Quantum Hall effect

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: May 09, 2013)

In 1980, Klitzing et al. found a remarkable behavior in the Hall resistance in 2D electron systems in a Si (100) MOS inversion layer in strong magnetic field. They realized that the value of the Hall resistance is quantized into a universal quantity (h/e^2) with a dramatic accuracy; $R_{yx} = n\ (h/e^2)$. This phenomenon has come to be known as the integral quantum Hall effect (IQHE); n is integer. Subsequently (Tsui et at al., 1982), it was discovered that there exist also some rational fraction values, around which the Hall plateau can be centered; $R_{yx} = \nu\,(h/e^2)$ with $\nu = 1/3$, 2/3, 2/5, 3/5, 4/5, 2/7, and so on. This phenomenon is called as fractional quantum Hall effect (FQHE). Klitzing was awarded the 1985 Nobel Prize in Physics for the discovery of IQHE.
Tsui, Störmer, and Laughlin were awarded the 1998 Nobel Prize in Physics for the discovery of FQHE.

Here we discuss only on the physics of IQHE.

Klitzing constant;

$$R_K = h/e^2 = 25{,}812.8074434\ \Omega$$

http://physics.nist.gov/cgi-bin/cuu/Value?rk|search_for=elecmag_in!

Klaus von Klitzing (28 June 1943 in Schroda) is a German physicist known for discovery of the integer quantum Hall Effect, for which he was awarded the 1985 Nobel Prize in Physics. In 1962, von Klitzing passed the Abitur at Artland Gymnasium in Quakenbrück, Germany, before studying physics at the Braunschweig University of Technology, where he received his diploma in 1969. He continued his studies at the University of Würzburg, completing his PhD thesis Galvanomagnetic Properties of Tellurium in Strong Magnetic Fields in 1972, and habilitation in 1978. This work was performed at the Clarendon Laboratory in Oxford and the Grenoble High Magnetic Field Laboratory in France, where he continued to work until becoming a professor at the Technical University of Munich in 1980. Von Klitzing has been a director of the Max Planck Institute for Solid State Research in Stuttgart since 1985. The von Klitzing constant, $R_K = h/e^2 = 25{,}812.807449(86)\ \Omega$, is named in honor of von Klitzing's discovery of the quantum Hall effect, and is listed in the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty. The inverse of the von Klitzing constant is equal to half that of the conductance quantum value.

http://en.wikipedia.org/wiki/Klaus_von_Klitzing

Daniel Chee Tsui (born February 28, 1939, Henan Province, China) is a Chinese-born American physicist whose areas of research included electrical properties of thin films and microstructures of semiconductors and solid-state physics. He was previously the Arthur LeGrand Doty Professor of Electrical Engineering at Princeton University and adjunct senior research scientist in the Department of Physics at Columbia University, where he was a visiting professor from 2006 to 2008. Currently, he is a research professor at Boston University. In 1998, along with Horst L. Störmer of Columbia and Robert Laughlin of Stanford, Tsui was awarded the Nobel Prize in Physics for his contributions to the discovery of the fractional quantum Hall effect.
http://en.wikipedia.org/wiki/Daniel_C._Tsui

Horst Ludwig Störmer (born April 6, 1949 in Frankfurt, Germany) is a German physicist who shared the 1998 Nobel Prize in Physics with Daniel Tsui and Robert Laughlin. The three shared the prize "for their discovery of a new form of quantum fluid with fractionally charged excitations" (the fractional quantum Hall effect). He and Tsui were working at Bell Labs at the time of the experiment cited by the Nobel committee, though the experiment itself was carried out in a laboratory at the Massachusetts Institute of Technology (Laughlin did not participate in the experiment but was later able to explain its results).

http://en.wikipedia.org/wiki/Horst_L._St%C3%B6rmer

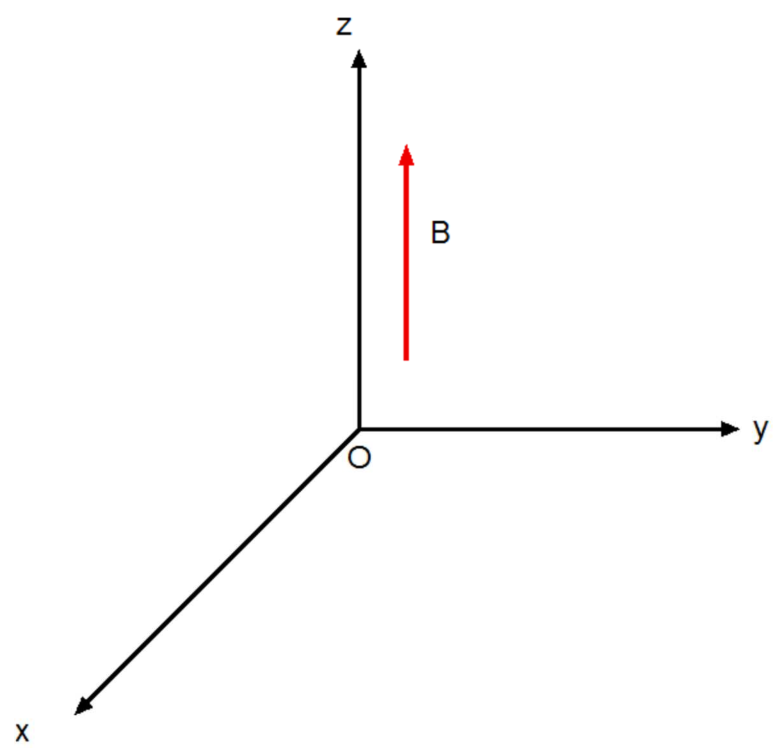
Robert Betts Laughlin (born November 1, 1950) is the Anne T. and Robert M. Bass Professor of Physics] and Applied Physics at Stanford University. Along with Horst L. Störmer of Columbia University and Daniel C. Tsui of Princeton University, he was awarded a share of the 1998 Nobel Prize in physics for their explanation of the fractional quantum Hall effect. Laughlin was born in Visalia, California. He earned a B.A. in Mathematics from UC Berkeley in 1972, and his Ph.D. in physics in 1979 at the Massachusetts Institute of Technology (MIT), Cambridge, Massachusetts, USA. Between 2004 and 2006 he served as the president of KAIST in Daejeon, South Korea.

http://en.wikipedia.org/wiki/Robert_Laughlin

1. Hall effect

We consider an electron (mass m and charge $-e$, relaxation time τ) inside the metal in the presence of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$.

$$\boldsymbol{E} = (E_x, E_y, 0), \qquad \boldsymbol{B} = (0, 0, B).$$



The equation of motion for the electron is

$$m(\frac{d}{dt} + \frac{1}{\tau})v_x = -e(E_x + \frac{B}{c}v_y),$$

$$m(\frac{d}{dt} + \frac{1}{\tau})v_y = -e(E_y - \frac{B}{c}v_x).$$

In the steady state ($d\boldsymbol{v}/dt = 0$) , we get

$$v_x = -\frac{e\tau}{m}E_x - \omega_c \tau v_y,$$

$$v_y = -\frac{e\tau}{m} E_y + \omega_c \tau v_x,$$

where

$$\omega_c = \frac{eB}{mc}. \qquad \text{(cyclotron frequency)}$$

Then we have

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{-\frac{e\tau}{m}}{1+\omega_c^2\tau^2} \begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}.$$

Since

$$\mathbf{J} = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = -ne\mathbf{v} = \frac{\sigma_0}{1+\omega_c^2\tau^2} \begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \sigma\mathbf{E},$$

with

$$\sigma_0 = \frac{ne^2\tau}{m},$$

and the conductivity tensor is given by

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_0}{1+\omega_0^2\tau^2}, \qquad \sigma_{xy} = -\sigma_{yx} = -\frac{\sigma_0\omega_0\tau}{1+\omega_0^2\tau^2}.$$

When $\omega_c\tau \gg 1$,

$$\sigma_{xx} = \sigma_{yy} = 0, \qquad \sigma_{xy} = -\sigma_{yx} = -\frac{\sigma_0}{\omega_0\tau} = \frac{nec}{B}.$$

The Onsager reciprocal relation is satisfied. We also note that

$$
\begin{aligned}
\sigma_{xy} - \frac{1}{\omega_0\tau}\sigma_{xx} &= -\frac{\sigma_0\omega_0\tau}{1+\omega_0{}^2\tau^2} - \frac{\sigma_0}{\omega_0\tau\left(1+\omega_0{}^2\tau^2\right)} \\
&= -\frac{\sigma_0(\omega_0\tau)^2}{\omega_0\tau\left(1+\omega_0{}^2\tau^2\right)} - \frac{\sigma_0}{\omega_0\tau\left(1+\omega_0{}^2\tau^2\right)} \\
&= -\frac{\sigma_0}{\omega_0\tau} = -\frac{nec}{B}
\end{aligned}
$$

where $e>0$.

2. Experimental configuration

(a) 3D case

Experimentally we need the following expression

$$
\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}^{-1} \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix},
$$

with

$$
\rho_{xx} = \rho_{yy} = \frac{\sigma_{xx}}{\sigma_{xx}{}^2 + \sigma_{xy}{}^2} = \frac{1}{\sigma_0}, \quad \rho_{xy} = -\rho_{yx} = \frac{\omega_c\tau}{\sigma_0} = \frac{B}{nec},
$$

or

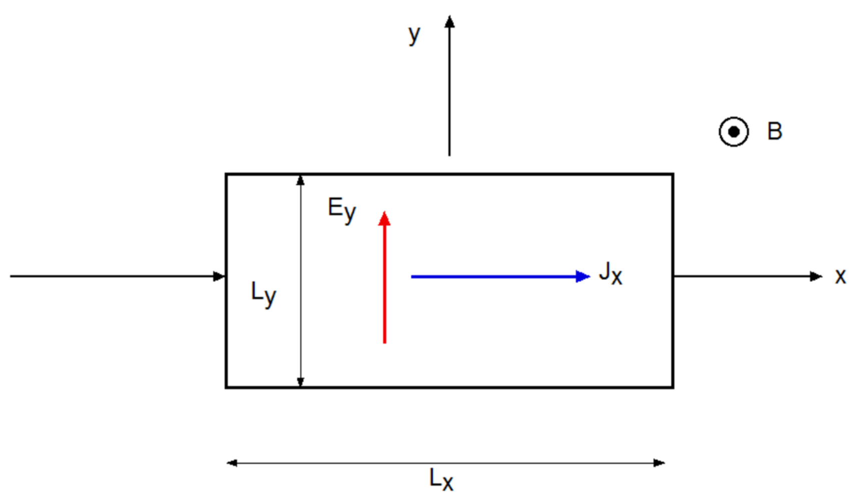


Fig. Experimental configuration for the measurement of Hall effect.

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_0} & \frac{B}{nec} \\ -\frac{B}{nec} & \frac{1}{\sigma_0} \end{pmatrix} \begin{pmatrix} J_x \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_0} J_x \\ -\frac{B}{nec} J_x \end{pmatrix}$$

Then we have

$$E_x = \frac{1}{\sigma_0} J_x \qquad \text{or} \qquad J_x = \sigma_0 E_x$$

and

$$E_y = -\frac{B}{nec} J_x$$

The Hall coefficient R_{H} is

$$R_H = \frac{E_y}{BJ_x} = -\frac{1}{nec}$$

where $-e$ is the charge of electron.

(b) 2D case

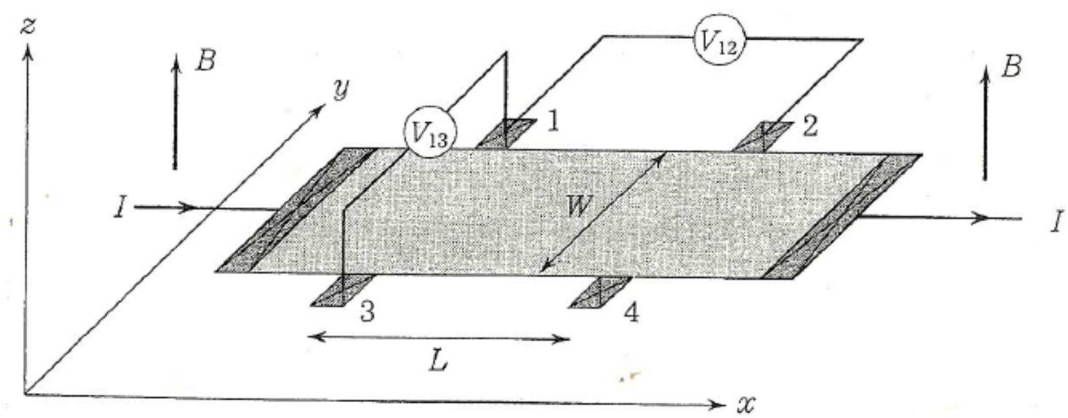


Fig. Measurement of the Hall and diagonal resistivities.

The Hall voltage is defined by

$$V_H = V_y = E_y L_y = -\frac{BL_y}{nec} J_x$$

where

$$n = \frac{N}{L_x L_y \delta t} = \frac{n_s}{\delta t},$$

where n_s is the surface concentration,

$$n_s = \frac{N}{L_x L_y},$$

$\delta t(\rightarrow 0)$ is a virtual thickness, and N is the total number of electrons in the system. Then we have

$$V_H = V_y = -\frac{B}{n_s ec}(L_y \delta t) J_x = -\frac{B}{n_s ec} I_x = R_{yx} I_x,$$

where

$$R_{yx} = -\frac{B}{n_s ec}$$

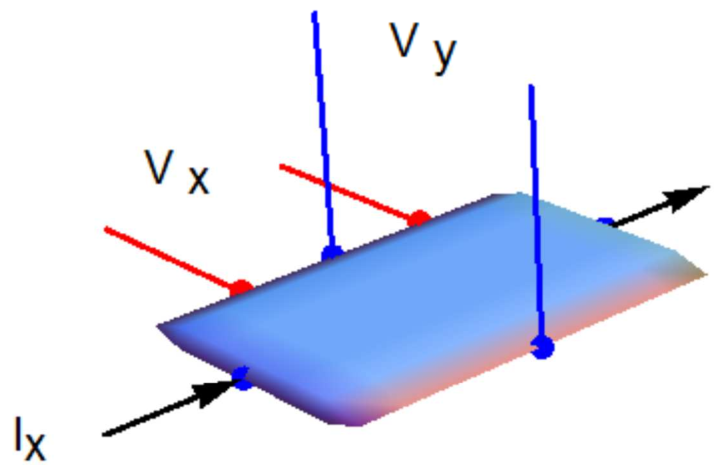
$$(L_y \delta t) J_x = I_x$$

This means that R_{yx} is in the units of Ω and is independent of the geometry of the system.

Next we consider V_x. It is obtained as

$$V_x = E_x L_x = \rho_{xx} J_x L_x = \frac{1}{\sigma_0} J_x L_x = \frac{m}{n_s e^2 \tau} \frac{L_x}{L_y} I_x$$

This means that the voltage V_x depends on the geometry of the system through L_x and L_y.



Four probes method

3. Physical meaning of σ_{xy}

The conductivity tensor is given by

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_0}{1+\omega_0{}^2\tau^2}, \qquad \sigma_{xy} = -\sigma_{yx} = -\frac{\sigma_0\omega_0\tau}{1+\omega_0{}^2\tau^2}.$$

Note that

$$\sigma_{xy} - \frac{1}{\omega_0\tau}\sigma_{xx} = -\frac{\sigma_0\omega_0\tau}{1+\omega_0{}^2\tau^2} - \frac{\sigma_0}{\omega_0\tau\left(1+\omega_0{}^2\tau^2\right)}$$

$$= -\frac{\sigma_0\left(\omega_0\tau\right)^2}{\omega_0\tau\left(1+\omega_0{}^2\tau^2\right)} - \frac{\sigma_0}{\omega_0\tau\left(1+\omega_0{}^2\tau^2\right)}$$

$$= -\frac{\sigma_0}{\omega_0\tau} = -\frac{nec}{B}$$

or

$$\sigma_{xy} = -\frac{nec}{B} + \frac{1}{\omega_0\tau}\sigma_{xx}$$

Suppose that the electric field is applied along the y direction. Then the current density along the x axis is given by

$$J_x = \sigma_{xy}E_y = -\frac{nec}{B}E_y + \frac{1}{\omega_0\tau}\sigma_{xx}E_y = -\frac{nec}{B}E_y + \frac{1}{\omega_0\tau}\sigma_{yy}E_y$$

The first term is due to the drift motion of electrons along the x axis. The drift velocity along the x axis is given by

$$\boldsymbol{v}_d = c\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} = c\frac{E_y}{B}\boldsymbol{e}_x$$

The drift current is

$$\boldsymbol{J}_d = -ne\boldsymbol{v}_d = -\frac{nec}{B}E_y\boldsymbol{e}_x$$

The second term is due to the scattering.

$$m(\frac{dv_x}{dt} + \frac{v_x}{\tau}) = F_x = (-e)E_x'$$

where τ is the relaxation time and E_x' is the effective electric field. In the steady state,

$$-\frac{mv_x}{\tau} = F_x = (-e)E_x'$$

Then the current density along the x direction is

$$\Delta J_x = \sigma_{xx} E_x' = \sigma_{xx} \frac{mv_x}{e\tau} = \sigma_{xx} \frac{m}{e\tau}\frac{c}{B} E_y = \frac{1}{\omega_0 \tau} \sigma_{xx} E_y$$

In the limit of $\omega_c \tau \to \infty$, we have

$$\sigma_{xx} = \sigma_{yy} = 0, \qquad \sigma_{xy} = -\sigma_{yx} = -\frac{\sigma_0}{\omega_0 \tau} = \frac{nec}{B}.$$

Experimentally we need the following general expression

$$\rho_{xx} = \rho_{yy} = \frac{\sigma_{xx}}{\sigma_{xx}{}^2 + \sigma_{xy}{}^2} = \frac{1}{\sigma_0}, \ \rho_{xy} = -\rho_{yx} = \frac{\omega_c \tau}{\sigma_0} = \frac{B}{nec},$$

The conductivity σ can be expressed in terms of ρ as

$$\sigma_{xx} = \sigma_{yy} = \frac{\rho_{xx}}{\rho_{xx}{}^2 + \rho_{xy}{}^2}, \qquad \sigma_{xy} = -\sigma_{xy} = -\frac{\rho_{xy}}{\rho_{xx}{}^2 + \rho_{xy}{}^2}$$

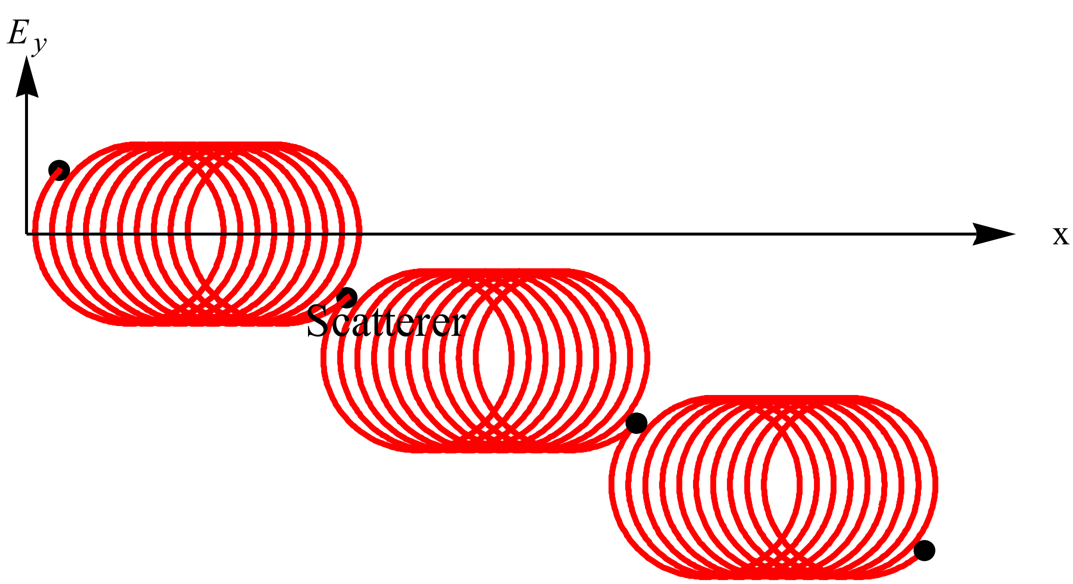


Fig. The 2D motion of electron under the strong magnetic field (the z direction). The electron undergoes a cyclotron motion. Due to the electric field along the y axis, the center of the circle shifts to the x direction at the velocity of eE_y/B. Each time the electrons collide with scatterer such as impurity, the center shift to the x-direction on the order of the radius of orbit, leading to the extra current along the x direction.

4. Landau quantization

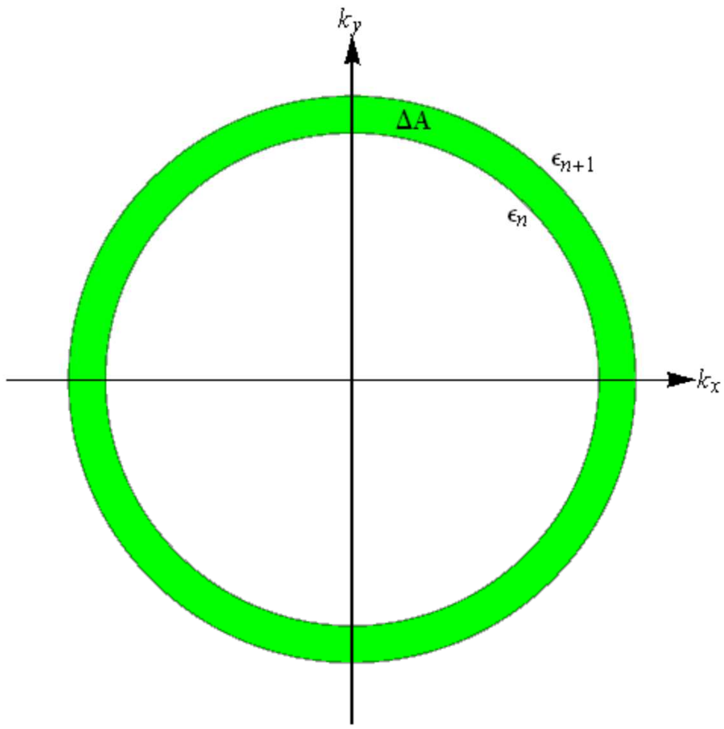


Fig. Energy contour plot of the Landau levels ε_n and ε_{n+1} in the ***k*** space.

$\Delta A(\varepsilon)$ is an area enclosed by the adjacent Landau levels in the k-space,

$$\Delta A(\varepsilon) = \hbar\omega_c \frac{\partial A(\varepsilon, k_B)}{\partial \varepsilon} = \frac{2\pi}{\hbar^2} m^* \frac{eB\hbar}{m^* c} = \frac{2\pi eB}{\hbar c},$$

The corresponding number of states is

$$D = \frac{L^2}{(2\pi)^2}\Delta A(\varepsilon) = \frac{eBL^2}{2\pi\hbar c} = \rho B$$

where

$$\rho = \frac{eB}{2\pi\hbar c} L^2 .$$

((**Note**))
Number of states per each Landau level

$$\varepsilon = \frac{\hbar^2}{2m} k^2 , \qquad d\varepsilon = \frac{\hbar^2}{m} kdk .$$

The number of states between ε and $\varepsilon +$ dε,

$$\frac{L^2}{(2\pi)^2} 2\pi kdk = \frac{mL^2}{2\pi\hbar^2} d\varepsilon = \frac{mL^2}{2\pi\hbar^2} \frac{e\hbar B}{mc} = \frac{eBL^2}{2\pi\hbar c} = \rho B$$

where

$$d\varepsilon = \hbar\omega_c = \frac{e\hbar B}{mc} ,$$ (the energy separation between the Landau levels).

and

$$\rho = \frac{eL^2}{2\pi\hbar c}$$

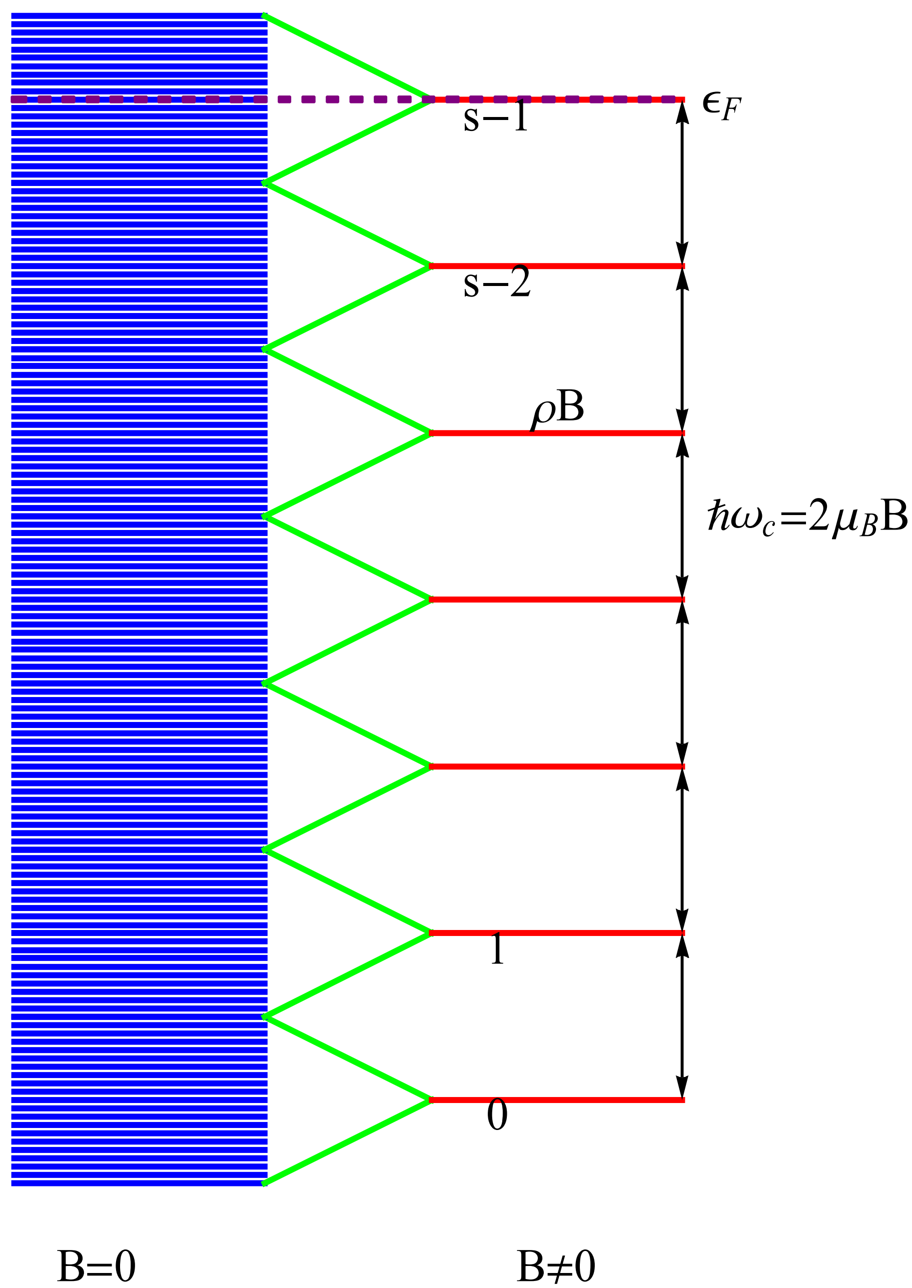


Fig. Schematic diagram of Landau levels in the presence of magnetic field *B*.

(i) The *s*-th Landau level crosses the Fermi energy when

$$\varepsilon_F = \hbar \frac{eB_s}{mc}(s+\frac{1}{2}), \qquad B_s = \frac{mc\varepsilon_F}{e\hbar}\frac{1}{s+\frac{1}{2}}.$$

(ii) The (*s*-1)-th Landau level crosses the Fermi energy (as shown in the above Fig.) when

$$\varepsilon_F = \hbar \frac{eB_{s-1}}{mc}(s - \frac{1}{2}), \qquad B_{s-1} = \frac{mc\varepsilon_F}{e\hbar}\frac{1}{s - \frac{1}{2}}$$

where

$$\omega_c = \frac{eB}{mc}$$

For $B_s < B < B_{s-1}$, or $\frac{1}{s + \frac{1}{2}} < \frac{B}{B_0} < \frac{1}{s - \frac{1}{2}}$,

the number of electrons per unit area is

$$\nu = \frac{s\rho B}{L^2} = \frac{eB}{2\pi\hbar c}s .$$

where B is the external magnetic field, which is independent of s.

$$B_0 = \frac{\varepsilon_F mc}{e\hbar}$$

Then we have the Hall resistance as

$$R_{yx} = \frac{B}{\nu ec} = \frac{2\pi\hbar}{e^2 s} = \frac{h}{e^2}\frac{1}{s},$$

Note that

$$\rho_{xx} = 0 \qquad \text{for} \qquad \frac{1}{s + \frac{1}{2}} < \frac{B}{B_0} < \frac{1}{s - \frac{1}{2}},$$

and shows a sharp peak (Dirac delta function-like) at

$$\frac{B}{B_0} = \frac{1}{s - \frac{1}{2}}, \quad \text{and} \quad \frac{B}{B_0} = \frac{1}{s + \frac{1}{2}} .$$

((Note)) B_0 is defined as

$$B_0 = \frac{\varepsilon_F mc}{e\hbar} = \frac{chn_s}{e}.$$

with

$$n_s = \frac{N}{L^2}$$

((Proof))
The total number of electron in the system with size L^2

$$N = \frac{L^2}{(2\pi)^2}\pi k_F{}^2, \qquad \varepsilon_F = \frac{\hbar^2}{2m}k_F{}^2.$$

Then we get

$$B_0 = \frac{\varepsilon_F m^* c}{e\hbar} = \frac{\hbar^2 ck_F{}^2}{2e\hbar} = \frac{c\hbar k_F{}^2}{2e} = \frac{c\hbar}{2e}\frac{(2\pi)^2 N}{L^2\pi} = \frac{ch}{e}\frac{N}{L^2} = \frac{ch}{e}n_s,$$

5. Klitzing constant

For $\hbar\omega_c >> k_B T$, the Landau levels are completely filled or completely empty. The number of electrons per unit area is

$$\nu = \frac{N}{L^2} = \frac{s\rho B}{L^2} = \frac{seBL^2}{2\pi\hbar cL^2} = \frac{seB}{2\pi\hbar c},$$

for

$$\frac{1}{s+\frac{1}{2}} < \frac{B}{B_0} < \frac{1}{s-\frac{1}{2}}$$

where s is an integer. Here we use the notation ν instead of n_s since n_s is constant but ν is dependent on the magnetic field. Then the Hall voltage is

$$\frac{V_H}{I_x} = R_{yx} = \frac{B}{vec} = \frac{B}{\frac{seB}{2\pi\hbar c}ec} = \frac{h}{se^2},$$

where s is an integer. Note that R_{yx} is free of geometrical corrections and

$$R_{yx} = \frac{25{,}813}{s}\Omega,$$

where 25,813 Ω is the value of h/e^2 expressed in ohms.

$$\Omega = \frac{V}{A} = \frac{\frac{1}{300}statV}{3\times 10^9 statA} = \frac{1}{9\times 10^{11}}\frac{s}{cm}.$$

More precisely we have

$$\frac{s}{cm} = 8.98752\times 10^{11}\Omega.$$

Then we have the von Klitzing constant,

$$R_{\text{K}} = \frac{h}{e^2}(cgs)\times(8.98752\times 10^{11}) = 25812.807449\ \Omega$$

((**Note-1**)) **Klitzing constant *R*K**: R_{K} = 25,812.807449(86) Ω,

Calculation of the Klitzing constant in SI Units and cgs units

```
Clear["Global`*"];
SIrule1 = {me → 9.1093821545 × 10^-31, eV → 1.602176487 × 10^-19,
   qe → 1.602176487 × 10^-19, ge → 2.0023193043622,
   c → 2.99792458 × 10^8, h → 6.62606896 × 10^-34,
   ℏ → 1.05457162853 × 10^-34};
CGSrule1 = {c → 2.99792 × 10^10, ℏ → 1.054571628 10^-27,
   h → 6.62606957 10^-27, me → 9.10938215 10^-28, qe → 4.8032068 × 10^-10,
   eV → 1.602176487 × 10^-12};
```

$$\text{RKSI} = \frac{h}{qe^2} \;/.\; \text{SIrule1}$$

25 812.8

$$\text{RKCGS} = \left(\frac{h}{qe^2} \;/.\; \text{CGSrule1}\right)$$

2.87206×10^{-8}

$$\text{ratio} = \frac{\text{RKSI}}{\text{RKCGS}}$$

8.98756×10^{11}

Note that
1 s/cm (cgs units) = 8.98752 x 10^{11} Ω (SI units)

((**Note-2**)) What is the practical units of R_K?

$$\frac{h}{e^2} = \frac{J \cdot s}{e^2} = \frac{[\frac{eV}{e}]}{\left[\frac{e}{s}\right]} = \frac{[V]}{[I]} = [\Omega]$$

where we use the relation $W=qV$ (energy) and $I = \frac{\Delta Q}{\Delta t}$.

6. Total number of electrons

The total number of electrons:

$$N = s\rho B = s\frac{eBL^2}{2\pi\hbar c} = s\frac{\Phi}{\frac{2\pi\hbar c}{e}} = s\frac{\Phi}{2\Phi_0} = s\frac{\Phi}{\phi_0}$$

The total charge Q in all the Landau levels below ε_F is given by

$$Q = -eN = -es\frac{\Phi}{\phi_0} = -\frac{se^2BL^2}{2\pi\hbar c},$$

where

$$\rho = \frac{eL^2}{2\pi\hbar c},$$

$\Phi = BL^2$ the total magnetic flux

$2\Phi_0 = \phi_0 = \frac{hc}{e} = \frac{2\pi\hbar c}{e}$ quantum flux

Then the charge in each Landau level is

$$\frac{Q}{s} = -e\frac{\Phi}{\phi_0} = -\frac{eBL^2}{\frac{2\pi\hbar c}{e}} = -\frac{e^2BL^2}{2\pi\hbar c}.$$

7. Simulation of the quantum Hall effect

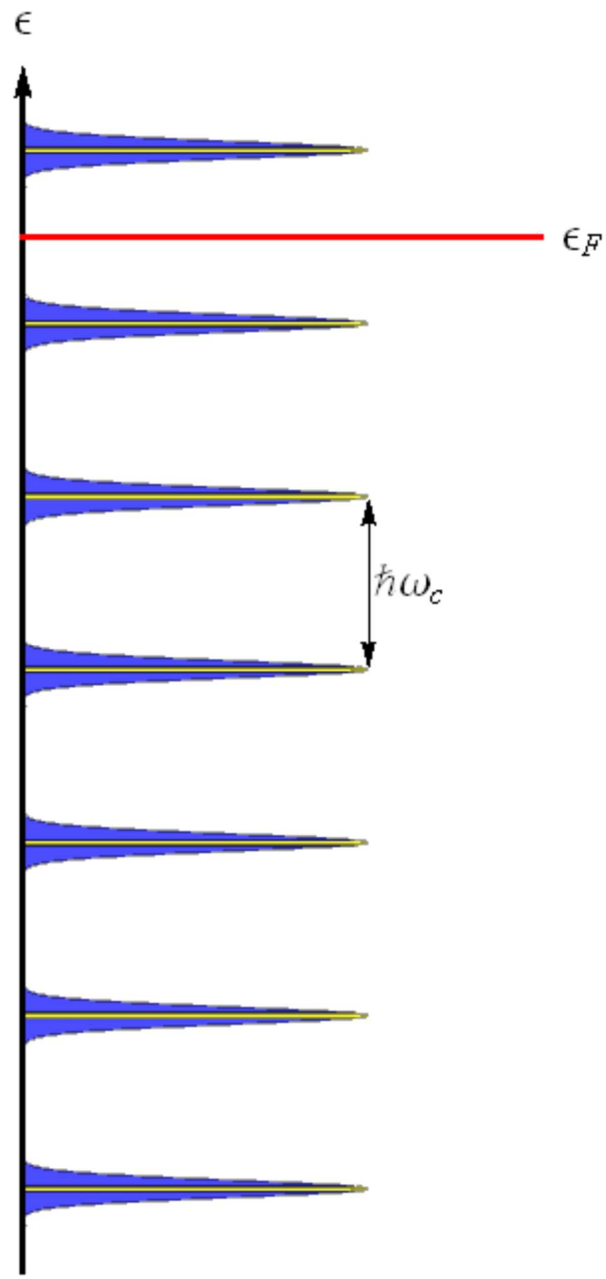


Fig. Density of states in real 2D system, with impurities and other imperfections.

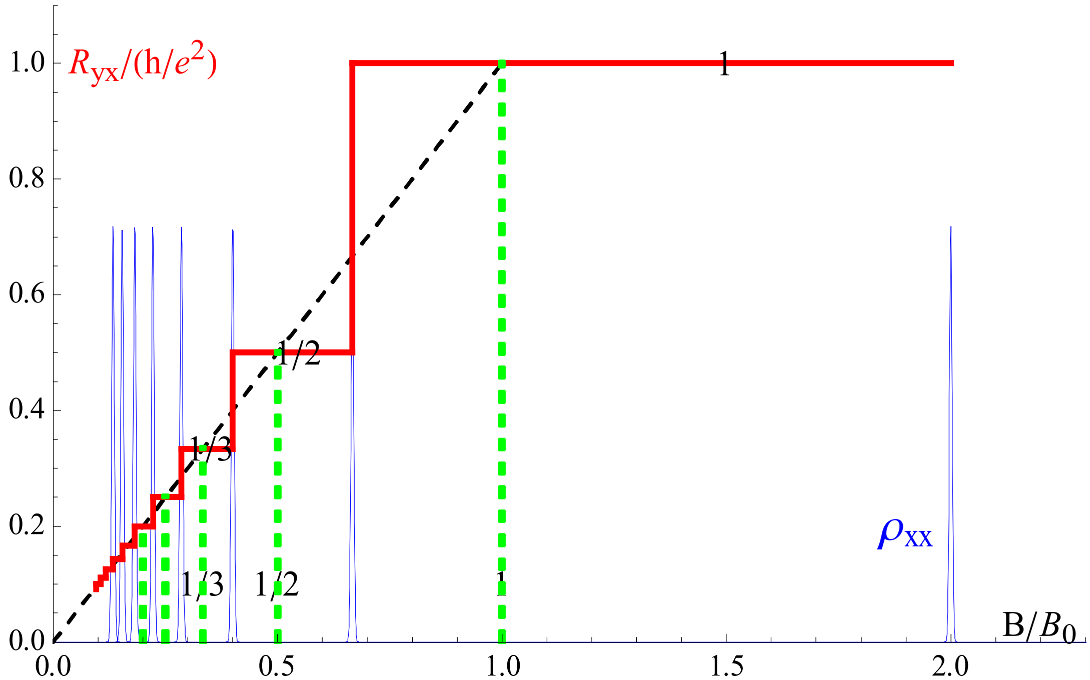


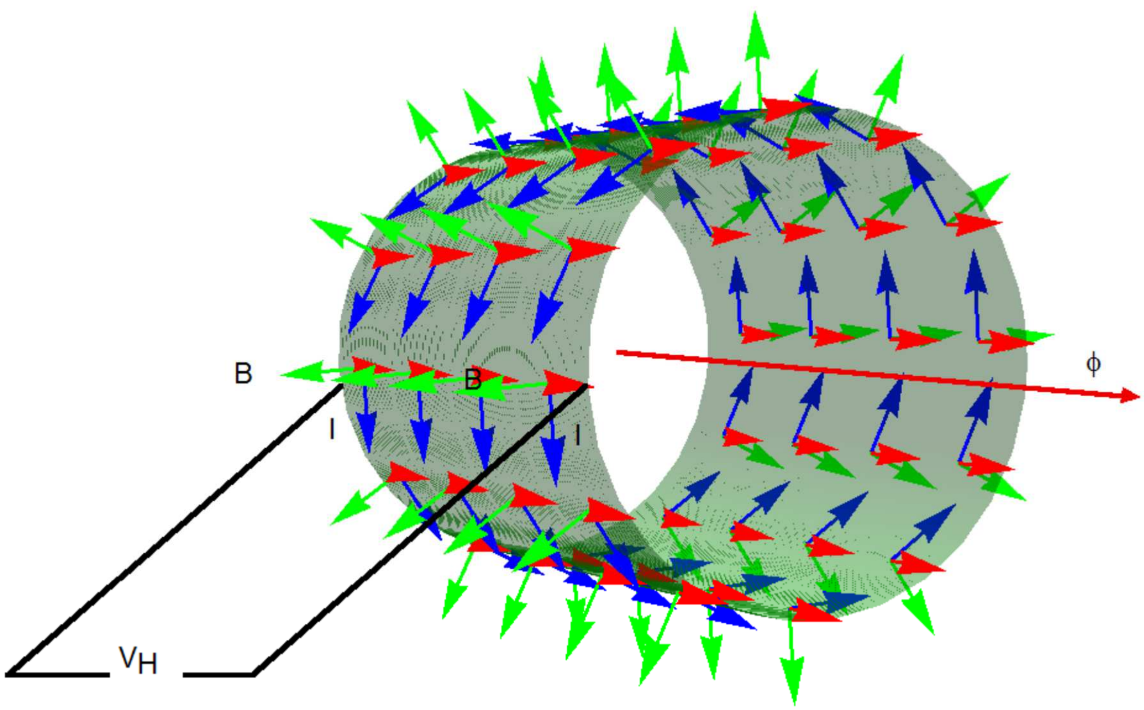
Fig. The integer quantum Hall effect which is illustrated schematically. $R_{yx}/(e^2/h)$ and ρ_{xx} (arbitrary scale) as a function of B/B_0. R_{yx} is the Hall resistance and ρ_{xx} is the longitudinal resistivity. The dashed line denotes the classical Hall resistance. ($R_{yx}/(e^2/h) = B/B_0$). The scale of the ρ_{xx} is arbitrary. $h/e^2 = 25.8128$ kΩ. $B_0 = chn_s/e$.

---

8. Laughlin's thought experiment

R.B. Laughlin, Phys. Rev. B**23**, 5632 (1981)

The 2D electron system is wrapped around to form a cylinder. The magnetic field is applied normal to the cylinder surface. The current I circles the loop. The Hall voltage V_H is produced between one edge of the cylinder and the other, perpendicular to both B and I. The circulating current I is accompanied by a small magnetic flux ϕ that threads the current loop.



Laughlin thought experiment

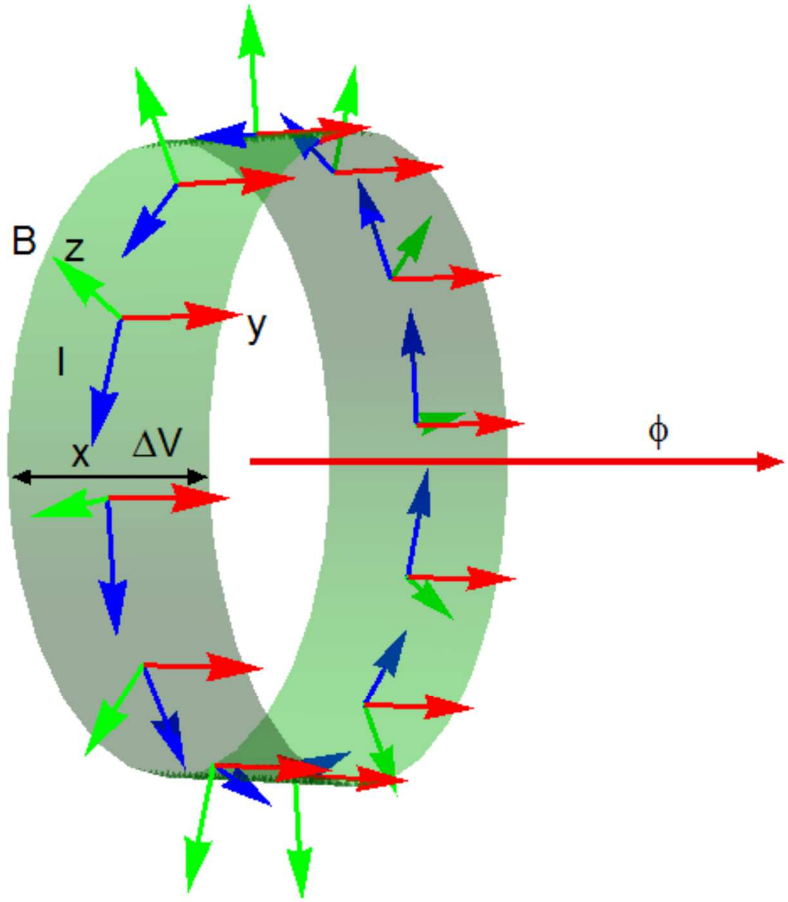


Fig. Model of a 2D metallic loop used for the derivation of the quantum Hall effect.

U: total electronic energy. The vector potential $\boldsymbol{A}$ has only the direction along the $\boldsymbol{e}_\theta$ axis (along the loop).

$$\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} = \int \boldsymbol{B}^0 \cdot d\boldsymbol{a} = \phi = \oint \boldsymbol{A} \cdot d\boldsymbol{r} = AL\,.$$

ϕ is the total magnetic flux and $\boldsymbol{B}^0$ is the magnetic field due to the current I along the ribbon.

$$\frac{\partial U}{\partial t} = -I_x V_x$$

$$V_x = \oint \mathbf{E} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = -\frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{B}^0 \cdot d\mathbf{a} = -\frac{1}{c}\frac{\partial}{\partial t}\phi$$

Then we get

$$\frac{\partial U}{\partial t} = \frac{I_x}{c}\frac{\partial \phi}{\partial t},$$

or

$$I_x = c\frac{\frac{\partial U}{\partial t}}{\frac{\partial \phi}{\partial t}} = c\frac{\partial U}{\partial \phi} = c\frac{\partial U}{\partial (AL)} = \frac{c}{L}\frac{\partial U}{\partial A}$$

9. Gauge transformation

In the presence of a magnetic field $\boldsymbol{B}$,

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B} \times \boldsymbol{r}) = \frac{1}{2}\begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 0 & 0 & B \\ x & y & z \end{vmatrix} = \frac{1}{2}(-By, Bx, 0) \qquad \text{(Symmetry gauge)}$$

Gauge transformation (I):

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$$

with

$$\chi = -\frac{1}{2}Bxy$$

Then we get a new vector potential as

$$\boldsymbol{A}' = (-By, 0, 0) = (A, 0, 0)$$

where

$A = -By$.

The Gauge transformation (II):

$$\boldsymbol{A}'' = \boldsymbol{A}' + \nabla\chi'$$

with

$$\chi' = Bxy = -Ax$$

Then we have

$$\mathbf{A}'' = (0, Bx, 0)\,,$$

In summary: vector potential and the wave function

(a) First gauge

$$\boldsymbol{A} = \frac{B}{2}(-y, x, 0)\,, \qquad \psi$$

(b) Second gauge (Landau gauge)

$$\mathbf{A}' = (-By, 0, 0) \qquad \psi' = \exp(-\frac{ie}{\hbar c}\chi)\psi$$

with

$$\chi = -\frac{B}{2}xy$$

(c) Third gauge

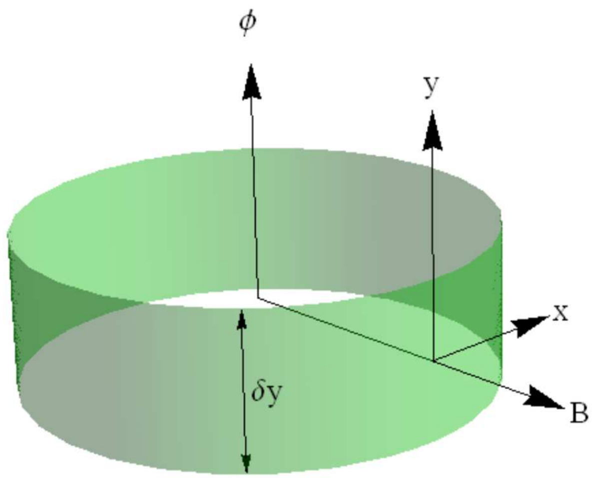
$$\mathbf{A}'' = (0, Bx, 0) \qquad \psi'' = \exp(-\frac{ie}{\hbar c}\chi')\psi'$$

with

$$\chi' = Bxy$$

Then we have

$$\psi' = \psi'' \exp(\frac{ie\chi'}{\hbar c}) = \psi'' \exp(\frac{ieBxy}{\hbar c})$$



At the fixed y, $\psi'(x,y)$ should satisfies the boundary condition as $\psi''(x, y)$.

$$\psi''(x, y) = \psi'(x, y) \exp(-\frac{ieBxy}{\hbar c}) = \psi'(x + L, y) \exp[-\frac{ieBy(x + L)}{\hbar c}] = \psi''(x + L, y)$$

and

$$\psi'(x, y) = \psi'(x + L, y)$$

Then we get

$$\exp(-\frac{ieByL}{\hbar c}) = 1, \qquad \text{or} \qquad \exp(\frac{ieAL}{\hbar c}) = 1$$

since A = -By. Thus we have

$$\frac{eAL}{\hbar c} = 2\pi n.$$

The magnetic flux ϕ is given by

$$\phi = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l} = AL = -ByL$$

or

$$\phi = AL = 2\pi \frac{\hbar c}{e} n = 2\frac{hc}{2e} n = 2\Phi_0 n = \phi_0 n$$

where Φ_0 is the quantum fluxoid. In other words, the magnetic flux is quantized. When the magnetic flux ϕ discretely changes by $2\Phi_0$, correspondingly, the value of y changes as follows.

$$\Delta\phi = -2\pi \frac{\hbar c}{e} = -2\Phi_0 = -BL\delta y$$

or

$$\delta y = \frac{2\Phi_0}{BL} = \frac{2\pi\hbar c}{eBL}$$

An increase δA that corresponds to the magnetic flux increases is equivalent to a displacement of an extended state by $\frac{\delta A}{B} = -\delta y$ in the y direction. Since $\delta\phi = L\delta A = -BL\delta y$, the change of $\delta\phi$ causes a motion of the entire electron gas to the y direction.

10. The expression of *U* (Landau gauge)

$$H = \frac{1}{2m}(\mathbf{p} + \frac{e}{c}\mathbf{A})^2 + eEy$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Landau gauge:

$$\boldsymbol{A} = -By\boldsymbol{e}_x$$

Schrödinger equation

$$H\psi = \varepsilon\psi$$

$$\frac{1}{2m}[(p_x - \frac{eB}{c}y)^2 + p_y{}^2 + eE_y y]\psi = \varepsilon\psi$$

We put

$$\psi = Y(y)e^{ik_x x}$$

$$\frac{1}{2m}[(\hbar k_x - \frac{eB}{c}y)^2 - \frac{\hbar^2}{2m}\frac{d^2}{dy^2} + eE_y y]Y(y) = \varepsilon Y(y)$$

Note that

$$\frac{1}{2m}(-\frac{eB}{c})^2(y - \frac{c\hbar}{eB}k_x)^2 = \frac{1}{2}m\omega_c{}^2(y - \frac{c\hbar}{eB}k_x)^2$$

Then we get

$$\frac{1}{2m}[(\hbar k_x - \frac{eB}{c}y)^2 - \frac{\hbar^2}{2m}\frac{d^2}{dy^2} + eE_y y]Y(y) = \varepsilon Y(y)$$

Here

$$[\frac{1}{2}m\omega_c{}^2(y - \frac{c\hbar}{eB}k_x)^2 - \frac{\hbar^2}{2m}\frac{d^2}{dy^2} + eE_y y]Y(y) = \varepsilon Y(y)$$

with

$$\begin{aligned}\frac{1}{2}m\omega_c{}^2(y - \frac{c\hbar}{eB}k_x)^2 + eEy &= \frac{1}{2}m\omega_c{}^2(y - \frac{c\hbar}{eB}k_x)^2 + \frac{2eE_y y}{m\omega_c{}^2} \\ &= \frac{1}{2}m\omega_c{}^2\{(y - \frac{c\hbar}{eB}k_x + \frac{eE_y}{m\omega_c{}^2})^2 + \frac{eE}{m\omega_c{}^2}(\frac{2c\hbar k_x}{eB} - \frac{eE_y}{m\omega_c{}^2})\}\end{aligned}$$

Then the Schrödinger equation can be rewritten as

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega_c{}^2(y - y_0)^2]Y(y) = [\varepsilon - \frac{eE_y}{2}(\frac{2c\hbar k_x}{eB} - \frac{eE_y}{m\omega_c{}^2})]Y(y)$$

where

$$y = \frac{c\hbar}{eB}k_x - \frac{eE_y}{m\omega_c^{\ 2}}$$

and

$$\varepsilon_{n,k_x} = (n + \frac{1}{2})\hbar\omega_c + eE_y y + \frac{1}{2}m(\frac{cE_y}{B})^2$$

which depends on the coordinate y. We put

$$U = (n + \frac{1}{2})\hbar\omega_c + eE_y y + \frac{1}{2}m(\frac{cE_y}{B})^2$$

When y increases, the energy U increases. Note that

$$\frac{\partial U}{\partial y} = eE_y$$

10. Quantum Hall effect

In summary we have

$$\frac{\partial U}{\partial y} = eE_y$$

and

$$\phi = AL = -BLy = 2\Phi_0 n \qquad (1)$$

When ϕ is constant (quantized), $\delta y = 0$.

$$I_x = 0.$$

Each Landau level contributes an energy change $\Delta U = eE_y \Delta y$. When ϕ discretely changes by $\Delta\phi = 2\Phi_0$, we have

$$I_x = c\frac{\partial U}{\partial \phi} = c\frac{\Delta U}{\Delta \phi} = \frac{ceE_y\Delta y}{\Delta \phi} = \frac{ceE_y\Delta y}{2\Phi_0} = \frac{ceE_y\Delta y}{2\frac{2\pi\hbar c}{2e}} = \frac{e^2E_y\Delta y}{h} = \frac{e^2V_H}{h}$$

for each Landau level. For s Landau levels, we have the total current given by

$$I_x = sc\frac{\partial U}{\partial \phi} = sc\frac{\Delta U}{\Delta \phi} = s\frac{ceE_y\Delta y}{\Delta \phi} = \frac{cseE_y\Delta y}{2\Phi_0} = \frac{cseE_y\Delta y}{2\frac{2\pi\hbar c}{2e}} = \frac{se^2E_y\Delta y}{h} = \frac{se^2V_H}{h}$$

or

$$R_H = \frac{V_H}{I_x} = \frac{h}{se^2}.$$

where

$$V_H = E_y\Delta y$$

((Note))
See the animation in the following URL
https://en.wikipedia.org/wiki/Quantum_Hall_effect
(Quantum Hall effect, Wikipedia)

11. Another method

The above expression for R_H can be also derived as follows.

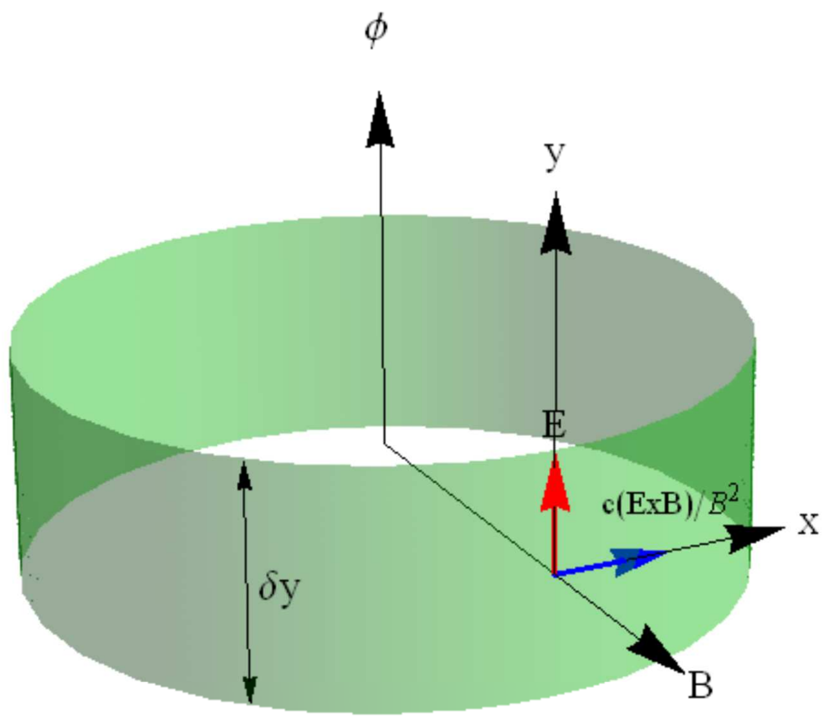


Fig. Drift velocity in the y direction.

For s full Landau levels, the electron density is

$$\frac{s\rho B}{L^2} = \frac{seL^2 B}{2\pi\hbar c L^2} = \frac{seB}{hc} = s\frac{BL^2}{\frac{2\pi\hbar c}{e}} = s\frac{\Phi}{2\Phi_0} = s\frac{\Phi}{\phi_0}.$$

The drift velocity is given by

$$v_d = c\frac{\boldsymbol{E}\times\boldsymbol{B}}{\boldsymbol{B}^2} = -\frac{cE_y}{B}\boldsymbol{e}_x.$$

Then the current along the x axis is

$$I_x = \Delta y[(-e)\frac{seB}{hc}](-\frac{cE_y}{B}) = \frac{se^2}{h}E_y\Delta y = \frac{se^2}{h}V_H \, .$$

or

$$R_H = \frac{V_H}{I_x} = \frac{h}{se^2}$$

11. Localized state and extended state

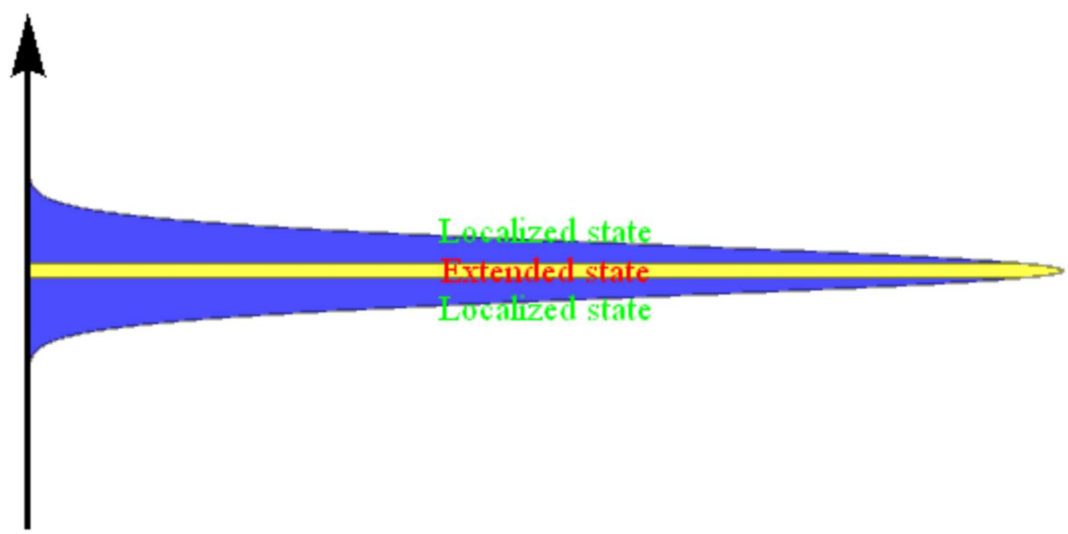


Fig. Landau level with localized state and extended state. The disorder produces localization of the states in the wings of each broadened level. Only the un-shaded states near the center of the level remain extended and capable of carrying a current. The localized states act as a kind of reservoir

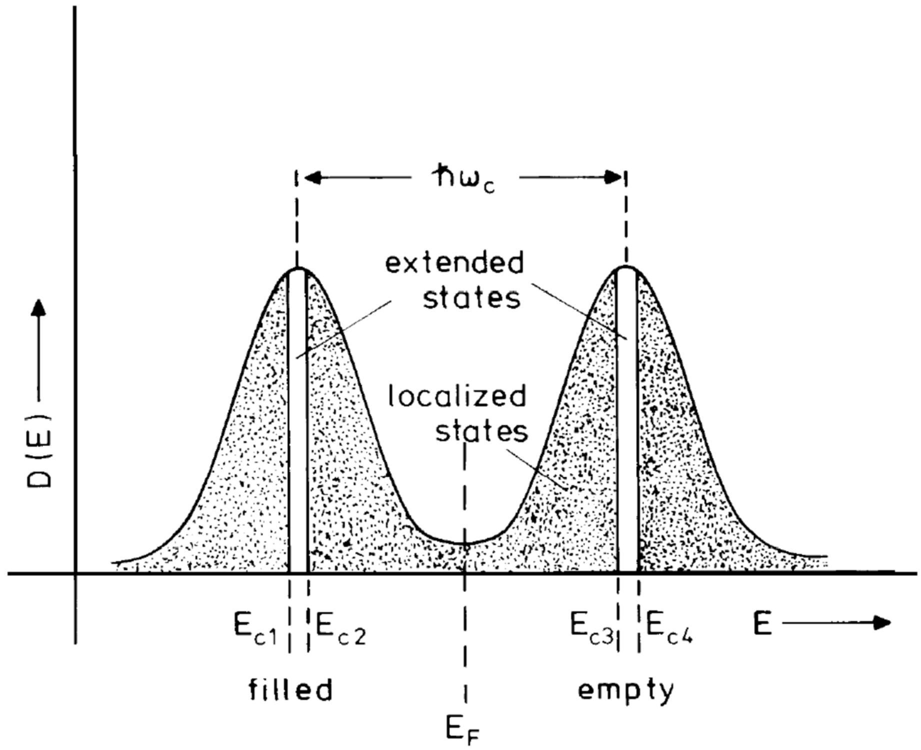
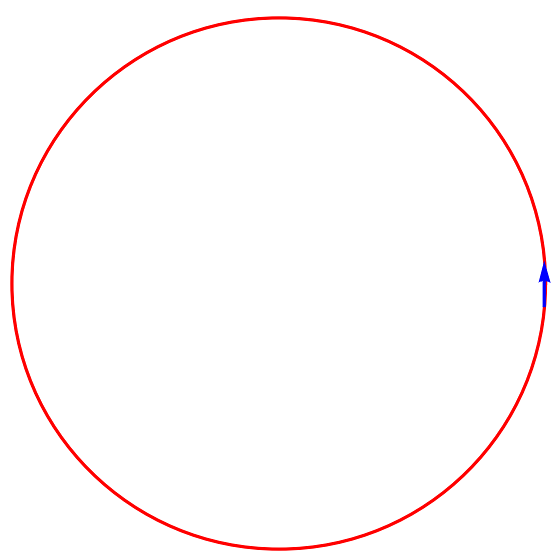


Fig. Model for the broadened density of states in a magnetic field. Mobility edges close to the center of the Landau levels separate extended states from the localized states. (from K. von Klitzing, Nobel Le3cture (12/9/1985); The Quantized Hall effect).

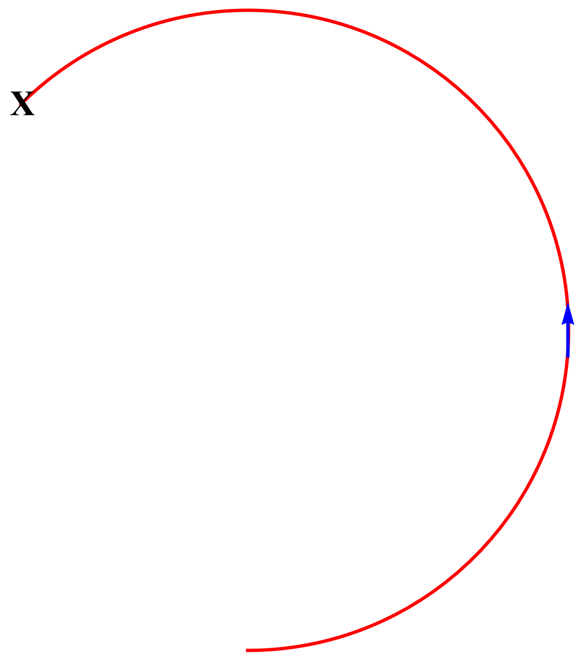
(a) Extended states: continuous around the loop



Extended state

The extended state encloses ϕ and their-energy may be changed.

(b) Localized states, which are not continuous around the loop



Localized state

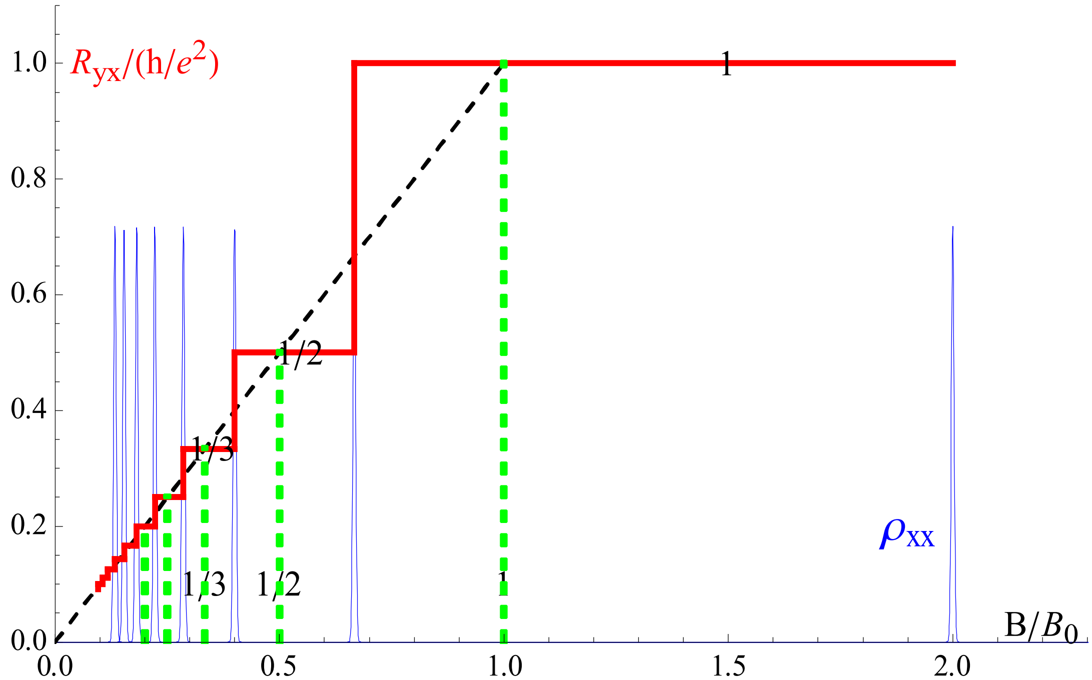
The localized states are unaffected to the first order because they do not enclose any significant part of ϕ.

12. Contribution of the extended state and the localized state to ρ_{xx} and V_{yx}

In the measurement of quantum Hall effect, it is observed that

$V_x = 0 \rightarrow$ $\rho_{xx} = 0$

$V_{yx} = \frac{h}{se^2}$ (s: integer). $\rightarrow$ $V_{yx} = \frac{h}{se^2}$



Klitzing has found that the plateau part of V_{yx} takes a universal constant which is independent of the systems used in the measurements. We note that

$$\sigma_{xx} = \sigma_{yy} = \frac{\rho_{xx}}{\rho_{xx}{}^2 + \rho_{xy}{}^2}, \qquad \sigma_{xy} = -\sigma_{xy} = -\frac{\rho_{xy}}{\rho_{xx}{}^2 + \rho_{xy}{}^2}$$

When $\rho_{xx} = 0$, the system is in the localized state.

$$\sigma_{xx} = \sigma_{yy} = 0, \qquad \sigma_{xy} = -\sigma_{xy} = -\frac{1}{\rho_{xy}} = \frac{1}{\rho_{yx}} = -\frac{se^2}{h}$$

When $\rho_{xx} \neq 0$ the system is in the extended state; ρ_{xx} shows a delta-function line peak, when the Landau level crosses the Fermi level. The effect of $\Delta\phi$ is equivalent to a translation in the y direction. In the ideal case, all the wave functions of the electrons are translated by

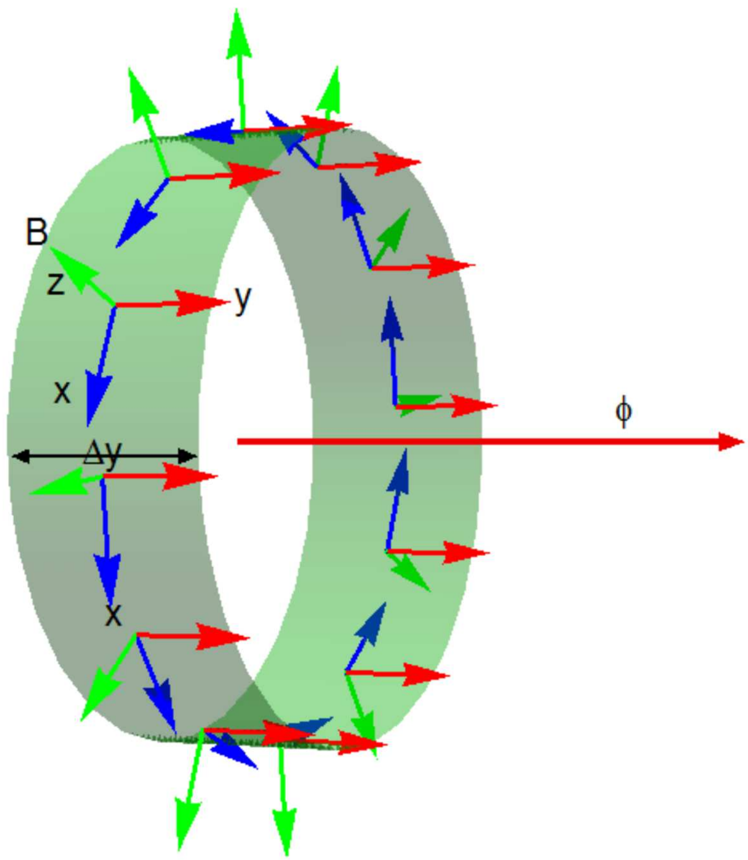
$$\Delta y = -\frac{\Delta\phi}{LB}$$

when the solenoid flux changes by $\Delta\phi$ of

$$2\Phi_0 = \frac{hc}{e}$$

in which case the gauge transformation is possible, corresponds to a shift of the wave functions by a distance equal to the separation between the center co-ordinates.

((**Note**))
The quantization of the real magnetic flux passing through the area $L\Delta y$ along the *z* direction.



When $\Delta y = \frac{2\Phi_0}{LB}$, the total magnetic flux passing through the cylinder surface (along the *z* axis, the direction of the magnetic field ***B***) is $2\Phi_0$. Note that total surface area of the cylinder is

$$\Delta S = L\Delta y .$$

The total magnetic flux (which is not the same as the magnetic flux ϕ) is given by

$$2\Phi_0 = B\Delta S \, .$$

This means that the real magnetic flux passing through the area $L\Delta y$ along the z direction is also quantized.

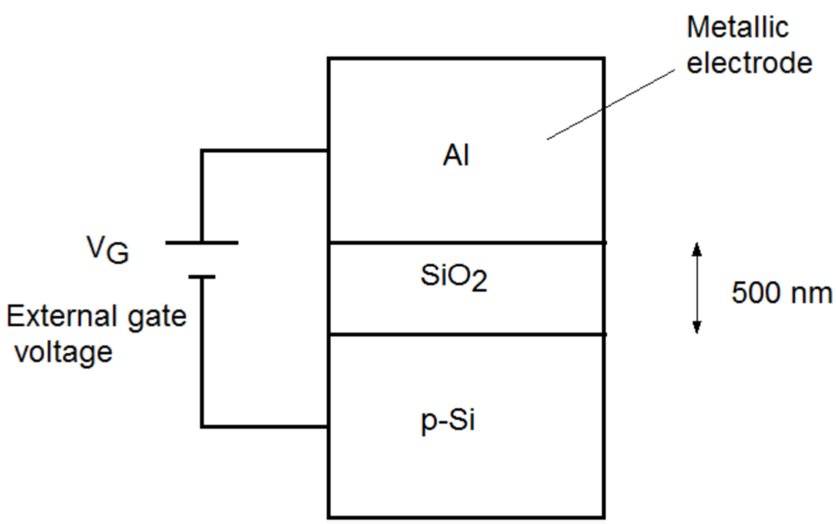
---

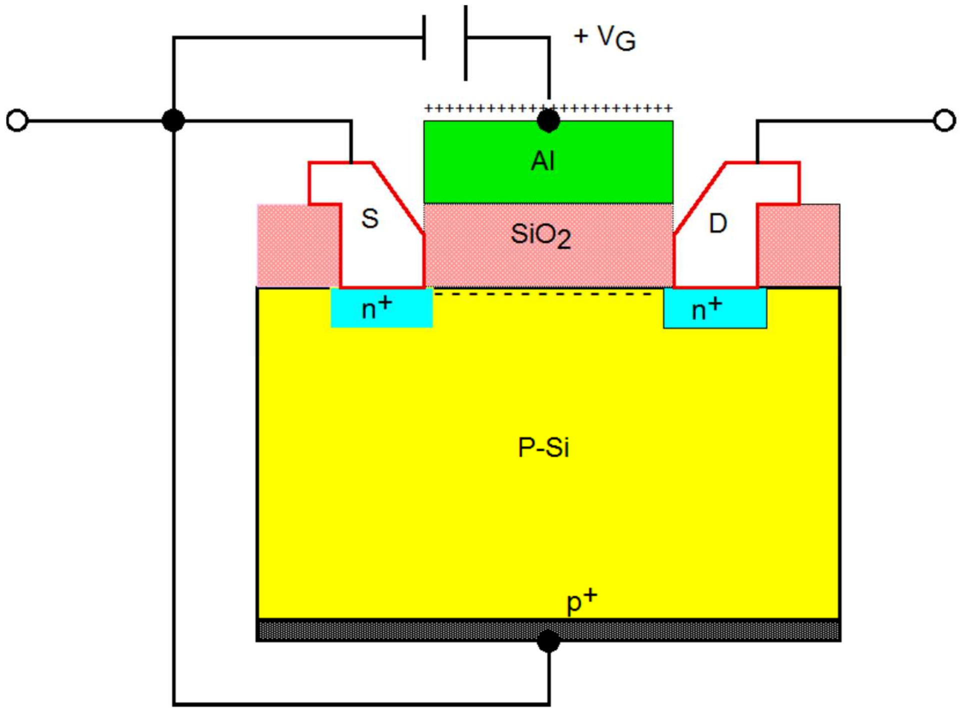
13. Systems for the experiments of Quantum Hall effect

(a) MOSFET (metal oxide semiconductor field electric transitor)

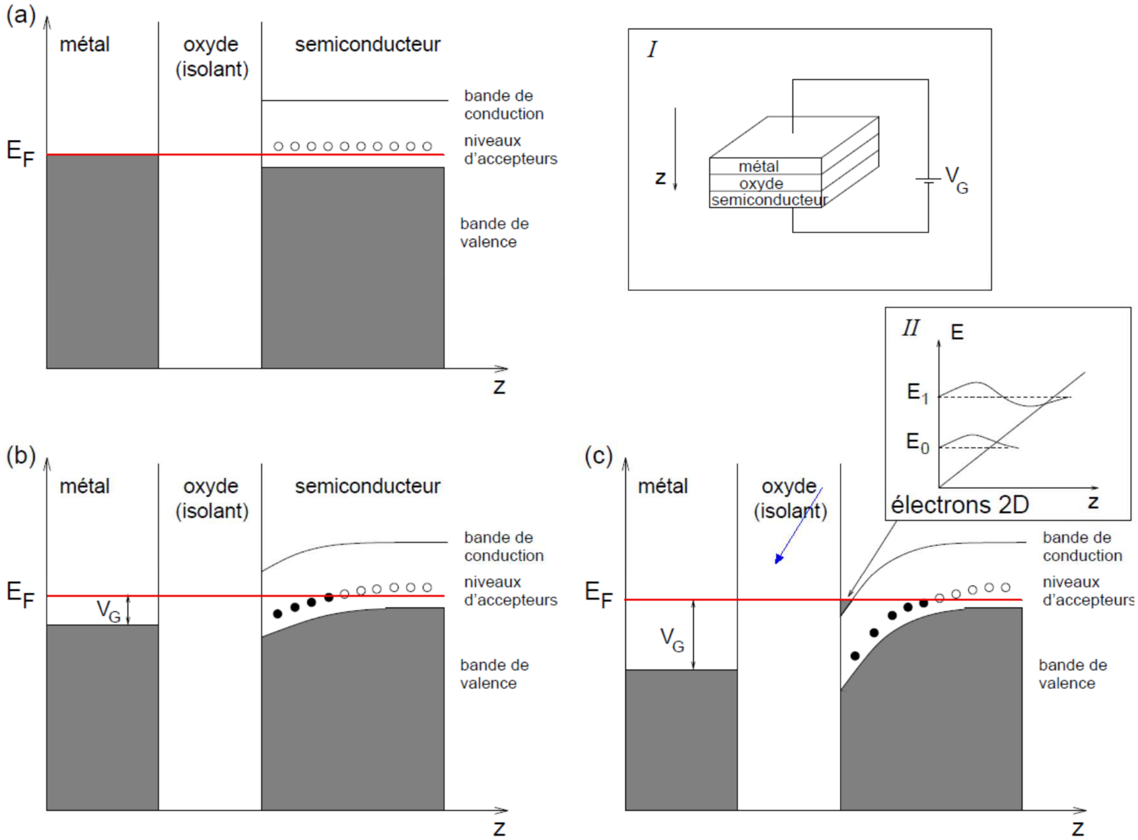
Al-SiO_2 (insulator) –Si (semiconductor)

The inversion layers are formed at the interface between a semiconductor and insulator, or between two semiconductors, with one of them acting as a insulator.





The source S and drain D contacts are heavily doped n+ regions with Al caps.

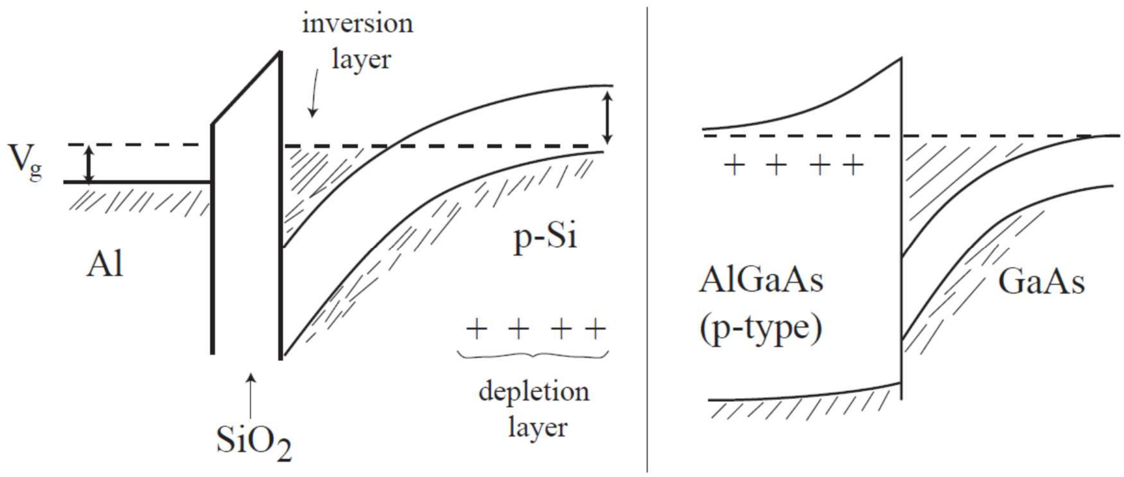


S

Fig. Metal-Oxide Field Effect Transistor (MOSFET). The inset I is a schematic view of a MOSFET. (a) Energy level structure. In the metallic part, the band states are occupied up to the Fermi level ε_F. The oxide is an insulating film. The Fermi level in the semiconductor falls in the gap between the valence band and the conduction band. There are acceptor states doped close to the valence band, but above the Fermi level ε_F. (b)The chemical potential in the metal is controlled by a gate bias V_G. The introduction of holes results in a band bending in the semiconducting part and (c) when the gate bias exceeds a certain value, the conduction band is filled close to the insulating interface, and a 2D electron gas is formed. The confining potential has a triangular profile with electric sub-bands which are represented in the inset II.

From

P. Lederer and M.O. Goerbig, Introduction to the Quantum Hall Effects (lecture notes, 2006). http://staff.science.uva.nl/~jcaux/DITP_QHE_files/LedererQHE.pdf



From
A.J. Leggett, The quantum Hall effect: general considerations.
http://online.physics.uiuc.edu/courses/phys598PTD/fall09/L16.pdf

(b) GaAs (semiconductor) – $Al_xGa_{1-x}As$

It is arranged that an electric field perpendicular to the interface attracts electrons from the semiconductor to it. These electrons sit in a quantum well created by this field and the interface. The motion perpendicular to the interface is quantized and thus has a fundamental rigidity which freezes out motional degrees of freedom in this direction.

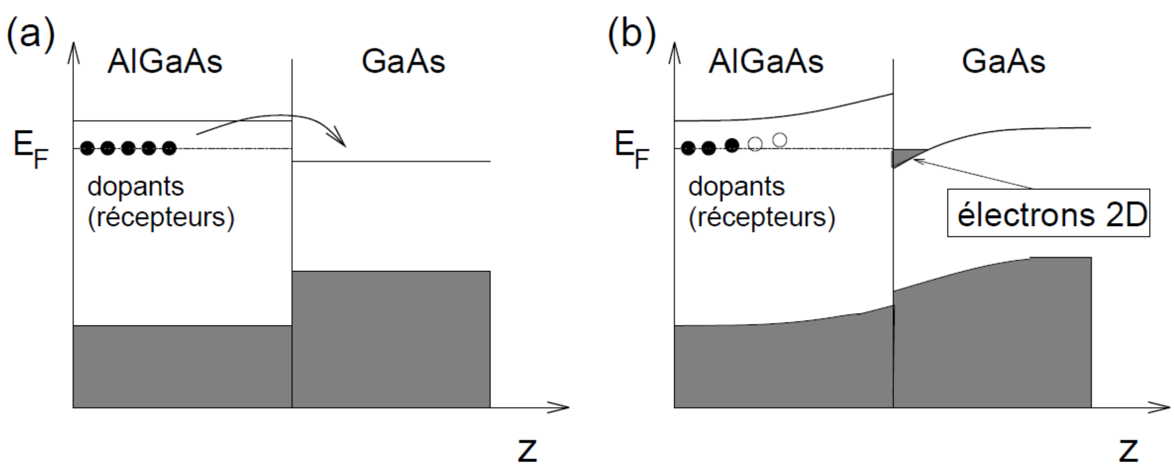


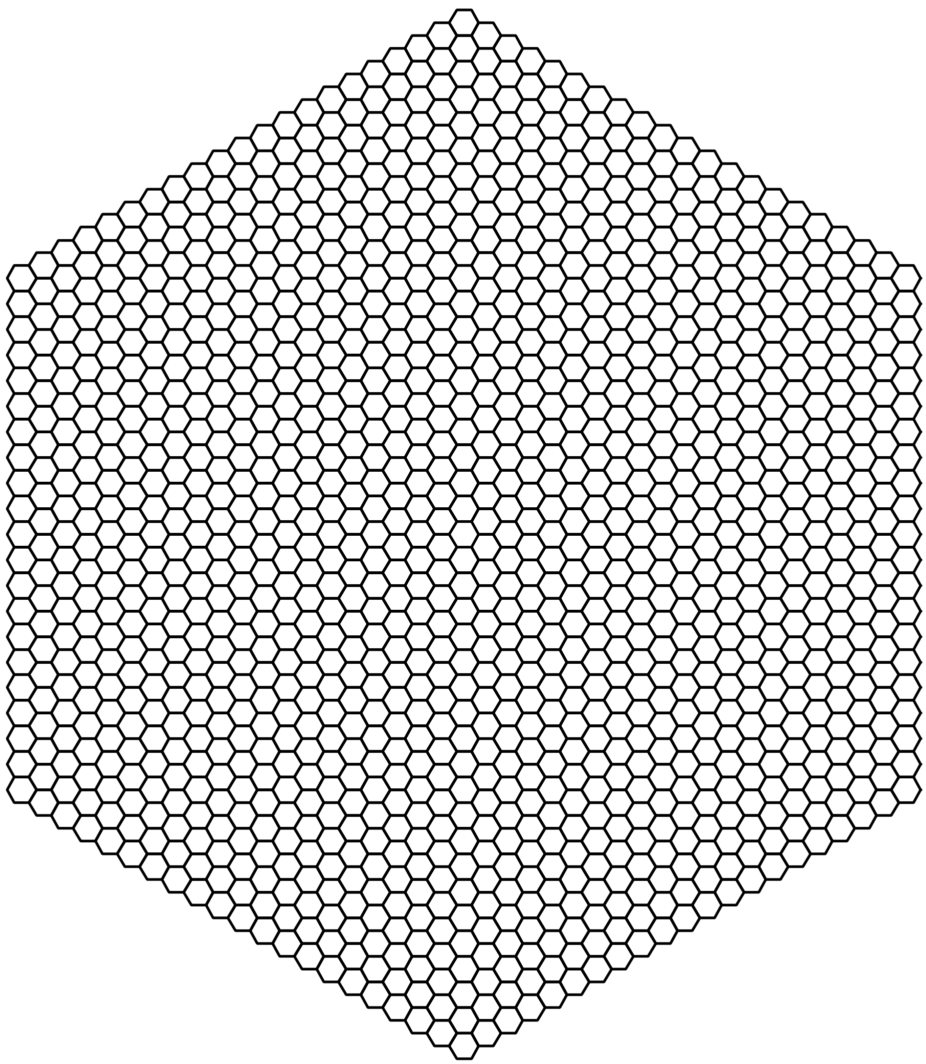
Fig. Semiconducting (GaAs/AlGaAs) heterostructure. (a) A layer of (receptor) dopants lies on the AlGaAs side, at a certain distance from the interface. The Fermi energy is locked to the dopant levels. The bottom of the GaAs conduction band lies lower

than those levels so electron close to the interface migrate to the GaAs conduction band. (b) This polarization leads to a band bending close to the interface, and a 2D electron gas forms, on the GaAs side.
From
P. Lederer and M.O. Goerbig, Introduction to the Quantum Hall Effects (lecture notes, 2006). http://staff.science.uva.nl/~jcaux/DITP_QHE_files/LedererQHE.pdf

(c) Graphene.

In a single atomic layer of carbon (grapheme), the quantum Hall effect can be measured reliably even at room temperature, which makes possible QHE resistance standards becoming available to a broader community, outside a few national institutions.

(see Novoselov et al, Science **9** Vol. 315 no. 5817 p. 1379, "Room-temperature quantum Hall effect in grapheme.")



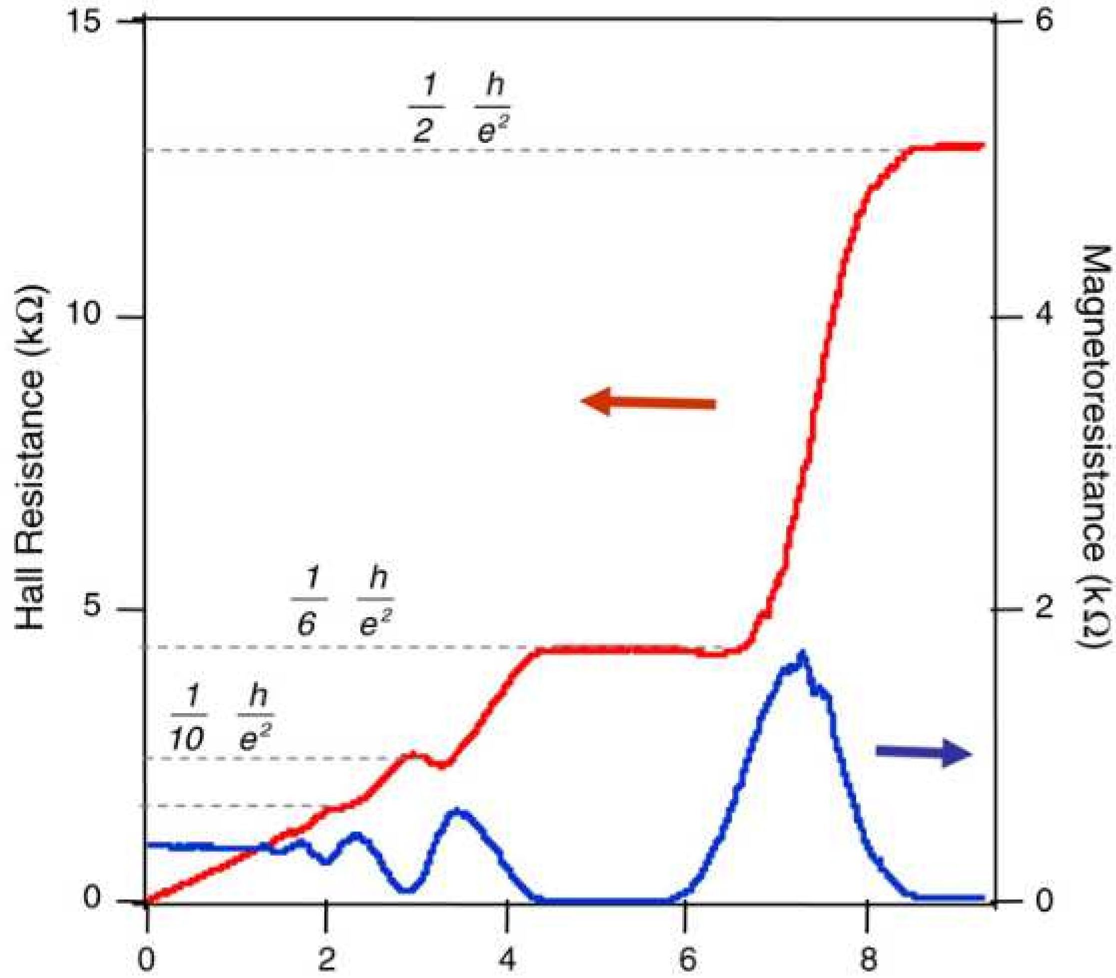


Fig. Quantized magnetoresistance and Hall resistance of a graphene device where $n = 10^{12}$ cm^2 and $T = 1.6$ K. Z. Jiang et al. Solid State Commun. 143, 14-19 (2007).

14. Kittel **Chapter 17 Problem 17-3**

Consider a two-dimensional electron gas with spin 1/2.

(a) Show that the number of orbitals per unit energy is given by

$$D(\varepsilon) = \frac{mL^2}{\pi\hbar^2}.$$

where L^2 is the area of the system.

(b) Show that the sheet density is related to the Fermi wavevector by

$$n_s = \frac{k_F^{\ 2}}{2\pi}$$

by taking into account of the spin freedom.

(c) Show that, in the Drude model, the sheet resistance, i.e., the resistance of a square segment of the 2D electron gas, can be written as

$$R_s = \frac{h}{e^2}\frac{1}{k_F l},$$

where $l = v_F\ \tau$ is the mean free path.

(a)

$$D(\varepsilon)d\varepsilon = 2\frac{L^2}{(2\pi)^2}2\pi k dk = \frac{mL^2}{\pi\hbar^2}d\varepsilon$$

since

$$\varepsilon = \frac{\hbar^2}{2m}k^2, \qquad d\varepsilon = \frac{\hbar^2}{m}kdk$$

Then we have the density of state for the 2D system with the area L^2,

$$D(\varepsilon) = \frac{mL^2}{\pi\hbar^2}$$

(b) The sheet density ns is defined by

$$n_s = \frac{N}{L^2} = \frac{1}{L^2}2\frac{L^2}{(2\pi)^2}\pi k_F^{\ 2} = \frac{k_F^{\ 2}}{2\pi}$$

where N is the total number of electrons below the Fermi energy ε_F,

$$N = 2\frac{L^2}{(2\pi)^2}\pi k_F^{\ 2}$$

(c) The sheet resistance is defined by

$$R_s = \frac{m}{n_s e^2 \tau}$$

Using the mean free path $l = v_F \tau$, we get

$$R_s = \frac{m}{\frac{k_F^{\ 2}}{2\pi} e^2 \tau} = \frac{h}{e^2} \frac{1}{k_F l},$$

since

$$m v_F = \hbar k_F .$$

15. de Broglie relation

Using the above expression of Rs with the de Broglie relation, we may derive the quantum Hall effect. We introduce the wavelength λ_F as

$$k_F = \frac{2\pi}{\lambda_F}$$

Then we get

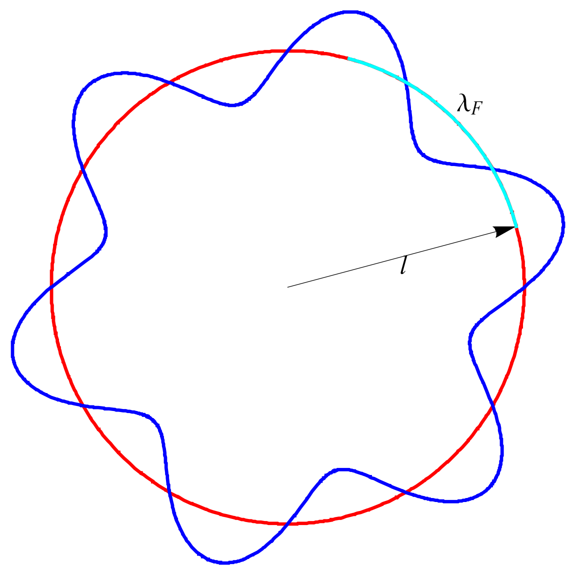
$$k_F l = \frac{2\pi l}{\lambda_F} .$$

Suppose that

$$2\pi l = s \lambda_F$$

where s is an integer. This relation corresponds to the de Broglie relation. Then we have

$$R_s = \frac{h}{e^2} \frac{1}{k_F l} = \frac{h}{s e^2} = \frac{25{,}812.8074434}{s} \Omega .$$



16. Minimum metallic conductivity

The conductivity is expressed by

$$\sigma = \frac{1}{R_s} = \frac{e^2}{h} k_F l \, .$$

When $k_F l = 1$, this conductivity is called the minimum metallic conductivity,

$$\sigma_{\min} = \frac{e^2}{h} \, .$$

This idea was proposed by Mott. However, this idea is found to be wrong from the scaling theory (E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979)).

REFERENCES

R.B. Laughlin, Phys. Rev. B **23**, 5632 (1981).
K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
D.D. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1562 (1982).
C. Kittel, *Introduction to Solid State Physics*, 8-th edition (Joh, Wiley & Sons, New York, 2005).
R.G. Chambers, *Electrons in Metals and Semiconductors* (Chapman and Hall, London,
B. Doucot, B. Duplantier, V. Pasquier, V. Rivasseau editors, *The Quantum Hall Effect, Poincare Seminar 2004* (Birkhauser Verlag, Basel, 2005).

H. Kamimura and H. Aoki, *The Physics of Interacting Electrons in Disordered Systems* (Oxford, 1989).

R.E. Prange, S.M. Girvin editors, The υantum Hall Effect 2nd edition (Springer-Verlag, New York, 1990).

Z.F. Ezawa, *Quantum Hall Effects*, 2nd edition (World Scientific, Singapore, 2008).

D. Yoshioka *The Quantum Hall Effect* (Springer. Berlin, 2002).

D.J. Thouless, *Topological Quantum Numbers in Nonrelativistic Physics* (World Scienctific, Singapore,1998).

P. Lederer and M.O. Goerbig, *Introduction to the Quantum Hall Effects* (lecture notes, 2006). http://staff.science.uva.nl/~jcaux/DITP_QHE_files/LedererQHE.pdf

A.J. Leggett, The quantum Hall effect: general considerations. http://online.physics.uiuc.edu/courses/phys598PTD/fall09/L16.pdf

K.S. Novoselov et al, Science **9** Vol. 315 no. 5817 p. 1379, "Room-temperature quantum Hall effect in grapheme."

N. Mott, *Conduction in Non-Crystalline Materials* (Clarendon Press, Oxford, 1987).

E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

APPENDIX

Experimental results of fractional quantum Hall effect

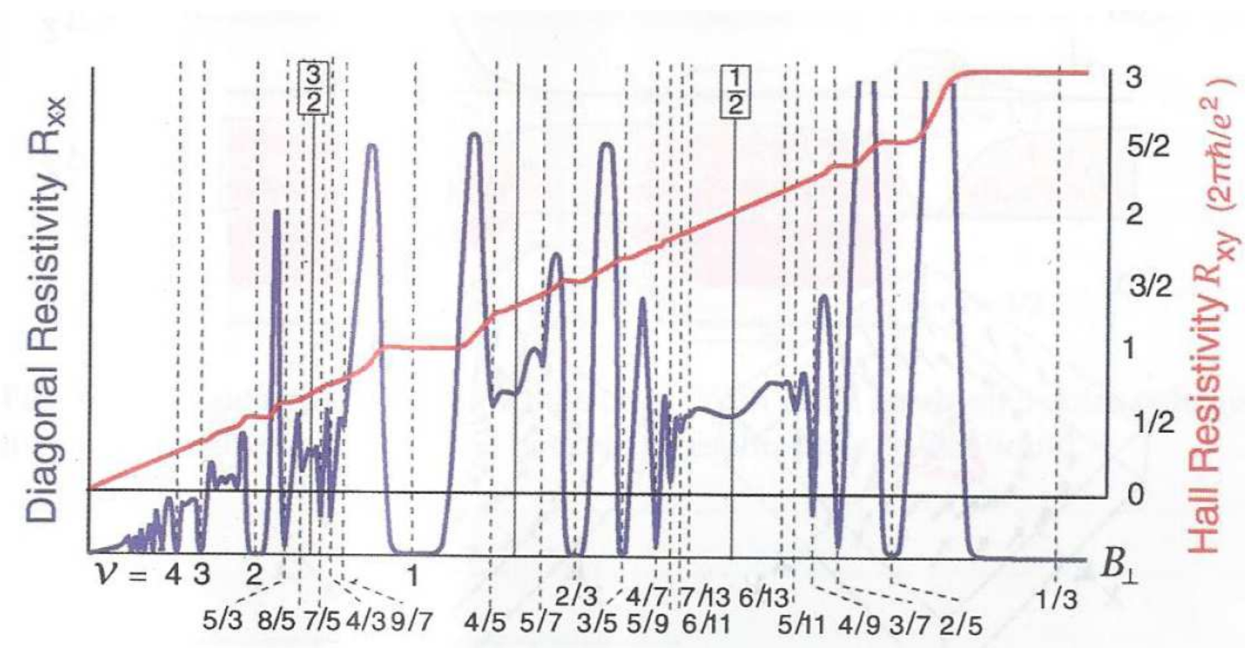


Fig. Quantum Hall effect which are detected by plateau developed by the Hall resistivity or dips in the diagonal resistivity. The numbers indicate the Landau level filling factors at which various features occur,