f}{2})},$$

(8.38)

and

$$J_{sx}(z) = -\frac{c}{4\pi}\frac{dB_y(z)}{dz} = -\frac{cfH}{4\pi\lambda}\frac{\sinh(\frac{fz}{\lambda})}{\cosh(\frac{f\varepsilon}{2})}.$$

(8.39)

Note that

$$|\psi| = \psi_\infty f, \qquad \psi_\infty^2 = \frac{m^*c^2}{4\pi q^{*2}\lambda^2},$$

(8.40)

$$v_{sx}(z) = \frac{J_{sx}(z)}{q^*|\psi|^2} = \frac{J_{sx}(z)}{q^*f^2\psi_\infty^2} = -\frac{q^*\lambda H}{m^*cf}\frac{\sinh(\frac{fz}{\lambda})}{\cosh(\frac{\varepsilon f}{2})},$$

(8.41)

$$\left\langle v_{sx}^2 \right\rangle = \frac{1}{d}\int_{-d/2}^{d/2}[v_{sx}(z)]^2\,dz = \frac{1}{2}\frac{H^2 q^{*2}\lambda^2}{m^{*2}c^2 f^2}\frac{(\frac{\sinh(\varepsilon f)}{\varepsilon f}-1)}{\cosh^2(\frac{\varepsilon f}{2})}.$$

(8.42)

Helmholtz free energy:

$$f_s = f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^2 + \frac{\mathbf{B}^2}{8\pi},$$

(8.43)

On minimizing this with respect to $|\psi|^2$,

$$\frac{\partial f_s}{\partial|\psi|^2} = 0,$$

(8.44)

$$\alpha + \beta|\psi|^2 + \frac{m^*}{2}v_s^2 = 0,$$

(8.45)

or

$$\frac{|\psi|^2}{\psi_\infty^2} = f^2 = \frac{|\alpha| - \frac{m^*}{2}\left\langle v_s^2 \right\rangle}{\beta\psi_\infty^2} = 1 - \frac{m^*}{2\beta\psi_\infty^2}\left\langle v_s^2 \right\rangle = 1 - \frac{m^*}{2|\alpha|}\left\langle v_s^2 \right\rangle,$$

(8.46)

$$f^2 = 1 - \frac{q^{*2}\lambda^2 H_c^2}{4|\alpha|m^*c^2}\left(\frac{H}{H_c}\right)^2 \frac{1}{f^2} \frac{(\frac{\sinh(\varepsilon f)}{\varepsilon f}-1)}{\cosh^2(\frac{\varepsilon f}{2})} = 1 - \frac{1}{4}\left(\frac{H}{H_c}\right)^2 \frac{1}{f^2} \frac{(\frac{\sinh(\varepsilon f)}{\varepsilon f}-1)}{\cosh^2(\frac{\varepsilon f}{2})},$$

(8.47)

$$\left(\frac{H}{H_c}\right)^2 = 4f^2(1-f^2)\frac{\cosh^2(\frac{\varepsilon f}{2})}{[\frac{\sinh(\varepsilon f)}{\varepsilon f}-1]}.$$

(8.48)

We now discuss the Helmholtz free energy

$$f_s = f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{m^*}{2}|\psi|^2\langle v_s^2\rangle + \frac{\langle\mathbf{B}\rangle^2}{8\pi},$$

(8.49)

with

$$\alpha + \beta|\psi|^2 + \frac{m^*}{2}\langle v_s\rangle^2 = 0.$$

(8.50)

Then we have

$$f_s = f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + |\psi|^2(-\alpha - \beta|\psi|^2) + \frac{\langle B^2\rangle}{8\pi},$$

$$= f_n - \frac{1}{2}\beta|\psi|^4 + \frac{\langle B^2\rangle}{8\pi}$$

(8.51)

or

$$f_s = f_n - \frac{H_c^2}{8\pi}f^4 + \frac{\langle B^2\rangle}{8\pi},$$

(8.52)

$$\langle B^2\rangle = \frac{1}{d}\int_{-d/2}^{d/2}[B_y(z)]^2 dz = \frac{\varepsilon f + \sinh(\varepsilon f)}{\varepsilon f[1+\cosh(\varepsilon f)]}H^2,$$

(8.53)

and

$$\langle B\rangle = \frac{1}{d}\int_{-d/2}^{d/2}B_y(z)dz = \frac{2H}{\varepsilon f}\tanh(\frac{\varepsilon f}{2}).$$

(8.54)

Gibbs free energy:

$$g_s = f_s - \frac{\langle B \rangle H}{4\pi} = f_n - \frac{H_c^2}{8\pi} f^4 + \frac{\langle B^2 \rangle}{8\pi} - \frac{\langle B \rangle H}{4\pi}, \tag{8.55}$$

$$g_s = f_n - \frac{H_c^2}{8\pi} f^4 + \frac{H^2}{8\pi} [\frac{\varepsilon f + \sinh(\varepsilon f)}{\varepsilon f [1 + \cosh(\varepsilon f)]} - \frac{4}{\varepsilon f} \tanh(\frac{\varepsilon f}{2})]. \tag{8.56}$$

Since

$$g_n = f_n + \frac{H^2}{8\pi} - \frac{BH}{4\pi} = f_n - \frac{H^2}{8\pi}, \tag{8.57}$$

where $B = H$. From the condition that $g_s = g_n$ at the critical field $H = H_l$,

$$-H_c^2 f^4 + H^2 \frac{\varepsilon f + \sinh(\varepsilon f)}{\varepsilon f [1 + \cosh(\varepsilon f)]} - 4H^2 \frac{1}{\varepsilon f} \tanh(\frac{\varepsilon f}{2}) = -H^2, \tag{8.58}$$

or

$$\left(\frac{H}{H_c}\right)^2 \{1 + \frac{\varepsilon f + \sinh(\varepsilon f)}{\varepsilon f [1 + \cosh(\varepsilon f)]} - \frac{4}{\varepsilon f} \tanh(\frac{\varepsilon f}{2})\} = f^4, \tag{8.59}$$

or

$$\left(\frac{H_l}{H_c}\right)^2 = \frac{f^4}{1 + \frac{\varepsilon f + \sinh(\varepsilon f)}{\varepsilon f [1 + \cosh(\varepsilon f)]} - \frac{4}{\varepsilon f} \tanh(\frac{\varepsilon f}{2})}. \tag{8.60}$$

In the limit of $f \rightarrow 0$,

$$\left(\frac{H_l}{H_c}\right)^2 = \frac{24}{\varepsilon^2} + 24(\frac{1}{5} - \frac{1}{\varepsilon^2})f^2 + (-\frac{24}{5} + \frac{81\varepsilon^2}{350})f^4 + O[f]^6. \tag{8.61}$$

From the above two equations, we get

$$4f^2(1 - f^2) \frac{\cosh^2(\frac{\varepsilon f}{2})}{[\frac{\sinh(\varepsilon f)}{\varepsilon f} - 1]} = \frac{f^4}{1 + \frac{\varepsilon f + \sinh(\varepsilon f)}{\varepsilon f [1 + \cosh(\varepsilon f)]} - \frac{4}{\varepsilon f} \tanh(\frac{\varepsilon f}{2})}, \tag{8.62}$$

or

$$1 + \frac{f^2}{6(1 - f^2)} = \frac{1}{3} \frac{\varepsilon f [\cosh(\varepsilon f) - 1]}{\sinh(\varepsilon f) - \varepsilon f}. \tag{8.63}$$

In the limit of $f \to 0$,

$$\frac{1}{6}(1 - \frac{\varepsilon^2}{5})f^2 + (\frac{1}{6} + \frac{\varepsilon^4}{12600})f^4 + O[f]^6. \tag{8.64}$$

Using Mathematica,

1. we determine the value of f as a function of ε.
2. we determine the value of $(H_l / H_c)^2$ as a function of f or f^2 for each ε.

There occurs the first order transition because of discontinuous change of order parameter.

((**Mathematica Program-12**))

Here we discuss the problem dscribed in the book of de Gennes, page 189

```
Clear["Global`*"]; hy[z_] = H (Cosh[z f/λ])/(Cosh[ε f/2]) ; rule1 = {ε → d/λ};
```

```
Jx[z_] = - c/(4 π) D[hy[z], z] // Simplify
```

$$- \frac{c\, f\, H\, \text{Sech}\left[\frac{f \epsilon}{2}\right]\, \text{Sinh}\left[\frac{f z}{\lambda}\right]}{4 \pi \lambda}$$

$$|\psi| = f\,\psi 0, \quad \psi 0^2 = \frac{m\, c^2\, \lambda^{-2}}{4 \pi\, q^2}$$

```
rule2 = {ψ0 → √(m c² λ⁻²)/(4 π q²) }; vsx[z_] = Jx[z]/(q f² ψ0²) /. rule2
```

$$- \frac{H\, q\, \lambda\, \text{Sech}\left[\frac{f \epsilon}{2}\right]\, \text{Sinh}\left[\frac{f z}{\lambda}\right]}{c\, f\, m}$$

```
Vavsq = 1/d ∫_{-d/2}^{d/2} vsx[z]² ⅆz // Simplify; eq1 = f² == 1 + m/(2 α) Vavsq;
```

```
rule3 = {H² → x}; eq2 = eq1 /. rule3
```

$$f^2 == 1 + \frac{q^2\, x\, \lambda^2\, \left(-d\, f + \lambda\, \text{Sinh}\left[\frac{d f}{\lambda}\right]\right)}{2\, c^2\, d\, f^3\, m\, \alpha\, (1 + \text{Cosh}[f \epsilon])}$$

```
eq3 = Solve[eq2, x] /. d → ε λ // Simplify
```

$$\left\{\left\{x \to - \frac{2\, c^2\, f^3\, \left(-1 + f^2\right)\, m\, \alpha\, \epsilon\, (1 + \text{Cosh}[f \epsilon])}{q^2\, \lambda^2\, (f \epsilon - \text{Sinh}[f \epsilon])}\right\}\right\}$$

$$Hc^2 = \frac{4\pi\alpha^2}{\beta} , \quad \psi\infty^2 = |\alpha| / \beta, \quad \frac{c^2 m \beta}{q^2 2\pi(-\alpha)\lambda^2} = 2$$

$$\text{Calculation of } H^2 / Hc^2, \quad \beta = -\frac{4\pi q^2 \alpha \lambda^2}{c^2 m}$$

$$\texttt{rule4} = \left\{\beta \rightarrow \frac{4\pi(-\alpha)q^2\lambda^2}{c^2 m}\right\}; \; \texttt{eq4} = \frac{\texttt{x /. eq3[[1]]}}{\frac{4\pi\alpha^2}{\beta}} \; \texttt{/. rule4}$$

$$\frac{2 f^3 \left(-1 + f^2\right) \in (1 + \text{Cosh}[f \in])}{f \in - \text{Sinh}[f \in]}$$

$$\texttt{Hsqave} = \frac{1}{d} \int_{-d/2}^{d/2} \texttt{hy[z]}^2 \, \mathbb{d}z \; \texttt{/. } d \rightarrow \in \lambda \; \texttt{// Simplify}$$

$$\frac{H^2 \, \text{Sech}\left[\frac{f \in}{2}\right]^2 (f \in + \text{Sinh}[f \in])}{2 f \in}$$

$$\texttt{B1} = \frac{1}{d} \int_{-d/2}^{d/2} \texttt{hy[z]} \, \mathbb{d}z \; \texttt{/. } d \rightarrow \in \lambda \; \texttt{// Simplify}$$

$$\frac{2 H \, \text{Tanh}\left[\frac{f \in}{2}\right]}{f \in}$$

$$\texttt{Fs} = \texttt{Fn} - \frac{Hc^2}{8\pi} f^4 + \frac{\texttt{Hsqave}}{8\pi} \; ;$$

```
Gs = Fs - (B1 H)/(4 π) ;
```

$$Gs = Fs - \frac{B1\ H}{4\ \pi}\ ;$$

$$Gn = Fn - \frac{H^2}{8\ \pi}\ ;$$

```
eq6 = Gs - Gn /. H² → y Hc²
```

$$-\frac{f^4\ Hc^2}{8\ \pi} + \frac{Hc^2\ y}{8\ \pi} + \frac{Hc^2\ y\ Sech\left[\frac{f\ \epsilon}{2}\right]^2\ (f\ \epsilon + Sinh[f\ \epsilon])}{16\ f\ \pi\ \epsilon} - \frac{Hc^2\ y\ Tanh\left[\frac{f\ \epsilon}{2}\right]}{2\ f\ \pi\ \epsilon}$$

```
eq7 = Solve[eq6 == 0, y]
```

$$\left\{\left\{y \rightarrow \frac{2\ f^5\ \epsilon}{2\ f\ \epsilon + f\ \epsilon\ Sech\left[\frac{f\ \epsilon}{2}\right]^2 + Sech\left[\frac{f\ \epsilon}{2}\right]^2\ Sinh[f\ \epsilon] - 8\ Tanh\left[\frac{f\ \epsilon}{2}\right]}\right\}\right\}$$

```
eq8 = y /. eq7[[1]] // FullSimplify
```

$$\frac{f^5\ \epsilon\ (1 + Cosh[f\ \epsilon])}{f\ \epsilon\ (2 + Cosh[f\ \epsilon]) - 3\ Sinh[f\ \epsilon]}$$

We now consider the solution of de Gennes model

```
K[f_, ε_] := 1 + f²/(6 (1 - f²)) - (f ε (Cosh[ε f] - 1))/(3 (-f ε + Sinh[f ε])) ;
```

$$K[f\_,\ \epsilon\_] := 1 + \frac{f^2}{6\ \left(1 - f^2\right)} - \frac{f\ \epsilon\ (Cosh[\epsilon\ f] - 1)}{3\ (-f\ \epsilon + Sinh[f\ \epsilon])}\ ;$$

Evaluation of critical magnetic field as a function of the thickness ϵ

```
Series[K[f, ε], {f, 0, 5}]
```

$$\left(\frac{1}{6} - \frac{\epsilon^2}{30}\right)\ f^2 + \frac{\left(2100 + \epsilon^4\right)\ f^4}{12\,600} + O[f]^6$$

```
L1[ε_] := Module[{f1, g1, ε1, f2}, ε1 = ε;
   g1 = FindRoot[K[f1, ε1] == 0, {f1, 0.2, 0.99}]; f2 = f1 /. g1[[1]]]

s1 = Table[{ε, L1[ε]}, {ε, 0.22, 10, 0.01}];
h1 = ListPlot[s1, PlotStyle → {Hue[0], Thick}, Background → LightGray,
   AxesLabel → {"ε", "f"}];
```

59

```mathematica
h2 = Plot[L1[ε], {ε, √5, 3}, PlotStyle → {Hue[0], Thickness[0.01]},
    Background → GrayLevel[0.7], PlotRange → {{2.1, 3}, {0, 0.7}},
    AxesLabel → {"ε", "f"}];
```

Evaluation of $(H/Hc)^2$ vs ϵ

Comparison with the approximation for $(H/Hc)^2$ vs ϵ : $24/\epsilon^2$

```mathematica
Hcrsq[ε_, f_] := (2 f^3 (-1 + f^2) ε (1 + Cosh[f ε])) / (f ε - Sinh[f ε])

Series[Hcrsq[ε, f], {f, 0, 5}]
```

$$\frac{24}{\epsilon^2} + \left(\frac{24}{5} - \frac{24}{\epsilon^2}\right) f^2 + \left(-\frac{24}{5} + \frac{81 \epsilon^2}{350}\right) f^4 + O[f]^6$$

```mathematica
h3 = Plot[{24/ε^2, Hcrsq[ε, L1[ε]]}, {ε, √5, 8},
    PlotStyle → {{Hue[0], Thickness[0.01]}, {Hue[0.7], Thickness[0.01]}},
    Background → GrayLevel[0.7], PlotRange → {{2, 8}, {0, 6}}];
```

Evaluation of f^2 vs $(H/Hc)^2$ with ϵ as a parameter.

Approximation for $(H/Hl)^2$ vs ϵ : $24/\epsilon^2$

```mathematica
<< "ErrorBarPlots`"; << "PlotLegends`"

fsq1[ε_, hsq1_] := Module[{f1, g1, ε1, f2, hsq2}, ε1 = ε; hsq2 = hsq1;
    g1 = FindRoot[hsq2 == Hcrsq[ε1, f1], {f1, 0.6, 1}]; f2 = f1 /. g1[[1]]]
```

```
h4 = Plot[fsq1[ϵ, 1], {ϵ, 2, 10}, AxesLabel → {"ϵ", "f"},
    PlotStyle → {Hue[0], Thickness[0.01]}, Background → GrayLevel[0.7]];
```

f^2 vs $(H/Hc)^2$ with $\epsilon = \sqrt{5}$, 2.5, 3, 4, 4, 5, 5, 7, 9, 11, 20, and 30

```
X1[ϵ_] := Table[{Hcrsq[ϵ, fsq1[ϵ, hsq1]], fsq1[ϵ, hsq1]^2},
    {hsq1, 0, 1, 0.01}];

h5 = ListPlot[{X1[√5], X1[2.5`], X1[3], X1[4], X1[4.5`], X1[5],
    X1[7], X1[9], X1[11], X1[20], X1[30]},
    AxesLabel → {"(H/Hc\!\(\*SuperscriptBox[\()\), \(2\)]\)",
    "\!\(\*SuperscriptBox[\(f\), \(2\)]\)"}];
```

$$hny[t\_, f\_, \epsilon\_] = \frac{Cosh\left[z\,\frac{f}{\lambda}\right]}{Cosh\left[\epsilon\,\frac{f}{2}\right]} \; /. \; \left\{\lambda \to \frac{d}{\epsilon}, z \to t\,d\right\};$$

$$Jnx[t\_, f\_, \epsilon\_] = f\,Sech\left[\frac{f\,\epsilon}{2}\right] Sinh\left[\frac{f\,z}{\lambda}\right] /. \; \left\{\lambda \to \frac{d}{\epsilon}, z \to t\,d\right\};$$

```
h6 = Plot[Evaluate[Table[hny[t, L1[ϵ], ϵ], {ϵ, 2.5, 10.0, 0.2}]],
    {t, -1/2, 1/2}, PlotStyle → Table[Hue[0.05 i], {i, 0, 20}],
    Prolog → AbsoluteThickness[1.5], Background → GrayLevel[0.7],
    AxesLabel → {"x/d", "h"}];
```

```
h7 = Plot[Evaluate[Table[Jnx[t, L1[ϵ], ϵ], {ϵ, 2.5, 10.0, 0.2}]],
    {t, -1/2, 1/2}, PlotStyle → Table[Hue[0.05 i], {i, 0, 20}],
    Prolog → AbsoluteThickness[1.5], Background → GrayLevel[0.7],
    AxesLabel → {"x/d", "J"}];

p1 = {h1, h2, h3, h4, h5, h6, h7};
Show[GraphicsGrid[Partition[p1, 3]]]
```


61
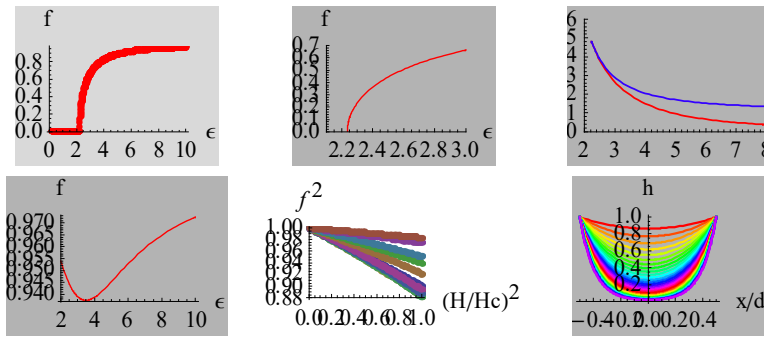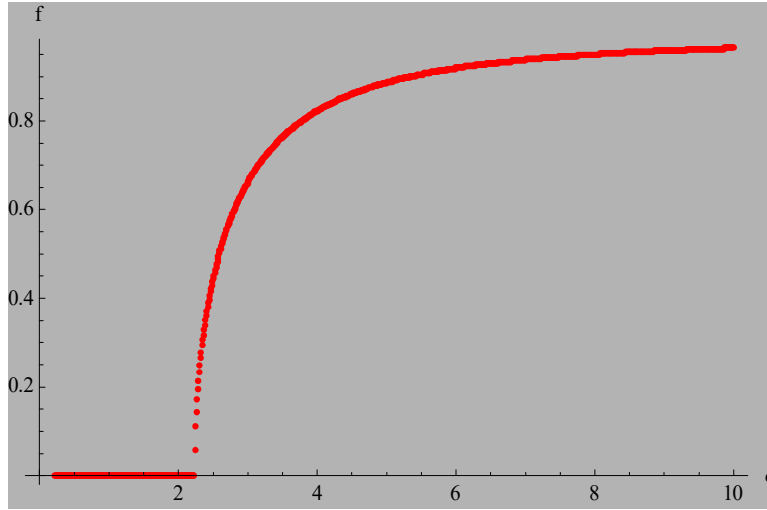
Fig.14(a)   The order parameter f vs ε $(= d/\lambda)$. f tends to 1 in the large limit of $\varepsilon \to \infty$.
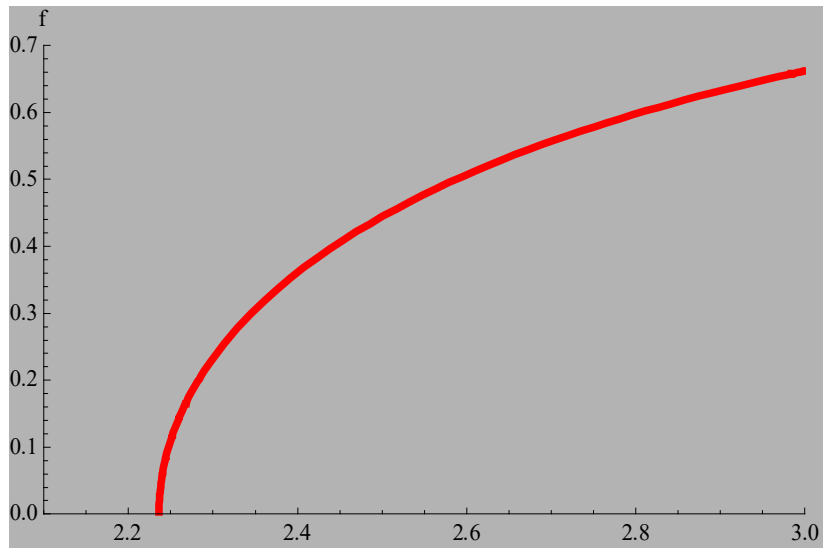
Fig.14(b)   Detail of Fig.14(a). The order parameter f vs ε $(= d/\lambda)$. f reduces to zero at $\varepsilon = \sqrt{5}$.
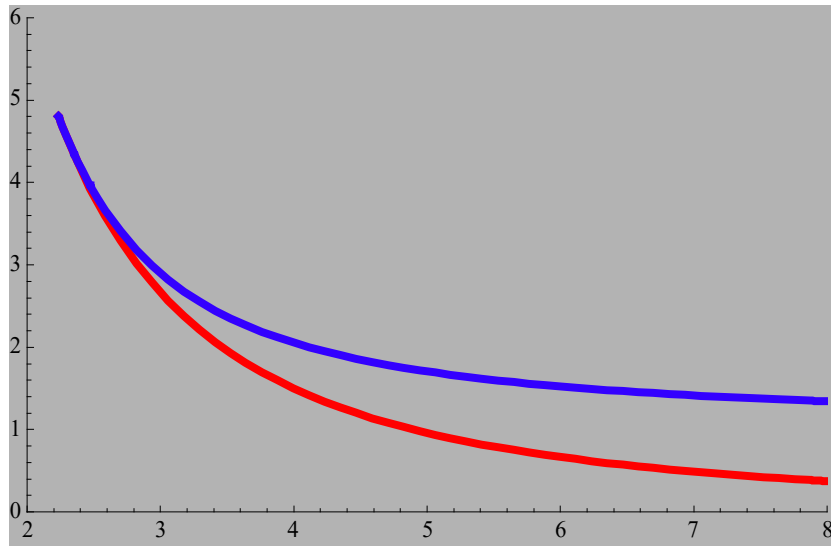
Fig.15 $(H/H_c)^2$ vs ε (red line) and the approximation $24/\varepsilon^2$ vs ε (blue line). These agree well only in the vicinity of $\varepsilon = \sqrt{5}$.
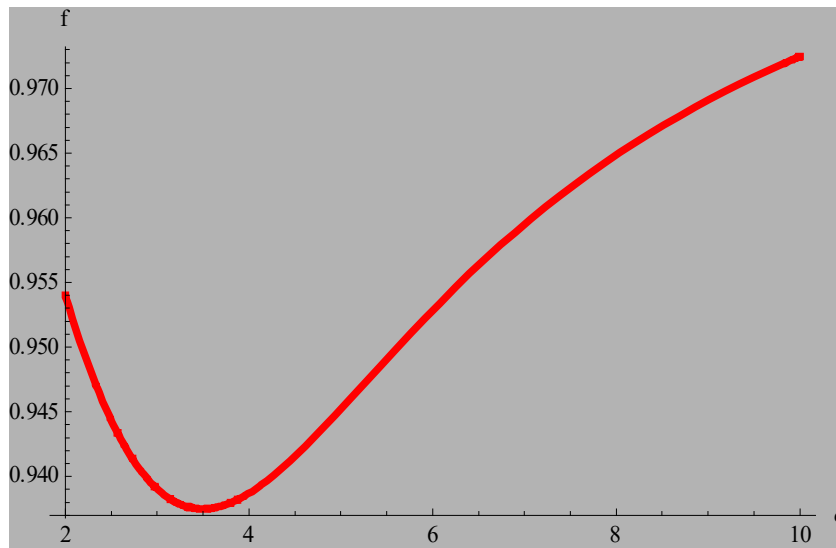
Fig.16   Plot of f at $H/H_c = 1$ as a function of ε.
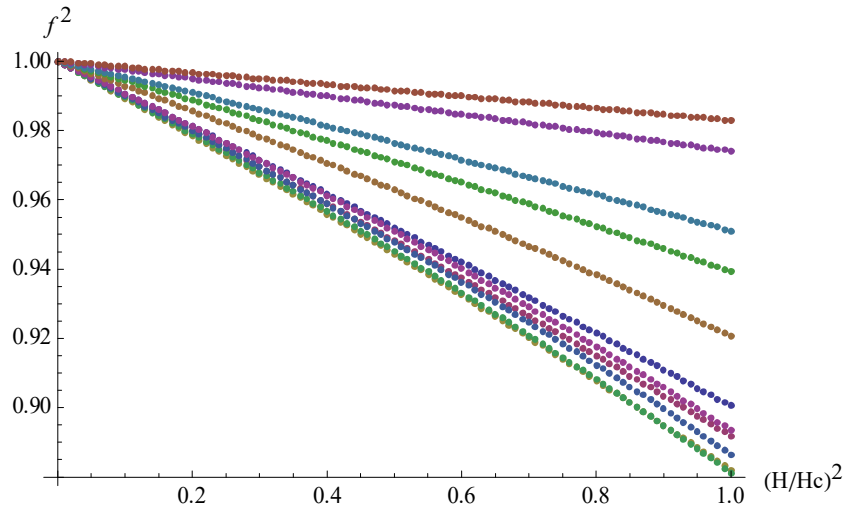
Fig.17   Plot of f^2 vs $(H/H_c)^2$ with $\varepsilon = \sqrt{5}$, 2.5, 3, 4, 4.5, 5, 7, 9, 11, 20, and 30.
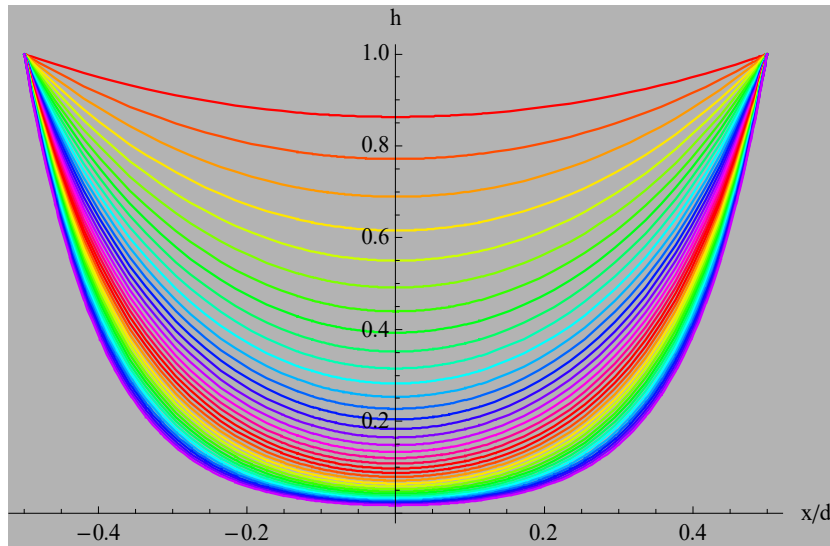
Fig.18   Plot of (a) the magnetic field distribution and (b) the current density as a function of x/λ, where e is changed as a parameter. $\varepsilon = 2.5 - 10.0$. $\Delta\varepsilon = 0.2$.

8.4   Superconducting cylindrical film[1]

We consider a superconducting cylindrical film (radius R) of thickness ($2d$), where $2d \ll R$. A small axial magnetic field is applied to the cylinder.
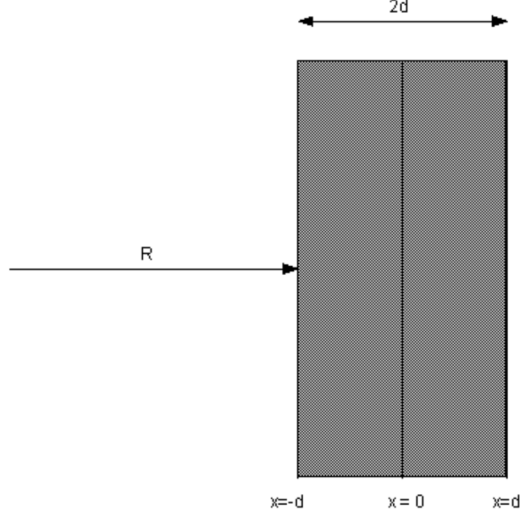
Fig.19 Geometrical configuration of a cylinder (a radius R) with small thickness. $R>>d$.

$$\mathbf{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\mathbf{A}) = q^*|\psi|^2\mathbf{v}_s. \tag{8.65}$$

Suppose that $n_s^* = |\psi|^2 = \psi_\infty^2$ =constant, then we have

$$\frac{c\hbar}{q^*}\nabla\theta = \frac{m^*c}{q^{*2}n_s^*}\mathbf{J}_s + \mathbf{A}. \tag{8.67}$$

Integrating the current density around a circle running along the inner surface of the cylinder,

$$-\frac{\Phi_0}{2\pi}\oint\nabla\theta\cdot d\mathbf{l} = \frac{4\pi\lambda^2}{c}\oint\mathbf{J}_s\cdot d\mathbf{l} + \oint\mathbf{A}\cdot d\mathbf{l}, \tag{8.68}$$

and

$$\oint\mathbf{A}\cdot d\mathbf{l} = \int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int\mathbf{B}\cdot d\mathbf{a} = \Phi_i, \tag{8.69}$$

where Φ_i is the flux contained within the core of the cylinder. The magnetic flux Φ_f is defined as

$$\Phi_f = -\frac{\Phi_0}{2\pi}\oint\nabla\theta\cdot d\mathbf{l}. \tag{8.70}$$

Then we have

$$\Phi_f = \frac{4\pi\lambda^2}{c} \oint \mathbf{J}_s \cdot d\mathbf{l} + \Phi_i, \tag{8.71}$$

or

$$\oint \mathbf{J}_s \cdot d\mathbf{l} = \frac{c}{4\pi\lambda^2} (\Phi_f - \Phi_i). \tag{8.72}$$

The phase θ of the wave function must be unique, or differ by a multiple of 2π at each point,

$$\Phi_f = \frac{2\pi c\hbar}{|q^*|} n. \tag{8.73}$$

In other words, the flux is quantized.

When the radius is much larger than the thickness $2d$, the magnetic field distribution is still described by

$$\frac{d^2}{dx^2} B_z(x) = \frac{1}{\lambda^2} B_z(x), \tag{8.74}$$

with the boundary condition, $B_z(-d) = H_i$ and $B_z(d) = H_0$. The origin $x = 0$ is shown in the above Figure.

$$B_z(x) = \frac{H_0 - H_i}{2} \frac{\sinh(\frac{x}{\lambda})}{\sinh(\frac{d}{\lambda})} + \frac{H_0 + H_i}{2} \frac{\cosh(\frac{x}{\lambda})}{\cosh(\frac{d}{\lambda})}. \tag{8.75}$$

The current density $J_y(x)$ is given by

$$J_y(x) = -\frac{c}{4\pi} \frac{dB_z(x)}{dx} = \frac{c}{4\pi\lambda} \frac{H_i \cosh(\frac{d-x}{\lambda}) - H_0 \cosh(\frac{d+x}{\lambda})}{\sinh(\frac{2d}{\lambda})}. \tag{8.76}$$

The value of current flowing near the inner surface is

$$J_y(-d) = \frac{c}{4\pi\lambda} \frac{H_i \cosh(\frac{2d}{\lambda}) - H_0}{\sinh(\frac{2d}{\lambda})}. \tag{8.77}$$

Then we have

$$\oint \mathbf{J}_s \cdot d\mathbf{l} = 2\pi R J_y(-d) = \frac{c}{4\pi\lambda^2}(\Phi_f - \Phi_i),$$ (8.78)

or

$$2\pi R \frac{c}{4\pi\lambda} \frac{H_i \cosh(\frac{2d}{\lambda}) - H_0}{\sinh(\frac{2d}{\lambda})} = \frac{c}{4\pi\lambda^2}(\Phi_f - \Phi_i),$$ (8.79)

or

$$H_i = \frac{\Phi_f + \dfrac{2H_0\pi R\lambda}{\sinh(\frac{2d}{\lambda})}}{\pi R^2 \left(1 + \dfrac{2\lambda}{R}\coth(\frac{2d}{\lambda})\right)}.$$ (8.80)

The Gibbs free energy is given by

$$\Delta G = \int d\mathbf{r}[\frac{\lambda^2}{8\pi}(\nabla \times \mathbf{B})^2 + \frac{1}{8\pi}(\mathbf{B} - \mathbf{H}_0)^2],$$ (8.81)

or

$$\Delta G = \int_{-d}^{d} \{\frac{\lambda^2}{8\pi}\left(\frac{\partial B_z(x)}{\partial x}\right)^2 + [B_z(x) - H_0]^2\}dx,$$ (8.82)

where the constant terms are neglected and H_i is given by Eq.(8.80). For a fixed H_0, we need to minimize the Gibbs free energy with respect to the fluxsoid Φ_f.

We find that the solution of $\partial\Delta G/\partial\Phi_f = 0$ Φ_f leads to $\Phi_f = \Phi_i$, where $R/\lambda = 10 - 50$ and $d/\lambda < 1$.

To prove this, we use the Mathematica for the calculation.

((**Mathematica Program-13**))

67

```mathematica
Clear["Global`*"]; Bz = (H0 - Hi)/2 Sinh[x/λ]/Sinh[d/λ] + (H0 + Hi)/2 Cosh[x/λ]/Cosh[d/λ];
```

```mathematica
rule1 = {Hi → (Φf + (2 H0 π R λ)/Sinh[(2 d)/λ])/(π R^2 (1 + (2 λ)/R Cosh[(2 d)/λ]/Sinh[(2 d)/λ]))}; Bz1 = Bz /. rule1 // Simplify;
```

```mathematica
C1 = D[Bz1, x] // Simplify; G1 = λ^2/(8 π) C1^2 + 1/(8 π) (Bz1 - H0)^2 // Simplify;
```

```mathematica
G2 = ∫_{-d/2}^{d/2} G1 ⅆx // Simplify;
```

```mathematica
G3 = D[G2, Φf] // Simplify;
```

```mathematica
G4 = Solve[G3 == 0, Φf] // Simplify; F1 = Φf /. G4[[1]] // Simplify;
rule1 = {d → ϵ λ, R → μ λ};
F2 = F1 / (H0 π R^2) /. rule1 // Simplify
```

$$\frac{\text{Sech}[2\,\epsilon] \left(\mu - \mu\,\text{Cosh}\left[\frac{3\,\epsilon}{2}\right] + \mu\,\text{Cosh}\left[\frac{5\,\epsilon}{2}\right] - 2\,\text{Sinh}\left[\frac{3\,\epsilon}{2}\right] + 2\,\text{Sinh}\left[\frac{5\,\epsilon}{2}\right]\right)}{\mu}$$

Φf is the local m inimum value of the Gibbs free energy. Plot
  of Φf / (H0 π R^2) vs ϵ = d / λ = 0 − 1, where μ = R / λ = 10 − 50

```mathematica
g1 = Plot[Evaluate[Table[F2, {μ, 10, 50, 10}]], {ϵ, 0, 1},
   PlotRange → {{0, 1}, {0.8, 1.5}}, PlotPoints → 100,
   PlotStyle → Table[{Thick, Hue[0.2 i]}, {i, 0, 5}],
   Background → LightGray , AxesLabel → {"ϵ", "F"}];
g2 = Graphics[{Text[Style["μ=10", Black, 12], {0.6, 1.2}],
    Text[Style["μ=50", Black, 12], {0.6, 1.07}]}];
g3 = Show[g1, g2]
```
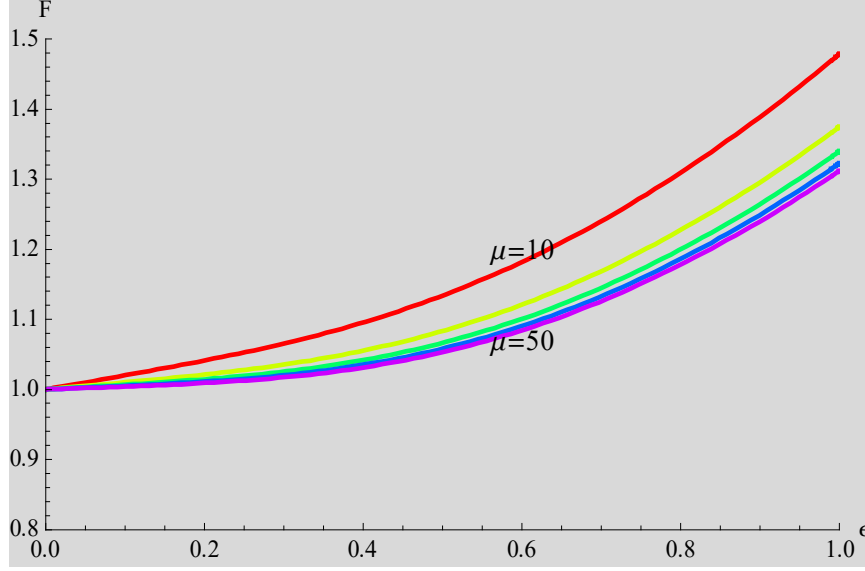
Fig.20   Plot of the normalized magnetic flux $F = \Phi/(\pi R^2 H_0)$ as a function of ε ($\varepsilon = 0 - 1$), where $\mu = 10 - 50$. $\Delta\mu = 10$.

8.5   Little Parks experiments[1,4]
8.5.1   The case of $d<<\lambda$ and $d<<\xi$,

The thickness d of the superconducting cylinder is much shorter than the radius R. It is also shorter than λ and ξ. Since $d \ll \lambda$, the change of B inside the superconducting layer is slight. The current density (proportional to derivative of B) is assumed to be constant. The absolute value of the order parameter is constant in the superconducting layer. The following three equations are valid in general.

$$\mathbf{J}_s = \frac{q^* \hbar}{m^*} |\psi|^2 (\nabla\theta - \frac{q^*}{c\hbar} \mathbf{A}) = q^* |\psi|^2 \mathbf{v}_s, \tag{8.83}$$

$$f_s - f_n = \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4 + \frac{1}{2m^*} \left| (\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi \right|^2 + \frac{B^2}{8\pi}, \tag{8.84}$$

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\theta(\mathbf{r})}. \tag{8.85}$$

We consider the case when

$$|\psi(\mathbf{r})|^2 = n_s^* = \text{constant} \qquad \text{or} \qquad \psi(\mathbf{r}) = |\psi| e^{i\theta(\mathbf{r})}. \tag{8.86}$$

Then we have

$$(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi = (\hbar\nabla\theta - \frac{q^*}{c}\mathbf{A})\psi = (m^* \mathbf{v}_s)\psi, \tag{8.87}$$

69

and

$$f_s = f_n + \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 + \frac{m^*}{2}|\psi|^2 v_s^{\ 2} + \frac{B^2}{8\pi}, \tag{8.88}$$

$$\mathbf{v}_s = \frac{\mathbf{J}_s}{q^* |\psi|^2}, \qquad\qquad \hbar \nabla \theta = \frac{q^*}{c}\mathbf{A} + m^* \mathbf{v}_s, \tag{8.89}$$

Then we have

$$\oint \hbar \nabla \theta \cdot d\mathbf{l} = \hbar[\theta] = \oint \frac{q^*}{c}\mathbf{A} \cdot d\mathbf{l} + \oint m^* \mathbf{v}_s \cdot d\mathbf{l}, \tag{8.90}$$

where $[\theta]$ represents the change in phase after a complete revolution about the cylinder. $[\theta] = 2\pi n$ (n; arbitrary integer).

$$\oint \mathbf{A} \cdot d\mathbf{l} = \iint_A (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \iint_A \mathbf{B} \cdot d\mathbf{a} = \Phi = \pi R^2 B. \tag{8.91}$$

Φ represents the flux contained in the interior of the cylinder.

$$v_s = \frac{\hbar}{m^* R}\left(n + \frac{\Phi}{\Phi_0}\right), \tag{8.92}$$

where

$$\Phi_0 = \frac{2\pi c \hbar}{|q^*|}. \tag{8.93}$$

Here we see from the free energy density that the free energy will be minimized by choosing n such that $|v_s|$ is minimum since f is proportional to $\dfrac{m^*}{2}|\psi|^2 v_s^{\ 2}$.

$$v_s = \min\left[\frac{\hbar}{m^* R}\left| n + \frac{\Phi}{\Phi_0}\right|\right]. \tag{8.94}$$
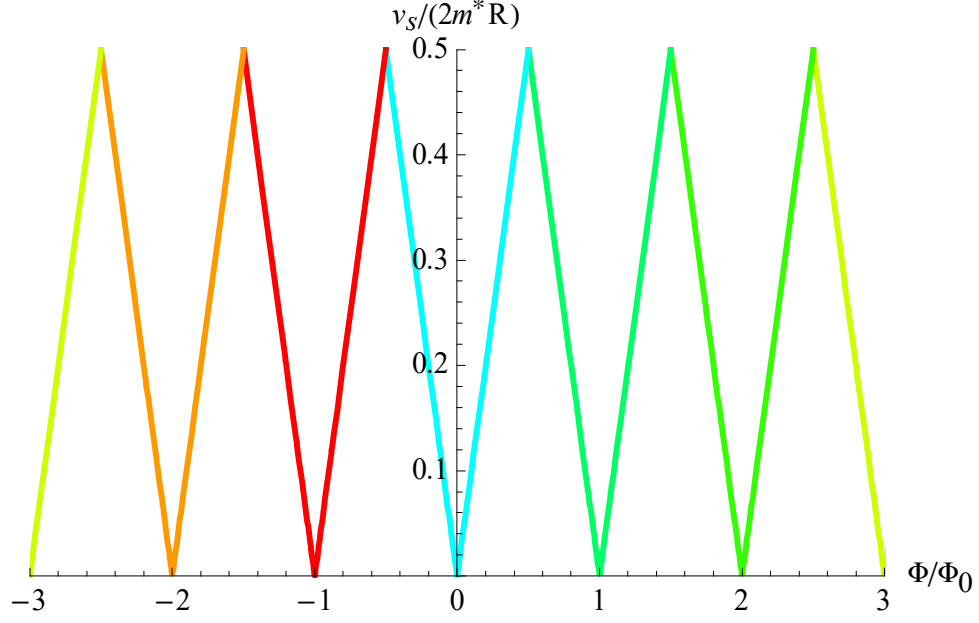
Fig.21   Plot of the velocity v_s as a function of Φ/Φ_0.

The current density is equal to zero when $\Phi = n\Phi_0$.

Since v_s is known, we can minimize the free energy with respect to $|\psi|$ and find

$$\partial f / \partial |\psi|^2 = 0, \tag{8.95}$$

or

$$\alpha + \beta |\psi|^2 + \frac{1}{2} m^* v_s^2 = 0, \tag{8.96}$$

or

$$|\psi|^2 = -(\alpha + \frac{1}{2} m^* v_s^2)/\beta \geq 0, \tag{8.97}$$

or

$$-\alpha \geq \frac{1}{2} m^* v_s^2 \qquad \text{or} \qquad |\alpha| \geq \frac{1}{2} m^* v_s^2. \tag{8.98}$$

71
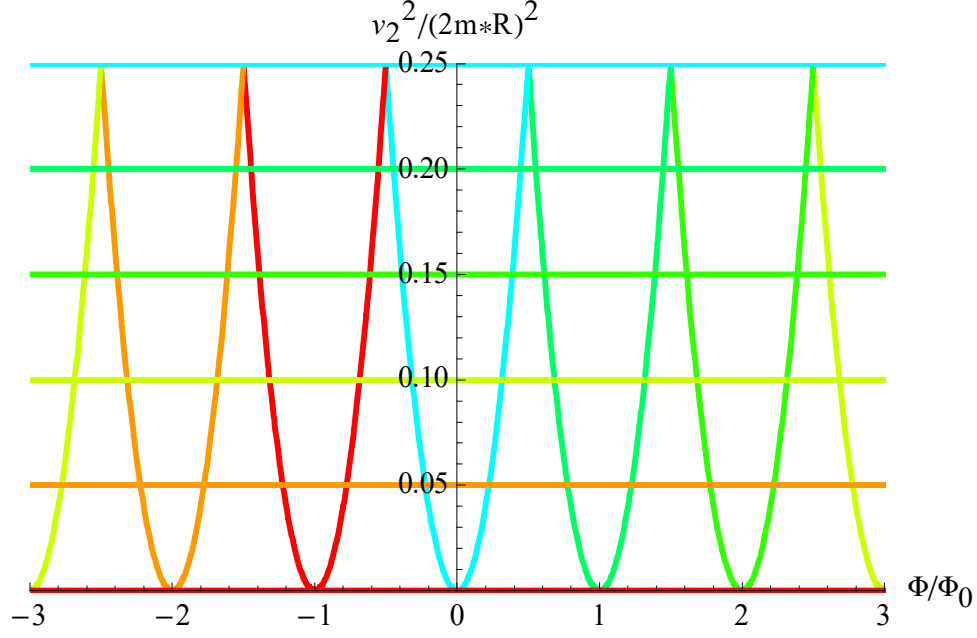
Fig.22 Plot of v_s^2 vs Φ/Φ_0. The horizontal lines denote $y = |\alpha|$ for several α.

The transition occurs when

$$|\alpha| = \frac{1}{2}m^*v_s^{\,2} = \alpha_0[1 - T(\frac{\Phi}{\Phi_0})/T_c],$$
(8.99)

where $\alpha_0 = 2\sqrt{\dfrac{\beta}{4\pi}}H_c(0)$.

In the above figure, the intersection of the horizontal line of $|\alpha|$ at fixed T and the curve $v_s^{\,2}$ gives a critical temperature. In the range of Φ/Φ_0 where the horizontal line is located above the curve $v_s^{\,2}$, the system is in the superconducting phase. In the range of Φ/Φ_0 where the horizontal line is located below the curve $v_s^{\,2}$, on the other hand, the system is in the normal phase.

8.5.2 The case of $d >> \lambda$ and $d >> \xi$.

The thickness d of the superconducting cylinder is longer than λ and ξ. The current density (proportional to derivative of B) is zero. The absolute value of the order parameter is constant in the superconducting layer.

$$\mathbf{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\mathbf{A}) = q^*|\psi|^2\mathbf{v}_s = 0,$$
(8.100)

or

$$\nabla \theta = \frac{q^*}{c\hbar} \mathbf{A} , \qquad (8.101)$$

$$\oint \hbar \nabla \theta \cdot d\mathbf{l} = \hbar[\theta] = \oint \frac{q^*}{c} \mathbf{A} \cdot d\mathbf{l} = \frac{q^*}{c} \oint \mathbf{B} \cdot d\mathbf{a} = \frac{q^*}{c} \Phi , \qquad (8.102)$$

or

$$\Phi = \frac{2\pi c\hbar}{|q^*|} n = n\Phi_0 . \qquad (8.103)$$

Then the magnetic flux is quantized. We consider a superconducting ring. Suppose that *A'*=0 inside the ring. The vector potential *A* is related to *A'* by

$$\mathbf{A'} = \mathbf{A} - \nabla \chi = 0 , \qquad (8.104)$$

or

$$\mathbf{A} = \nabla \chi . \qquad (8.105)$$

The scalar potential χ is described by

$$\chi(L) - \chi(0) = \int d\mathbf{l} \cdot \mathbf{A} = \int da \cdot (\nabla \times \mathbf{A}) = \int da \cdot \mathbf{B} = \Phi = \Phi_{ext} + \Phi_i , \qquad (8.106)$$

where Φ is the total magnetic flux.

We now consider the gauge transformation. *A'* and *A* are the new and old vector potentials, respectively. ψ' and ψ are the new and old wave functions, respectively.

$$\mathbf{A'} = \mathbf{A} + \nabla(-\chi) = 0 , \qquad (8.107)$$

$$\psi'(\mathbf{r}) = \exp(\frac{iq^*\chi}{\hbar c})\psi(\mathbf{r}) . \qquad (8.108)$$

Since *A'* = 0, ψ' is the field-free wave function and satisfies the GL equation

$$\alpha\psi' + \beta|\psi'|^2\psi' - \frac{\hbar^2}{2m^*}\nabla^2\psi' = 0 . \qquad (8.109)$$

In summary, we have

$$\psi(\mathbf{r}) = \psi'(\mathbf{r})\exp[-\frac{iq^*}{\hbar c}\chi] = \psi'(\mathbf{r})\exp[-\frac{iq^*}{\hbar c}\Phi] = \psi'(\mathbf{r})\exp[2\pi i\frac{\Phi}{\Phi_0}], \ (8.110)$$

since $\dfrac{|q^*|}{\hbar c} = \dfrac{2\pi}{\Phi_0}$ and $q^*<0$.

9. Isolated filament: the field for first penetration[2]
9.1 Critical field H_{c1}
We have the following equations;

$$\left(\frac{1}{i\kappa}\nabla_\rho + \widetilde{\mathbf{A}}\right)^2 f = f - |f|^2 f, \tag{9.1}$$

$$\nabla_\rho \times (\nabla_\rho \times \widetilde{\mathbf{A}}) = \frac{i}{2\kappa}[f^*\nabla_\rho f - f\nabla f^*] - |f|^2\widetilde{\mathbf{A}}, \tag{9.2}$$

$$\mathbf{h} = \nabla_\rho \times \widetilde{\mathbf{A}}. \tag{9.3}$$

The vector potential in cylindrical co-ordinates takes the form

$$\mathbf{A} = (0, A_\phi, 0), \quad \mathbf{h} = \frac{\mathbf{B}}{\sqrt{2}H_c}, \quad \widetilde{\mathbf{A}} = \frac{2\pi\xi}{\Phi_0}\mathbf{A}, \quad \mathbf{r} = \lambda\boldsymbol{\rho} \tag{9.4}$$

Stokes theorem:

$$\oint \mathbf{B}\cdot d\mathbf{a} = \oint(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{r} = 2\pi r A_\phi = \Phi, \tag{9.5}$$

or

$$A_\phi = \frac{\Phi}{2\pi r} = \frac{\Phi_0}{2\pi\xi}\widetilde{A}_\phi \tag{9.6}$$

Far from the center of the vortex,

$$\widetilde{A}_\phi = \frac{\xi}{r}\frac{\Phi}{\Phi_0} = \frac{\xi}{\lambda\rho}\frac{\Phi}{\Phi_0} = \frac{1}{\kappa}\frac{\Phi}{\Phi_0}\frac{1}{\rho} = \frac{1}{\kappa\rho}, \tag{9.7}$$

for $\Phi = \Phi_0$.

Near the center of the vortex,

$$A_\phi 2\pi r = H(0)\pi r^2, \quad \text{or} \quad \widetilde{A}_\phi = \frac{\pi\kappa\rho}{\Phi_0}H(0). \tag{9.8}$$

We assume that

$$f = \psi(\rho)e^{-i\phi}, \tag{9.9}$$

where $\psi(\rho)$ is a real positive function of ρ.

$$\mathbf{h} = \nabla_\rho \times \widetilde{\mathbf{A}} = \mathbf{e}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\widetilde{A}_\phi) = \mathbf{e}_z \frac{1}{\rho}(\widetilde{A}_\phi + \rho\frac{\partial\widetilde{A}_\phi}{\partial\rho}), \tag{9.10}$$

$$\nabla_\rho \times \mathbf{h} = \mathbf{e}_\phi[\frac{1}{\rho^2}A_\phi - \frac{1}{\rho}(\frac{\partial\widetilde{A}_\phi}{\partial\rho} + \rho\frac{\partial^2\widetilde{A}_\phi}{\partial^2\rho})] = -\mathbf{e}_\phi\frac{\partial}{\partial\rho}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\widetilde{A}_\phi)]. \tag{9.11}$$

GL equation can be described by

$$-\frac{1}{\kappa^2}\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d\psi}{d\rho}) + (-\frac{1}{\kappa\rho} + \widetilde{A}_\phi)^2\psi = \psi - \psi^3. \tag{9.12}$$

The current equation: $\mathbf{e}_\phi$ component.

$$-\frac{d}{d\rho}[\frac{1}{\rho}\frac{d}{d\rho}(\rho\widetilde{A}_\phi)] + (-\frac{1}{\kappa\rho} + \widetilde{A}_\phi)\psi^2 = 0. \tag{9.13}$$

The above equations can be treated approximately when $\kappa \gg 1$.
We assume that $\psi(\rho) = 1$ in the region $\rho > 1/\kappa$.

$$[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - (1 + \frac{1}{\rho^2})](-\frac{1}{\kappa\rho} + \widetilde{A}_\phi) = 0 \tag{9.14}$$

The solution of this differential equation is

$$-\frac{1}{\kappa\rho} + \widetilde{A}_\phi = c_1 K_1(\rho) \tag{9.15}$$

or

$$\widetilde{A}_\phi = \frac{1}{\kappa\rho} + c_1 K_1(\rho) \tag{9.16}$$

75

$$h_z = \frac{1}{\rho}(\tilde{A}_\phi + \rho\frac{\partial\tilde{A}_\phi}{\partial\rho}) = \frac{1}{\rho}[\frac{1}{\kappa\rho} + c_1 K_1(\rho) + \rho(-\frac{1}{\kappa\rho^2} + c_1\frac{dK_1(\rho))}{d\rho}], \quad (9.17)$$

or

$$h_z = \frac{c_1}{\rho}[K_1(\rho) + \rho\frac{dK_1(\rho))}{d\rho}] = \frac{c_1}{\rho}[-\rho K_0(\rho)] = -c_1 K_0(\rho). \quad (9.18)$$

or

$$h_z = -c_1 K_0(\rho), \quad (9.19)$$

from the recursion formula

$$\rho\frac{dK_1(\rho)}{d\rho} + K_1(x) = -\rho K_0(\rho), \quad (9.20)$$

What is the value of c_1? When $\rho \approx 0$, $\tilde{A}_\phi$ should be equal to zero.

$$\tilde{A}_\phi = \frac{1}{\kappa\rho} + c_1 K_1(\rho) \approx \frac{1}{\kappa\rho} - \frac{c_1}{\rho} = 0, \quad (9.21)$$

because $K_1(\rho) \approx 1/\rho$. Then we have

$$c_1 = \frac{1}{\kappa}. \quad (9.22)$$

In fact,

$$h = \frac{B}{\sqrt{2}H_c} = \frac{\frac{\Phi_0}{2\pi\lambda^2}}{\sqrt{2}H_c}K_0(\rho) = \frac{\Phi_0}{2\pi\sqrt{2}H_c\lambda^2}K_0(\rho) = \frac{1}{\kappa}K_0(\rho), \quad (9.23)$$

or

$$h = \frac{1}{\kappa}K_0(\rho), \quad (9.24)$$

since

$$\sqrt{2}H_c\lambda^2 = \frac{\Phi_0}{2\pi}\kappa. \quad (9.25)$$

In summary, we have

$$\tilde{A}_\phi = \frac{1}{\kappa\rho} - \frac{1}{\kappa}K_1(\rho)$$
$$h = \frac{1}{\kappa}K_0(\rho)$$

(9.26)

((**Note**))
The total magnetic flux through a circle of radius of $\rho \gg 1$ is given in the following way

$$\tilde{A}_\phi = \frac{1}{\kappa\rho} - \frac{1}{\kappa}K_1(\rho) \approx \frac{1}{\kappa\rho} .$$

(9.27)

Then we have

$$\Phi = 2\pi(\lambda\rho)A_\phi = 2\pi\lambda\rho\frac{\Phi_0}{2\pi\xi}\tilde{A}_\phi = 2\pi\lambda\rho\frac{\Phi_0}{2\pi\xi}\frac{1}{\kappa\rho} = \Phi_0 .$$

(9.28)

9.2   The center of vortex[4]
First we consider the behavior at the center of vortex, as $\rho \to 0$.

$$-\frac{1}{\kappa^2}\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d\psi}{d\rho}) + (-\frac{1}{\kappa\rho} + \tilde{A}_\phi)^2\psi = \psi - \psi^3 ,$$

(9.29)

with

$$\tilde{A}_\phi = \frac{\pi\kappa\rho\xi^2}{\Phi_0}H(0) ,$$

(9.30)

or

$$-\frac{1}{\kappa^2}\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d\psi}{d\rho}) + [-\frac{1}{\kappa\rho} + \frac{\pi\kappa\xi^2\rho}{\Phi_0}H(0)]^2\psi = \psi - \psi^3 .$$

(9.31)

We assume that ψ can be described by series

$$\psi = \rho^k(a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3 + ...) ,$$

(9.32)

where $a_0 \neq 0$. Using the Mathematica 5.2, we determine the parameters;

$$k = 1$$

$$a_1 = 0$$

$$a_2 = -a_0 \frac{\kappa^2}{8}[1 + \frac{2\pi\xi^2 H(0)}{\Phi_0}] = -a_0 \frac{\kappa^2}{8}[1 + \frac{H(0)}{H_{c2}}], \tag{9.33}$$

$$a_3 = 0$$

where

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2}.$$

Then we have

$$\psi(\rho) = a_0\rho[1 - \frac{\kappa^2\rho^2}{8}(1 + \frac{H(0)}{H_{c2}})]. \tag{9.34}$$

The rise of $\psi(\rho)$ starts to saturate at $\rho = 2/\kappa$.

((**Summary**))

If the vortex contains 1 flux quantum, we have

$$f(\rho) = e^{-i\phi}\psi(\rho), \tag{9.35}$$

$$\psi(\rho) = a_0\rho\{1 - \frac{\kappa^2\rho^2}{8}[1 + \frac{H(0)}{H_{c2}}] + ...\}. \tag{9.37}$$

9.3   Far from the center of the vortex

$$-\frac{1}{\kappa^2}\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d\psi}{d\rho}) = \psi - \psi^3, \tag{9.38}$$

with

$$\tilde{A}_\phi = \frac{1}{\kappa\rho}. \tag{9.39}$$

This differential equation is simplified as

$$\frac{1}{\kappa^2}[\psi''(\rho) + \frac{1}{\rho}\psi'(\rho)] + \psi(\rho) - \psi(\rho)^3 = 0. \tag{9.40}$$

In the region where $\psi(\rho) \to 1$, for simplicity we assume that

$$\psi(\rho) = 1 - \varphi(\rho), \tag{9.41}$$

$$[1 - \varphi(\rho)]^3 = 1 - \varphi(\rho) - \frac{\varphi'(\rho) + \rho\varphi''(\rho)}{\kappa^2 \rho}. \tag{9.42}$$

This equation can be approximated by

$$1 - 3\varphi(\rho) = 1 - \varphi(\rho) - \frac{\varphi'(\rho) + \rho\varphi''(\rho)}{\kappa^2 \rho}, \tag{9.43}$$

or

$$\varphi''(\rho) + \frac{1}{\rho}\varphi'(\rho) - 2\kappa^2 \varphi(\rho) = 0. \tag{9.44}$$

The solution is given by

$$\varphi(\rho) = C_1 I_0(\sqrt{2}\rho\kappa) + C_2 K_0(\sqrt{2}\rho\kappa). \tag{9.45}$$

Since $I_0(\sqrt{2}\rho\kappa)$ increases with increasing ρ, C_1 should be equal to zero. Then we have

$$\varphi(\rho) = C_2 K_0(\sqrt{2}\rho\kappa). \tag{9.46}$$

For large ρ,

$$K_0(x) \approx \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}, \qquad K_1(x) \approx \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}. \tag{9.47}$$

10. Properties of one isolated vortex line ($H = H_{c1}$)
10.1 The method of the Green's function

$$\mathbf{J}_s = q^* |\psi|^2 \mathbf{v}_s, \tag{10.1}$$

$$f_s = f_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)\psi\right|^2 + \frac{1}{8\pi}B^2, \tag{10.2}$$

When $\psi = |\psi|e^{i\theta}$ and $|\psi|^2 = n_s^* = $ independent of $\mathbf{r}$,

$$f_s - f_n = \alpha n_s{}^* + \frac{1}{2}\left(\beta n_s{}^*\right)^2 + \frac{m^*}{2}n_s{}^* v_s{}^2 + \frac{\mathbf{B}^2}{8\pi}. \tag{10.3}$$

$$\mathbf{v}_s = \frac{\mathbf{J}_s}{q^* n_s{}^*} \qquad \text{and} \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_s. \tag{10.4}$$

Then the net energy can be written by

$$E = E_0 + \frac{1}{8\pi}\int dr[\mathbf{B}^2 + \lambda^2(\nabla \times \mathbf{B})^2]. \tag{10.5}$$

When

$$\mathbf{B} \to \mathbf{B} + \delta\mathbf{B} \, , \, , \, E \to E + \delta E, \tag{10.6}$$

$$\delta E = \frac{1}{4\pi}\int d\mathbf{r}[\mathbf{B} + \lambda^2 \nabla \times (\nabla \times \mathbf{B})] \cdot \delta\mathbf{B}. \tag{10.7}$$

The integrand can be calculated by using VariationalD of the Mathematica 5.2.

((**Mathematica Program-14**)) Variation method

```
Clear["Global`*"];

<< "VariationalMethods`"

Needs["VectorAnalysis`"];

h = {h1[x, y, z], h2[x, y, z], h3[x, y, z]};
eq1 = h.h + λ² (Curl[h]).(Curl[h])  // Expand

h1[x, y, z]² + h2[x, y, z]² + h3[x, y, z]²

eq3 = VariationalD[eq1, h2[x, y, z], {x, y, z}] / 2 // Expand;
eq4 = VariationalD[eq1, h3[x, y, z], {x, y, z}] / 2 // Expand;
eq2 = VariationalD[eq1, h1[x, y, z], {x, y, z}] / 2 // Expand;
```

London's equation

```
eq5 = h + λ² Curl[Curl[h]];

A1 = {eq2 - eq5[[1]], eq3 - eq5[[2]], eq3 - eq5[[2]]} // Simplify

{0, 0, 0}
```

Demanding that E be a minimum leads to the London equation; $\delta E = 0$,

$$\mathbf{B} + \lambda^2 \nabla \times (\nabla \times \mathbf{B}) = 0 . \tag{10.8}$$

This is called the second London's equation.

((Note))

$$\mathbf{B}^2 \to \mathbf{B}^2 + 2\mathbf{B} \cdot \delta\mathbf{B} , \tag{10.9}$$

$$(\nabla \times \mathbf{B})^2 \to (\nabla \times \mathbf{B})^2 + 2(\nabla \times \mathbf{B}) \cdot (\nabla \times \delta\mathbf{B}) . \tag{10.10}$$

Then we have

$$\delta E = \frac{1}{4\pi} \int dr [\mathbf{B} \cdot \delta\mathbf{B} + \lambda^2 (\nabla \times \mathbf{B}) \cdot (\nabla \times \delta\mathbf{B})]. \tag{10.11}$$

Since

$$(\nabla \times \mathbf{B}) \cdot (\nabla \times \delta\mathbf{B}) = \delta\mathbf{B} \cdot [\nabla \times (\nabla \times \mathbf{B})] + \nabla \cdot [\delta\mathbf{B} \times (\nabla \times \mathbf{B})], \tag{10.12}$$

$$\delta E = \frac{1}{4\pi}\int d\mathbf{r}[\mathbf{B} + \lambda^2 \nabla \times (\nabla \times \mathbf{B})] \cdot \delta \mathbf{B} + \frac{1}{4\pi}\int d\mathbf{a} \cdot [\delta \mathbf{B} \times (\nabla \times \mathbf{B})]. \qquad (10.13)$$

The second term is equal to zero.
Here we assume that
(i) $\quad \lambda \gg \xi$.
(ii) $\quad$ The hard core of radius ξ is very small, and we shall neglect completely its contribution to the energy
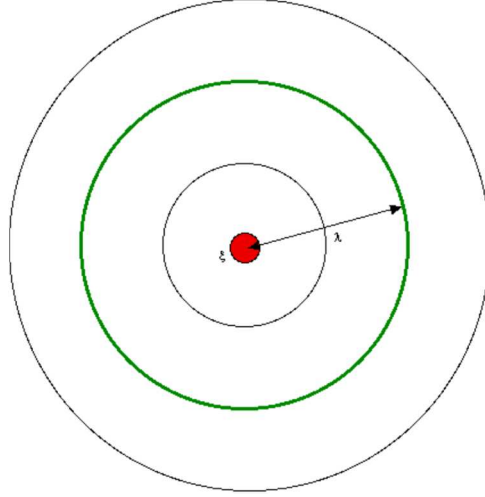
Fig.23 Top view of the geometrical configuration of a single vortex.

The line energy is given by

$$\varepsilon = \int_{r>\xi} d^2\mathbf{r}\frac{1}{8\pi}[\mathbf{B}^2 + \lambda^2(\nabla \times \mathbf{B})^2]. \qquad (10.14)$$

Demanding that ε be a minimum leads to the London's equation.

$$\mathbf{B} + \lambda^2[\nabla \times (\nabla \times \mathbf{B})] = 0, \qquad (10.15)$$

for $|\mathbf{r}| > \xi$. In the interior of hard core, we can try to replace the corresponding singularity by a 2D delta function.

$$\mathbf{B} + \lambda^2[\nabla \times (\nabla \times \mathbf{B})] = \Phi_0 \mathbf{e}_z \delta_2(\mathbf{r}), \qquad (10.16)$$

$$\int \mathbf{B} \cdot d\mathbf{a} + \lambda^2 \int [\nabla \times (\nabla \times \mathbf{B})] \cdot d\mathbf{a} = \int \Phi_0 \mathbf{e}_z \delta_2(\mathbf{r}) \cdot d\mathbf{a} = \Phi_0, \qquad (10.17)$$

Using Stokes theorem,

$$\int \mathbf{B} \cdot d\mathbf{a} + \lambda^2 \int (\nabla \times \mathbf{B}) \cdot d\mathbf{l} = \int \Phi_0 \mathbf{e}_z \delta_2(\mathbf{r}) \cdot d\mathbf{a} = \Phi_0 . \qquad (10.18)$$
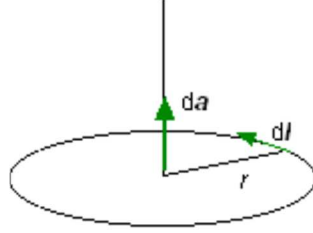
Fig.24   The path of line integral

If the circle of the path integral has a radius r ($\xi << r << \lambda$), the first term I_1 is on the order of $\Phi_0 r^2 / \lambda^2$ and is negligibly small. In the second term I_2,

$$d\mathbf{l} = \mathbf{e}_\phi r d\phi , \qquad (10.19)$$

$$\nabla \times \mathbf{B} = -\mathbf{e}_\phi \frac{d}{dr} B(r) . \qquad (10.20)$$

Then we have

$$I_2 = \lambda^2 \int (\nabla \times \mathbf{B}) \cdot d\mathbf{l} = \lambda^2 \int_0^{2\pi} [-\mathbf{e}_\phi \frac{d}{dr} B(r)] \cdot \mathbf{e}_\phi r d\phi$$
$$= -\lambda^2 2\pi r \frac{d}{dr} B(r) = \Phi_0 \qquad (10.21)$$

or

$$\frac{dB(r)}{dr} = -\frac{\Phi_0}{2\pi \lambda^2 r} , \qquad (10.22)$$

or

$$B(r) = \frac{\Phi_0}{2\pi \lambda^2} [\ln(\frac{\lambda}{r}) + const] , \qquad (10.23)$$

for $\xi < r << \lambda$. We now consider the exact solution

$$\mathbf{B} + \lambda^2 [\nabla \times (\nabla \times \mathbf{B})] = \Phi_0 \mathbf{e}_z \delta_2(\mathbf{r}) , \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0 . \qquad (10.24)$$

Note that

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B}.$$ (10.25)

Thus we have

$$\nabla^2 \mathbf{B} - \frac{1}{\lambda^2}\mathbf{B} = -\frac{\Phi_0}{\lambda^2}\mathbf{e}_z\delta_2(\mathbf{r}),$$ (10.26)

where $\delta_2(r)$ is a 2D delta function.

((**Note**))   2D case Green function

(i)   Modified Helmholtz

$$(\nabla_1^2 - k^2)G(\mathbf{r}_1,\mathbf{r}_2) = -\delta_2(\mathbf{r}_1 - \mathbf{r}_2),$$ (10.27)

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{2\pi}K_0(k|\mathbf{r}_1-\mathbf{r}_2|).$$ (10.28)

(ii)   Modified Bessel function

$$(\nabla^2 - k^2)\psi(\mathbf{r}) = -f(\mathbf{r}),$$ (10.29)

$$\psi(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\mathbf{r}',$$ (10.30)

$$\begin{aligned}(\nabla^2 - k^2)\psi(\mathbf{r}) &= \int(\nabla^2 - k^2)G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\mathbf{r}' \\ &= \int -\delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')d\mathbf{r}' = -f(\mathbf{r})\end{aligned},$$ (10.31)

The solution of the equation

$$\nabla^2 \mathbf{B} - \frac{1}{\lambda^2}\mathbf{B} = -\frac{\Phi_0}{\lambda^2}\mathbf{e}_z\delta_2(\mathbf{r}),$$ (10.32)

is that

$$\mathbf{B}(\mathbf{r}) = \mathbf{e}_z\int G(\mathbf{r},\mathbf{r}')\frac{\Phi_0}{\lambda^2}\delta_2(\mathbf{r}') = \mathbf{e}_z\frac{\Phi_0}{2\pi\lambda^2}K_0(\frac{r}{\lambda}),$$ (10.33)

or

$$B(r) = \frac{\Phi_0}{2\pi\lambda^2}K_0(\frac{r}{\lambda}) \text{ (in general).}$$ (10.34)

((**Asymptotic form**))
For $\xi < r << \lambda$,

$$B(r) = \frac{\Phi_0}{2\pi\lambda^2}[\ln(\frac{r}{\lambda}) + 0.115932]. \qquad (10.35)$$

For $r >> \lambda$,

$$B(r) = \frac{\Phi_0}{2\pi\lambda^2}\sqrt{\frac{\pi\lambda}{2r}}\exp(-\frac{r}{\lambda}). \qquad (10.36)$$

The magnetic flux $\Phi(r)$ inside the circle with a radius r is

$$\Phi(r) = \int_0^r 2\pi r' dr' B(r') = \frac{\Phi_0}{\lambda^2}\int_0^r r' dr' K_0(\frac{r'}{\lambda}) = \Phi_0 \int_0^{r/\lambda} x dx K_0(x) = \Phi_0[1 - \frac{r}{\lambda}K_1(\frac{r}{\lambda})]. \qquad (10.37)$$

The current density

$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B} = -\mathbf{e}_\phi \frac{c}{4\pi}\frac{d}{dr}B(r). \qquad (10.38)$$
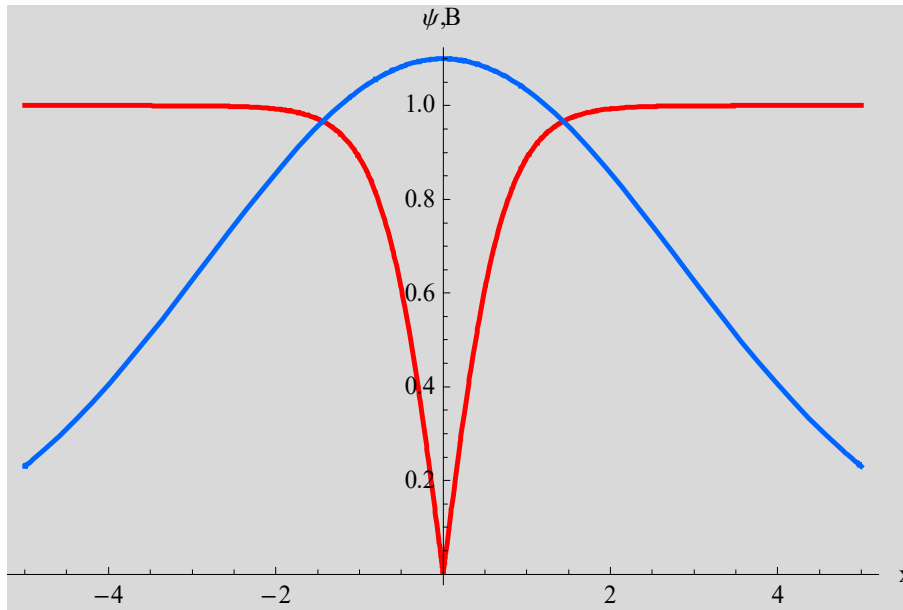
Fig.25   Schematic diagram for the plots of the magnetic induction B (blue line) and the magnitude of the order parameter (red line) as a function of x (the distance from the center of the vortex) for an isolated vortex line in a mixed phase of type-II superconductor.

((Mathematica Program-15))

```
Clear["Global`*"];
h1 = Plot[BesselK[0, x], {x, 0, 2}, PlotPoints → 100,
    PlotStyle → {Thick, Red}, Background → LightGray,
    AxesLabel → {"r/λ", "h/(Φ0/2πλ²"}];
```

$$\Phi = \int_0^x BesselK[0, y]\ y\ \mathbb{d}y\ //\ Simplify[\#,\ x > 9]\ \&$$

```
1 - x BesselK[1, x]
```

```
h2 = Plot[Φ, {x, 0, 10}, PlotPoints → 100, PlotStyle → {Red, Thick},
    PlotRange → {{0, 10}, {0, 1}}, Background → LightGray,
    AxesLabel → {"r/λ", "Φ/Φ0"}];
```
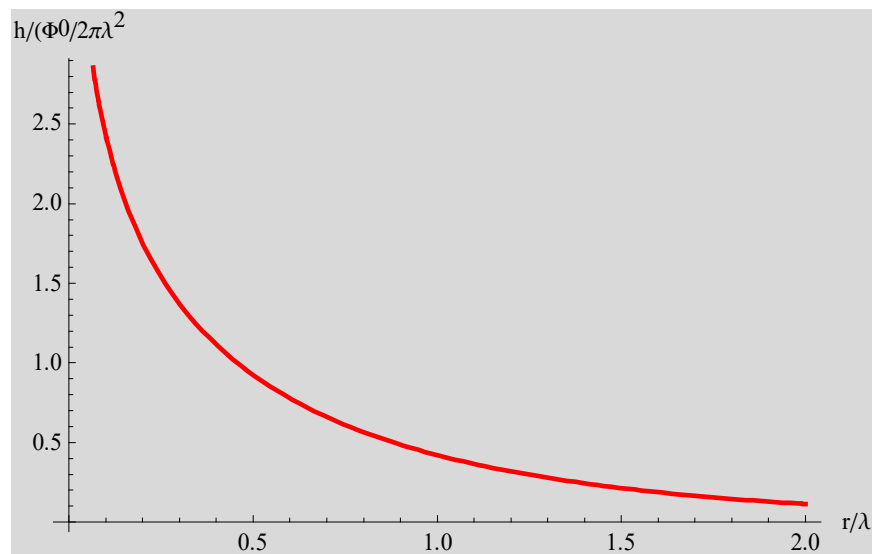

Fig.26   Plot of the normalized magnetic induction $B(r)/\dfrac{\Phi_0}{2\pi\lambda^2} = K_0(r/\lambda)$ as a function of r/λ.

86
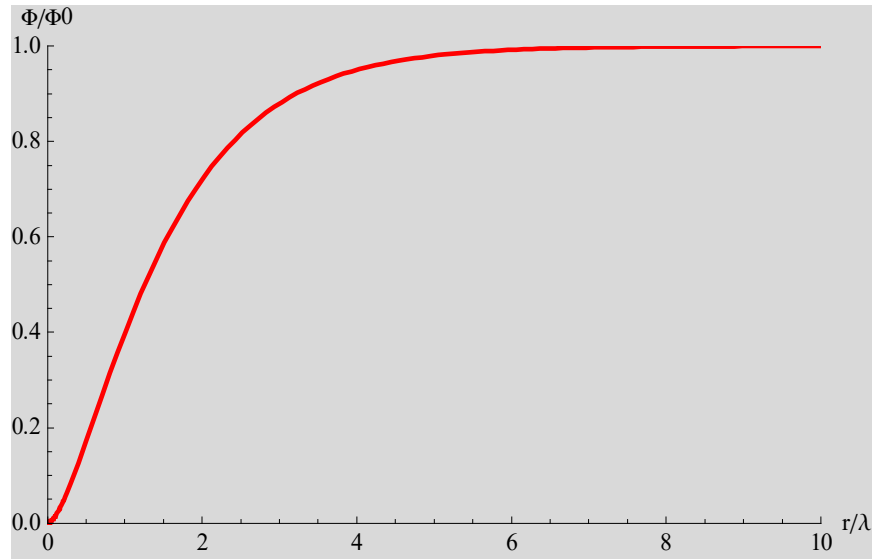
Fig.27 Magnetic flux $\Phi(r)/\Phi_0 = [1 - \frac{r}{\lambda} K_1(\frac{r}{\lambda})]$ as a function of $x = r/\lambda$. $\Phi(r)/\Phi_0$ reduces to 1 as $x \to \infty$.

10.2. Vortex line energy
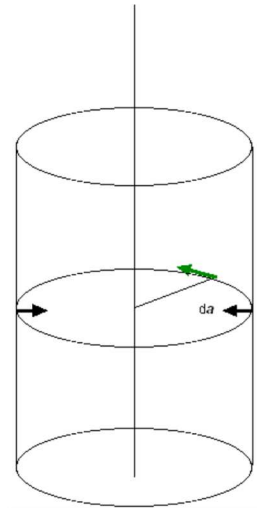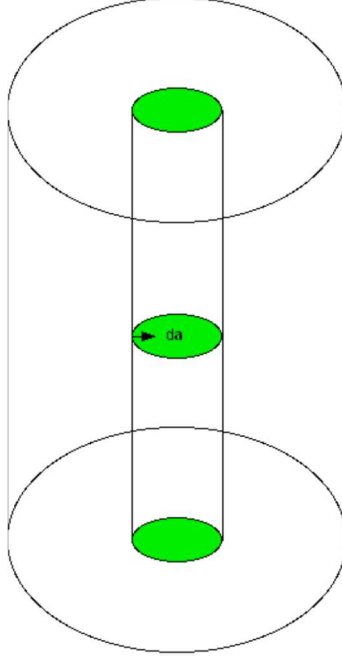
Cylinder with a radius ξ.

Fig.28   Geometrical configuration of a single vortex. The vortex is approximated by a cylinder.

Now let us find the line tension or free energy per unit length. neglecting the core, we have only the contributions

$$\varepsilon_1 = \int d^3\mathbf{r} \frac{1}{8\pi}[\mathbf{B}^2 + \lambda^2(\nabla \times \mathbf{B})^2]. \tag{10.39}$$

Vector identity

$$(\nabla \times \mathbf{B})^2 = \mathbf{B} \cdot [\nabla \times (\nabla \times \mathbf{B})] + \nabla \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})]. \tag{10.40}$$

$$\varepsilon_1 = \frac{1}{8\pi}\int d^3\mathbf{r}[\mathbf{B} + \lambda^2 \nabla \times (\nabla \times \mathbf{B})] \cdot \mathbf{B} + \frac{1}{8\pi}\int d^3\mathbf{r}\lambda^2 \nabla \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})], \tag{10.41}$$

or

$$\varepsilon_1 = \frac{1}{8\pi}\int d^3\mathbf{r}[\Phi_0\mathbf{e}_z\delta_2(\mathbf{r})] \cdot \mathbf{B} + \frac{\lambda^2}{8\pi}\int d\mathbf{a} \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})]. \tag{10.42}$$

We use the Stokes theorem for the second term. We now neglect the core contribution (the first term). The integral $\int d\mathbf{a}$ is to be taken over the surface of the hard core (cylinder of radius ξ).

$$\varepsilon_1 = \frac{\lambda^2}{8\pi} \int d\mathbf{a} \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})].$$ (10.43)

Note that

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = B_z(\rho)\frac{\partial B_z(\rho)}{\partial \rho}\mathbf{e}_\rho \qquad \text{and} \qquad d\mathbf{a} = -\mathbf{e}_\rho da$$ (10.44)

$$\varepsilon_1 = \frac{\lambda^2}{8\pi} \int [B_z(\rho)\frac{\partial B_z(\rho)}{\partial \rho}\mathbf{e}_\rho \cdot (-\mathbf{e}_\rho)]da \cdot = -\frac{\lambda^2}{8\pi}[B_z(\rho)\frac{\partial B_z(\rho)}{\partial \rho}2\pi\rho]_{\rho=\xi} \, .$$
(10.45)

Since

$$\frac{dB_z(\rho)}{d\rho} = -\frac{\Phi_0}{2\pi\lambda^2}\frac{1}{\rho}$$ (10.46)

we have

$$\varepsilon_1 = \frac{\Phi_0}{8\pi}B_z(\xi) = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 [\ln(\frac{\lambda}{\xi}) + 0.115932].$$ (10.47)

More exactly

$$\varepsilon_1 = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \frac{\xi}{\lambda}K_0(\frac{\xi}{\lambda})K_1(\frac{\xi}{\lambda}).$$ (10.48)

(see the derivation of this equation in the Mathematica Program-16).

((**Mathematica Program-16**))

```
Clear["Global`*"]; << "VectorAnalysis`"; SetCoordinates[Cylindrical[ρ, ϕ, z]];
h = {0, 0, hz[ρ]}; Cross[h, Curl[h]] // Simplify;
```

$$\text{rule1} = \left\{ \text{hz} \rightarrow \left(\frac{\Phi 0}{2\,\pi\,\lambda^2}\ \text{BesselK}\left[0, \frac{\#}{\lambda} \right] \& \right) \right\};$$

$$\epsilon 1 = -\frac{\lambda^2}{8\,\pi}\ 2\,\pi\,\xi\ \text{hz}[\xi]\ \text{D}[\text{hz}[\xi],\,\xi]\ /.\ \text{rule1}\ //\ \text{Simplify}$$

$$\frac{\xi\,\Phi 0^2\ \text{BesselK}\left[0, \frac{\xi}{\lambda} \right]\ \text{BesselK}\left[1, \frac{\xi}{\lambda} \right]}{16\,\pi^2\,\lambda^3}$$

$$\text{f1} = \frac{1}{x}\ \text{BesselK}\left[0, \frac{1}{x} \right]\ \text{BesselK}\left[1, \frac{1}{x} \right];\ \text{Limit}[\text{f1} - \text{Log}[x],\ x \rightarrow \infty]\ //\ \text{N}$$

```
0.115932
```

```
h1 = Plot[{Log[x] + 0.115932, f1}, {x, 1, 10}, PlotPoints → 100,
    PlotRange → {{0, 10}, {0, 2.5}}, PlotStyle → {{Red, Thick}, {Blue, Thick}},
    Background → LightGray, AxesLabel → {"λ/ξ", "ϵ₁/(Φ₀²/16π²λ²"}];
```
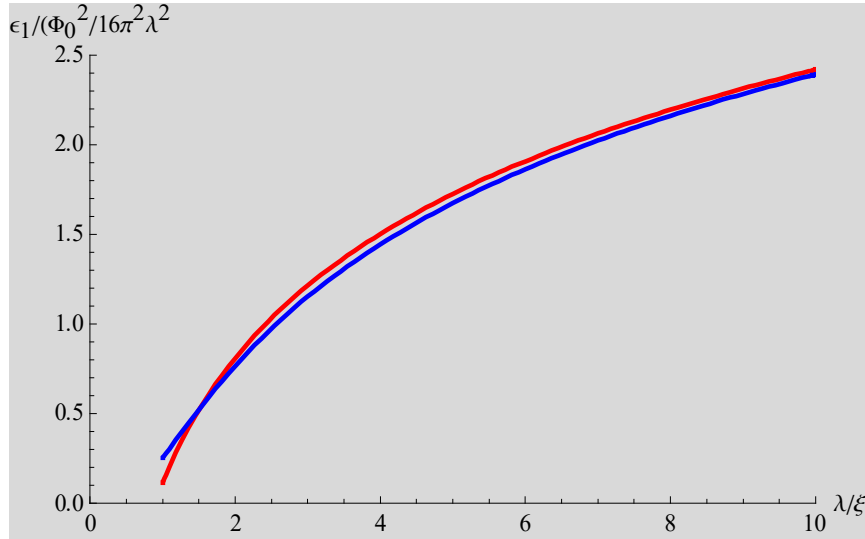

Fig.29   Plot of $\varepsilon_1 / \left(\Phi_0 / 4\pi\lambda \right)^2 = \frac{1}{x} K_0(1/x) K_1(1/x)$ [blue line] as a function of $x = \lambda / \xi$. For comparison, the approximation given by $(\ln x + 0.115932)$ is also shown by a red line.

10.3.   Derivation of H_{c1}

$$G = F - \frac{H}{4\pi} \int h d^3\mathbf{r} \,. \tag{10.49}$$

By definition, when $H = H_{c1}$, the Gibbs free energy must have the same value whether the first vortex is in or out of the system;

$$G_s|_{\text{no flux}} = G_s|_{\text{first vortex},} \tag{10.50}$$

where

$$G_s|_{\text{no flux}} = F_s \,. \tag{10.51}$$

$$G_s|_{\text{first vortex}} = F_s + \varepsilon_1 L - \frac{H_{c1}}{4\pi} L \Phi_0 \,, \tag{10.52}$$

where L is the length of the vortex line in the system and

$$\int B d^3\mathbf{r} = L \int B da = L\Phi_0 \,. \tag{10.53}$$

Then we have

$$F_s = F_s + \varepsilon_1 L - \frac{H_{c1}}{4\pi} L\Phi_0 \,, \tag{10.54}$$

or

$$H_{c1} = \frac{4\pi\varepsilon_1}{\Phi_0} \,, \tag{10.55}$$

or

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda^2} \ln(\frac{\lambda}{\xi}) = \frac{H_c}{\sqrt{2}\kappa} \ln(\kappa) \,. \tag{10.56}$$

10.4. Interaction between vortex lines

Suppose that there are two vortices in the 2 D plane.

$$\nabla^2 B - \frac{1}{\lambda^2} B = -\frac{\Phi_0}{\lambda^2} [\delta_2(\mathbf{r} - \mathbf{r}_1) + \delta_2(\mathbf{r} - \mathbf{r}_2)] \,. \tag{10.57}$$

Using the Green function

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} K_0(\frac{|\mathbf{r} - \mathbf{r}'|}{\lambda}) \,. \tag{10.58}$$

Then we have

$$B = \int d\mathbf{r}' G(\mathbf{r},\mathbf{r}') \frac{\Phi_0}{\lambda^2} [\delta_2(\mathbf{r} - \mathbf{r}_1) + \delta_2(\mathbf{r} - \mathbf{r}_2)], \qquad (10.59)$$

or

$$B = \frac{\Phi_0}{2\pi\lambda^2} \int d\mathbf{r}' K_0(\frac{|\mathbf{r} - \mathbf{r}'|}{\lambda}) [\delta_2(\mathbf{r} - \mathbf{r}_1) + \delta_2(\mathbf{r} - \mathbf{r}_2)], \qquad (10.60)$$

or

$$B(\mathbf{r}) = \frac{\Phi_0}{2\pi\lambda^2} [K_0(\frac{|\mathbf{r} - \mathbf{r}_1|}{\lambda}) + K_0(\frac{|\mathbf{r} - \mathbf{r}_2|}{\lambda})] = B_1(\mathbf{r}) + B_2(\mathbf{r}). \qquad (10.61)$$

where

$$B_1(\mathbf{r}) = \frac{\Phi_0}{2\pi\lambda^2} K_0(\frac{|\mathbf{r} - \mathbf{r}_1|}{\lambda}), \qquad (10.62)$$

$$B_2(\mathbf{r}) = \frac{\Phi_0}{2\pi\lambda^2} K_0(\frac{|\mathbf{r} - \mathbf{r}_2|}{\lambda}). \qquad (10.63)$$

$$B_1(\mathbf{r}_2) = B_2(\mathbf{r}_1). \qquad (10.64)$$

The total increase in free energy per unit length can be rewritten as

$$\varepsilon = \frac{\Phi_0}{8\pi} [\{B_1(\mathbf{r}_1) + B_2(\mathbf{r}_1)\} + \{B_1(\mathbf{r}_2) + B_2(\mathbf{r}_2)\}]$$
$$= 2\frac{\Phi_0}{8\pi} B_1(\mathbf{r}_1) + 2\frac{\Phi_0}{8\pi} B_1(\mathbf{r}_2) \qquad (10.65)$$

The first tem is the energy of individual vortex line. We have the interaction energy

$$U_{12} = \frac{\Phi_0}{4\pi} B_1(\mathbf{r}_2) = \frac{\Phi_0^2}{8\pi^2\lambda^2} K_0(\frac{r_{12}}{\lambda}), \qquad (10.66)$$

where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$

The interaction is repulsive, in which the flux has the same sense in both vortices.

11.   Magnetization curves
11.1   Theory of *M* vs *H*
We consider the Gibbs function given by

$$G - G_{s0} = n_L \varepsilon_1 + \sum_{i>j} U_{ij} - \frac{BH}{4\pi} . \tag{11.1}$$

The first term is an individual energy of the lines. n_L is the number of lines per unit area (cm^2).

$$B = n_L \Phi_0 . \tag{11.2}$$

The second term is a repulsive interaction between vortices.

$$U_{ij} = \frac{\Phi_0^2}{8\pi^2 \lambda^2} K_0(\frac{r_{ij}}{\lambda}) , \tag{11.3}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$

One can distinguish three regimes between H_{c1} and H_{c2}.

(a) $H \approx H_{c1}$: $n_L \lambda^2 \ll 1$, $r > \lambda$

(b) $n_L \lambda^2 \gg 1$ $\xi < r < \lambda$

(c) $H \approx H_{c2}$: $n_L \xi^2 \approx 1$ $r \approx \xi$

$$G - G_{s0} = \frac{B}{\Phi_0} \varepsilon_1 + \sum_{i>j} U_{ij} - \frac{BH}{4\pi} \tag{11.4}$$

(a)
At very low densities (low B) the interaction is small and we neglect it completely.

$$G - G_{s0} = (\frac{\varepsilon_1}{\Phi_0} - \frac{H}{4\pi}) B . \tag{11.5}$$

When $\dfrac{\varepsilon_1}{\Phi_0} > \dfrac{H}{4\pi}$ or $H < H_{c1}$, the lowest G is obtained for $B = 0$. When $\dfrac{\varepsilon_1}{\Phi_0} < \dfrac{H}{4\pi}$, we can lower G by choosing $B \neq 0$. There is some flux penetration.

$$H_{c1} = \frac{4\pi\varepsilon_1}{\Phi_0} = \frac{\Phi_0}{4\pi\lambda^2} \ln(\frac{\lambda}{\xi}) . \tag{11.6}$$

(b) When $H > H_{c1}$
d: distance between neighboring lines ($d > \lambda$).
(i)   Square lattice

$$z_s = 4$$

$$n_L = \frac{B}{\Phi_0} = \frac{1}{d_s^2} \tag{11.7}$$

or

$$d_s = \sqrt{\frac{\Phi_0}{B}} \tag{11.8}$$

(ii)   Triangular lattice

$$z_t = 6$$

$$n_L = \frac{B}{\Phi_0} = \frac{1}{\frac{\sqrt{3}}{2} d_t^2} \tag{11.9}$$

or

$$d_t = \left(\frac{4}{3}\right)^{1/4} \sqrt{\frac{\Phi_0}{B}} = 1.07457 \sqrt{\frac{\Phi_0}{B}} \tag{11.10}$$

We neglect all contributions except those of the nearest neighbors.

$$\sum_{i>j} U_{ij} = \left(\frac{B}{\Phi_0}\right) \frac{z}{2} \frac{\Phi_0^2}{8\pi^2 \lambda^2} K_0(\frac{d}{\lambda}) = \frac{Bz\Phi_0}{16\pi^2 \lambda^2} K_0(\frac{d}{\lambda}) \tag{11.11}$$

Then we have

$$G - G_{s0} = \frac{B}{4\pi}[(H_{c1} - H) + \frac{1}{2} z \frac{\Phi_0}{2\pi\lambda^2} K_0(\frac{d}{\lambda})] \tag{11.12}$$

or

$$\frac{G - G_{s0}}{\frac{1}{2} z \frac{\Phi_0}{2\pi\lambda^2}} = \frac{B}{4\pi}[-\frac{(H - H_{c1})}{\frac{1}{2} z \frac{\Phi_0}{2\pi\lambda^2}} + K_0(\frac{d}{\lambda})] = \frac{B}{4\pi}[-\frac{\sqrt{2}\pi\kappa H_{c1}}{zH_c}(\frac{H}{H_{c1}} - 1) + K_0(\frac{d}{\lambda})], \tag{11.13}$$

or

$$\frac{G - G_{s0}}{\frac{zH_c}{\sqrt{2}\pi\kappa}} = \frac{B}{4\pi}[-\frac{\sqrt{2}\pi\kappa H_{c1}}{zH_c}(\frac{H}{H_{c1}} - 1) + K_0(\frac{d}{\lambda})], \tag{11.14}$$

94

or

$$\frac{G - G_{s0}}{\frac{zH_c}{\sqrt{2}\pi\kappa}} = \frac{B}{4\pi}[-\frac{\sqrt{2}\pi\kappa H_{c1}}{zH_c}(\frac{H}{H_{c1}} - 1) + K_0(\rho_0(\frac{\Phi_0}{B})^{1/2})],$$ (11.15)

where

$$\frac{\Phi_0}{\lambda^2} = \frac{2\sqrt{2}H_c}{\kappa}.$$ (11.16)

The B vs H curve can be obtained from

$$\frac{\partial G}{\partial B} = 0.$$ (11.17)

or

$$-\frac{\sqrt{2}\pi\kappa H_{c1}}{zH_c}(\frac{H}{H_{c1}} - 1) + K_0(\frac{d}{\lambda}) - \frac{1}{2}\frac{d}{\lambda}K_0'(\frac{d}{\lambda})\frac{1}{B} = 0$$ (11.18)

or

$$-\frac{\sqrt{2}\pi\kappa H_{c1}}{zH_c}(\frac{H}{H_{c1}} - 1) + K_0(\rho_0(\frac{\Phi_0}{B})^{1/2}) + \frac{1}{2}\rho_0\frac{1}{B}(\frac{\Phi_0}{B})^{1/2}K_1(\rho_0(\frac{\Phi_0}{B})^{1/2}) = 0$$ (11.19)

Here we note that

$$\alpha_0 = \frac{\sqrt{2}\pi\kappa H_{c1}}{zH_c},$$ (11.20)

and

$$\frac{d}{\lambda} = \rho_0(\frac{\Phi_0}{B})^{1/2}$$ (11.21)

where $\rho_0 = 1$ for the square lattice and 1.07457 for the triangular lattice.

$$K_0'(x) = -K_1(x)$$ (11.22)

11.2   Calculation of M vs H using Mathematica

95

We make a plot of the free energy vs *B* and *B* vs *H* using the Mathematica 5.2.

((Mathematica Program-17))

```
F[a_, b_] := B/(4 π) (BesselK[0, a/√B] - b (h - 1));

K[a_, b_] := Plot[Evaluate[Table[F[a, b], {h, 1, 3, 0.1}]], {B, 0, 5},
    PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}], Background → LightGray];

G[a_, b_] := D[B/(4 π) (BesselK[0, a/√B] - b (h - 1)), B] // Simplify;

p11 = ContourPlot[Evaluate[G[1, 1] == 0, {h, 1, 1.15}, {B, 0.01, 0.15}],
    PlotPoints → 50, ContourStyle → {Hue[0], Thick}, Background → LightGray,
    AxesLabel → {"h", "B"}];
p12 = ContourPlot[Evaluate[G[1.2, 1] == 0, {h, 1, 1.03}, {B, 0.01, 0.08}],
    PlotPoints → 50, ContourStyle → {Hue[0], Thick}, Background → LightGray,
    AxesLabel → {"h", "B"}];
```

(a) $a = 0.4$
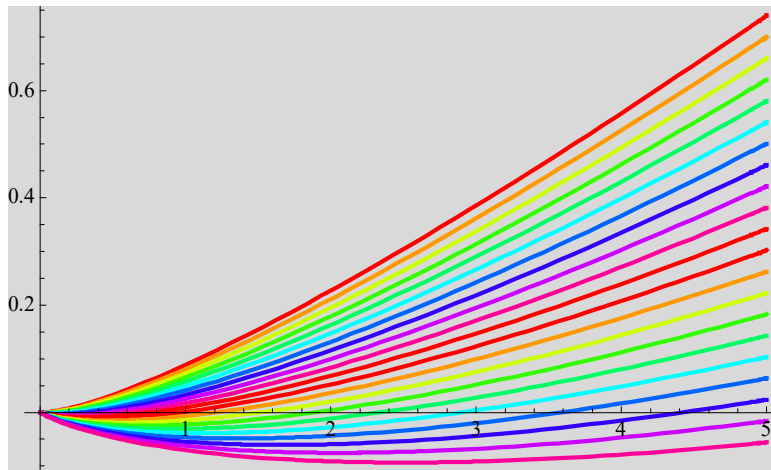
(b) $a = 0.4$
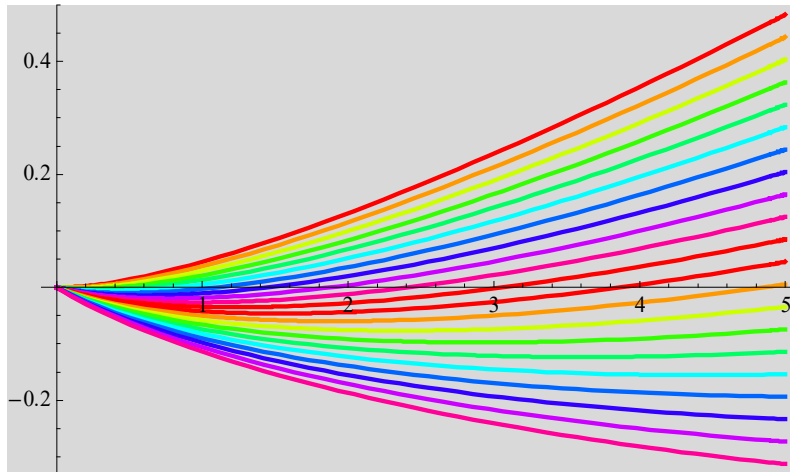
(b) $a = 1.2$
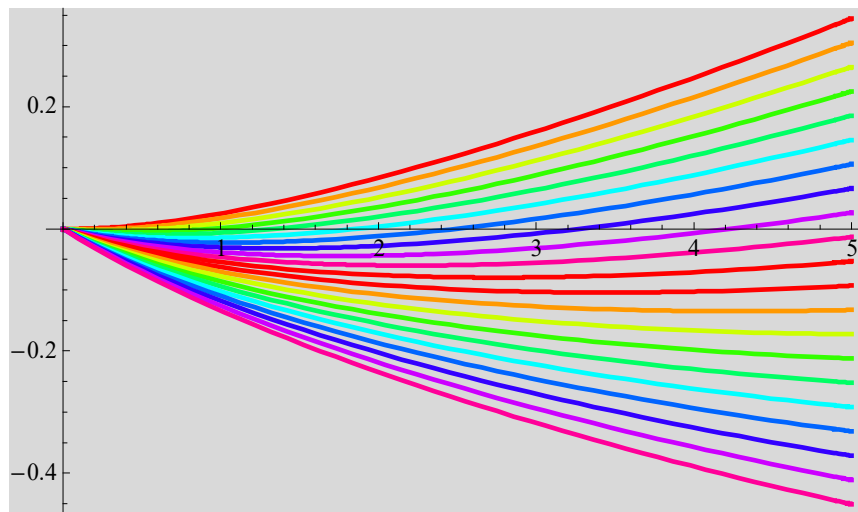
Fig.30   Gibbs free energy given by $\dfrac{B}{4\pi}[-b(h-1)+K_0(\dfrac{a}{\sqrt{B}})]$ as a function of B, where h $(=H/H_{c1})$ is changed as a parameter ($h = 1 - 3$, $\Delta h = 0.1$). b $(= 1)$ is fixed.. The parameter a is chosen appropriately. (a) $a = 0.4$, (b) $a = 0.8$, and (c) $a = 1.2$.
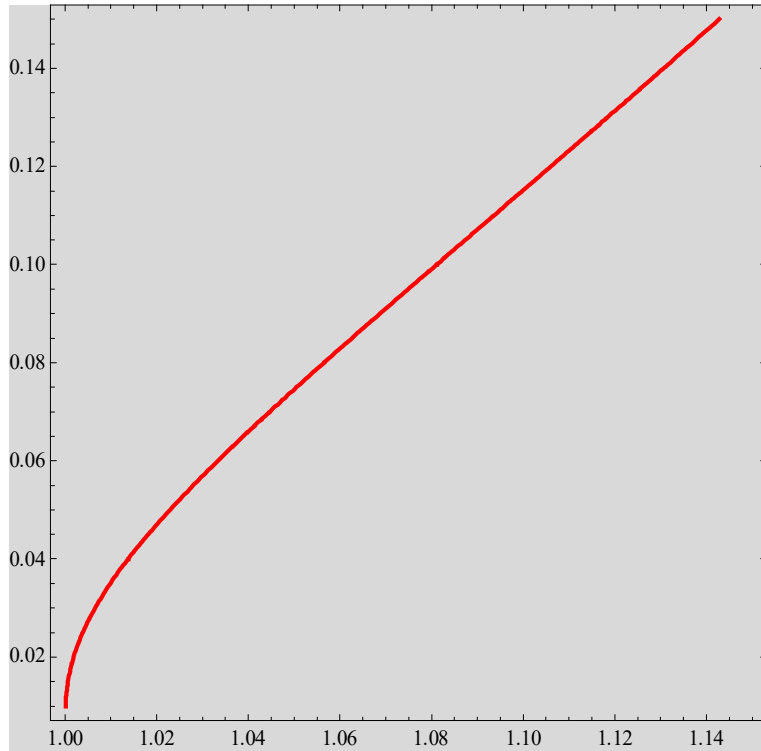
Fig.31   The magnetic induction B vs h $(=H/H_{c1})$ above $h = 1$. $h = 1$ $(H = H_{c1})$.
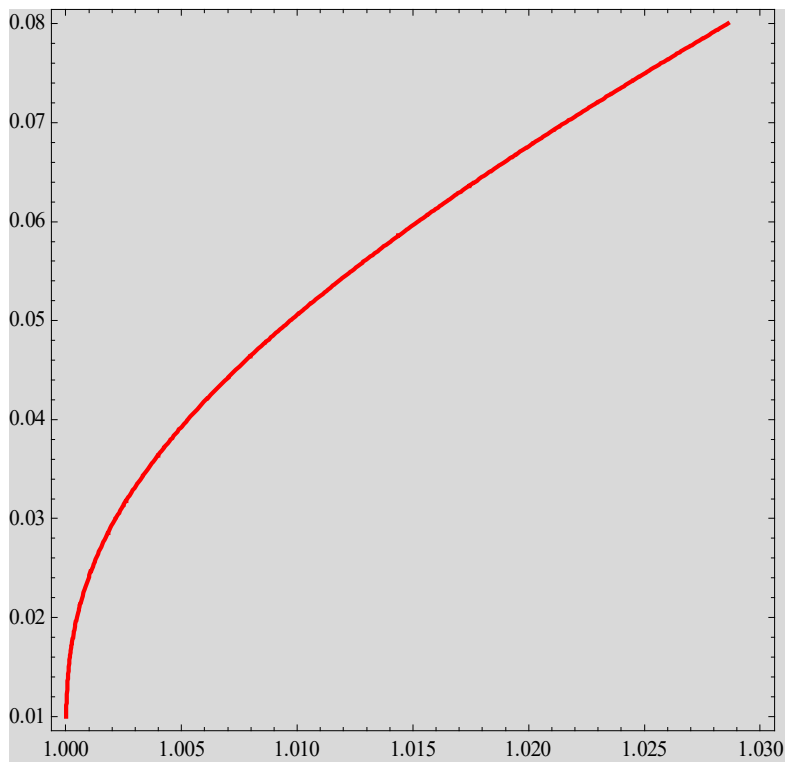
Fig.32   The detail of the magnetic induction B vs h $(=H/H_{c1})$ near $h = 1$ $(H = H_{c1})$.

12.   The linearized GL equation ($H = H_{c2}$).
12.1.   Theory

When the term $\beta|\psi|^2\psi$ can be dropped in the GL equation, we obtain linearized GL equation

$$\alpha\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi = 0. \tag{12.1}$$

This omission will be justified only if $|\psi|^2 << \psi_\infty^2 = |\alpha|/\beta$.

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi = \frac{\hbar^2}{2m^*}\frac{2m^*|\alpha|}{\hbar^2}\psi = \frac{\hbar^2}{2m^*\xi^2}\psi = E\psi, \tag{12.2}$$

where

$$\xi^2 = \frac{\hbar^2}{2m^*|\alpha|}. \text{ and } E = \frac{\hbar^2}{2m^*\xi^2}.$$

When $B = H$ (//z), we choose the gauge

$$\mathbf{A} = (0, Hx, 0). \tag{12.3}$$

Note that we use $\mathbf{A} = (\mathbf{H}\times\mathbf{r})/2 = (-Hy/2, Hx/2, 0)$. $\chi = Hxy/2$,

$$\mathbf{A} \to \mathbf{A} + \nabla\chi = (0, Hx, 0) \text{ (gauge transformation)}.$$

This problem is reduced to the eigenvalue problem in the quantum mechanics.

$$\hat{H}\psi = E\psi, \tag{12.4}$$

$$\hat{H} = \frac{1}{2m}[\hat{p}_x^2 + (\hat{p}_y - \frac{q^*}{c}H\hat{x})^2 + \hat{p}_z^2]. \tag{12.5}$$

This Hamiltonian $\hat{H}$ commutes with $\hat{p}_y$ and $\hat{p}_z$.

$$[\hat{H}, \hat{p}_y] = 0 \quad \text{and} \quad [\hat{H}, \hat{p}_z] = 0 \tag{12.6}$$

$$\hat{H}|n, k_y, k_z\rangle = E_n|n, k_y, k_z\rangle, \tag{12.7}$$

and

$$\hat{p}_y \left| n, k_y, k_z \right\rangle = \hbar k_y \left| n, k_y, k_z \right\rangle, \quad \hat{p}_z \left| n, k_y, k_z \right\rangle = \hbar k_z \left| n, k_y, k_z \right\rangle, \qquad (12.8)$$

$$\left\langle x, y, z \middle| \hat{p}_y \middle| n, k_y, k_z \right\rangle = \hbar k_y \left\langle x, y, z \middle| n, k_y, k_z \right\rangle,$$

$$\left\langle x, y, z \middle| \hat{p}_z \middle| n, k_y, k_z \right\rangle = \hbar k_z \left\langle x, y, z \middle| n, k_y, k_z \right\rangle, \qquad (12.9)$$

or

$$\frac{\hbar}{i} \frac{\partial}{\partial y} \left\langle y \middle| n, k_y, k_z \right\rangle = \hbar k_y \left\langle y \middle| n, k_y, k_z \right\rangle, \qquad \left\langle y \middle| n, k_y, k_z \right\rangle \approx e^{ik_y y}. \qquad (12.10)$$

$$\frac{\hbar}{i} \frac{\partial}{\partial z} \left\langle x, y, z \middle| n, k_y, k_z \right\rangle = \hbar k_z \left\langle x, y.z \middle| n, k_y, k_z \right\rangle \quad \left\langle x, y, z \middle| n, k_y, k_z \right\rangle \approx e^{ik_z z}.$$

$$(12.11)$$

Schrödinger equation for $\psi(x, y, z) = \left\langle x, y, z \middle| n, k_y, k_z \right\rangle$,

$$\frac{1}{2m^*} [(\frac{\hbar}{i} \frac{\partial}{\partial x})^2 + (\frac{\hbar}{i} \frac{\partial}{\partial y} + \frac{|q^*|}{c} Hx)^2 + (\frac{\hbar}{i} \frac{\partial}{\partial y})^2] \psi(x, y, z) = E \psi(x, y, z), \qquad (12.12)$$

$$\psi(x, y, z) = e^{ik_y y + ik_z z} \phi(x), \qquad (12.13)$$

$$x = \frac{\varsigma}{\beta}, \quad \text{with} \quad \beta = \sqrt{\frac{m^* \omega_c^*}{\hbar}} = \sqrt{\frac{|q^*| H}{\hbar c}} = \frac{1}{\ell_H} \quad \text{and} \quad \omega_c^* = \frac{|q^*| H}{m^* c}, \qquad (12.14)$$

$$\varsigma_0 = -\beta \frac{c \hbar k_y}{|q^*| H} = -\sqrt{\frac{c \hbar}{|q^*| H}} k_y = -\ell_H k_y. \qquad (12.15)$$

We assume the periodic boundary condition along the y axis.

$$\psi(x, y + L_y, z) = \psi(x, y, z), \qquad (12.16)$$

or

$$e^{ik_y L_y} = 1, \qquad (12.17)$$

or

$$k_y = \frac{2\pi}{L_y} n_y \quad (n_y\text{: integers}).$$ (12.18)

Then we have

$$\phi''(\varsigma) = [(\varsigma - \varsigma_0)^2 + \frac{c}{|q^*|\hbar H}(-2m^*E + \hbar^2 k_z{}^2)]\phi(\varsigma).$$ (12.19)

We put

$$E = \hbar\omega_c^*(n + \frac{1}{2}) + \frac{\hbar^2 k_z{}^2}{2m^*}.$$ (12.20)

where $n = 0, 1, 2, 3, \ldots$ (Landau level),

or

$$2m^*E = \hbar^2 k_z{}^2 + 2m^*\hbar\omega_c^*(n + \frac{1}{2}) = \hbar^2 k_z{}^2 + \frac{2|q|^*\hbar H}{c}(n + \frac{1}{2}),$$ (12.21)

or

$$E = \frac{\hbar^2}{2m^*}k_z{}^2 + \frac{|q|^*\hbar H}{m^*c}(n + \frac{1}{2}).$$ (12.22)

Finally we get a differential equation for $\phi(\varsigma)$

$$\phi''(\varsigma) + [2n + 1 - (\varsigma - \varsigma_0)^2]\phi(\varsigma) = 0.$$ (12.23)

The solution of this differential equation is

$$\phi_n(\varsigma) = (\sqrt{\pi}\, 2^n n!)^{-1/2} e^{-\frac{(\varsigma - \varsigma_0)^2}{2}} H_n(\varsigma - \varsigma_0),$$ (12.24)

with

$$\varsigma_0 = -\sqrt{\frac{c\hbar}{|q^*|H}}k_y = -\ell_H k_y,$$ (12.25)

$$\ell_H = \sqrt{\frac{c\hbar}{|q|^* H}},$$ (12.26)

$$x_0 = \frac{\varsigma_0}{\beta} = \ell_H \varsigma_0 = -\ell_H{}^2 k_y \tag{12.27}$$

The coordinate x_0 is the center of orbits. Suppose that the size of the system along the x axis is L_x. The coordinate x_0 should satisfy the condition, $0 < x_0 < L_x$. Since the energy of the system is independent of x_0, this state is degenerate.

$$0 < x_0 = \frac{\varsigma_0}{\beta} = \ell_H \varsigma_0 = -\ell_H{}^2 k_y < L_x, \tag{12.28}$$

or

$$\ell_H{}^2 k_y = \frac{2\pi}{L_y} \ell_H{}^2 n_y < L_x, \tag{12.29}$$

or

$$n_y < \frac{L_x L_y}{2\pi \ell^2}, \tag{12.30}$$

$$E = \frac{\hbar^2}{2m^* \xi^2} = \frac{\hbar^2}{2m^*} k_z{}^2 + \frac{|q|^* H\hbar}{m^* c}(n + \frac{1}{2}) = \frac{\hbar^2}{2m^*} k_z{}^2 + \frac{|q|^* H\hbar}{m^* c}(n + \frac{1}{2}). \tag{12.31}$$

$$\frac{1}{\xi^2} - k_z{}^2 = \frac{2|q|^* H}{c\hbar}(n + \frac{1}{2}) = \frac{4\pi H}{\Phi_0}(n + \frac{1}{2}). \tag{12.32}$$

or

$$\frac{1}{\xi^2} - k_z{}^2 = \frac{2}{\ell_H{}^2}(n + \frac{1}{2}), \tag{12.33}$$

where

$$\Phi_0 = \frac{2\pi\hbar c}{|q^*|},$$

$$\ell_H{}^2 = \frac{\hbar c}{|q^*| H} = \frac{\Phi_0}{2\pi H}. \tag{12.34}$$

Then we have

$$H = \frac{\Phi_0}{4\pi(n+1/2)}\left(\frac{1}{\xi^2} - k_z^{\,2}\right). \tag{12.35}$$

((**Ground state**))

If k_z=0 and $n = 0$, this has its highest value at $H = HH_{c2} = \dfrac{\Phi_0}{2\pi\xi^2}$, (12.36)

or

$$H_{c2} = \kappa\sqrt{2}H_c. \tag{12.37}$$

This is the highest field at which superconductivity can nucleate in the interior of a large sample in a decreasing external field.

$$\frac{1}{\xi^2} = \frac{1}{\ell_H^{\,2}} = \frac{2\pi H}{\Phi_0}. \tag{12.38}$$

12.2　Numerical calculation of wave finctions

((**Mathematica Program-18**))

Simple Harmonics wave function: plot of φn[ζ]

```
In[31]:= Clear["Global`*"]; conjugateRule = {Complex[re_, im_] :> Complex[re, -im]};
        Unprotect[SuperStar]; SuperStar /: exp_* := exp /. conjugateRule;
        Protect[SuperStar];

        ψ[n_, ζ_, ζ0_] := 2^(-n/2) π^(-1/4) (n!)^(-1/2) Exp[-((ζ - ζ0)^2)/2] HermiteH[n, ζ - ζ0]

In[34]:= pt1 = Plot[Evaluate[Table[ψ[0, ζ, ζ0], {ζ0, -2, 2, 0.4}]], {ζ, -4, 4},
          PlotLabel → {0}, PlotPoints → 100, PlotRange → All,
          PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 10}], Background → LightGray];
        pt2 = Plot[Evaluate[Table[ψ[1, ζ, ζ0], {ζ0, -2, 2, 0.4}]], {ζ, -4, 4},
          PlotLabel → {1}, PlotPoints → 100, PlotRange → All,
          PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 10}], Background → LightGray];
        pt3 = Plot[Evaluate[Table[ψ[2, ζ, ζ0], {ζ0, -2, 2, 0.4}]], {ζ, -6, 6},
          PlotLabel → {2}, PlotPoints → 100, PlotRange → All,
          PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 10}], Background → LightGray];
```

(a)　　$n = 0$ (even parity)
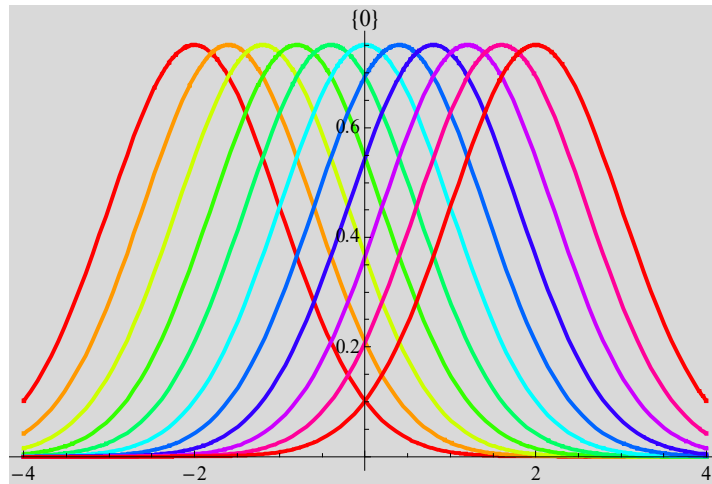
(b) $n = 1$ (odd parity)
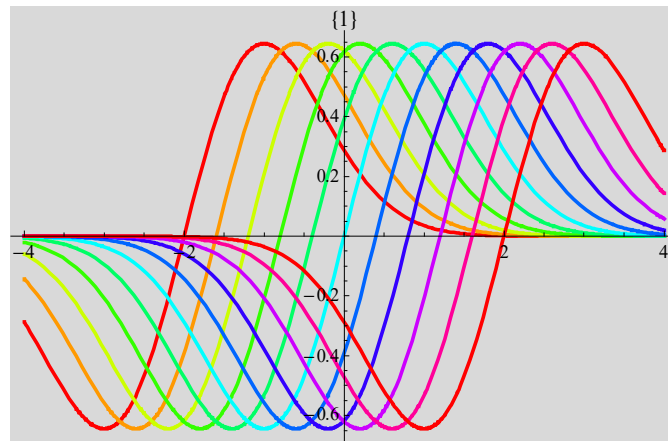
(c) $n = 2$ (even parity)

Fig.33.   Plot of the wave function with the quantum number (a) $n = 0$ (an even parity), (b) $n = 1$ (an odd parity), and (c) $n = 2$ (even parity) centered around $\zeta = \zeta_0$, as a function of ζ. The position ζ_0 is changed as aparameter between -2 and 2.

13.   Nucleation at surfaces: H_{c3}
13.1   Theory

$$\phi''(\varsigma) = [(\varsigma - \varsigma_0)^2 + \ell_H{}^2(-\frac{1}{\xi^2} + k_z{}^2)]\phi(\varsigma),$$ (13.1)

and the boundary condition: $J_s|_n = 0$ at the boundary ($\zeta = 0$).

$$d\phi(\varsigma)/d\varsigma = 0 \text{ at } \zeta = 0.$$ (13.2)

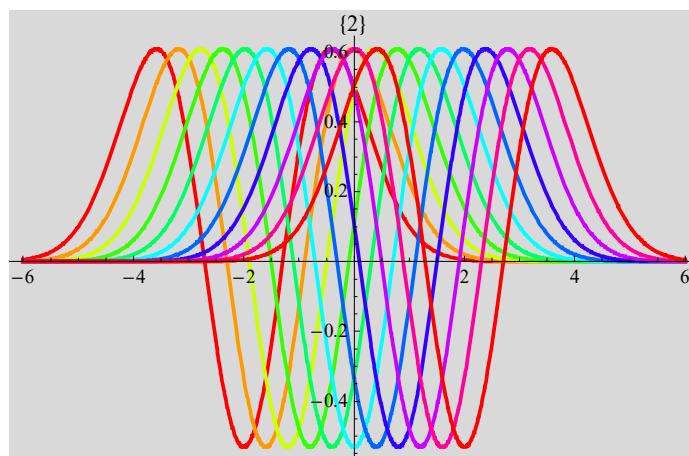For $x_0 = \ell_H \varsigma_0 \gg \xi$, the harmonic oscillator function is nearly zero at $\zeta = 0$ and the boundary condition is satisfied.

The problem is reduced to a model with a 1D parabolic potential well, where x_0 is a constant corresponding to the center of orbit. Let us now consider a double well consisting of two equal parabolic well lying symmetrically to the plane $\zeta = 0$: the wave function is either an even or an odd function). The corresponding wave function should be an even function since $d\phi(\varsigma)/d\varsigma = 0$ at $\zeta = 0$. The ground level of a particle in the double well is below that in the single wall. The eigenvalue associated with the double well potential is smaller than the eigenvalues associated with the potential $(\varsigma - \varsigma_0)^2$.

$$\phi''(\varsigma) = [(\varsigma - \varsigma_0)^2 + g_0)]\phi(\varsigma).$$ (13.3)

$$\phi'(\varsigma) = 0 \text{ at } \zeta = 0.$$ (13.3)

where

$$g_0 = \ell_H{}^2(-\frac{1}{\xi^2} + k_z{}^2).$$ (13.4)

When $k_z = 0$,

$$g_0 = -\frac{\ell_H{}^2}{\xi^2} = -(2n+1),$$ (13.5)

((**Mathematica Program-19**))

```
Clear["Global`*"]; f1 = (x - 1)²;

f2[x_] := (x - 1)² /; x > 0

f2[x_] := (x + 1)² /; x < 0

Plot[{f1, f2[x]}, {x, -2, 2},
 PlotStyle → {{Thick, Hue[0]}, {Hue[0.7], Thick}},
 PlotRange → {{-2, 2}, {0, 2}}, AxesLabel → {"ζ/ζ₀", "V(ζ)"},
 Background → LightGray]
```
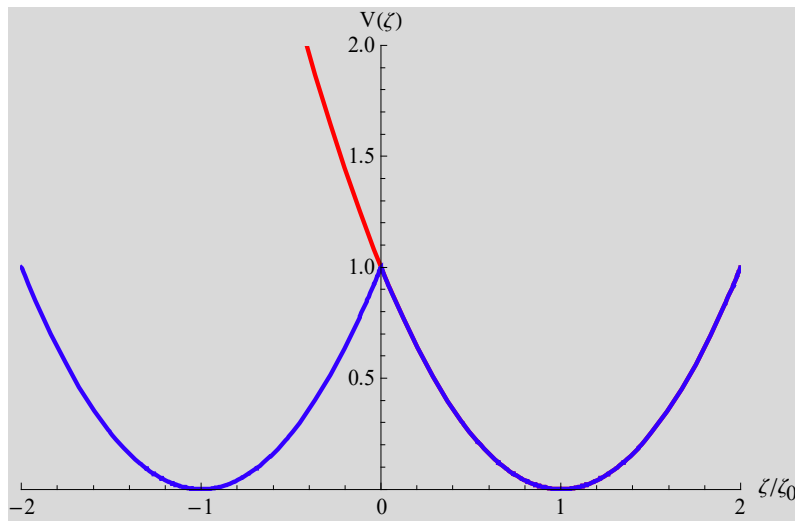

Fig.34　　　The harmonic potential (denoted by red line) and the symmetrical potential (denoted by dark blue line).

From the book of Eugene Merzbacher,[16] we have

$$\varphi''(\varsigma) + [2v + 1 - (\varsigma - \varsigma_0)^2]\varphi(\varsigma) = 0 .$$

(13.6)

We put

$$z = \sqrt{2}(\varsigma - \varsigma_0) ,$$

(13.7)

where $\varsigma_0 > 0$. Then the differential equation can be rewritten as

$$\varphi''(z) + (v + \frac{1}{2} - \frac{z^2}{4})\varphi(z) = 0 ,$$

(13.8)

where v is generally not an integer.

106

The boundary condition:

(1) $\varphi(z = \infty) = 0$.

(2) $\varphi'(z = -\sqrt{2}\varsigma_0) = 0$.   (13.9)

The solution of this differential equation is

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4} \left[\frac{\Gamma(1/2)}{\Gamma[(1-\nu)/2]} {}_1F_1(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}) + \frac{z}{\sqrt{2}} \frac{\Gamma(-1/2)}{\Gamma(-\nu/2)} {}_1F_1(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}) \right],$$
     (13.10)

where ${}_1F_1$ is the confluent hypergeometric (or Kummer) function. For even $\varphi(z)$ the boundary condition is given by

$$D_\nu'(z = -\sqrt{2}\varsigma_0) = 0.$$ (13.11)

Imagine the potential well about $\zeta = \varsigma_0$ to be extended by a mirror image outside the surface. Lowest eigenfunction of a symmetric potential is itself symmetric; $\varphi'(\varsigma) = 0$ at the surface.

(i) $\varsigma_0 \to \infty$

 Wave function $\varphi(z)$ will be localized around $\zeta = \varsigma_0$, and nearly zero at the surface. Therefore the boundary condition will automatically be satisfied.

(ii) $\varsigma_0 \to 0$.

 The function still satisfies the boundary condition and Eq.(13.8).

13.2　Numerical solution by Mathematica

((Mathematica Program-20))

```
ψ[z_] =

   2^(ν/2) Exp[- z²/4 ]

      ( Gamma[1 / 2]/Gamma[ (1-ν)/2 ]  Hypergeometric1F1[- ν/2 , 1/2 , z²/2 ] +

         z/√2  Gamma[-1 / 2]/Gamma[ -ν/2 ]  Hypergeometric1F1[ (1 - ν)/2 , 3/2 , z²/2 ] );

eq1 = ψ'[z] // Simplify;

p11 = ContourPlot[Evaluate[eq1 == 0 /. z → -√2 ξ0, {ξ0, 0, 3}, {ν, -0.3, 0.1}],
   PlotPoints → 50, ContourStyle → {Red, Thick}, Background → LightGray,
   AspectRatio → 1, AxesLabel → {"ξ0", "ν"}];
```
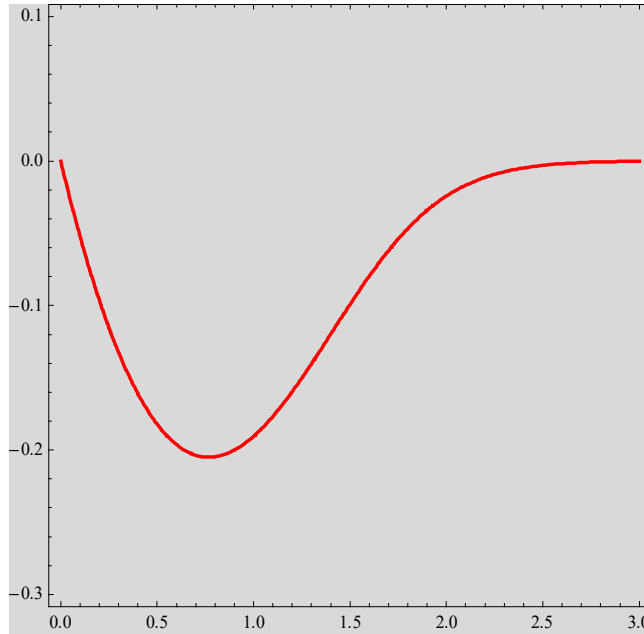

Fig.35   Plot of v as a function of ζ_0, satisfying the condition. $D_v'(z = -\sqrt{2}\zeta_0) = 0$.

$v = 0$ at $\zeta_0 = 0$, corresponding to the ground-state wave function for the simple harmonic potential.
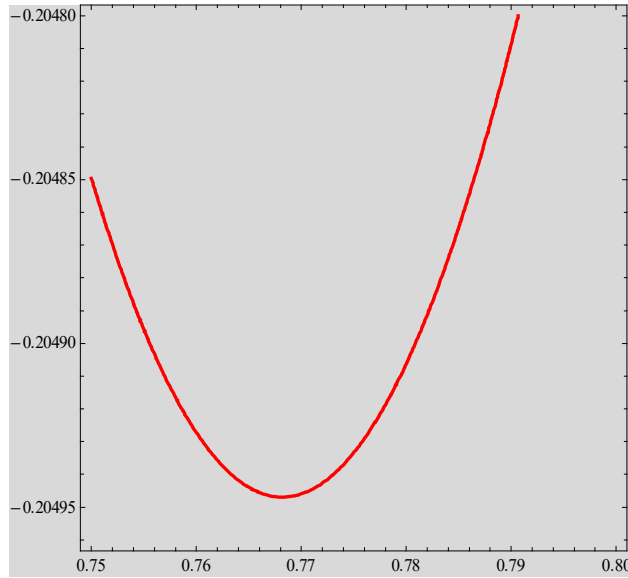
Fig.36   Detail of Fig.38. The plot of ν as a function of ζ_0 around $\zeta_0 = 0.768$ ($\nu = \nu$ = -0.20494), satisfying the condition $D_\nu{}'(z = -\sqrt{2}\zeta_0) = 0$.
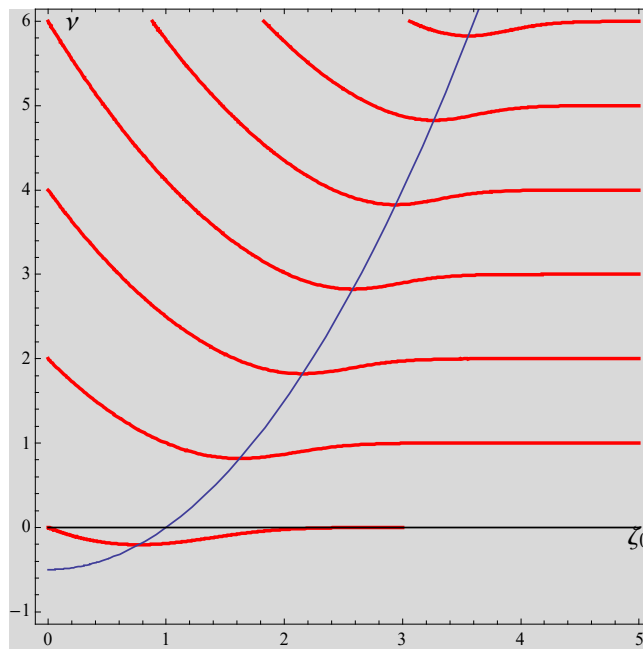
Fig.37   Plot of ν as a function of ζ_0, satisfying the condition. $D_\nu{}'(z = -\sqrt{2}\zeta_0) = 0$. $\nu = 2$ and $\nu = 4$ at $\zeta_0 = 0$, corresponding to ν for the wave function ($n = 2$ and $n = 4$). The solid line denotes the curve given by . $\nu = (\zeta_0{}^2 - 1)/2$.
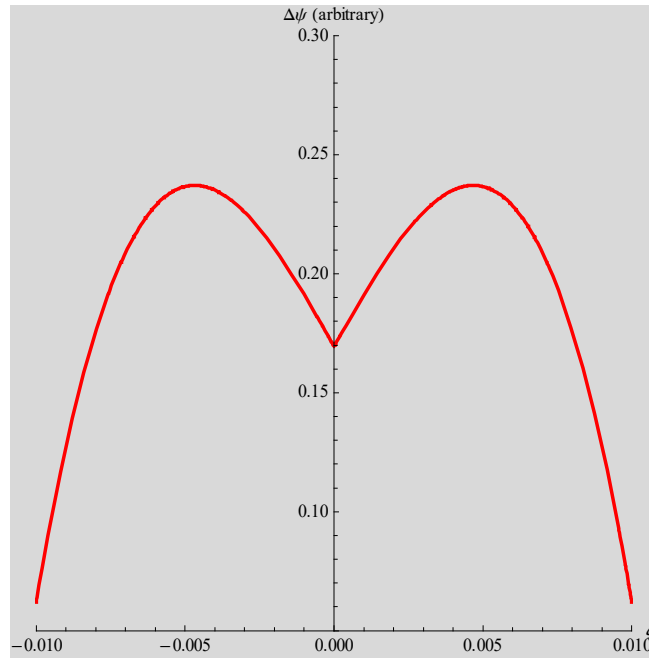
Fig.38 The plot of the ground state wave function with $\varsigma_0 = 0.768$ and $v = -0.20494$.
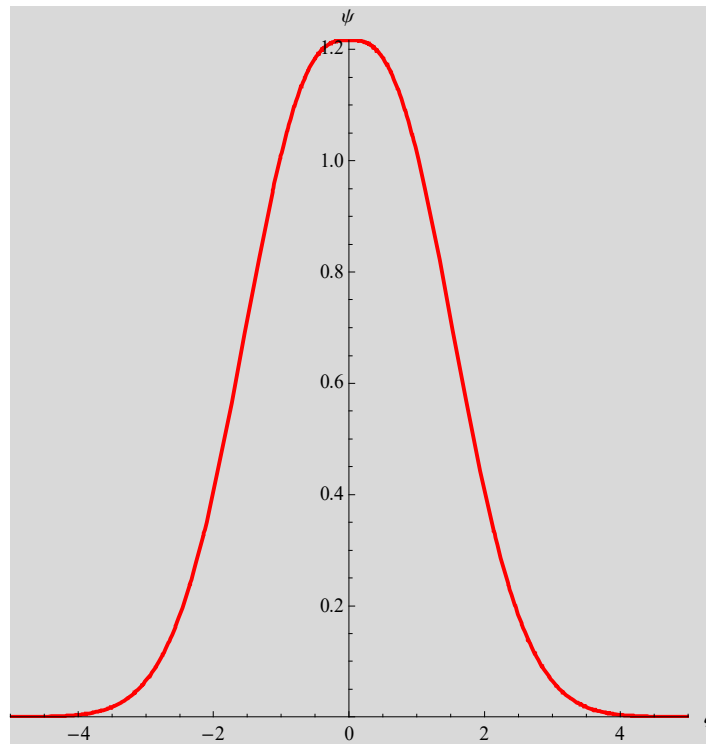
Fig.39 Overview of Fig.41 for the same wave function with $\varsigma_0 = 0.768$, $v = -0.20494$

It is clear by inspection that this new surface eigenfunction must have a lower eigenfunction that the interior ones, because it arises from a potential curve that is lower and broader than the simple parabora about $\varsigma = \varsigma_0$.

The exact solution shows that this eigenvalue is lower by a factor of 0.59.

$$H_{c3} = \frac{H_{c2}}{0.59} = 1.695 H_{c2} = 1.695(\sqrt{2}\kappa H_c). \tag{13.12}$$

$$E = \frac{\hbar^2}{2m^*\xi^2} = \hbar\omega^*(v + \frac{1}{2}) = \frac{|q^*|\hbar H}{m^*c}(v + \frac{1}{2}). \tag{13.13}$$

When $v = n = 0$,

$$\frac{\hbar^2}{2m^*\xi^2} = \frac{|q^*|\hbar H_{c2}}{2m^*c} \tag{13.14}$$

or

$$H_{c2} = \frac{c\hbar}{|q^*|\xi^2} = \frac{\Phi_0}{2\pi\xi^2} \tag{13.15}$$

where

$$\Phi_0 = \frac{2\pi c\hbar}{|q^*|}.$$

((**Surface superconductivity**))
When $v = -0.20494$, $\zeta_0 = 0.768$,

$$\frac{\hbar^2}{2m^*\xi^2} = \frac{|q^*|\hbar H}{m^*c}(-0.20494 + 0.5) = \frac{|q^*|\hbar H_{c3}}{m^*c}0.29506 \tag{13.16}$$

or

$$\frac{c\hbar}{|q^*|\xi^2} = (0.29506 \times 2)H_{c3} = 0.5901 H_{c3} \tag{13.17}$$

or

$$\frac{c\hbar}{|q^*|\xi^2} = H_{c2} = 0.5901 H_{c3}, \tag{13.18}$$

or

$$H_{c3} = 1.695 H_{c2}.$$ (13.19)

13.3 Variational method by Kittel

We assume a trial function given by

$$\psi(x,y) = e^{ik_y y} \exp(-ax^2),$$ (13.20)

The parameter a and k_y are determined variationally so as to minimize the Gibbs free energy per unit area

$$g_s = f_n + \alpha|\psi|^2 + + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2,$$ (13.21)

or

$$g_s \approx f_n + \frac{\hbar^2}{2m^*}[-\frac{1}{\xi^2}|\psi|^2 + \left|(\frac{1}{i}\nabla + \frac{2\pi}{\Phi_0}\mathbf{A})\psi\right|^2],$$ (13.22)

with $\mathbf{A} = (0, Hx, 0)$.

$$x_0 = -\frac{k_y \Phi_0}{2\pi H}.$$ (13.23)

We now calculate

$$\Delta G = \frac{\hbar^2}{2m^*}\int_0^\infty [-\frac{1}{\xi^2}|\psi|^2 + \left|(\frac{1}{i}\nabla + \frac{2\pi}{\Phi_0}\mathbf{A})\psi\right|^2]dx.$$ (13.24)

After the calculation of ΔG, we put

$$a = \frac{1}{2\xi^2}.$$ (13.25)

$$\Delta G = -\frac{\sqrt{\pi}}{4\xi\Phi_0^2}\{-4H^2[-4\pi^{3/2}x_0\xi^3 + \pi^2(2x_0^2\xi^2 + \xi^4)] + \Phi_0^2\}$$

We take a derivative of ΔG with respect to x_0.

$$\frac{\partial \Delta G}{\partial x_0} = \frac{4H^2 \pi^2 (\sqrt{\pi} x_0 - \xi)\xi}{\Phi_0{}^2}$$

We have

$$x_0 = \frac{\xi}{\sqrt{\pi}}, \tag{13.26}$$

$$H_{c3} = \frac{\Phi_0}{2\pi\xi^2}\sqrt{\frac{\pi}{\pi-2}} = 1.6589 H_{c2}. \tag{13.27}$$

((**Mathematica Program-21**))

Kittel's method for the surface superconductivity: Variational method

```
Clear["Global`*"];
conjugateRule = {Complex[re_, im_] :> Complex[re, -im]};
Unprotect[SuperStar]; SuperStar /: exp_* := exp /. conjugateRule;
Protect[SuperStar];
```

$$\pi x := \left(\frac{1}{\dot{\mathbb{i}}} \, D[\#, x] \, \& \right); \quad \pi y := \left(\frac{1}{\dot{\mathbb{i}}} \, D[\#, y] + \frac{2\pi}{\Phi 0} \, H \, x \, \# \, \& \right);$$

```
ψ[x_, y_] := Exp[-a x²] Exp[i ky y]; K1 = πx[ψ[x, y]] // Simplify;
K2 = πy[ψ[x, y]] // Simplify;
```

$$K3 = -\frac{1}{\xi^2} \, \psi[x, y]^* \, \psi[x, y] + \left(K1 \, K1^* + K2 \, K2^*\right) \; /. \; \left\{ky \to \frac{-2\pi H x0}{\Phi 0}\right\} \; //$$

  Simplify;

$$G1 = \int_0^\infty K3 \, dx; \; G2 = Simplify[G1, a > 0]; \; G3 = G2 \; /. \; a \to \frac{1}{2\,\xi^2} \; // \; Simplify;$$

```
G4 = Simplify[G3, ξ > 0]; G5 = D[G4, x0] // Simplify;
```

$$G6 = G4 \; /. \; x0 \to \frac{\xi}{\sqrt{\pi}} \; // \; Simplify$$

$$\frac{\sqrt{\pi} \, \left(4 \, H^2 \, (-2 + \pi) \, \pi \, \xi^4 - \Phi 0^2\right)}{4 \, \xi \, \Phi 0^2}$$

```
Solve[G6 == 0, H] // Simplify
```

$$\left\{ \left\{ H \to -\frac{\Phi 0}{2\,\sqrt{(-2 + \pi)\,\pi}\,\xi^2} \right\}, \; \left\{ H \to \frac{\Phi 0}{2\,\sqrt{(-2 + \pi)\,\pi}\,\xi^2} \right\} \right\}$$

$$\sqrt{\frac{\pi}{\pi - 2}} \; // \; N$$

```
1.6589
```

14.   Abrikosov vortex states[14]
14.1   Theory

(1)   All ψ_k are orthogonal because of the different e^{iky} factor.

(2)   Each of these solutions is equally valid exactly at $H = H_{c2}$ and all give the same H_{c2}.

(3) $\beta|\psi|^4$ term is not longer negligible. We expect a crystalline array of vortices to have lower energy than a random one.

$$\psi_L(x,y) = \sum_{k_y} C_{k_y} e^{ik_y y} \exp\left[-\frac{(\varsigma - \varsigma_{k_y})^2}{2}\right] = \sum_{k_y} C_{k_y} e^{ik_y y} \exp\left[-\frac{(x - x_{k_y})^2}{2l_H^2}\right],$$

$$(14.1)$$

with

$$x = \frac{\varsigma}{\beta} = l_H \varsigma, \qquad l_H^2 = \frac{c\hbar}{|q^*|H} = \frac{\Phi_0}{2\pi H} = \xi^2, \tag{14.2}$$

$$x_{k_y} = l_H \varsigma_{k_y} \qquad \varsigma_{k_y} = l_H k_y, \qquad x_{k_y} = l_H^2 k_y = l_H^2 kn = l_H^2 \frac{2\pi n}{b}, \tag{14.3}$$

$$k_y = \frac{2\pi}{b} n = kn, \tag{14.4}$$

where

$$k = \frac{2\pi}{b}, \tag{14.5}$$

$$\psi_L(x,y) = \sum_{k_y} C_{k_y} e^{ikny} \exp\left[-\frac{(x - l_H^2 kn)^2}{2l_H^2}\right], \tag{14.6}$$

$$\psi_L(x,y) = \sum_n C_n \exp\left[-\frac{(x - l_H^2 nk)^2}{2l_H^2} + inky\right], \tag{14.7}$$

Here we assume that C_n is independent of n. We define the period in the x direction.

$$a = l_H^2 k = \frac{2\pi}{b} l_H^2, \tag{14.9}$$

$$\psi_L(x,y) = C\sum_n \exp\left[-\frac{(x - na)^2}{2l_H^2} + inky\right], \tag{14.10}$$

$$\psi_L(x+a,y) = C\sum_n \exp\left[-\frac{(x + a - na)^2}{2l_H^2} + inky\right], \tag{14.11}$$

$$\psi_L(x+a,y) = Ce^{iky} \sum_n \exp\left[-\frac{1}{2l_H^2}(x - (n-1)a)^2 + i(n-1)ky\right] = e^{iky}\psi_L(x,y),$$

(14.12)

$$\psi_L(x,y) = C\sum_n \exp[-\frac{(x-na)^2}{2l_H^2} + \frac{2\pi i n y}{b}].$$

(14.13)

((**Mathematica Program 22**)) Square vortex lattice

```
Clear["Global`*"]; rule1 = {a → 1, b → 1, LH → 0.40};

f[m_] := Sum[Exp[-(x - n a)^2/(2 LH^2) + (2 π i n y)/b], {n, -m, m}] /.
  rule1

ContourPlot[
 Evaluate[Table[Abs[f[200]]^2 == α, {α, 0, 1, 0.05}]],
 {x, -1.5, 1.5}, {y, -1.5, 1.5}, AspectRatio → 1,
 ContourStyle → Table[{Hue[0.05 i], Thick}, {i, 0, 20}],
 PlotPoints → 100]
```
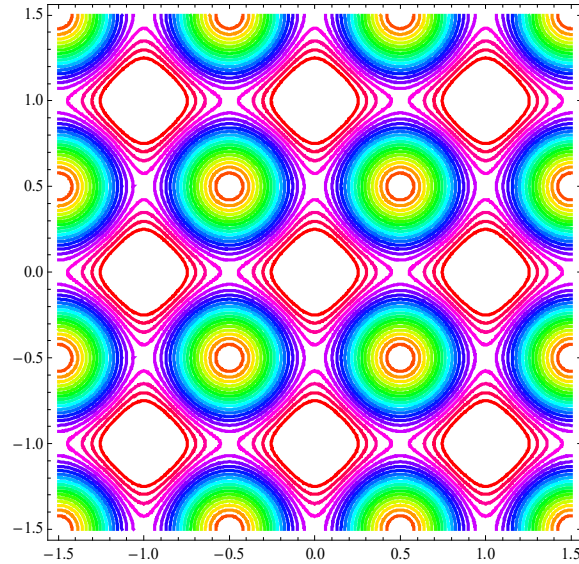

Fig.40   The spatial configuration of $|\psi|^2$ near $H = H_{c2}$ for a square vortex lattice. a = 1. b = 1. l_H = 0.4

This is a general solution to the linearized GL equation at $H = H_{c2}$, periodic by construction. The form of condition chosen by Abrikosov is given by (see also the paper by Kleiner et al.)[17]

116

$$\psi_L(x,y) = \sum_n C_n \exp[-\frac{(x - x_n)^2}{2l_H^2} + inky] \tag{14.14}$$

$$C_{n+\nu} = C_n \ (\nu = \text{integer}), \tag{14.15}$$

$$\psi_L(x + x_\nu, y) = \sum_n C_n \exp[-\frac{(x + x_\nu - x_n)^2}{2l_H^2} + inky], \tag{14.16}$$

or

$$\psi_L(x + x_\nu, y) = e^{i\nu ky} \sum_{n-\nu} C_{n-\nu} e^{i(n-\nu)ky} \exp[-\frac{(x - x_{n-\nu})^2}{2l_H^2}] = e^{i\nu ky}\psi_L(x,y),$$

$$\tag{14.17}$$
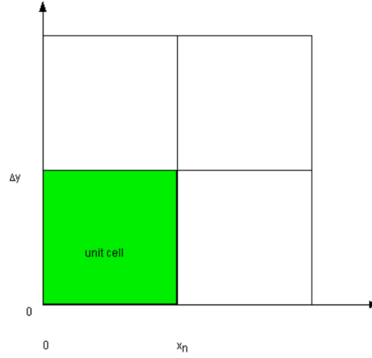
Fig.41   Rectangular vortex lattice

Each unit cell of the periodic array carriers ν quanta of flux.

$$Hx_\nu b = \nu \frac{k\Phi_0}{2\pi} \cdot \frac{2\pi}{k} = \nu\Phi_0 \tag{4.18}$$

$$\ell_H{}^2 = \frac{\Phi_0}{2\pi H} \tag{4.19}$$

$$\frac{\ell_H{}^2}{\lambda^2} = \frac{\Phi_0}{2\pi\lambda^2 H} \tag{4.20}$$

$$\frac{1}{\xi^2} = \frac{1}{\ell_H{}^2} \tag{4.21}$$

Triangular lattice:

$$C_{n+2} = C_n, \ C_1 = iC_0 = iC \tag{4.22}$$

$$\psi_\Delta(x,y) = C[\sum_{n=even} \exp[-\frac{1}{2l_H^{\ 2}}(x - l_H^{\ 2}nk)^2 + inky]$$

$$+ i\sum_{n=odd} \exp[-\frac{1}{2l_H^{\ 2}}(x - l_H^{\ 2}nk)^2 + inky]$$

(4.23)

where $k = 2\pi/b$. We define the period in the x direction

$$a = 2l_H^{\ 2}k = \frac{4\pi}{b}l_H^{\ 2}$$

(14.25)

We note that $b = \sqrt{3}a$ for a triangular lattice.

Fig.42   Triangular vortex lattice, where red circles denote the centers of vortices.

$$\psi_\Delta(x,y) = C[\sum_{n=even} \exp[-\frac{1}{2l_H^{\ 2}}(x - l_H^{\ 2}nk)^2 + inky] +$$

$$i\sum_{n=even} \exp[-\frac{1}{2l_H^{\ 2}}(x - l_H^{\ 2}(n+1)k)^2 + i(n+1)ky]$$

(14.26)

or

$$\psi_\Delta(x,y) = C\sum_{n=even}\{\exp[-\frac{1}{2l_H{}^2}(x - l_H{}^2 nk)^2 + inky]$$

$$+ i\exp[-\frac{1}{2l_H{}^2}(x - l_H{}^2(n+1)k)^2 + i(n+1)ky]\}$$

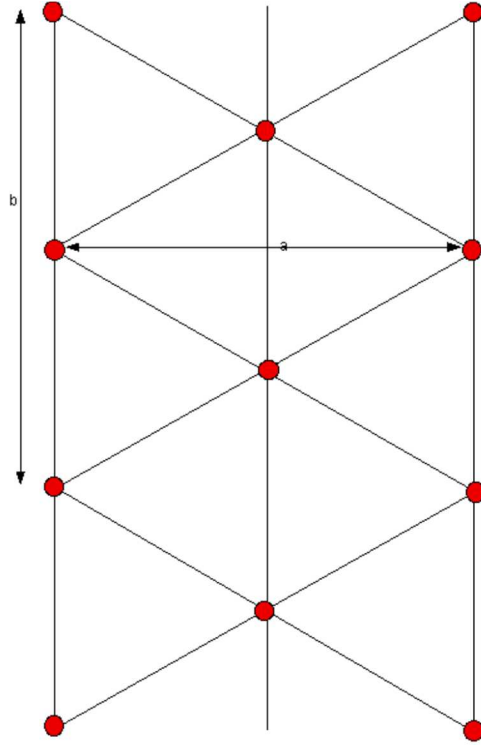$$\psi_\Delta(x,y) = C\sum_{n=even}\{\exp[-\frac{1}{2l_H{}^2}(x - \frac{na}{2})^2 + inky]$$

$$+ i\exp[-\frac{1}{2l_H{}^2}(x - \frac{(n+1)a}{2})^2 + i(n+1)ky]\}$$

$$\psi_\Delta(x,y) = C\sum_{n}\{\exp[-\frac{1}{2l_H{}^2}(x - \frac{2na}{2})^2 + i2nky]$$

$$+ i\exp[-\frac{1}{2l_H{}^2}(x - \frac{(2n+1)a}{2})^2 + i(2n+1)ky]\}$$

$$\psi_\Delta(x,y) = C\sum_{n}\exp(i2nky)\{\exp[-\frac{1}{2l_H{}^2}(x - na)^2]$$

$$+ i\exp(iky)\exp[-\frac{1}{2l_H{}^2}(x - (n+1/2)a)^2]\}$$

(14.27)

or

$$\psi_\Delta(x,y) = C\sum_{n}\exp(\frac{4\pi iny}{b})\{\exp[-\frac{1}{2l_H{}^2}(x - na)^2]$$

$$+ i\exp(\frac{2\pi iy}{b})\exp[-\frac{1}{2l_H{}^2}(x - (n+1/2)a)^2]\}$$

(14.28)

((**Mathematica Program 23**)) Triangular vortex lattice

```
Clear["Global`*"]; rule1 = {a → 1, b → √3 , LH → 0.31};
f[m_] :=
  Sum[Exp[4 π i n y / b] (Exp[-(x - n a)^2 / (2 LH^2)] + i Exp[2 π i y / b] Exp[-(x - (n + 1/2) a)^2 / (2 LH^2)]),
    {n, -m, m}] /. rule1

ContourPlot[Evaluate[Table[Abs[f[200]]^2 == α, {α, 0, 2, 0.1}]], {x, -1.5, 1.5},
  {y, -1.5, 1.5}, AspectRatio → 1,
  ContourStyle → Table[{Hue[0.05 i], Thick}, {i, 0, 20}], PlotPoints → 100]
```
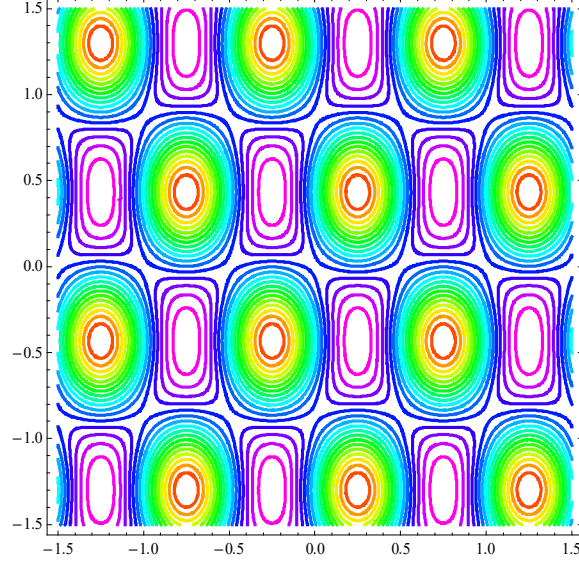
Fig.43 The spatial configuration of $|\psi|^2$ near $H = H_{c2}$ for a triangular vortex . $a = 1$, $b = \sqrt{3}$, and $l_H = 0.31$.

14.2 Structure of the vortex phase near $H = H_{c2}$

We start with the following equation. This equation was derived previously.

$$\int d\mathbf{r}[\alpha|\psi_L|^2 + \beta|\psi_L|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi_L\right|^2] = 0, \qquad (14.29)$$

where ψ_L is a linear combination of the solutions of the linearized GL equation

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1.$$

A_0 is the vector potential which would exist in the presence of the field H_{c2}. Ginzburg-Landau equation for ψ_L is given by

$$\alpha\psi_L + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0\right)^2\psi_L = 0, \qquad (14.30)$$

and

$$\mathbf{J}_L = \frac{q^*\hbar}{2m^*i}[\psi_L{}^*\nabla\psi_L - \psi_L\nabla\psi_L{}^*] - \frac{q^{*2}|\psi_L|^2}{m^*c}\mathbf{A}_0, \qquad (14.31)$$

or

$$\mathbf{J}_L = \frac{q^*\hbar}{2m^*i}[\psi_L^*(\nabla\psi_L - \frac{iq^*}{\hbar c}\mathbf{A}_0\psi_L) - \psi_L(\nabla\psi_L^* + \frac{iq^*}{\hbar c}\mathbf{A}_0\psi_L^*)], \qquad (14.32)$$

or

$$\mathbf{J}_L = \frac{q^*}{2m^*}[\psi_L^*(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0)\psi_L + \psi_L(-\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0)\psi_L^*]. \qquad (14.33)$$

J_L is the current associated with the unperturbed solution. Denoting quantities of the form $\frac{1}{V}\int\ldots$ by a bar (V is a volume),

$$\overline{\alpha|\psi_L|^2} + \overline{\beta|\psi_L|^4} + \frac{1}{2m^*}\overline{\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi_L\right|^2} = 0. \qquad (14.34)$$

Expanding Eq.(14.34) to the first order in A_1, we obtain

$$\overline{\alpha|\psi_L|^2} + \overline{\beta|\psi_L|^4} + \frac{1}{2m^*}\overline{\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0)\psi_L\right|^2} - \frac{1}{c}\overline{\mathbf{A}_1\cdot\mathbf{J}_L} = 0. \qquad (14.35)$$

((**Note**))

$$\frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}(\mathbf{A}_0 + \mathbf{A}_1))\psi_L\right|^2 - \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0)\psi_L\right|^2 + \frac{1}{c}\mathbf{A}_1\cdot\mathbf{J}_L = \frac{q^{*2}}{2m^*}\mathbf{A}_1^2\psi_L^*\psi_L$$

.

$$(14.35')$$

If the right term is negligibly small, we get

$$\frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}(\mathbf{A}_0 + \mathbf{A}_1))\psi_L\right|^2 - \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0)\psi_L\right|^2 = -\frac{1}{c}\mathbf{A}_1\cdot\mathbf{J}_L.$$

((**Mathematica Program 24**))

```
Clear["Global`*"]; << "VariationalMethods`";

Needs["VectorAnalysis`"];

A0 = {A01[x, y, z], A02[x, y, z], A03[x, y, z]};
A1 = {A11[x, y, z], A12[x, y, z], A13[x, y, z]};
eq1 =
```

$$\frac{1}{2\,m} \left(\frac{\hbar}{i}\ \text{Grad}[\psi[x,\,y,\,z]] - \frac{q}{c}\ (A0 + A1)\ \psi[x,\,y,\,z]\right) .$$

$$\left(-\frac{\hbar}{i}\ \text{Grad}[\psi c[x,\,y,\,z]] - \frac{q}{c}\ (A0 + A1)\ \psi c[x,\,y,\,z]\right)\ \texttt{// Expand;}$$

```
eq2 =
```

$$\frac{1}{2\,m} \left(\frac{\hbar}{i}\ \text{Grad}[\psi[x,\,y,\,z]] - \frac{q}{c}\ A0\ \psi[x,\,y,\,z]\right) .$$

$$\left(-\frac{\hbar}{i}\ \text{Grad}[\psi c[x,\,y,\,z]] - \frac{q}{c}\ A0\ \psi c[x,\,y,\,z]\right)\ \texttt{// Expand;}$$

```
eq12 = eq1 - eq2 // Simplify;
```

Current density

```
Jeq1 =
```

$$\left(-\frac{q^2}{m\,c}\ \psi[x,\,y,\,z]\ \psi c[x,\,y,\,z]\ A0\ +\right.$$

$$\frac{q\,\hbar}{i\,2\,m}\ (\psi c[x,\,y,\,z]\ \text{Grad}[\psi[x,\,y,\,z]] -$$

$$\left.\psi[x,\,y,\,z]\ \text{Grad}[\psi c[x,\,y,\,z]])\right)\ \texttt{// Simplify;}$$

$$\texttt{seq2} = -\frac{1}{c}\ \texttt{Jeq1.A1 // Simplify;}$$

```
eq12 - seq2 // Simplify
```

$$\frac{q^2 \left(A11[x,\,y,\,z]^2 + A12[x,\,y,\,z]^2 + A13[x,\,y,\,z]^2\right)\ \psi[x,\,y,\,z]\ \psi c[x,\,y,\,z]}{2\,c^2\,m}$$

((Note))
If one multiplies the linearized GL equation by $\psi_L{}^*$ and integrates over all the volume by parts, one obtain the identity

$$\int[\alpha|\psi_L|^2 + \frac{1}{2m^*}\psi_L^*\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0\right)^2\psi_L]d\mathbf{r} = 0 \tag{14.36}$$

or

$$\int[\alpha|\psi_L|^2 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0\right)\psi_L\right|^2]d\mathbf{r} = 0 \tag{14.37}$$

or

$$\overline{\alpha|\psi_L|^2} + \frac{1}{2m^*}\overline{\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}_0)\psi_L\right|^2} = 0. \tag{14.38}$$

Then we obtain

$$\overline{\beta|\psi_L|^4} - \frac{1}{c}\overline{\mathbf{A}_1\cdot\mathbf{J}_L} = 0, \tag{14.39}$$

$$-\frac{1}{c}\mathbf{A}_1\cdot\mathbf{J}_L = -\frac{q^*\hbar}{2m^*ci}[\psi_L^*(\nabla\psi_L - \frac{iq^*}{\hbar c}\mathbf{A}_0) - \psi_L(\nabla\psi_L^* + \frac{iq^*}{\hbar c}\mathbf{A}_0)]. \tag{14.40}$$

Note that

$$\nabla\times\mathbf{B}^{(s)} = \frac{4\pi}{c}\mathbf{J}_L, \qquad \mathbf{B}^{(1)} = \nabla\times\mathbf{A}_1, \tag{14.41}$$

$$-\frac{1}{c}\mathbf{A}_1\cdot\mathbf{J}_L = -\frac{1}{4\pi}\mathbf{A}_1\cdot\nabla\times\mathbf{B}^{(s)} = -\frac{1}{4\pi}[\mathbf{B}^{(s)}\cdot(\nabla\times\mathbf{A}_1) - \nabla\cdot(\mathbf{A}_1\times\mathbf{B}^{(s)})], \tag{14.42}$$

or

$$-\frac{1}{c}\mathbf{A}_1\cdot\mathbf{J}_L = -\frac{1}{4\pi}\mathbf{B}^{(s)}\cdot\mathbf{B}^{(1)} + \frac{1}{4\pi}\nabla\cdot(\mathbf{A}_1\times\mathbf{B}^{(s)})]. \tag{14.43}$$

We use the formula of vector analysis

$$\nabla\cdot(\mathbf{A}_1\times\mathbf{B}^{(s)}) = \mathbf{B}^{(s)}\cdot(\nabla\times\mathbf{A}_1)\cdot -\mathbf{A}_1\cdot\nabla\times\mathbf{B}^{(s)} \tag{14.44}$$

We note that

$$\int \nabla \cdot (\mathbf{A}_1 \times \mathbf{B}^{(s)}) \cdot d\mathbf{r} = \int (\mathbf{A}_1 \times \mathbf{B}^{(s)}) \cdot d\mathbf{a} \qquad \text{(Stokes theorem)}.$$

This value becomes zero on the boundary surface of the system. Then we have

$$\overline{\beta |\psi_L|^4} - \frac{1}{4\pi} \overline{\mathbf{B}^{(s)} \cdot \mathbf{B}^{(1)}} = 0. \tag{14.45}$$

What are the expressions for $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(s)}$?

$$\mathbf{B}^{(1)} = \mathbf{H} - \mathbf{H}_{c2} + \mathbf{B}^{(s)}, \tag{14.46}$$

where $\mathbf{B}^{(s)}$ gives the effect of the supercurrent.
We introduce the canonical operator,

$$\hat{\mathbf{\Pi}} = \mathbf{p} - \frac{q^*}{c} \mathbf{A}_0, \tag{14.47}$$

which has the commutation relation

$$[\hat{\Pi}_x, \hat{\Pi}_y] = \frac{iq^* \hbar}{c} H. \tag{14.48}$$

We introduce the raising (creation) and lowering (destruction) operators.

$$\hat{\Pi}_+ = \hat{\Pi}_x + i\hat{\Pi}_y, \qquad \hat{\Pi}_- = \hat{\Pi}_x - i\hat{\Pi}_y, \tag{14.49}$$

$$[\hat{\Pi}_-, \hat{\Pi}_+] = 2i[\hat{\Pi}_x, \hat{\Pi}_y] = -2\frac{q^* \hbar}{c} B_z. \tag{14.50}$$

The linearized GL equation can be described by

$$\hat{\Pi}_+ \hat{\Pi}_- \psi_L = 0. \tag{14.51}$$

((Note))

$$\alpha \psi_L + \frac{1}{2m^*} \left(\frac{\hbar}{i} \nabla - \frac{q^*}{c} \mathbf{A}_0 \right)^2 \psi_L = 0, \tag{14.52}$$

$$\alpha \psi_L + \frac{1}{2m^*} (\hat{\Pi}_x^{\;2} + \hat{\Pi}_y^{\;2}) \psi_L = 0. \tag{14.53}$$

Since

$$\hat{\Pi}_x^2 + \hat{\Pi}_y^2 = \hat{\Pi}_-\hat{\Pi}_+ + \frac{\hbar q^*}{c}B_z, \qquad (14.54)$$

$$\hat{\Pi}_x^2 + \hat{\Pi}_y^2 = \hat{\Pi}_+\hat{\Pi}_- - \frac{\hbar q^*}{c}B_z, \qquad (14.55)$$

we have

$$\frac{1}{2m^*}\hat{\Pi}_+\hat{\Pi}_-\psi_L + \frac{1}{2m^*}(-\frac{\hbar q^*}{c}B_z + 2m^*\alpha)\psi_L = 0, \qquad (14.56)$$

with

$$-\frac{\hbar q^*}{2m^*c}B_z + \alpha = \frac{\hbar|q^*|}{2m^*c}(B_z - \frac{2m^*c|\alpha|}{\hbar|q^*|}). \qquad (14.57)$$

We also note that

$$\frac{2m^*c|\alpha|}{\hbar|q^*|} = \frac{c\hbar}{|q^*|\xi^2} = \frac{\Phi_0}{2\pi\xi^2} = H_{c2}, \qquad (14.58)$$

$$\frac{1}{2m^*}\hat{\Pi}_+\hat{\Pi}_-\psi_L + \frac{\hbar|q^*|}{2m^*c}(B_z - H_{c2})\psi_L = 0. \qquad (14.59)$$

When $B_z - H_{c2} = 0$,

$$\hat{\Pi}_+\hat{\Pi}_-\psi_L = 0. \qquad (14.60)$$

The ground state has the property
$$\Pi_-\psi_L = 0. \qquad (14.61)$$

or

$$\Pi_-\psi_L = (\frac{\hbar}{i}\frac{\partial\psi_L}{\partial x} - \frac{q^*}{c}A_0{}^x\psi_L) - i(\frac{\hbar}{i}\frac{\partial\psi_L}{\partial y} - \frac{q^*}{c}A_0{}^y\psi_L) = 0, \qquad (14.62)$$

or

$$-i\hbar\frac{\partial\psi_L}{\partial x} - \frac{q^*}{c}A_0{}^x\psi_L - \hbar\frac{\partial\psi_L}{\partial y} + i\frac{q^*}{c}A_0{}^y\psi_L = 0. \qquad (14.63)$$

The substitution of $\psi_L = |\psi_L| e^{i\theta}$ into the above equation leads to

$$-\frac{q^*}{c} A_0^x |\psi_L| + \frac{iq^*}{c} A_0^y |\psi_L| - i\hbar |\psi_L| \frac{\partial\theta}{\partial y} - \hbar \frac{\partial |\psi_L|}{\partial y} + \hbar |\psi_L| \frac{\partial\theta}{\partial x} - i\hbar \frac{\partial |\psi_L|}{\partial x} = 0 .$$

(14.64)

The real part of Eq.(14.64) is

$$-\frac{q^*}{c} A_0^x |\psi_L| - \hbar \frac{\partial |\psi_L|}{\partial y} + \hbar |\psi_L| \frac{\partial\theta}{\partial x} = 0 ,$$

(14.65)

or

$$|\psi_L| \left(\frac{\partial\theta}{\partial x} - \frac{q^*}{c\hbar} A_0^x \right) = \frac{\partial |\psi_L|}{\partial y} .$$

(14.66)

The imaginary part of Eq.(14.64) is

$$\frac{q^*}{c} A_0^y |\psi_L| - \hbar \frac{\partial |\psi_L|}{\partial x} - \hbar |\psi_L| \frac{\partial\theta}{\partial y} = 0 ,$$

(14.67)

or

$$|\psi_L| \left(\frac{\partial\theta}{\partial y} - \frac{q^*}{c\hbar} A_0^y \right) = -\frac{\partial |\psi_L|}{\partial x} .$$

(14.68)

Then the current density is given by

$$J_{Lx} = \frac{q^* \hbar}{m^*} |\psi_L|^2 \left(\frac{\partial\theta}{\partial x} - \frac{q^*}{c\hbar} A_0^x \right) = \frac{q^* \hbar}{m^*} |\psi_L| \frac{\partial |\psi_L|}{\partial y} = \frac{q^* \hbar}{2m^*} \frac{\partial |\psi_L|^2}{\partial y} ,$$

(14.69)

$$J_{Ly} = \frac{q^* \hbar}{m^*} |\psi_L|^2 \left(\frac{\partial\theta}{\partial y} - \frac{q^*}{c\hbar} A_0^y \right) = -\frac{q^* \hbar}{m^*} |\psi_L| \frac{\partial |\psi_L|}{\partial x} = -\frac{q^* \hbar}{2m^*} \frac{\partial |\psi_L|^2}{\partial x} .$$

(14.70)

We see that the current lines are the lines of constant ($|\psi_L|^2$=const). We note that $\nabla |\psi_L|^2$ is perpendicular to the lines of constant ($|\psi_L|^2$=const).

$$\nabla |\psi_L|^2 = \mathbf{e}_x \frac{\partial}{\partial x} |\psi_L|^2 + \mathbf{e}_y \frac{\partial}{\partial y} |\psi_L|^2 = -\frac{2m^*}{\hbar q\hbar} (J_{Ly}\mathbf{e}_x - L_{Lx}\mathbf{e}_y)$$

(14.71)

and

$$\mathbf{J}_L \cdot \nabla |\psi_L|^2 = -\frac{2m^*}{\hbar q \hbar}(J_{Lx}\mathbf{e}_x + L_{Ly}\mathbf{e}_y) \cdot (J_{Ly}\mathbf{e}_x - L_{Lx}\mathbf{e}_y) = -\frac{2m^*}{\hbar q \hbar}(J_{Lx}J_{Ly} - L_{Lx}L_{Ly}) = 0.$$

(14.72)

Since

$$\nabla \times \mathbf{B}_s = \frac{4\pi}{c}\mathbf{J}_L = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & B_s \end{vmatrix} = \left(\frac{\partial B_s}{\partial y}, \quad -\frac{\partial B_s}{\partial x}, \quad 0 \right),$$

(14.73)

we have

$$\frac{q^*\hbar}{2m^*}\frac{\partial |\psi_L|^2}{\partial y}\frac{4\pi}{c} = \frac{\partial B_s}{\partial y}$$

$$-\frac{q^*\hbar}{2m^*}\frac{\partial |\psi_L|^2}{\partial x}\frac{4\pi}{c} = -\frac{\partial B_s}{\partial x}$$

(14.74)

or

$$B_s = \frac{2\pi q^*\hbar}{m^*c}|\psi_L|^2$$

(14.75)

or

$$B_s = -\frac{2\pi |q^*|\hbar}{m^*c}|\psi_L|^2 = -\frac{H_{c2}}{2\kappa^2}|f_L|^2$$

(14.76)

where

$$f_L = \frac{\psi_L}{\psi_\infty}$$

(14.77)

$$\frac{2m^*c|\alpha|}{\hbar|q^*|} = H_{c2}. \quad \text{and} \quad \frac{H_{c2}}{2\kappa^2} = -\frac{2\pi|q^*|\hbar|\alpha|}{m^*c\beta},$$

(14.78)

$$B_1 = H - H_{c2} + B_s,$$

(14.79)

$$\beta \overline{|\psi_L|^4} - \frac{1}{4\pi}\overline{B_s \cdot [H - H_{c2} + B_s]} = 0,$$

(14.80)

$$\beta \psi_\infty^4 \overline{|f_L|^4} - \frac{1}{4\pi}\overline{(H - H_{c2})B_s + B_s^2} = 0.$$

(14.81)

Since

$$B_s = -\frac{H_{c2}}{2\kappa^2}|f_L|^2, \tag{14.82}$$

we have

$$\beta\psi_\infty^4\overline{|f_L|^4} - \frac{1}{4\pi}[-(H - H_{c2})\frac{H_{c2}}{2\kappa^2}\overline{|f_L|^2} + \frac{H_{c2}^2}{4\kappa^4}\overline{|f_L|^4}] = 0, \tag{14.83}$$

or

$$\frac{\beta}{H_{c2}^2}\psi_\infty^4\overline{|f_L|^4} + \frac{1}{4\pi}(\frac{H}{H_{c2}} - 1)\frac{1}{2\kappa^2}\overline{|f_L|^2} - \frac{1}{4\pi}\frac{1}{4\kappa^4}\overline{|f_L|^4}] = 0, \tag{14.84}$$

or

$$(8\pi\kappa^2\frac{\beta}{H_{c2}^2}\psi_\infty^4 - \frac{1}{2\kappa^2})\overline{|f_L|^4} - (1 - \frac{H}{H_{c2}})\overline{|f_L|^2} = 0. \tag{14.85}$$

Noting that

$$8\pi\kappa^2\beta\psi_\infty^4 = 8\pi\kappa^2\beta\frac{|\alpha|^2}{\beta^2} = 4\pi\frac{|\alpha|^2}{\beta}2\kappa^2 = H_c^2 2\kappa^2, \quad \text{and} \quad \frac{H_c}{H_{c2}} = \frac{1}{\sqrt{2}\kappa}, \tag{14.86}$$

we get the final form

$$(1 - \frac{1}{2\kappa^2})\overline{|f_L|^4} - (1 - \frac{H}{H_{c2}})\overline{|f_L|^2} = 0. \tag{14.87}$$

The quantities $\overline{|f_L|^2}$ and $\overline{|f_L|^4}$ may be calculated, depending on the vortex lattice symmetry and the lattice spacing. It turns out that near H_{c2}, the ratio

$$\beta_A = \frac{\overline{|f_L|^4}}{\left(\overline{|f_L|^2}\right)^2} \tag{14.88}$$

is independent of H. Then the above equation can be rewritten by

$$\overline{|f_L|^2} = \frac{1 - \dfrac{H}{H_{c2}}}{\beta_A(1 - \dfrac{1}{2\kappa^2})} \qquad (14.89)$$

and

$$\overline{|f_L|^4} = \beta_A\left(\overline{|f_L|^2}\right)^2 = \frac{\left(1 - \dfrac{H}{H_{c2}}\right)^2}{\beta_A(1 - \dfrac{1}{2\kappa^2})^2} \qquad (14.90)$$

((**Note**)) Here we follow the discussion given by Tinkham. We assume that

$$|\psi| = c_1|f_L| \qquad (14.91)$$

where c_1 ia an adjustable parameter. Then the free energy is expressed by

$$\overline{(f_s - f_N)} = \alpha\overline{|\psi|^2} + \frac{1}{2}\beta\overline{|\psi|^4} = \alpha c_1^2\overline{|f_L|^2} + \frac{1}{2}\beta c_1^4\overline{|f_L|^4} \qquad (14.92)$$

Minimizing this with respect to c_1^2, we find that

$$c_1^2 = -\frac{\alpha}{\beta}\frac{\overline{|f_L|^2}}{\overline{|f_L|^4}} \qquad (14.93)$$

Then the minimum free energy is given by

$$\overline{(f_s - f_N)} = -\frac{\alpha^2}{2\beta}\frac{\left(\overline{|f_L|^2}\right)^2}{\overline{|f_L|^4}} = -\frac{\alpha^2}{2\beta}\frac{1}{\beta_A} = -\frac{H_c^2}{8\iota}\frac{1}{\beta_A} \qquad (14.94)$$

If f_L is constant, $\beta_A = 1$. If f_L is not constant, β_A is larger than 1. The more β_A increases, the less favorable is the energy. It is found that $\beta_A = 1.18$ for the square lattice and $\beta_A = 1.16$ for the triangular lattice. The latter case is slightly more stable.

14.3 Magnetic properties near $H = H_{c2}$[1]

We consider the magnetic properties near $H = H_{c2}$ based on the discussion given in the previous section.

(a)   The magnetic induction $\overline{B}$. It is given by

$$\overline{B} = H + \overline{B_s} = H - \frac{H_{c2}}{2\kappa^2}\overline{|f_L|^2} = H - \frac{H_c}{\sqrt{2}\kappa}\overline{|f_L|^2},$$ (14.,95)

or

$$\overline{B} = H - \frac{1}{2\kappa^2}\frac{H_{c2} - H}{\beta_A(1 - \frac{1}{2\kappa^2})} = H - \frac{H_{c2} - H}{\beta_A(2\kappa^2 - 1)}.$$ (14.96)

Note that $\overline{B} = H$ at $H = H_{c2}$,

(b) The magnetization $\overline{M}$. It is described by

$$-4\pi\overline{M} = H - \overline{B} = \frac{H_{c2} - H}{\beta_A(2\kappa^2 - 1)}.$$ (14.97)

The magnetization $\overline{M}$ is negative as expected (since superconductors are diamagnetic) for $H < H_{c2}$. It vanishes at $H = H_{c2}$. The transition at $H = H_{c2}$ is of second order.

(c) The deviation $(\Delta B) = \sqrt{\overline{B^2} - \overline{B}^2}$

$\overline{B^2}$, $\overline{B}^2$ and $\overline{B^2} - \overline{B}^2$ are expressed as follows.

$$\overline{B^2} = \overline{(H - \frac{H_{c2}}{2\kappa^2}|f_L|^2)^2} = H^2 - \frac{HH_{c2}}{\kappa^2}\overline{|f_L|^2} + \frac{H_{c2}^2}{4\kappa^4}\overline{|f_L|^4},$$ (14.98)

$$\overline{B}^2 = (H - \frac{H_{c2}}{2\kappa^2}\overline{|f_L|^2})^2 = H^2 - \frac{HH_{c2}}{\kappa^2}\overline{|f_L|^2} + \frac{H_{c2}^2}{4\kappa^4}\left(\overline{|f_L|^2}\right)^2.$$ (14.99)

Then the deviation ΔB is calculated as

$$(\Delta B) = \sqrt{\overline{B^2} - \overline{B}^2} = \frac{H_{c2}}{2\kappa^2}\sqrt{\beta_A - 1}\left(\overline{|f_L|^2}\right),$$ (14.100)

or

$$(\Delta B) = \sqrt{\beta_A - 1}\frac{H_{c2} - H}{\beta_A(2\kappa^2 - 1)}.$$ (14.101)

(d) The Helmholtz free energy F is given by

$$F = F_0 + \int d\mathbf{r}[\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{1}{8\pi}\mathbf{B}^2],$$ (14.102)

or

$$\tilde{F} = \frac{F - F_0}{V} = -\frac{1}{2}\beta\overline{|\psi_L|^4} + \frac{1}{8\pi}\overline{B^2} \tag{14.103}$$

$$\tilde{F} = -\frac{1}{2}\beta\psi_\infty^4\overline{|\psi_L|^4} + \frac{1}{8\pi}\overline{B^2} = -\frac{1}{8\pi}H_c^2\overline{|f_L|^4} + \frac{1}{8\pi}\overline{B^2}, \tag{14.104}$$

Here we also note that

$$\overline{B} - H_{c2} = H - H_{c2} - \frac{H_{c2}}{2\kappa^2}\overline{|f_L|^2} = H_{c2}[-(1 - \frac{H}{H_{c2}}) - \frac{1}{2\kappa^2}\overline{|f_L|^2}]. \tag{14.105}$$

Since

$$\beta_A(1 - \frac{1}{2\kappa^2})\overline{|f_L|^2} = 1 - \frac{H}{H_{c2}} \tag{14.106}$$

we have

$$\overline{B} - H_{c2} = -\frac{H_{c2}}{2\kappa^2}[1 + \beta_A(2\kappa^2 - 1)]\overline{|f_L|^2} \tag{14.107}$$

or

$$\left(\overline{|f_L|^2}\right)^2 = \frac{4\kappa^4\left(\overline{B} - H_{c2}\right)^2}{H_{c2}^2[1 + \beta_A(2\kappa^2 - 1)]^2}. \tag{14.108}$$

Using this relation, we obtain the expression

$$\tilde{F} = -\frac{1}{8\pi}\frac{H_{c2}^2}{2\kappa^2}\beta\left(\overline{|f_L|^2}\right)^2 + \frac{1}{8\pi}[\overline{B}^2 + \frac{H_{c2}^2}{4\kappa^4}(\beta - 1)\left(\overline{|f_L|^2}\right)^2]$$

$$= \frac{1}{8\pi}[\overline{B}^2 - \frac{H_{c2}^2}{4\kappa^4}\left(\overline{|f_L|^2}\right)^2(-\beta + 1 + 2\kappa^2\beta)], \tag{14.109}$$

or

$$\tilde{F} = \frac{1}{8\pi}[\overline{B}^2 - \frac{\left(\overline{B} - H_{c2}\right)^2}{1 + \beta_A(2\kappa^2 - 1)}]. \tag{14.110}$$

Using this relation, we have

131

$$4\pi \frac{\partial}{\partial \bar{B}} \tilde{F} = \bar{B} - \frac{\left(\bar{B} - H_{c2}\right)}{1 + \beta_A(2\kappa^2 - 1)} = \frac{\beta_A(2\kappa^2 - 1)\bar{B} + H_{c2}}{1 + \beta_A(2\kappa^2 - 1)} = H \qquad (14.111)$$

since

$$\beta_A(2\kappa^2 - 1)\bar{B} + H_{c2} = [1 + \beta_A(2\kappa^2 - 1)]H \qquad (14.112)$$

So we can verify the thermodynamic relation as expected.

(e)   The Gibbs free energy $\tilde{G}$
The Gibbs free energy $\tilde{G}$ is related to the Helmholtz free energy $\tilde{F}$ by

$$\tilde{G} = \tilde{F} - \frac{H}{4\pi}\bar{B} = \frac{1}{8\pi}[\bar{B}^2 - \frac{\left(\bar{B} - H_{c2}\right)^2}{1 + \beta_A(2\kappa^2 - 1)}] - \frac{H}{4\pi}\bar{B}. \qquad (14.113)$$

Using the relation

$$\bar{B} = H - \frac{H_{c2} - H}{\beta_A(2\kappa^2 - 1)}, \qquad (14.114)$$

we find that

$$\tilde{G} = -\frac{1}{8\pi}[\frac{(H - H_{c2})^2}{\beta_A(2\kappa^2 - 1)} + H^2]. \qquad (14.115)$$

It is clear that the lower the value of β_A, the lower the value of $\tilde{G}$. Thus the triangular lattice is more stable than the square lattice.

14.4   The wall energy at $\kappa = 1/\sqrt{2}$.[1]

We show that the wall energy in a type I superconductor vanishes for $\kappa = 1/\sqrt{2}$. We consider the 2D GL equation, where the magnetic field is directed along the z axis.

$$\frac{1}{2m^*}\left(\hat{\Pi}_x^2 + \hat{\Pi}_y^2\right)\psi + \alpha\psi + \beta|\psi|^2\psi = 0, \qquad (14.116)$$

where

$$\hat{\Pi}_x = \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{q^*}{c}A_x\right) = \left(\hat{p}_x - \frac{q^*}{c}A_x\right), \quad \hat{\Pi}_y = \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - \frac{q^*}{c}A_y\right) = \left(\hat{p}_y - \frac{q^*}{c}A_y\right),$$

$$(14.117)$$

$$\hat{\Pi}_+ = \hat{\Pi}_x + i\hat{\Pi}_y, \qquad \hat{\Pi}_- = \hat{\Pi}_x - i\hat{\Pi}_y. \tag{14.118}$$

We note that

$$[\hat{\Pi}_-, \hat{\Pi}_+] == 2i[\hat{\Pi}_x, \hat{\Pi}_y], \tag{14.119}$$

or

$$[\hat{\Pi}_x, \hat{\Pi}_y] = [\hat{p}_x - \frac{q^*}{c}A_x, \hat{p}_y - \frac{q^*}{c}A_y] = -\frac{q^*}{c}[\hat{p}_x, A_y] + \frac{q^*}{c}[\hat{p}_y, A_x]. \tag{14.120}$$

For any arbitrary function f, we have,

$$[\hat{p}_x, A_y]f = \left[\frac{\hbar}{i}\frac{\partial}{\partial x}, A_y\right]f = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}A_y\right)f - \left(A_y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)f$$
$$= \frac{\hbar}{i}A_y\frac{\partial f}{\partial x} + \frac{\hbar}{i}\frac{\partial A_y}{\partial x}f - A_y\frac{\hbar}{i}\frac{\partial f}{\partial x} = \frac{\hbar}{i}\frac{\partial A_y}{\partial x}f. \tag{14.121}$$

Similarly, we have

$$[\hat{p}_y, A_x]f = \frac{\hbar}{i}\frac{\partial A_x}{\partial y}f. \tag{14.122}$$

Then we get the relation

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -\frac{q^*}{c}\frac{\hbar}{i}\left(\frac{\partial A_y}{\partial x} - \frac{\hbar}{i}\frac{\partial A_x}{\partial y}\right) = \frac{i\hbar q^*}{c}B_z, \tag{14.123}$$

where

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

Using this commutation relation, we get

$$[\hat{\Pi}_-, \hat{\Pi}_+] = 2i[\hat{\Pi}_x, \hat{\Pi}_y] = -2\frac{q^*\hbar}{c}B_z. \tag{14.124}$$

Since

$$\hat{\Pi}_-\hat{\Pi}_+ = (\hat{\Pi}_x - i\hat{\Pi}_y)(\hat{\Pi}_x + i\hat{\Pi}_y) == \hat{\Pi}_x^2 + \hat{\Pi}_y^2 + i[\hat{\Pi}_x, \hat{\Pi}_y], \tag{14.125}$$

we get

$$\hat{\Pi}_x^2 + \hat{\Pi}_y^2 = \hat{\Pi}_- \hat{\Pi}_+ + \frac{\hbar q^*}{c} B_z . \tag{14.126}$$

Then the GL equation can be rewritten as

$$\frac{1}{2m^*} \hat{\Pi}_- \hat{\Pi}_+ \psi + \frac{\hbar q^*}{2m^* c} B_z \psi + \alpha \psi + \beta |\psi|^2 \psi = 0 . \tag{14.127}$$

We now look for the solution such that $\hat{\Pi}_+ \psi = 0$.

$$\hat{\Pi}_+ \psi = \left(\frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{q^*}{c} A_x \right) \psi + i \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{q^*}{c} A_y \right) \psi = 0 , \tag{14.128}$$

or

$$\frac{\partial \psi}{\partial y} - i \frac{\partial \psi}{\partial x} = \frac{q^*}{c\hbar} (A_x + iA_y) \psi . \tag{14.129}$$

When $\hat{\Pi}_+ \psi = 0$, the GL eq. reduces to

$$\frac{\hbar q^*}{2m^* c} B_z + \alpha + \beta |\psi|^2 = 0 . \tag{14.130}$$

Note that the current density is given by

$$\mathbf{J}_s = \frac{q^*}{2m^*} [\psi^* (\frac{\hbar}{i} \nabla - \frac{q^*}{c} \mathbf{A}) \psi + \psi (-\frac{\hbar}{i} \nabla - \frac{q^*}{c} \mathbf{A}) \psi^*] \tag{14.131}$$

Then we have

$$\nabla \times \mathbf{B} = \left(\frac{\partial B_z}{\partial y}, \quad -\frac{\partial B_z}{\partial x}, \quad 0 \right) = \frac{2\pi \hbar q^*}{m^* c} [\psi^* (\frac{1}{i} \nabla - \frac{q^*}{c\hbar} \mathbf{A}) \psi + \psi (-\frac{1}{i} \nabla - \frac{q^*}{c\hbar} \mathbf{A}) \psi^*]) , \tag{14.132}$$

or

$$\frac{\partial B_z}{\partial y} = \frac{2\pi \hbar q^*}{m^* c} [\psi^* (\frac{1}{i} \frac{\partial}{\partial x} - \frac{q^*}{c\hbar} A_x) \psi + \psi (-\frac{1}{i} \frac{\partial}{\partial x} - \frac{q^*}{c\hbar} A_x) \psi^*] , \tag{14.133}$$

and

$$-i\frac{\partial B_z}{\partial x} = \frac{2\pi\hbar q^*}{m^*c}i[\psi^*(\frac{1}{i}\frac{\partial}{\partial y} - \frac{q^*}{c\hbar}A_y)\psi + \psi(-\frac{1}{i}\frac{\partial}{\partial y} - \frac{q^*}{c\hbar}A_y)\psi^*]. \qquad (14.134)$$

Then we have

$$\frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x} = \frac{2\pi\hbar q^*}{m^*c}[\psi^*(\frac{\partial\psi}{\partial y} - i\frac{\partial\psi}{\partial x}) + \psi(i\frac{\partial\psi^*}{\partial x} - \frac{\partial\psi^*}{\partial y}) - \frac{2q^*}{c\hbar}(A_x + iA_y)\psi^*\psi],$$

$$(14.135)$$

or

$$\frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x} = \frac{2\pi\hbar q^*}{m^*c}[-\psi^*(\frac{\partial\psi}{\partial y} - i\frac{\partial\psi}{\partial x}) - \psi(\frac{\partial\psi^*}{\partial y} - i\frac{\partial\psi^*}{\partial x})], \qquad (14.136)$$

This relation must agree with the relation given by
Using the relation (14.130), we have

$$\frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x} = -\frac{\partial}{\partial y}[\frac{2m^*c}{\hbar q^*}(\alpha + \beta\psi^*\psi)] + i\frac{\partial}{\partial x}[\frac{2m^*c}{\hbar q^*}(\alpha + \beta\psi^*\psi)], \qquad (14.137)$$

or

$$\frac{\hbar q^*}{2m^*c}\left(\frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x}\right) + \beta\psi^*(\frac{\partial}{\partial y}\psi - i\frac{\partial}{\partial x}\psi) + \beta\psi(\frac{\partial}{\partial y}\psi^* - i\frac{\partial}{\partial x}\psi^*)] = 0, \qquad (14.138)$$

From the comparison between Eqs.(14.137) and (14.138), we find that

$$\beta = \pi\frac{\hbar^2}{c^2}\frac{q^{*2}}{m^{*2}}. \qquad (14.139)$$

Since κ is defined by

$$\kappa = \frac{\lambda}{\xi} = \frac{cm^*\sqrt{\beta}}{\sqrt{2\pi}\hbar|q^*|}, \qquad (14.140)$$

we obtain

$$\kappa = \frac{cm^*}{\sqrt{2\pi}\hbar|q^*|}\sqrt{\pi}\frac{\hbar}{c}\frac{|q^*|}{m^*} = \frac{1}{\sqrt{2}}. \qquad (14.141)$$

((**Note**))

$$B_z = -\frac{2m^*c}{\hbar q^*}(\alpha + \beta|\psi|^2) = -\frac{2m^*c}{\hbar|q^*|}(-|\alpha| + \beta|\psi|^2) = \frac{2m^*c}{\hbar|q^*|}(\frac{\hbar^2}{2m^*\xi^2} - \beta|\psi|^2), \quad (14.142)$$

or

$$B_z = \frac{2m^*c}{\hbar|q^*|}(\frac{\hbar^2}{2m^*\xi^2} - \beta|\psi|^2) = \sqrt{2}\kappa H_c - \frac{4\pi\hbar|q^*|}{m^*c}\kappa^2|\psi|^2, \quad (14.143)$$

where

$$\frac{2m^*c}{\hbar|q^*|}\frac{\hbar^2}{2m^*\xi^2} = \frac{1}{|q^*|}\frac{c\hbar}{\xi^2} = \frac{\Phi_0}{2\pi c\hbar}\frac{c\hbar}{\xi^2} = \frac{\Phi_0}{2\pi\xi^2} = H_{c2} = \sqrt{2}\kappa H_c, \quad (14.145)$$

and

$$\frac{2m^*c}{\hbar|q^*|}\beta|\psi|^2 = \frac{2m^*c}{\hbar|q^*|}\frac{2\pi\hbar^2|q^*|^2}{c^2m^{*2}}\kappa^2|\psi|^2 = \frac{4\pi\hbar|q^*|}{m^*c}\kappa^2|\psi|^2. \quad (14.146)$$

One can estimate the surface energy given by

$$\gamma = \int_{-\infty}^{\infty} dx[-\frac{1}{2}\beta|\psi|^4 + \frac{1}{8\pi}(B_z - H_c)^2]. \quad (14.147)$$

In our case,

$$B_z = \sqrt{2}\kappa H_c - \frac{4\pi\hbar|q^*|}{m^*c}\kappa^2|\psi|^2, \quad (14.148)$$

with

$$\beta = 2\kappa^2\frac{\pi\hbar^2|q^*|^2}{m^{*2}c^2}. \quad (14.149)$$

Then the integrand of γ is described by

$$\frac{(B_z - H_c)^2}{8\pi} - \frac{\beta}{2}|\psi|^4 = \frac{H_c^2}{8\pi}\left((\sqrt{2}\kappa - 1) - \frac{4\pi\hbar|q^*|}{m^*cH_c}\kappa^2|\psi|^2\right)^2 - \kappa^2\frac{\pi\hbar^2|q^*|^2}{m^{*2}c^2}|\psi|^4. \quad (14.150)$$

This is equal to 0 when $\kappa = 1/\sqrt{2}$, leading to $\gamma = 0$.

15. Conclusion

We show that the phenomena of the superconductivity can be well explained in terms of the Ginzburg-Landau theory. The superconductors are classed into two types of superconductors, type I and type-II. The GL parameter κ has a limiting value $\kappa = 1/\sqrt{2}$ separating superconductors with positive surface energy ($\kappa < 1/\sqrt{2}$) (type-I superconductors) from those with negative surface energy at large $\kappa > 1/\sqrt{2}$ (type-II superconductors. The instability noticed by Ginzburg and Landau allowed the magnetic field to enter the superconductor without destroying it. This leads to the appearance of the mixed phase (the Abrikosov phase), where superconducting and normal regions coexist. The normal regions appear in the cores (of size ξ) of vortices binding individual magnetic flux quanta $\Phi_0 = hc/2e$ on the scale λ, with the charge '$2e$' appearing in Φ_0 a consequence of the pairing mechanism; since $\lambda > \xi$, the vortices repel and arrange in a stable lattice. In his 1957 paper, Abrikosov derived the periodic vortex structure near the upper critical field H_{c2}. In 2003, Abrikosov and Ginzburg got the Nobel prize in physics for their "pioneering contributions to the theory of superconductors.

References

1. P. G. Gennes, *Superconductivity of Metals and Alloys* (W.A. Benjamin, Inc. 1966).
2. D. Saint-Janes, E.J. Thomas, and G. Sarma, *Type II Superconductivity* (Pergamon Press, Oxford, 1969).
3. S. Nakajima, *Introduction to Superconductivity* (in Japanese) (Baifukan, Tokyo, 1971).
4. M. Tinkham, *Introduction to Superconductivity* (Robert E. Grieger Publishing Co. Malabar, Florida 1980).
5. J.B. Ketterson and S.N. Song, *Superconductivity* (Cambridge University Press 1999).
6. L.M. Lifshitz and Pitaefski, Statistical Physics part 2 Theory of the Condensed State, Vol.9 of Course of Theoretical Physics (Pergamon Press, Oxford, 1991).
7. T. Tsuneto, *Superconductivity and Superfluidity* (Cambridge University Press 1998).
8. M. Cyrot, Reports on Prog. Phys. **36**, 103 (1973).
9. G. Blatter, M.V. Feigel'man, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur, Rev. Mod. Physics **66**, 1125 (1994).
10. M. Trott, The Mathematica Guide Book Vol.**1-4** (Springer Verlag, 2006).
11. V.L. Ginzburg and L.D. Landau, Zh. Exp. Theor. Fiz **20**, 1064 (1950).
12. L.D. Landau, Phys. Z. der Sowjet Union , **11**, 26 (1937); ibid. **11**, 129 (1937).
13. L.P. Gorkov, Sov. Phys. JETP **9**, 1364 (1959).
14. A.A.A. Abrikosov, Soviet Physics, JETP **5**, 1174 (1957).
15. J.J. Sakurai, *Modern Quantum Mechanics* Revised Edition (Addison-Wesley, Reading, Massachusetts, 1994).
16. E. Merzbacher, *Quantum Mechanics* 3rd edition (John Wiley & Sons, New York 1998).
17. W.H. Kleiner, L.M. Roth, and S.H. Autler, Phys. Rev. **113**, A1226 (1964).

Appendix
A.   Superconducting parameters

The definition of the parameters are given in the text.

Thermodynamic field
$$H_c = \sqrt{\frac{4\pi\alpha^2}{\beta}} \ .$$

Order parameter
$$\psi_\infty{}^2 = n_s^* = |\alpha|/\beta \ .$$

The parameters
$$|\alpha| = \frac{H_c{}^2}{4\pi n_s^*}, \qquad \beta = \frac{H_c{}^2}{4\pi n_s^{*2}} \ .$$

Quantum fluxoid
$$\Phi_0 = \frac{2\pi\hbar c}{|q^*|} \ .$$

Magnetic field penetration depth
$$\lambda = \sqrt{\frac{m^* c^2 \beta}{4\pi q^{*2}|\alpha|}}, \qquad \lambda = \sqrt{\frac{m^* c^2}{4\pi n_s^* q^{*2}}} \ .$$

Coherence length
$$\xi = \frac{\hbar}{\sqrt{2m^*|\alpha|}}, \qquad \xi = \frac{\sqrt{2\pi n_s^*}\,\hbar}{H_c\sqrt{m^*}} \ .$$

Ginzburg-Landau parameter
$$\kappa = \frac{\lambda}{\xi} = \frac{cm^*\sqrt{\beta}}{\sqrt{2\pi}\hbar|q^*|}, \qquad \kappa = \frac{cm^* H_c}{2\sqrt{2\pi}\hbar n_s^*|q^*|} \ .$$

Some relations between H_c, H_{c1}, and H_{c2}

$$\frac{\lambda\xi}{\Phi_0} = \frac{1}{2\sqrt{2}\pi H_c}, \qquad \frac{\kappa}{\lambda^2} = \frac{\sqrt{2}|q^*| H_c}{c\hbar} \ .$$

$$\frac{\kappa}{\lambda^2}\Phi_0 = 2\sqrt{2}\pi H_c, \qquad \kappa = \frac{2\sqrt{2}\pi\lambda^2 H_c}{\Phi_0} \ .$$

$$\sqrt{2}H_c = \frac{\Phi_0}{2\pi\lambda\xi}, \qquad \frac{H_c}{H_{c2}} = \frac{1}{\sqrt{2}\kappa} \ .$$

Upper critical field (type II)
$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2} \ .$$

Lower critical field (type II)
$$H_{c1} = \frac{\Phi_0}{4\pi\lambda^2}\ln(\frac{\lambda}{\xi}) = \frac{H_c}{\sqrt{2}\kappa}\ln(\kappa) \ .$$

Surface-sheath field $\qquad H_{c3} = 1.695 H_{c2}$.

A.2.   Modified Bessel functions
The differential equation for the modified Bessel function is given by

$$x^2 y'' + xy' - (x^2 + n^2)y = 0,$$

$$y'' + \frac{1}{x}y' - (1 + \frac{n^2}{x^2})y = 0,$$

The solution of this equation is

$$y = c_1 I_n(x) + c_2 K_n(x),$$

where

$$I_n(x) = i^{-n} J_n(ix),$$

$$K_n(x) = \frac{\pi}{2} i^{n+1} [J_n(ix) + iN_n(x)] = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix),$$

Recursion formula:

$$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x} K_n(x)$$

$$K_n'(x) = -K_{n-1}(x) - \frac{n}{x} K_n(x)$$

$$K_n'(x) = -K_{n+1}(x) + \frac{n}{x} K_n(x) \ ,$$

$$\frac{d}{dx}[x^n K_n(x)] = -x^n K_{n-1}(x)$$

$$\frac{d}{dx}[x^{-n} K_n(x)] = -x^{-n} K_{n+1}(x)$$

The parameter γ is the Euler-Mascheroni constant: $\gamma = 0.57721$.

$$K_0(x) = -\ln(\frac{x}{2}) - \gamma = -\ln x + \ln 2 - \gamma = -\ln x + 0.11593718$$
,
$$K_1(x) = \frac{1}{x}$$

for $x \approx 0$.

```
Clear["Global`*"];
p1 = Plot[Evaluate[Table[BesselI[n, x], {n, 0, 5}]], {x, 0.1, 4},
  PlotPoints → 50, AxesLabel → {"x", "BesselI[n,x]"},
  PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 5}], Background → LightGray,
  Epilog → {Text[Style["n=0", Black, 12], {1.8, 2}],
    Text[Style["n=1", Black, 12], {2.3, 2}],
    Text[Style["n=2", Black, 12], {3, 2}],
    Text[Style["n=3", Black, 12], {3.5, 2}],
    Text[Style["n=4", Black, 12], {3.5, 0.8}],
    Text[Style["n=5", Black, 12], {3.5, 0.2}]}];
p2 = Plot[Evaluate[Table[BesselK[n, x], {n, 0, 1}]], {x, 0, 1.5},
  PlotPoints → 50, PlotStyle → Table[{Thick, Hue[0.2 i]}, {i, 0, 5}],
  Background → LightGray, AxesLabel → {"x", "BesselK[n,x]"},
  Epilog → {Text[Style["n=0", Black, 12], {0.4, 0.8}],
    Text[Style["n=1", Black, 12], {0.4, 3}]}];
```
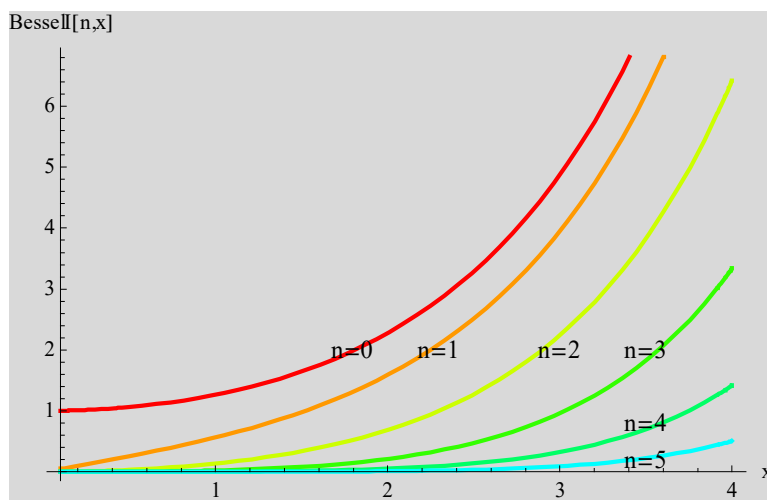

Fig.   Modified Bessel function $I_n(x)$ with $n = 0, 1, 2, 3, 4,$ and 5.
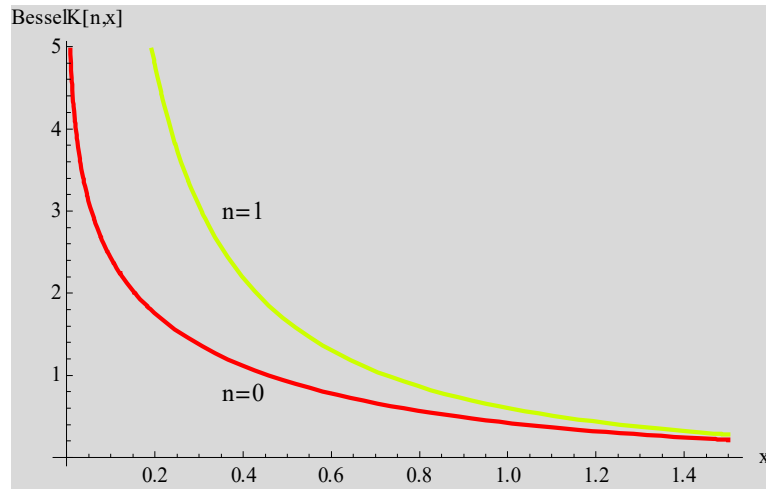
Fig.   Modified Bessel function $K_n(x)$ with $n = 0$ and 1.