Langevin function

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The classical theory of paramagnetism, which is the limit of quantum theory when $J \to \infty$, was worked out in 1905 by Paul Langevin. Colloidal ferromagnetic minerals, usually magnetite, dispersed in a rock are examples of the systems where the classical theory is expected to apply.

Each atom or particle has a macroscopic magnetic moment $\boldsymbol{\mu}$, which can take any orientation relative to the field applied in the direction (the *z* axis).

We consider a magnetic moment (vector $\boldsymbol{\mu}$) whose direction is arbitrary. When the angle between the direction of $\boldsymbol{\mu}$ and the *z* axis is an angle θ, the Zeeman energy is obtained as

$$H = -\mu B \cos\theta$$

The one-particle partition Z_1 is given by

$$\begin{aligned} Z_1 &= \int_0^{\pi} e^{\beta\mu B\cos\theta} d\Omega \\ &= \int_0^{\pi} e^{\beta\mu B\cos\theta} 2\pi \sin\theta d\theta \\ &= 4\pi \frac{\sinh(\beta\mu B)}{\beta\mu B} \end{aligned}$$

where Ω is the solid angle. The *N*-particle partition function is expressed by

$$Z_N = Z_1^N$$

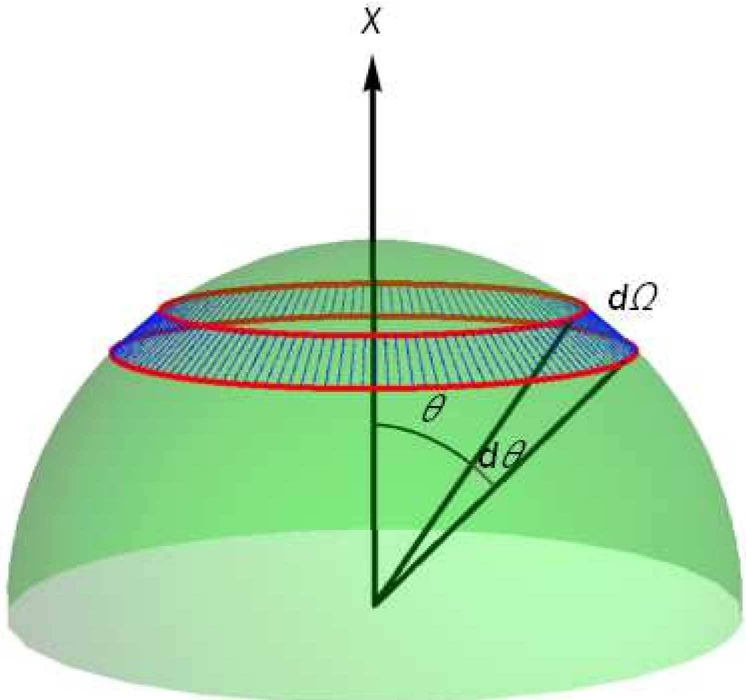


Fig. Solid angle. $d\Omega = 2\pi \sin\theta d\theta$

We note that the total magnetization is given by

$$\begin{aligned}\langle M \rangle &= k_B T \frac{\partial \ln Z^N}{\partial B} \\ &= N k_B T \frac{\partial \ln Z_1}{\partial B} \\ &= N \mu L(x)\end{aligned}$$

where $x = \beta \mu B$ and N is the total number of atoms,

$$L(x) = \coth(x) - \frac{1}{x} \qquad \text{(Langevin function).}$$

In the limit of $x \to 0$, we have

$$L(x) \approx \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + O(x^9)\,.$$

In the limit of $x \to \infty$, we have $\qquad L(\infty) = 1$

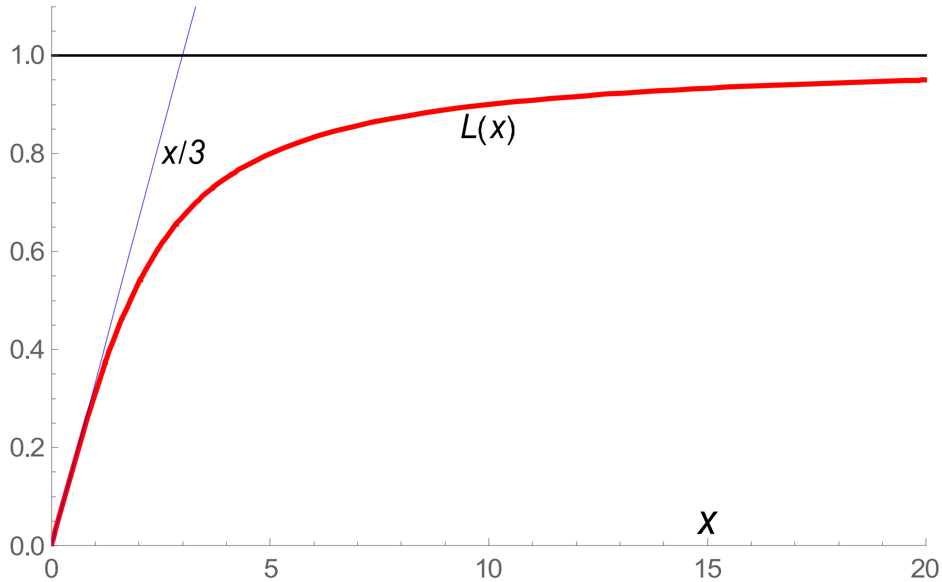


Fig. Langevin function $L(x)$ as a function of $x = \beta\mu B$.

((**Note**)) Derivation of magnetization using the partition function

$$Z = Tr[e^{-\beta H}]$$

$$\frac{\partial Z}{\partial B} = Tr[(-\beta)\frac{\partial H}{\partial B} e^{-\beta H}]$$
$$= \beta Tr[M e^{-\beta H}]$$

$$\frac{1}{Z}\frac{\partial Z}{\partial B} = \beta \frac{Tr[M e^{-\beta H}]}{Tr[e^{-\beta H}]} = \beta \langle M \rangle$$

or

$$\langle M \rangle = k_B T \frac{\partial \ln Z}{\partial B}$$

2. Another method

$$\langle\cos\theta\rangle = \frac{\int_0^\pi e^{\beta\mu B\cos\theta}\cos\theta d\Omega}{\int_0^\pi e^{\beta\mu B\cos\theta} d\Omega} = \frac{\int_0^\pi e^{\beta\mu B\cos\theta} 2\pi\sin\theta\cos\theta d\theta}{\int_0^\pi e^{\beta\mu B\cos\theta} 2\pi\sin\theta d\theta}$$

$$= L(x)$$

where $d\Omega = 2\pi\sin\theta d\theta$. Note that the probability $P(\theta)$ is expressed by

$$P(\theta)d\theta = Ae^{\beta\mu B\cos\theta} 2\pi\sin\theta d\theta$$

where A is constant,

$$\int_0^\pi P(\theta)d\theta = 1$$

((**Mathematica**))

```
Clear["Global`*"];
f1 = Exp[β μ B Cos[θ]] 2 π Sin[θ];

Z1 = Integrate[f1, {θ, 0, π}]
```

$$\frac{4\,\pi\,\mathrm{Sinh}[B\,\beta\,\mu]}{B\,\beta\,\mu}$$

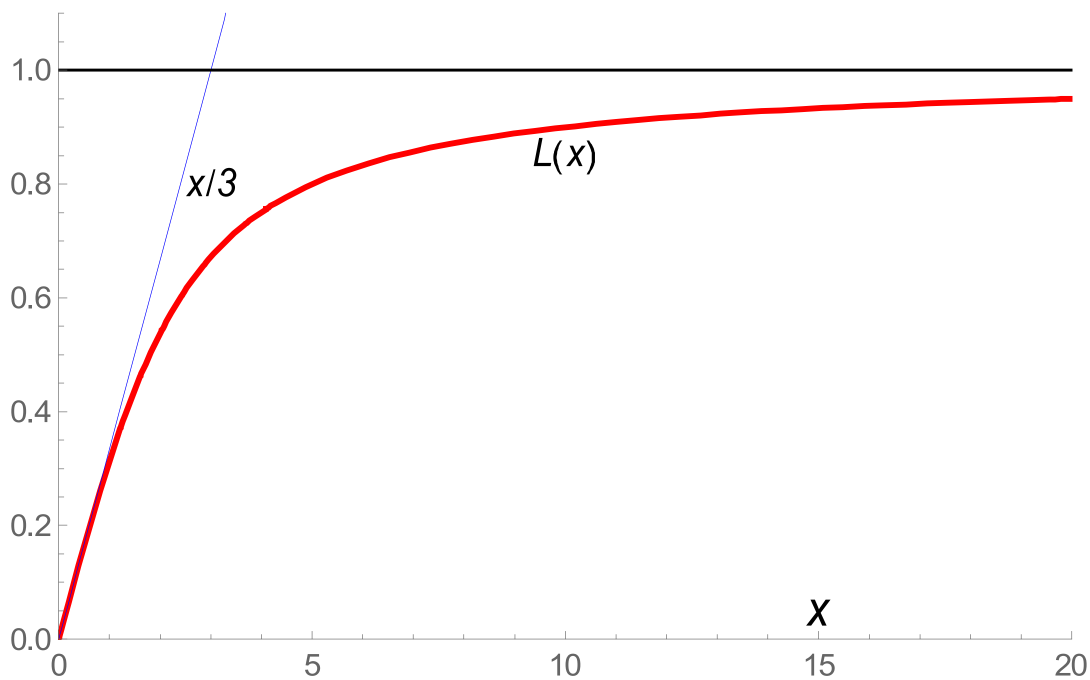
```
N1/β D[Log[Z1], B] // FullSimplify
```

$$-\frac{N1}{B\,\beta}+N1\,\mu\,\mathrm{Coth}[B\,\beta\,\mu]$$

```
L1[x_] := Coth[x] - 1/x; Series[L1[x], {x, 0, 10}]
```

$$\frac{x}{3}-\frac{x^3}{45}+\frac{2\,x^5}{945}-\frac{x^7}{4725}+\frac{2\,x^9}{93\,555}+O[x]^{11}$$

```
h1 = Plot[{L1[x], x / 3}, {x, 0, 20},
  PlotStyle → {{Red, Thick}, {Blue, Thin}},
  PlotRange → {{0, 20}, {0, 1.1}}];
h2 = Graphics[{Line[{{0, 1}, {20, 1}}],
   Text[Style["x/3", Black, 12, Italic], {3, 0.8}] ,
   Text[Style["x", Black, 15, Italic], {15, 0.05}] ,
   Text[Style["L(x)", Black, 12, Italic],
    {10, 0.85}]}];
Show[h1, h2]
```



```
Limit[L1[x], x → ∞]
```

1

```
f11 = Exp[β μ B Cos[θ]] 2 π Sin[θ];
f12 = Exp[β μ B Cos[θ]] 2 π Sin[θ] Cos[θ];
```

$$\frac{\text{Integrate[f12, \{}\theta\text{, 0, }\pi\text{\}]}}{\text{Integrate[f11, \{}\theta\text{, 0, }\pi\text{\}]}} \text{ // Simplify}$$

$$-\frac{1}{B\,\beta\,\mu} + \mathrm{Coth}[B\,\beta\,\mu]$$