Josephson effect
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Brian David Josephson, FRS (born 4 January 1940; Cardiff, Wales) is a Welsh physicist. He became a Nobel Prize laureate in 1973 for the prediction of the eponymous Josephson effect. As of late 2007, he was a retired professor at the University of Cambridge, where he is the head of the Mind–Matter Unification Project in the Theory of Condensed Matter (TCM) research group. He is also a fellow of Trinity College, Cambridge.

http://en.wikipedia.org/wiki/Brian_David_Josephson

1. DC Josephson junction

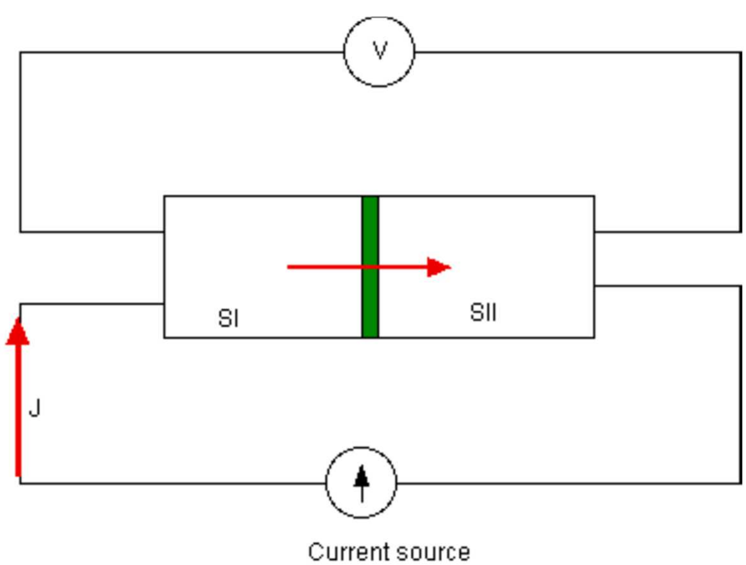

Fig. Schematic diagram for experiment of DC Josephson effect. Two superconductors SI and SII (the same metals) are separated by a very thin insulating layer (denoted by green). A DC Josphson supercurrent (up to a maximum value I_c) flows without dissipation through the insulating layer.

Let ψ_1 be the probability amplitude of electron pairs on one side of a junction. Let ψ_2 be the probability amplitude of electron pairs on the other side. For simplicity, let both superconductors be identical.

$$i\hbar\frac{\partial\psi_1}{\partial t}=\hbar T\psi_2,$$

$$i\hbar\frac{\partial\psi_2}{\partial t}=\hbar T\psi_1$$

where $\hbar T$ is the effect of the electron-pair coupling or (transfer interaction across the insulator). T(1/s) is the measure of the leakage of ψ_1 into the region 2, and of ψ_2 into the region 1.
Let

$$\psi_1=\sqrt{n_1}e^{i\theta_1},$$

$$\psi_2=\sqrt{n_2}e^{i\theta_2}$$

where

$$|\psi_1|^2=n_1$$

$$|\psi_2|^2=n_2$$

Then we have

$$\frac{1}{2}\frac{\partial n_1}{\partial t}+in_1\frac{\partial\theta_1}{\partial t}=-iT\sqrt{n_1n_2}e^{i\delta}$$

$$\frac{1}{2}\frac{\partial n_2}{\partial t}+in_2\frac{\partial\theta_2}{\partial t}=-iT\sqrt{n_1n_2}e^{-i\delta}$$

where

$$\delta=\theta_2-\theta_1.$$

Now equate the real and imaginary parts of Eqs.(3) and (4),

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1 n_2}\,\sin\delta$$

$$\frac{\partial n_2}{\partial t} = -2T\sqrt{n_1 n_2}\,\sin\delta$$

$$\frac{\partial \theta_1}{\partial t} = -T\sqrt{\frac{n_2}{n_1}}\cos\delta$$

$$\frac{\partial \theta_2}{\partial t} = -T\sqrt{\frac{n_1}{n_2}}\cos\delta$$

If $n_1 \approx n_2$ as for identical superconductors 1 and 2, we have

$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \qquad \text{or} \qquad \frac{\partial(\theta_2 - \theta_1)}{\partial t} = 0\,.$$

$$\frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t}$$

The current flow from the superconductor S_1 and to the superconductor S_2 is proportional to $\frac{\partial n_2}{\partial t}$. J is the current of superconductor pairs across the junction

$$J = J_0 \sin(\theta_2 - \theta_1)\,,$$

where J_0 is proportional to T (transfer interaction).

$$I = I_0 \sin\phi\,. \tag{5}$$

2. AC Josephson effect

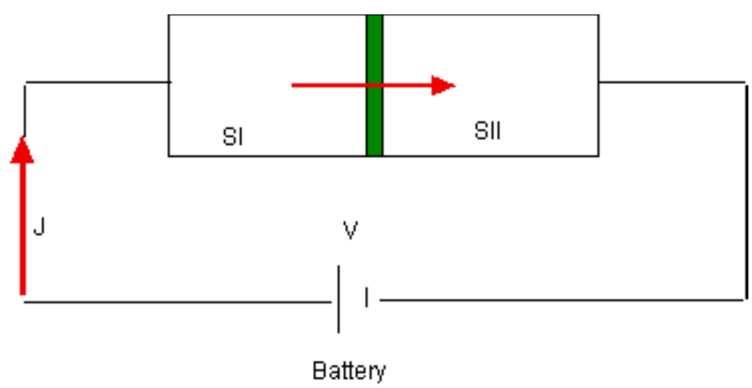

Fig. Schematic diagram for experiment of AC Josephson effect. A finite DC voltage is applied across both the ends.

Let a dc voltage V be applied across the junction. An electron pair experiences a potential energy difference qV on passing across the junction (q = -2e). We can say that a pair on one side is at $-eV$ and a pair on the other side is at eV.

$$i\hbar\frac{\partial\psi_1}{\partial t} = \hbar T\psi_2 - eV\psi_1$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \hbar T\psi_1 + eV\psi_2$$

or

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + in_1\frac{\partial\theta_1}{\partial t} = \frac{ieVn_1}{\hbar} - iT\sqrt{n_1n_2}e^{i\delta},$$

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + in_2\frac{\partial\theta_2}{\partial t} = -\frac{ieVn_2}{\hbar} - iT\sqrt{n_1n_2}e^{-i\delta}.$$

This equation breaks up into the real part and imaginary part,

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1n_2}\sin\delta$$

$$\frac{\partial n_2}{\partial t} = -2T\sqrt{n_1n_2}\sin\delta$$

$$\frac{\partial\theta_1}{\partial t} = \frac{eV}{\hbar} - T\sqrt{\frac{n_2}{n_1}}\cos\delta \quad .$$

$$\frac{\partial\theta_2}{\partial t} = -\frac{eV}{\hbar} - T\sqrt{\frac{n_1}{n_2}}\cos\delta$$

From these two equations with $n_1 = n_2$,

$$\frac{\partial(\theta_2-\theta_1)}{\partial t} = \frac{\partial\delta}{\partial t} = -\frac{2eV}{\hbar}.$$

$$J = J_0\sin[\delta(t)].$$

with

$$\delta(t) = \delta(0) - \frac{2e}{\hbar}\int Vdt .$$

When $V = V_0$ = constant, we have

$$\delta(t) = \delta(0) - \frac{2e}{\hbar} V_0 t \ .$$

$$J = J_0 \sin[\delta(0) - \frac{2e}{\hbar} V_0 t] .$$

The current oscillates with frequency

$$\omega_0 = \frac{2e}{\hbar} V \ .$$

A DC voltage of 1 eV produces a frequency of 483.5935 MHz.

$$\dot{\phi} = \frac{2e}{\hbar} V \ .$$

((Note))
Suppose that $V = V_0 = 1\ V$. The corresponding frequency is estimated from the relation,

$$\frac{2eV_0}{\hbar} = 2\pi\nu_0 \ ,$$

or

$$\nu_0 = \frac{2eV_0}{2\pi\hbar} = \frac{2 \times (1.60219 \times 10^{-12}) \times 10^{-6}}{2\pi \times 1.05459 \times 10^{-27}} = 483.5935 MHz \ .$$

3. *I-V* characteristic of Josephson tunneling junction

We now consider the *I-V* characteristic of the Josephson tunneling junction where a insulating layer is sandwiched between two superconducting layers (the same type). A capacitor is formed by these two superconductors. In this type of Josephson junctions, one can see the quasiparticle *I-V* curve which is different with increasing voltage and decreasing voltage (hysteresis). There are two voltage states, 0 V and 2Δ/e, where Δ is an energy gap of each superconductor. The *I-V* curve is characterized by (i) maximum Josephson tunneling current of Cooper pairs at $V = 0$ and (ii) Quasi-particle tunneling current ($V>2\Delta/e$).

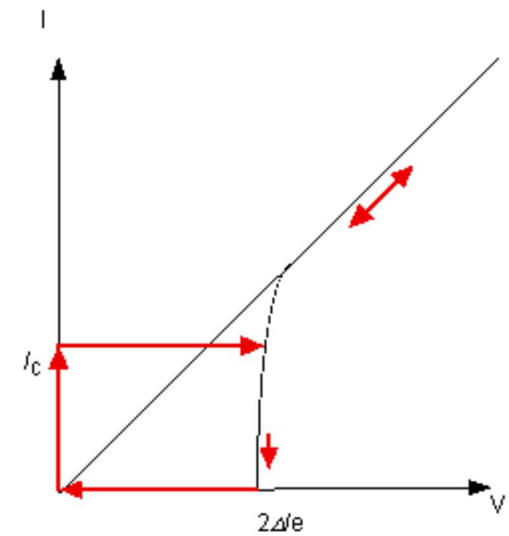

Fig. Schematic diagram of quasiparticle *I-V* characteristic (usually observed in a S-I-S Josephson *tunneling-type*). Josephson current (up to a maximum value I_c) flows at $V = 0$. Δ is an energy gap of the superconductor . The DC Josepson supercurrent flows under $V = 0$. For $V>2\Delta/e$ the quasiparticle tunneling current is seen.

The strong nonlinearity in the quasiparticle *I-V* curve of a tunneling junction is not an appropriate to the application to the SQUID element. This nonlineraity can be removed by the use of thin normal film deposited across the electrodes. In this effective resistance is a parallel combination of the junction. The *I-V* characteristic has no hysteresis. Such behavior is often observed in the bridge-type Josephson junction where two superconducting thin films are bridged by a very narrow superconducting thin film.

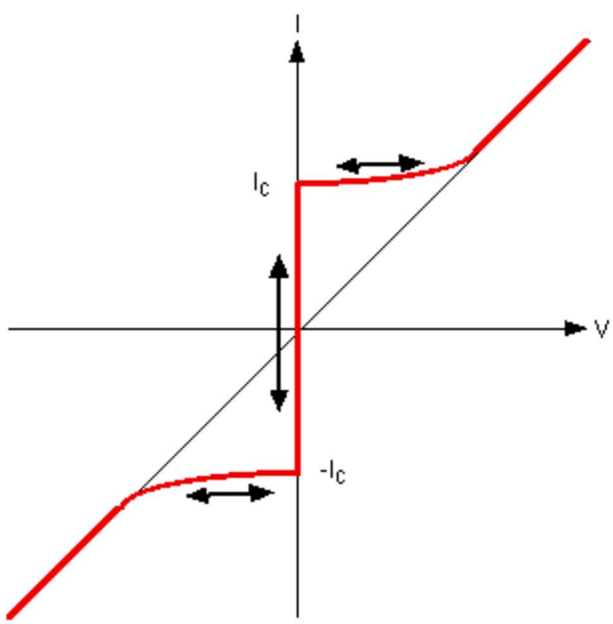

Fig. Schematic diagram of *I-V* characteristic of a Josephson junction (usually observed in *bridge-type* junction), which is reversible on increasing and decreasing *V*. A Josephson supercurrent flows up to I_c at $V = 0$. A transition occurs from the $V = 0$ state to a finite voltage state for $I>I_c$.. Above this voltage the *I-V* characteristic exhibits an Ohm's law with a finite resistance of the Junction. The current has an oscillatory component of angular frequency ω (= $2eV/\hbar$) (the AC Josephson effect). Note that in order to avoid the hysteresis of *I-V* characteristic, the normal metal is used as a shunt resistance *R*. Two superconductors are connected in parallel with this resistance (resistively shunted junction (RSJ), see below).

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4. RSJ (Resistively shunted junction) model: Josephson junction circuit application[7]

Here we discuss the *I-V* characteristics of a Josephson tunneling junction using an equivalent circuit shown below. This circuit includes the effect of various dissipative processes and the distributed capacity with so-called lumped circuit parameters (connection of *R* and *C* in parallel).

4.1 Fundamental equation

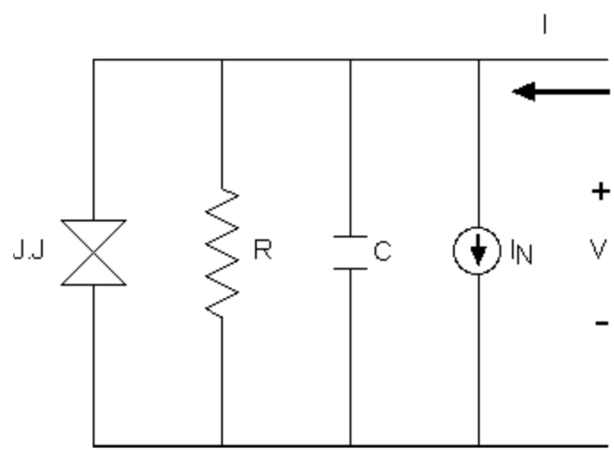

Fig. Equivalent circuit of a real Josephson junction with a current noise source. (RSJ model). J.J. stands for the Josephson junction. $I_N(t)$ is the noise current source.

We now consider an equivalent circuit for the Josephson junction, which is described above. *J.J.* stands for the Josephson junction.

$$I_c \sin\phi + \frac{V}{R} + C\dot{V} + I_N(t) = I ,$$

where the first term is a Josephson current, the second term is an ohmic current, the third term is a displacement current, and $I_N(t)$ is the noise current source. Here we neglect this term. Since

$$\dot{\phi} = \frac{2e}{\hbar} V ,$$

we get a second-order differential equation for the phase ϕ

$$\frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} = I - I_c \sin\phi ,$$

with $\kappa = I/I_c$, where Φ_0 is a magnetic quantum flux.

4.2 Differential equation for ϕ

For the sake of simplicity, we use the dimensionless quantities. Here we assume that

$$\eta = \frac{V}{I_c R}, \qquad \kappa = \frac{I}{I_c}, \qquad \tau = \omega_J t ,$$

where ω_J is the Josephson plasma frequency and is defined by

$$\omega_J = \left(\frac{2eI_c}{\hbar C}\right)^{1/2},$$

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{d\phi}{d\tau}\omega_J .$$

Similarly we have

$$\frac{d^2\phi}{dt^2} = \omega_J{}^2 \frac{d^2\phi}{d\tau^2} .$$

Then we get

$$\frac{\hbar C}{2eI_c}\omega_J{}^2 \frac{d^2\phi}{d\tau^2} + \frac{\hbar}{2eRI_c}\omega_J \frac{d\phi}{d\tau} + \sin\phi = \kappa ,$$

or

$$\frac{d^2\phi}{d\tau^2} + \beta_J \frac{d\phi}{d\tau} + \sin\phi = \kappa ,$$

where

$$\beta_J = \frac{\hbar\omega_J}{2eRI_c} = \frac{\hbar\omega_J{}^2}{2eRI_c\omega_J} = \frac{\hbar}{2eRI_c\omega_J}\frac{2eI_c}{\hbar C} = \frac{1}{\omega_J RC} .$$

The normalized voltage is described by

$$\eta(\tau) = \frac{V}{I_c R} = \frac{\hbar}{2e}\frac{1}{I_c R}\frac{d\phi}{dt} = \frac{\hbar\omega_J}{2eRI_c}\frac{d\phi}{d\tau} = \beta_J \frac{d\phi}{d\tau},$$

since

$$\dot{\phi} = \frac{2e}{\hbar}V .$$

4.3. *I-V* characteristic for $\beta_J >> 1$

For the special case β_J>>1 (small capacitance limit), the above equation is reduced to

$$\beta_J \frac{d\phi}{d\tau} + \sin\phi = \kappa ,$$

$$\langle V \rangle = \left\langle I_c R \beta_J \frac{d\phi}{d\tau} \right\rangle = I_0 R \beta_J \frac{1}{T}\int_0^T \frac{d\phi}{d\tau} d\tau = 2\pi I_0 R \frac{\beta_J}{T},$$

where T is a period;

$$T = \frac{2\pi}{\sqrt{\kappa^2 - 1}}\beta_J .$$

Here

$$T = \int_0^T d\tau = \int_0^{2\pi} \frac{d\phi}{\frac{d\phi}{d\tau}} = \beta_J \int_0^{2\pi} \frac{d\phi}{\kappa - \sin\phi},$$

or

$$\frac{T}{\beta_J} = \int_0^{2\pi} \frac{d\phi}{\kappa - \sin\phi} = \int_0^{\pi} \frac{d\phi}{\kappa - \sin\phi} + \int_0^{\pi} \frac{d\phi}{\kappa + \sin\phi} = \int_0^{\pi} [\frac{1}{\kappa - \sin\phi} + \frac{1}{\kappa + \sin\phi}] d\phi ,$$

or

$$\frac{T}{\beta_J} = \frac{2\pi}{\sqrt{\kappa^2 - 1}} \qquad \text{for } \kappa > 1,$$

$$\frac{T}{\beta_J} = 0 \qquad \text{for } \kappa < 1,$$

So we get

$$\langle V\rangle = 2\pi I_c R\frac{\beta_J}{T} = I_c R\sqrt{\kappa^2-1},$$

or

$$\frac{\langle V\rangle}{I_c R} = \sqrt{\kappa^2-1},$$

or

$$\kappa = \sqrt{1+\left(\frac{\langle V\rangle}{I_c R}\right)^2}.$$

((Mathematica))

```
Clear["Global`*"];
K1[x_] := Integrate[(1/(x - Sin[ϕ]) + 1/(x + Sin[ϕ])), {ϕ, 0, π},
  Assumptions → x > 1]

K1[x]
```

$$\frac{2\pi}{\sqrt{-1+x^2}}$$

```
eq1 = V1 == (2 π Ic R)/K1[x] ; eq2 = Solve[eq1, x]; I1 = x /. eq2[[2]] /. {R → 1, Ic → 1};
Plot[{I1, V1}, {V1, 0, 4}, PlotStyle → {{Red, Thick}, {Blue, Thick}},
 AxesLabel → {"V/IcR", "I/Ic"}]
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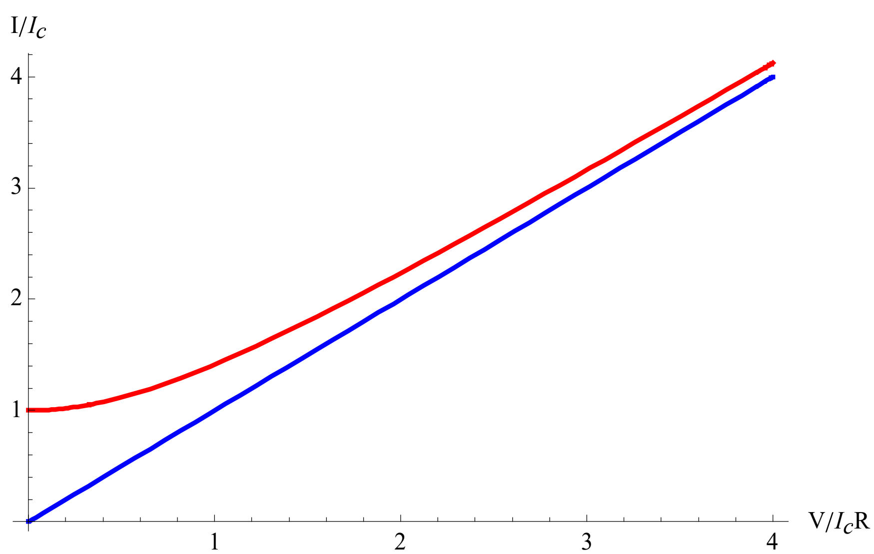

Fig. I/I_c vs $V/(I_cR)$ curve for β_J>>1 (red curve). The blue curve shows the Ohm's law: $I/I_c = V/(I_cR)$

5. Flux quantization

We start with the current density

$$\mathbf{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\mathbf{A}) = q^*|\psi|^2\mathbf{v}_s .$$

Suppose that $n_s = |\psi|^2$ =constant. Then we have

$$\nabla\theta = \frac{m^*}{q^*\hbar n_s}\mathbf{J}_s + \frac{q^*}{c\hbar}\mathbf{A} ,$$

or

$$\oint\nabla\theta\cdot d\mathbf{l} = \frac{m^*}{q^*\hbar n_s^*}\oint\mathbf{J}_s\cdot d\mathbf{l} + \frac{q^*}{c\hbar}\oint\mathbf{A}\cdot d\mathbf{l} .$$

The path of integration can be taken inside the penetration depth where $\mathbf{J}_s$=0.

$$\oint\nabla\theta\cdot d\mathbf{l} = \frac{q^*}{c\hbar}\oint\mathbf{A}\cdot d\mathbf{l} = \frac{q^*}{c\hbar}\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \frac{q^*}{c\hbar}\int\mathbf{B}\cdot d\mathbf{a} = \frac{q^*}{c\hbar}\Phi ,$$

where Φ is the magnetic flux. Then we find that

$$\Delta\theta = \theta_2 - \theta_1 = 2\pi n = \frac{q^*}{c\hbar}\Phi ,$$

where n is an integer. The phase θ of the wave function must be unique, or differ by a multiple of 2π at each point,

$$\Phi = \frac{2\pi c\hbar}{|q^*|}n .$$

The flux is quantized. When $|q^*| = 2|e|$, we have a magnetic quantum fluxoid;

$$\Phi_0 = \frac{2\pi c\hbar}{2|e|} = \frac{\pi c\hbar}{|e|} = \frac{ch}{2|e|} = 2.06783372 \times 10^{-7} \text{ Gauss cm}^2$$

((**Note**))

The current flows along the ring. However, this current flows only on the surface boundary (region from the surface to the penetration depth λ). Inside of the system (region far from the surface boundary), there is no current since $\nabla \times \mathbf{H} = 4\pi \mathbf{J}/c$ and $\boldsymbol{H} = 0$.

6. DC SQUID (double junctions): quantum mechanics

DC SQUID consists of two points contacts in parallel, forming a ring. Each contact forms a Josephson junctions of superconductor 1, insulating layer, and superconductor 2 (S_1-I-S_2). Suppose that a magnetic flux Φ passes through the interior of the loop.

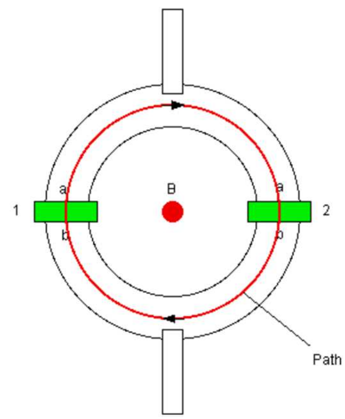

Fig. Schematic diagram of superconducting quantum interference device. δ_1 and δ_2 refer to two point-contact weak links. The rest of the circuit is strongly superconducting.

Here we have

$$\oint \nabla\theta \cdot d\mathbf{l} = \theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b}.$$

or

$$\theta_{2a} - \theta_{1a} + \theta_{1b} - \theta_{2b} = 2\pi \frac{\Phi}{\Phi_0}$$

or

$$\delta_1 - \delta_2 = 2\pi \frac{\Phi}{\Phi_0}$$

where $\delta_1 (= \theta_{1b} - \theta_{1a})$ is the phase difference between the superconductors a and b through the junction 1 and $\delta_2 = (\theta_{2b} - \theta_{2a})$ are is the phase difference between the superconductors a and b through the junction 2.

When $\boldsymbol{B} = 0$ (or $\Phi = 0$), we have $\delta_1 - \delta_2 = 0$. In general, we put the form

$$\delta_1 = \delta_0 + \frac{e}{\hbar c}\Phi, \qquad \delta_2 = \delta_0 - \frac{e}{\hbar c}\Phi.$$

The total current is given by

$$\begin{aligned} I &= I_1 + I_2 = I_c[\sin(\delta_1) + \sin(\delta_2)] \\ &= I_c[\sin(\delta_0 + \frac{e}{\hbar c}\Phi) + \sin(\delta_0 - \frac{e}{\hbar c}\Phi)] \\ &= 2I_c \sin(\delta_0)\cos(\frac{e}{\hbar c}\Phi) \end{aligned}$$

or

$$I = 2I_c \sin(\delta_0)\cos(\pi\frac{\Phi}{\Phi_0}).$$

since

$$\pi\frac{\Phi}{\Phi_0} = \frac{e\Phi}{\hbar c}.$$

The current varies with Φ and has a maximum of $2I_c$ when $\frac{e}{\hbar c}\Phi = s\pi$ (s: integers),
or

$$\Phi = \frac{\hbar c\pi}{e}s = \frac{hc}{2e}s = \Phi_0 s.$$

The simple two point contact device corresponds to a two-slit interference pattern, for which the physically interesting quantity is the modulus of the amplitude rather than the square modulus, as it is for optical interference patterns.

7. Critical current

The maximum of I (called as the critical current) is given by

$$I_{\max} = 2I_c\left|\cos(\pi\frac{\Phi}{\Phi_0})\right|.$$

Note that

$$I_{\max} = 2I_c$$

when $\Phi = 0$. This means that the critical current is a periodic function of the magnetic flux Φ.

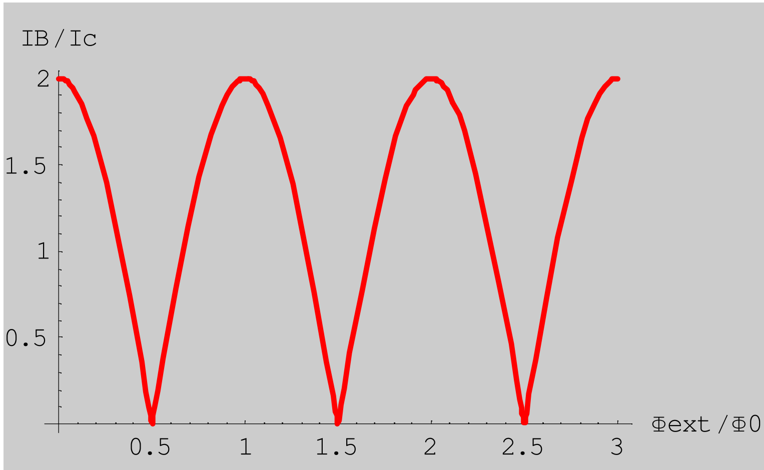

Fig. Ideal case for the I_B/I_c vs Φ_{ext}/Φ_0 curve in the DC SQUID, where I_B is the maximum supercurrent. $I_B = 2\ I_c$ when $\Phi_{ext}/\Phi_0 = n$ (integer) and $I_B = 0$ for $\Phi_{ext}/\Phi_0 = n + 1/2$.

8. Analogy of the diffraction with double slits and single slit

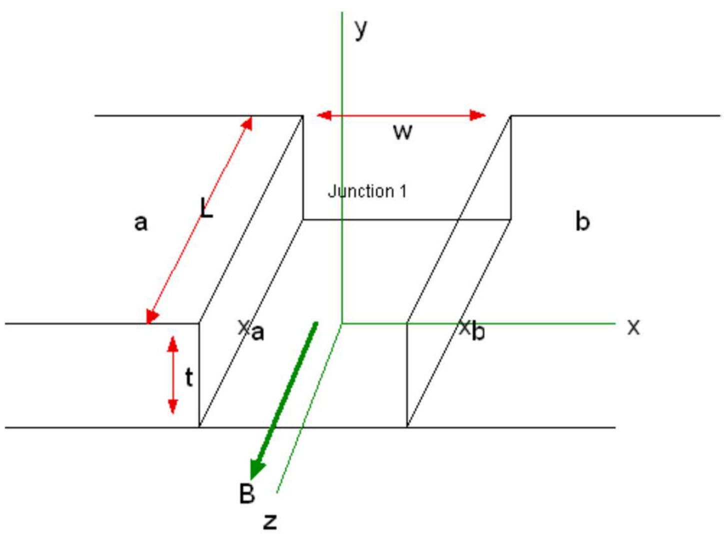

Fig. Diffraction effect of Josephson junction. A magnetic field ***B*** along the z direction, which is penetrated into the junction (in the normal phase).

We consider a junction (1) of rectangular cross section with magnetic field ***B*** applied in the plane of the junction, normal to an edge of width *w*,

$$J = J_0 \sin[\delta_1 + \frac{q}{\hbar c}\int_1^2 \mathbf{A}\cdot d\mathbf{l}] = J_0 \sin[\delta_1 - \frac{2e}{\hbar c}\int_1^2 \mathbf{A}\cdot d\mathbf{l}],$$

with q = -2e. We use the vector potential $\boldsymbol{A}$ given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r}) = (-\frac{By}{2},\frac{Bx}{2},0)\,,$$

$$\mathbf{A}' = \mathbf{A} + \nabla\chi = (-By,0,0)\,,$$

where

$$\chi = -\frac{Bxy}{2}\,.$$

Then we have

$$J(y) = J_0\sin[\delta_1 - \frac{2e}{\hbar c}\int_{x_a}^{x_b}(-By)dx] = J_0\sin[\delta_1 + \frac{2eB}{\hbar c}yW]\,,$$

$$dI_1 = J(y)Ldy = J_0L\sin[\delta_1 + \frac{2eB}{\hbar c}yW]dy\,,$$

or

$$\begin{aligned}
I_1 &= J_0L\int_{-t/2}^{t/2}\sin[\delta_1 + \frac{2eB}{\hbar c}yW]dy \\
&= 2J_0L\sin\delta_1\int_0^{t/2}\cos(\frac{2eB}{\hbar c}yW)dy \\
&= 2J_0L\sin\delta_1\frac{1}{\frac{2eB}{\hbar c}W}\left[\sin(\frac{2eB}{\hbar c}yW)\right]_0^{t/2}. \\
&= J_0L\sin\delta_1\frac{1}{\frac{eB}{\hbar c}W}\left[\sin(\frac{2eB}{\hbar c}yW)\right]_0^{t/2}
\end{aligned}$$

Then we have

$$I_1 = J_0L\frac{\hbar c}{eBW}\sin(\delta_1)\sin(\frac{eB}{\hbar c}Wt)\,.$$

Here we introduce the total magnetic flux passing through the area Wt ($\Phi_W = BWt$),

$$\frac{\Phi_W}{\Phi_0} = \frac{BWt}{\frac{2\pi\hbar c}{2e}} = \frac{eBWt}{\pi\hbar c}, \qquad \text{or} \qquad \pi\frac{\Phi_W}{\Phi_0} = \frac{eBWt}{\hbar c}.$$

Therefore we have

$$I_1 = I_c \sin(\delta_1)\frac{\sin(\frac{\pi\Phi_W}{\Phi_0})}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)}.$$

where

$$I_c = J_0 L t.$$

The total current is given by

$$\begin{aligned} I &= I_1 + I_2 \\ &= [I_c \sin(\delta_1) + I_c \sin(\delta_2)]\frac{\sin(\frac{\pi\Phi_W}{\Phi_0})}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)}, \\ &= 2I_c \sin(\delta_0)\cos(\pi\frac{\Phi}{\Phi_0})\frac{\sin(\frac{\pi\Phi_W}{\Phi_0})}{\left(\frac{\pi\Phi_W}{\Phi_0}\right)} \end{aligned}$$

. (32)

where

$$I_c[\sin(\delta_1) + \sin(\delta_2)] = 2I_c \sin\delta_0 \cos(\pi\frac{\Phi}{\Phi_0})$$

The short period variation is produced by interference from the two Josephson junctions, while the long period variation is a diffraction effect and arises from the finite dimensions of each junction. The interference pattern of $|I|^2$ is very similar to the intensity of the Young's double slits experiment. If the slits have finite width, the intensity must be multiplied by the diffraction pattern of a single slit, and for large angles the oscillations die out.

((**Example**))

The pattern of $|I|^2$ vs Φ/Φ_0 is very similar to the pattern of the Young's double slit experiments.

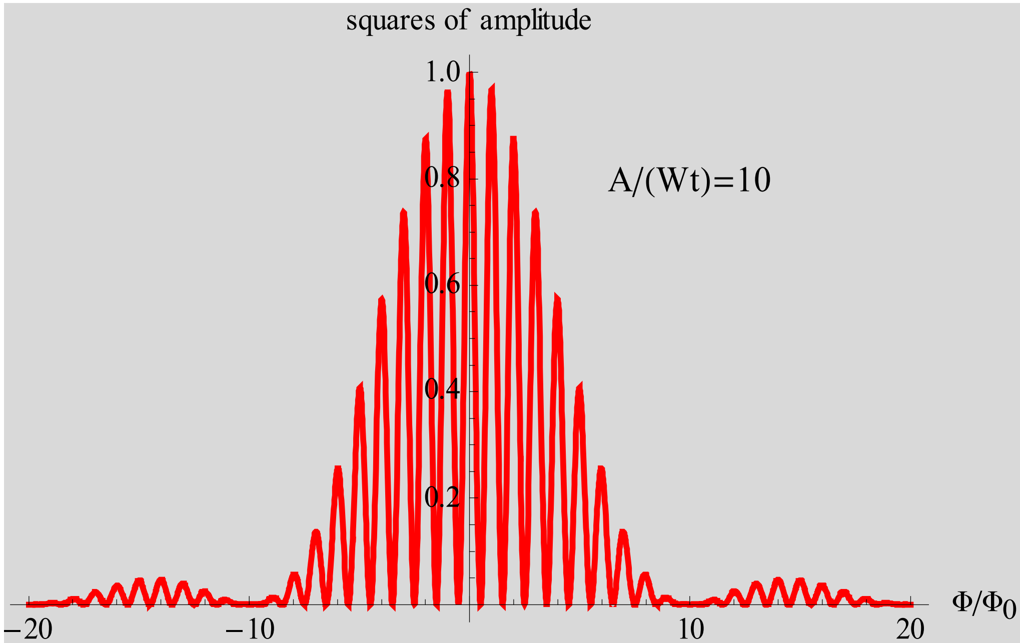

Fig. Plot of $[\frac{I}{2I_c \sin(\delta_0)}]^2$ vs $\frac{\Phi}{\Phi_0}$, where $A/(Wt)$ = 10. The envelope arises from $\sin^2(\frac{\pi\Phi_W}{\Phi_0})/\left(\frac{\pi\Phi_W}{\Phi_0}\right)^2$ and the oscillation with short period arises from $\cos^2(\pi\frac{\Phi}{\Phi_0})$

9. Young's double slit experiment

We consider the Young's double slits (the slits are separated by *d*). Each slit has a finite width *a*.

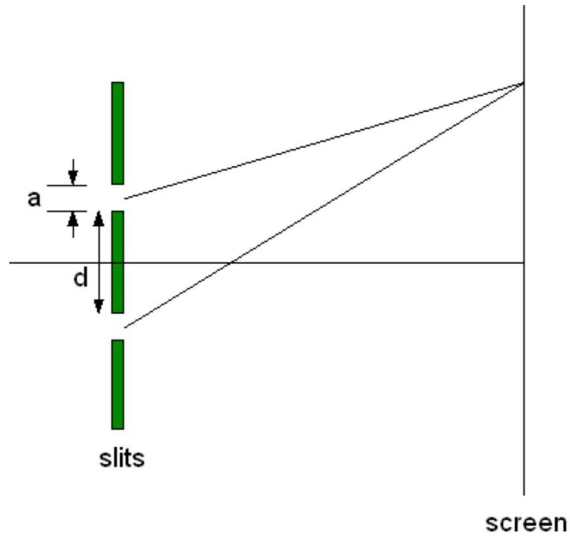

Fig. Geometric construction for describing the Young's double-slit experiment (not to scale).

((double slits))
E is the electric field of a light with the wavelength λ. d is the separation distance between the centers of the slits.

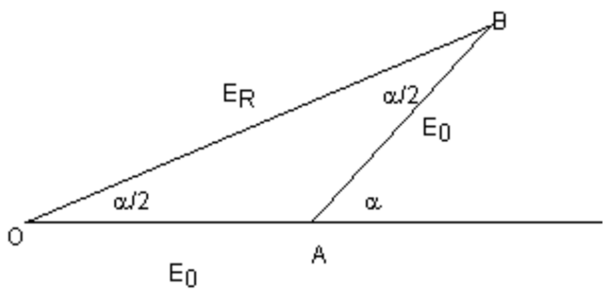

Fig. A reconstruction of the resultant phasor E_R which is the combination of two phasors (E_0).

$$E_R = 2E_0 \cos\frac{\alpha}{2}.$$

The intensity:

$$I \propto E_R{}^2 = 4E_0{}^2 \cos^2\frac{\alpha}{2} = 2E_0{}^2(1+\cos\alpha) .],$$

where the phase difference α is given by

$$\alpha = \frac{2\pi d}{\lambda}\sin\theta .$$

((single slit))
We assume that each slit has a finite width a.

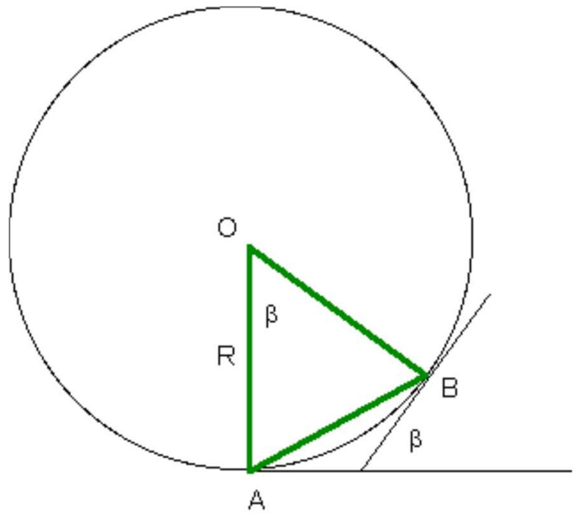

Fig. Phaser diagram for a large number of coherent sources. All the ends of phasors lie on the circular arc of radius R. The resultant electric field magnitude E_R equals the length of the chord.

$$E_0 = R\beta ,$$

$$E_R = 2R\sin\frac{\beta}{2} = 2\frac{E_0}{\beta}\sin\frac{\beta}{2} = E_0\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}} .$$

where the phase difference β is given by $\beta = \frac{2\pi a}{\lambda}\sin\theta$. Then the resultant intensity I for the double slits (the distance d) (each slit has a finite width a) is given by

$$I = I_m \cos^2\frac{\alpha}{2}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2 = \frac{I_m}{2}(1+\cos\alpha)\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2 .$$

10. Principle of DC SQUID

For the special case β_J>>1 (small capacitance limit), we have

$$\beta_J \frac{d\phi}{d\tau} + \sin\phi = \kappa ,$$

for the RSJ model. In the DC SQUID, the two Josephson junctions (junctions 1 and 2) are connected in parallel.

$$\beta_J \frac{d\phi_1}{d\tau} + \sin\phi_1 = \frac{\kappa_B}{2} - \kappa_s, \qquad \text{for the Josephson junction-1}$$

$$\beta_J \frac{d\phi_2}{d\tau} + \sin\phi_2 = \frac{\kappa_B}{2} + \kappa_s, \qquad \text{for the Josephson junction}$$

where

$$\phi_2 = \phi_1 - 2\pi \frac{\Phi_{ext}}{\Phi_0}. \tag{3}$$

From these three equations, we have

$$\beta_J \frac{d}{d\tau}(\phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0}) + \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \sin(\phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0}) = \frac{\kappa_B}{2}.$$

When we introduce a new parameter

$$\varphi_1 = \phi_1 - \pi \frac{\Phi_{ext}}{\Phi_0},$$

we have the final form

$$\beta_J \frac{d\varphi_1}{d\tau} + \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \sin\varphi_1 = \frac{\kappa_B}{2}.$$

We are interested in the DC current-voltage characteristic so we need to determine the time averaged voltage

$$V = \left\langle \frac{\hbar}{2e} \frac{d\phi_1}{dt} \right\rangle = \left\langle I_c R \beta_J \frac{d\phi_1}{d\tau} \right\rangle = \frac{I_c R \beta_J}{T} \int_0^T \frac{d\phi_1}{d\tau} d\tau = I_c R \beta_J \frac{2\pi}{\tau}.$$

Here

$$\tau = \int_0^{2\pi} \frac{d\phi_1}{\frac{d\phi_1}{d\tau}} = \int_0^{2\pi} \frac{\beta_J d\phi_1}{\frac{\kappa_B}{2} - \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \sin\varphi_1} = \frac{\beta_J}{\cos(\pi \frac{\Phi_{ext}}{\Phi_0})} \int_0^{2\pi} \frac{d\phi_1}{\kappa' - \sin\varphi_1} = \frac{1}{\cos(\pi \frac{\Phi_{ext}}{\Phi_0})} \tau',$$

$$\kappa' = \frac{\kappa_B}{2\cos(\pi \frac{\Phi_{ext}}{\Phi_0})},$$

where

$$\tau' = \int_0^{2\pi} \frac{d\phi}{\kappa' - \sin\phi} = \int_0^{\pi} \frac{d\phi}{\kappa' - \sin\phi} + \int_0^{\pi} \frac{d\phi}{\kappa' + \sin\phi} = \int_0^{\pi} [\frac{1}{\kappa' - \sin\phi} + \frac{1}{\kappa' + \sin\phi}] d\phi,$$

or

$$\tau' = \frac{2\pi}{\sqrt{\kappa'^2 - 1}} \quad \text{for } \kappa > 1,$$

$$\tau' = 0 \qquad\qquad \text{for } \kappa < 1.$$

Then we have

$$V = 2\pi I_c R \beta_J \frac{1}{\tau} = \frac{2\pi I_c R}{\tau'} \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) = I_c R \sqrt{\kappa'^2 - 1} \cos(\pi \frac{\Phi_{ext}}{\Phi_0}),$$

or

$$\frac{V}{I_c R} = \left| \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \right| \sqrt{\kappa'^2 - 1},$$

or

$$|\kappa'| = \sqrt{1 + \left(\frac{V}{I_c R \cos(\pi \frac{\Phi_{ext}}{\Phi_0})} \right)^2} = \frac{1}{2\left| \cos(\pi \frac{\Phi_{ext}}{\Phi_0}) \right|} \frac{I_B}{I_c},$$

or

$$\frac{I_B}{2I_c} = \sqrt{\cos^2(\pi \frac{\Phi_{ext}}{\Phi_0}) + (\frac{V}{RI_c})^2},$$

or

$$\frac{\langle V \rangle}{RI_c} = \sqrt{(\frac{I_B}{2I_c})^2 - \cos^2(\pi \frac{\Phi_{ext}}{\Phi_0})},$$

or

$$< V >= \frac{R}{2}\sqrt{I_B{}^2 - 4I_c{}^2 \cos^2(\pi \frac{\Phi_{ext}}{\Phi_0})} . \qquad (45)$$

When this equation for the voltage is compared with that for one Josephson junction with β_J»1

$$\langle V \rangle = R\sqrt{I^2 - I_c{}^2} .$$

We find that the critical current is $2I_c \left|\cos(\pi \frac{\Phi_{ext}}{\Phi_0})\right|$. This means that the critical current is $2I_c$ for $\Phi_{ext}/\Phi_0 = n$ (integer) and zero for $\Phi_{ext}/\Phi_0 = n+1/2$. In other words, the critical current is a periodic function of Φ with a period Φ_0. However, the actual critical current does not oscillate between 0 and $2I_c$ because of the finite self-inductance L. In the above model, L (or $\beta = 0$) is assumed to be zero. The critical current varies between $2I_c$ and finite value depending on the value of β (see the detail in Sec.7.1).

When the total current I_B is constant, the voltage across the DC SQUID periodically changes with the external magnetic flux. This is the phenomenon one exploit to create the most sensitive magnetic field detection.

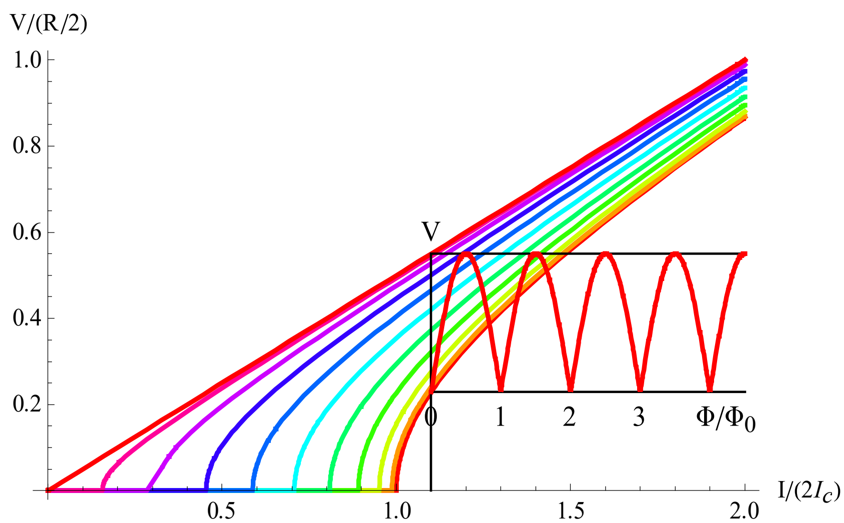

Fig. The critical current of the *I-V* characteristic changes periodically with the magnetic flux Φ, with the period of the quantum fluxoid Φ_0. Note that here we neglect the effect of the penetration of magnetic field into the insulation layers. Suppose that the current in the circuit is fixed. Then the voltage across the DC SQUID periodically changes with increasing Φ with the period of the quantum fluxoid Φ_0.

11. Experimental procedure for Mr. SQUID

This figure is obtained from the Instruction manual of Mr. SQUID.[9]

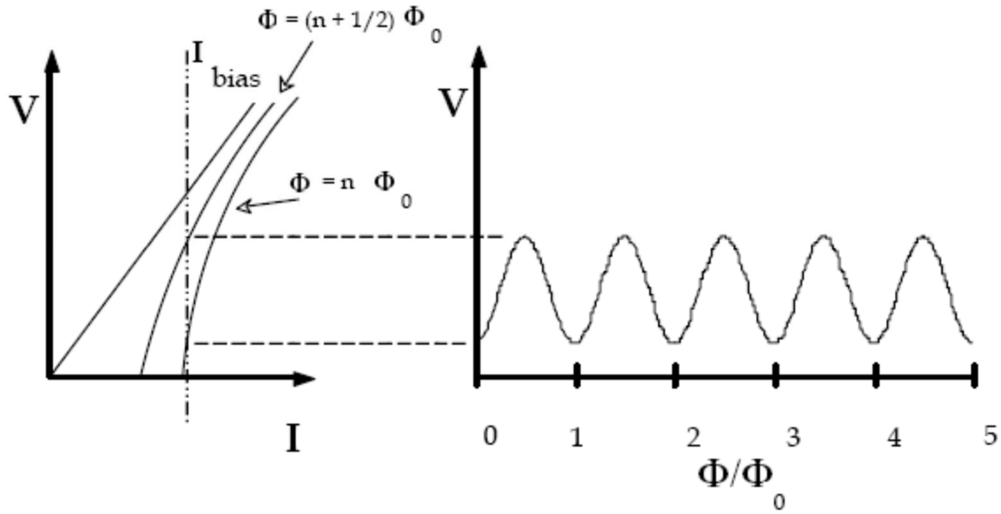

Fig. Detected voltage vs the magnetic flux Φ/Φ_0. The current I_B is kept at fixed value which is a little larger than $2I_c$. The detected voltage shows a maximum for $\Phi = (n+1/2)\Phi_0$, and a minimum for $\Phi = n\Phi_0$. The detected voltage is a periodic function of Φ with a period of Φ_0. (This figure is copied from the User Guide of Mr SQUID[9]).

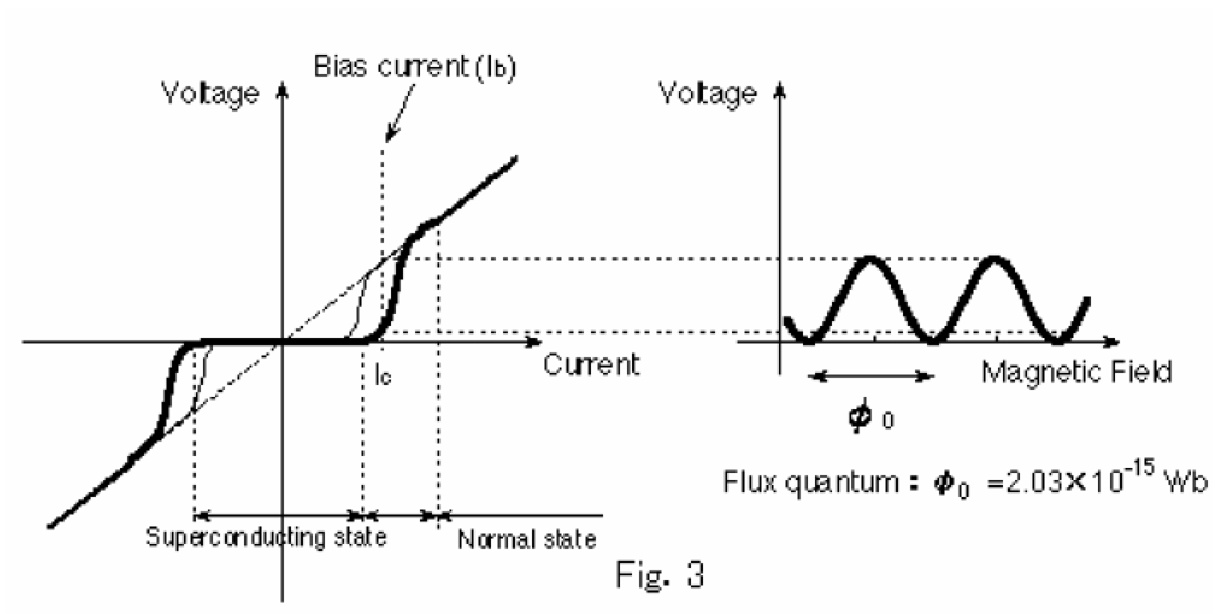

Fig. 3

12. Experimental result from Mr. SQUID (Advanced Lab. Binghamton University)

We use Mr SQUID for the measurement on the properties of the Josephson effect.

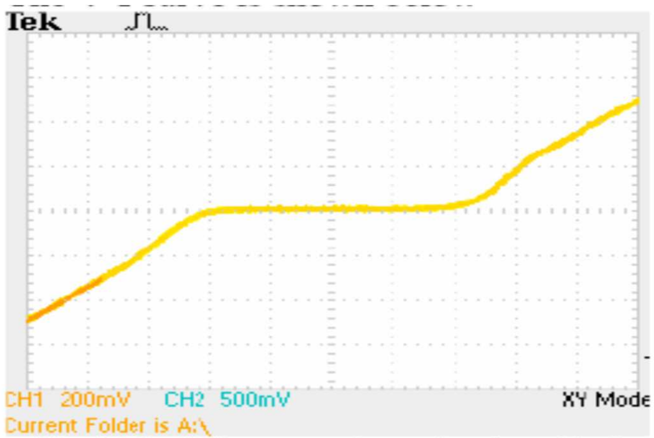

Fig. 5 The V-I curve to determine the critical current

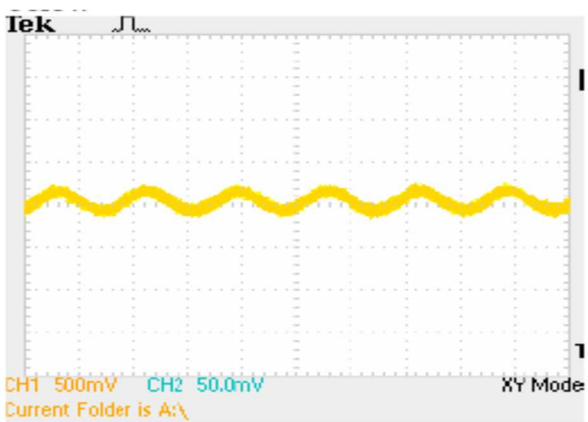

Fig. 8 The $V-\Phi$ curve to determine the modulation depth

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APPENDIX-1
Current density for the superconductors

We consider the current density for the superconductor. ψ is the order parameter of the superconductor and m^* and q^* are the mass and charge of the Cooper pairs. The current density is invariant under the gauge transformation.

$$\mathbf{J}_s = \frac{q^*}{m^*}\mathrm{Re}[\langle\psi|\hat{\mathbf{p}} - \frac{q^*}{c}\mathbf{A}|\psi\rangle],$$

This can be rewritten as

$$\begin{aligned}\mathbf{J}_s &= \frac{q^*}{m^*}\mathrm{Re}[\psi^*(\frac{\hbar}{i}\nabla\psi - \frac{q^*}{c}\mathbf{A}\psi)] \\ &= \frac{q^*}{2m^*}[(\psi^*\frac{\hbar}{i}\nabla\psi - \frac{q^*}{c}\mathbf{A}\psi^*\psi) + (-\psi\frac{\hbar}{i}\nabla\psi^* - \frac{q^*}{c}\mathbf{A}\psi^*\psi)] \\ &= \frac{q^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^{*2}|\psi|^2}{m^*c}\mathbf{A}\end{aligned}$$

The density is also gauge independent.

$$\rho_s = |\langle\mathbf{r}|\psi\rangle|^2.$$

Now we assume that

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})|e^{i\theta(\mathbf{r})}.$$

We note that

$$\psi^*(\mathbf{p}-\frac{q^*}{c}\mathbf{A})\psi(\mathbf{r}) - = \psi^*\frac{\hbar}{i}\nabla[e^{i\theta(\mathbf{r})}|\psi(\mathbf{r})|] - \frac{q^*}{c}\mathbf{A}|\psi(\mathbf{r})|^2$$

$$= |\psi(\mathbf{r})|e^{-i\theta(\mathbf{r})}\frac{\hbar}{i}[i|\psi(\mathbf{r})|e^{i\theta(\mathbf{r})}\nabla\theta(\mathbf{r}) + e^{i\theta(\mathbf{r})}\nabla|\psi(\mathbf{r})|] - \frac{q^*}{c}\mathbf{A}|\psi(\mathbf{r})|^2 .$$

$$= \hbar|\psi(\mathbf{r})|^2[\nabla\theta(\mathbf{r}) - \frac{q^*}{c\hbar}\mathbf{A}] - i\hbar|\psi(\mathbf{r})|\nabla|\psi(\mathbf{r})|$$

The last term is pure imaginary. Then the current density is obtained as

$$\mathbf{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\mathbf{A}) = q^*|\psi|^2\mathbf{v}_s$$

or

$$\hbar\nabla\theta = \frac{q^*}{c}\mathbf{A} + m^*\mathbf{v}_s.$$

Since

$$\boldsymbol{\pi} = m^*\mathbf{v}_s = \mathbf{p} - \frac{q^*}{c}\mathbf{A}$$

we have

$$\mathbf{p} = \frac{q^*}{c}\mathbf{A} + m^*\mathbf{v}_s = \hbar\nabla\theta$$

Note that $\boldsymbol{J}_s$ (or $\boldsymbol{v}_s$) is gauge-invariant. Under the gauge transformation, the wave function is transformed as

$$\psi'(\mathbf{r}) = \exp(\frac{iq^*\chi}{\hbar c})\psi(\mathbf{r}).$$

This implies that

$$\theta \to \theta' = \theta + \frac{q^*\chi}{\hbar c},$$

Since $\mathbf{A}' = \mathbf{A} + \nabla\chi$, we have

$$
\begin{aligned}
\mathbf{J}_s' &= \hbar(\nabla\theta' - \frac{q^*}{c\hbar}\mathbf{A}') \\
&= \hbar[\nabla(\theta + \frac{q^*\chi}{\hbar c}) - \frac{q^*}{c\hbar}(\mathbf{A} + \nabla\chi)]\,. \\
&= \hbar(\nabla\theta - \frac{q^*}{c\hbar}\mathbf{A})
\end{aligned}
$$

So the current density is invariant under the gauge transformation.