Overview on fundamentals of magnetism

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1. Diamagnetism and Paramagnetism

Magnetism is inseparable from quantum mechanics. A strictly classical system in thermal equilibrium can display no magnetic moment, even in a magnetic field.

Langevin paramagnetism	
Van Vleck paramagnetism	
Pauli paramagnetism	Spin magnetic moment of the conduction electrons in metals
Diamagnetism (Landau diamagnetism)	Orbital magnetic moment of the conduction electrons in metals

2. Magnetic susceptibilty

$\chi = \frac{M}{B}$; Magnetic susceptibility per unit volume

where M is defined as the magnetization per unit volume and B is the macroscopic magnetic field intensity.

χ (emu/cm^3)

χ (emu/g)

χ (emu/mol)

3. Magnetic moment and angular momentum

Angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

or

$$L_z = mvr$$

The magnetic moment:

$$\mu_z = -\frac{1}{c} I_\theta A = -\frac{1}{c}\frac{\Delta q}{\Delta t} A = -\frac{1}{c}\frac{e}{T} A = \frac{1}{c}\frac{e}{\frac{2\pi r}{v}}\pi r^2 = -\frac{evr}{2c}$$

where T is the period.

The ratio of angular momentum to the magnetic moment

$$\frac{\mu_z}{L_z} = -\frac{\frac{evr}{2c}}{mvr} = -\frac{e\hbar}{2mc}\frac{1}{\hbar} = -\frac{\mu_B}{\hbar}$$

The orbital magnetic moment is defined as

$$\boldsymbol{\mu} = -\frac{\mu_B}{\hbar}\boldsymbol{L}$$

4. Langevin diamagnetism

Next we consider the effect of external magnetic field $\boldsymbol{B}$

$$m\boldsymbol{v} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A} = \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}$$

with

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

Then the orbital magnetic moment is described as

$$\boldsymbol{\mu}_O = -\frac{\mu_B}{\hbar}(\boldsymbol{r} \times m\boldsymbol{v}) = -\frac{\mu_B}{\hbar}\boldsymbol{r} \times (\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}) = -\frac{\mu_B}{\hbar}[\boldsymbol{r} \times \boldsymbol{p} + \frac{e}{c}(\boldsymbol{r} \times \boldsymbol{A})]$$

The vector potential:

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B} \times \boldsymbol{r})$$

Then we get

$$\boldsymbol{r}\times\boldsymbol{A}=\frac{1}{2}[\boldsymbol{r}\times(\boldsymbol{B}\times\boldsymbol{r})]=\frac{1}{2}[r^2B-(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{r}]$$

The magnetic moment is given by

$$\begin{aligned}\boldsymbol{\mu}_O&=-\frac{\mu_B}{\hbar}(\boldsymbol{r}\times\boldsymbol{p})-\frac{e^2}{4mc^2}[r^2B-(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{r}]\\&=-\frac{\mu_B}{\hbar}\boldsymbol{L}-\frac{e^2}{4mc^2}[r^2B-(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{r}]\end{aligned}$$

The latter term is the magnetic moment induced by magnetic field (based on the Lentz law)

When ***B*** is directed along the z axis, the interaction energy is calculated as

$$\begin{aligned}H'&=-\int_0^B\boldsymbol{\mu}_o\cdot d\boldsymbol{B}\\&=\frac{\mu_B}{\hbar}\boldsymbol{L}\cdot\boldsymbol{B}+\frac{e^2}{4mc^2}\int_0^B(x^2+y^2)BdB\\&=\frac{\mu_B}{\hbar}L_zB+\frac{e^2B^2}{8mc^2}(x^2+y^2)\end{aligned}$$

5. Evaluation of Langevin diamagnetism (core diamagnetism)

In the presence of magnetic field, the current for each electron due to its angular frequency change (Larmor theorem, see the Appendix).

$$I=Zef=\frac{Ze}{2\pi}\omega=\frac{Ze}{2\pi}\frac{eB}{2mc}=\frac{Ze^2B}{4\pi mc}$$

where ω is the Larmor angular frequency, $\omega=\dfrac{eB}{2mc}$. The magnetic moment is defined by

$$\mu_{core}=-\frac{1}{c}IA=-\frac{Ze^2B}{4\pi mc^2}\pi[\langle x^2\rangle+\langle y^2\rangle]$$

The core-diamagnetism (in the presence of external magnetic field)

$$\mu_{core} = -\frac{e^2 B}{4mc^2}(\langle x^2\rangle + \langle y^2\rangle)$$

$(\langle x^2\rangle + \langle y^2\rangle)$ is the mean square of the perpendicular distance of the electron from an axis through the atom. The mean square distance of the electrons from the nucleus is

$$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle$$

For a spherically symmetrical distribution of charge, we have

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle$$

$$\langle x^2\rangle + \langle y^2\rangle = 2\langle x^2\rangle = \frac{2}{3} < r^2 >$$

$$\mu_{core} = -\frac{Ze^2 B}{6mc^2} < r^2 >$$

If N_A is the number of atoms per mole

$$\chi_M = N_A \frac{\mu}{B} = -\frac{e^2 Z N_A}{6mc^2} < r^2 > = \text{-2.82839 x } 10^{10} < r^2 > Z$$

where Z is the atomic number. Note that

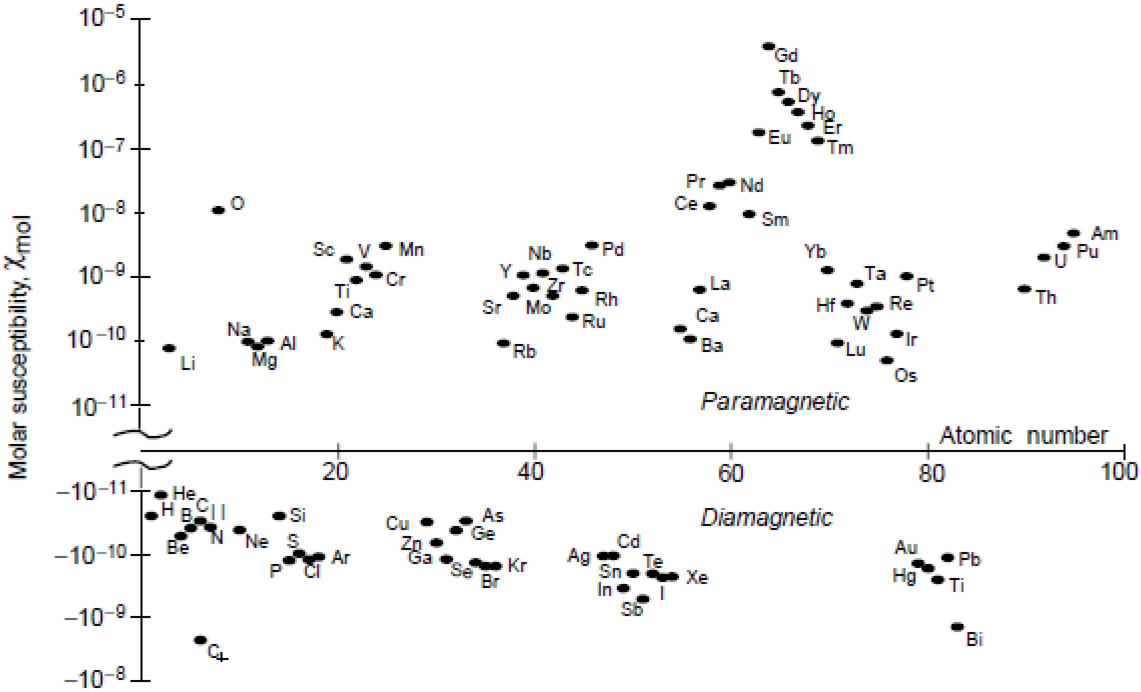
emu=erg/G = G^2 cm^3/G = G cm^3.

Then the units of χ_M is

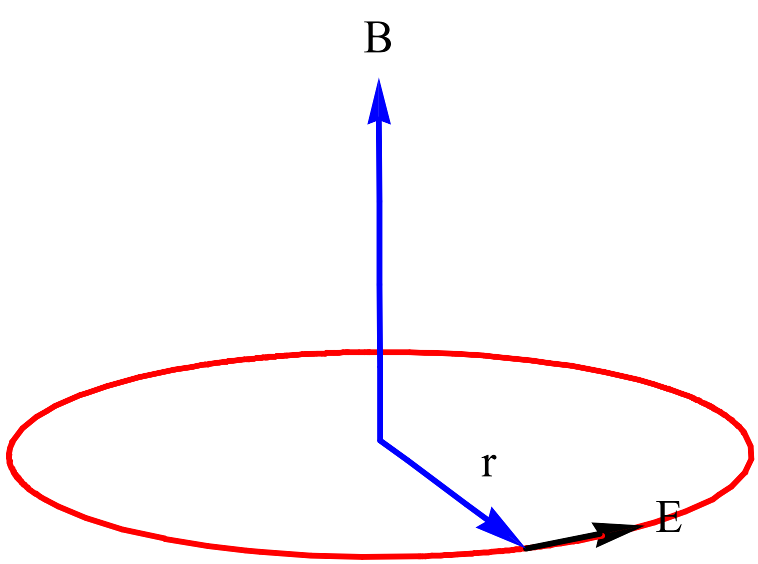
emu/(G.mol) = cm^3/mol

	$\chi_M (10^{-6} emu/mol)$
He	-1.9
Ne	-7.2
Ar	-19.4
Kr	-28.0

Xe	-43.0



6. Lenz's law (Feynman)



Maxwell's equation:

$$V = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = \oint (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = \int (-\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}) \cdot d\boldsymbol{a} = -\frac{1}{c}\frac{\partial \Phi}{\partial t},$$

with the magnetic flux,

$$\Phi = \pi r^2 B$$

Note that $\boldsymbol{E}$ is the electric field along the orbit,

$$\boldsymbol{E} = E_\theta \boldsymbol{e}_\theta$$

$$\boldsymbol{B} = B\boldsymbol{e}_z$$

Then we get

$$V = 2\pi r E_\theta = -\frac{1}{c}\pi r^2 \frac{dB}{dt}$$

or

$$E_\theta = -\frac{r}{2c}\frac{dB}{dt}. \qquad \text{(Lentz law)}$$

In other words, we have

For $\frac{dB}{dt} > 0$, $E_\theta < 0$; the current flows in a clock-wise direction.

For $\frac{dB}{dt} < 0$, $E_\theta > 0$; the current flows in a counter clock-wise direction.

The induced electric field acting on an electron in the atoms produces a torque equal to

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = r\boldsymbol{e}_r \times (-eE_\theta \boldsymbol{e}_\theta) = -erE_\theta \boldsymbol{e}_z$$

Newton's second law for rotation;

$$\tau_z = \frac{dL_z}{dt} = -erE_\theta = \frac{er^2}{2c}\frac{dB}{dt}$$

Integrating with respect to time from zero field,

$$\Delta L_z = \frac{er^2}{2c} B$$

The magnetic moment is given by

$$\Delta\mu_z = -\frac{\mu_B}{\hbar}\Delta L_z = -\frac{e}{2mc}\Delta L_z = -\frac{e}{2mc}\frac{er^2}{2c}B = -\frac{e^2r^2}{4mc^2}B$$

5. Method using the quantum mechanics

We consider the Hamiltonian H of non-relativistic electron in the presence of an external magnetic field,

$$H = \frac{1}{2m}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 = \frac{1}{2m}(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2,$$

with the charge of electron

$$q = -e.$$

The vector potential is given by

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}.$$

Then we get

$$\begin{aligned} H &= \frac{1}{2m}(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2 \\ &= \frac{1}{2m}\boldsymbol{p}^2 + \frac{e^2B^2}{8mc^2}(x^2+y^2) + \frac{eB}{2mc}\frac{\hbar}{i}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \\ &= \frac{1}{2m}\boldsymbol{p}^2 + \frac{e^2B^2}{8mc^2}(x^2+y^2) - (-\frac{\mu_B}{\hbar}L_z)B \end{aligned}$$

where

$$\boldsymbol{p} = \frac{\hbar}{i}\nabla$$

$$L_z = \frac{\hbar}{i}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

where

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} .$$

The third term is the Zeeman energy for the orbital magnetic moment

$$\boldsymbol{\mu}_o = -\frac{\mu_B}{\hbar}\boldsymbol{L} .$$

in the presence of an external magnetic field. This term gives rise only to paramagnetism.

((**Mathematica**))

```
Clear["Global`*"]; p1 := (ℏ/ⅈ D[#, x] - (e B)/(2 c) y #) &;
p2 := (ℏ/ⅈ D[#, y] + (e B)/(2 c) x #) &;
p3 := (ℏ/ⅈ D[#, z]) &;

f1 =
 1/(2 m) (Nest[p1, ψ[x, y, z], 2] + Nest[p2, ψ[x, y, z], 2] +
     Nest[p3, ψ[x, y, z], 2]) // Expand
```

$$\frac{B^2 e^2 x^2 \psi[x, y, z]}{8 c^2 m} + \frac{B^2 e^2 y^2 \psi[x, y, z]}{8 c^2 m} - \frac{\hbar^2 \psi^{(0,0,2)}[x, y, z]}{2 m} - \frac{i B e x \hbar \psi^{(0,1,0)}[x, y, z]}{2 c m} - \frac{\hbar^2 \psi^{(0,2,0)}[x, y, z]}{2 m} + \frac{i B e y \hbar \psi^{(1,0,0)}[x, y, z]}{2 c m} - \frac{\hbar^2 \psi^{(2,0,0)}[x, y, z]}{2 m}$$

APPENDIX **Larmor frequency** $\dfrac{eB}{2mc}$

(L.D. Landau and E.M. Lifshitz; The Classical Theory of Fields)

We determine the angular frequency of vibration of a charged spatial oscillator, placed in a constant uniform magnetic field; proper frequency of vibration of the oscillator (in the absence of the field) is ω_0

The equations of forced vibration of the oscillator in a magnetic field (directed along the z axis) are

$$m(\ddot{\boldsymbol{r}} + \omega_0^2 \boldsymbol{r}) = \boldsymbol{F} = -\frac{e}{c}(\dot{\boldsymbol{r}} \times \boldsymbol{B})$$

or

$$\ddot{x} + \omega_0^2 x = -\frac{eB}{mc}\dot{y}, \qquad \ddot{y} + \omega_0^2 y = \frac{eB}{mc}\dot{x}, \qquad \ddot{z} + \omega_0^2 z = 0$$

Multiplying the second equation by a purely imaginary (i) and combining with the first equation, we have

$$(\ddot{x} + i\ddot{y}) + \omega_0^{\,2}(x + iy) = i\frac{eB}{mc}(\dot{x} + i\dot{y})$$

or

$$\ddot{\varsigma} + \omega_0^{\,2}\varsigma = i\frac{eB}{mc}\dot{\varsigma}$$

where $\varsigma = x + iy$. We assume that ς has a form of $\varsigma = \varsigma_0 e^{i\omega t}$. Thus we have

$$\omega^2 - \frac{eB}{mc}\omega - \omega_0^{\,2} = 0$$

or

$$\omega = \frac{eB}{2mc} \pm \sqrt{\left(\frac{eB}{2mc}\right)^2 + \omega_0^{\,2}}$$

If $\left(\dfrac{eB}{2mc}\right) \ll \omega_0$, we have

$$\omega = \frac{eB}{2mc} \pm \omega_0$$

The $\pm$ sign on ω_0 means that those electrons whose orbital moments were parallel to the field are slowed down and those whose moments were antiparallel are speeded up, by an amount $\dfrac{eB}{2mc}$. The result is called the Larmor theorem.

APPENDIX-II

Lagrangian L of a charged particle in in the presence of electronmagnetic fields,

The action for a charge in an electromagnetic field has the form

$$S = \int (-mcds + \frac{q}{c} A^{\mu} dx_{\mu})$$
$$= \int (-mc\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c} \boldsymbol{A} \cdot \boldsymbol{v} - q\phi) dt$$

where $A^{\mu} = (\boldsymbol{A}, \phi)$, and $dx_{\mu} = (d\boldsymbol{r}, -cdt)$. $\boldsymbol{A}$ is the vector potential and ϕ is the scalar potential. The integrand is the Lagrangian for a charge q in the electromagnetic field;

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c} \boldsymbol{A} \cdot \boldsymbol{v} - q\phi$$
$$= -mc^2 (1 - \frac{v^2}{c^2})^{1/2} + \frac{q}{c} \boldsymbol{A} \cdot \boldsymbol{v} - q\phi$$

For $v \ll c$, L is approximated by

$$L = -mc^2 (1 - \frac{\boldsymbol{v}^2}{2c^2}) + \frac{q}{c} \boldsymbol{A} \cdot \boldsymbol{v} - q\phi$$
$$= -mc^2 + \frac{m\boldsymbol{v}^2}{2} + \frac{q}{c} \boldsymbol{A} \cdot \boldsymbol{v} - q\phi$$

The canonical momentum $\boldsymbol{p}$ is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} \ \boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{q}{c} \boldsymbol{A}$$

or

$$\boldsymbol{v} = \frac{1}{m} (\boldsymbol{p} - \frac{q}{c} \boldsymbol{A})$$

The Hamiltonian H is obtained as

$$
\begin{aligned}
H &= \boldsymbol{p}\cdot\boldsymbol{v} - L \\
&= (m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A})\cdot\boldsymbol{v} + mc^2 - \frac{m\boldsymbol{v}^2}{2} - \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v} + q\phi \\
&= \frac{m\boldsymbol{v}^2}{2} + mc^2 + q\phi \\
&= \frac{1}{2m}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 + q\phi + mc^2
\end{aligned}
$$

When $q = -e$ for electron (e>0), we find the Hamiltonian

$$
H = \frac{1}{2m}(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2 - e\phi
$$