Stoner model: magnetism in metals

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1. Introduction

So far we have concentrated on localized magnetic moments on the lattice sites. The conduction electrons in metals are delocalized and can move freely over the whole system. It is are called itinerant electrons. In some cases, the magnetic moments in metals associated with the conduction electrons, in other cases the magnetic moment remain localized. Here we discuss the Stoner model on the magnetism of metals. Ferromagnetism is possible under certain conditions (Stoner criterion). To this end, we will use problems which appear in the standard textbook of Solid State Physics.

In itinerant ferromagnets, there are two kinds of interactions between two electrons with spin magnetic moments. One is the exchange interaction between two electrons (the direction of spin magnetic moment for the first electron is parallel to that for the second electron). It is called the exchange interaction -V (V>0) (which is attractive). The other is the repulsive Coulomb interaction U (U>0) between two electrons (the direction of spin magnetic moment for one electron is antiparallel to that for the other electron). U is called the on-site Hubbard interaction. According to the Pauli's exclusion principle, these two electrons are in the same orbital state (of the same atomic site).

2. ((Kittel problem 14-5)) Pauli spin susceptibility in the case of $U = V = 0$

The spin susceptibility of a conduction electron gas at $T = 0$ K may be approached as follows. Let

$$N^{+} = \frac{1}{2}N(1+\varsigma), \qquad N^{-} = \frac{1}{2}N(1-\varsigma),$$

be the concentration of spin-up and spin-down electrons.

(a) Show that in a magnetic field B the total energy of the spin-up band in free electron gas is

$$U^{+} = E_0(1+\varsigma)^{5/3} - \frac{1}{2}N\mu_B B(1+\varsigma)$$

where $E_0 = \dfrac{3}{10}N\varepsilon_F$, in terms of the Fermi energy ε_F in zero magnetic field. Find a similar expression for U^{-}.

$$U^{-} = E_0(1-\varsigma)^{5/3} + \frac{1}{2}N\mu_B B(1-\varsigma)$$

(b) Minimize $U_{total} = U^{+} + U^{-}$ with respect to ς and solve for the equilibrium value of ς in the approximation $\varsigma \ll 1$. Go on to show that the magnetization is

$$M = \frac{3N\mu_B{}^2 B}{2\varepsilon_F}.$$

((**Solution**))

The total energy of the spin-up band in a free electron gas is

$$U^{+} = \int^{\infty} D^{+}(\varepsilon)\varepsilon f(\varepsilon)d\varepsilon$$

$$= \frac{1}{2}\int_{-\mu_B B}^{\infty} D(\varepsilon+\mu_B B)\varepsilon f(\varepsilon)d\varepsilon$$

The total energy of the spin-up band in a free electron gas is

$$U^{-} = \int^{\infty} D^{-}(\varepsilon)\varepsilon f(\varepsilon)d\varepsilon$$

$$= \frac{1}{2}\int_{\mu_B B}^{\infty} D(\varepsilon-\mu_B B)\varepsilon f(\varepsilon)d\varepsilon$$

where the density of states is given by

$$D^{+}(\varepsilon) = \frac{1}{2}D(\varepsilon+\mu_B B), \qquad D^{-}(\varepsilon) = \frac{1}{2}D(\varepsilon-\mu_B B)$$

for the up-spins and the down spins, respectively. The number of up-spins and down-spins are given

$$N^{+} = \frac{1}{2}\int_{-\mu_B B}^{\infty} d\varepsilon D^{+}(\varepsilon)f(\varepsilon) = \frac{1}{2}\int_{0}^{\infty} d\varepsilon D(\varepsilon)f(\varepsilon-\mu_B B)$$

$$N^{-} = \frac{1}{2}\int_{\mu_B B}^{\infty} d\varepsilon D^{-}(\varepsilon) f(\varepsilon) = \frac{1}{2}\int_{0}^{\infty} d\varepsilon D(\varepsilon) f(\varepsilon + \mu_B B),$$

respectively. The magnetization is obtained as

$$\begin{aligned} M &= \mu_B N^{+} - \mu_B N^{-} \\ &= -\frac{1}{2}\mu_B \int_{0}^{\infty} D(\varepsilon)[f(\varepsilon + \mu_B \Delta) - f(\varepsilon - \mu_B \Delta)] \end{aligned}$$

We assume that

$$N^{+} = \frac{1}{2}N(1+\varsigma), \qquad N^{-} = \frac{1}{2}N(1-\varsigma),$$

At $B = 0$,

$$N^{+} = N^{-} = \frac{1}{2}N$$

Then we have

$$N^{+} = \frac{1}{2}N(1+\varsigma)$$

$$N^{-} = \frac{1}{2}N(1-\varsigma)$$

and

$$\begin{aligned} N_{tot} &= N^{+} + N^{-} \\ &= \frac{1}{2}N[(1+\varsigma) + (1-\varsigma)] \\ &= N \end{aligned}$$

(the conservation of the total number of spins). We also

$$\begin{aligned} U^{+} &= E_0(1+\varsigma)^{5/3} - N^{+}\mu_B B \\ &= E_0(1+\varsigma)^{5/3} - \frac{1}{2}N\mu_B B(1+\varsigma) \end{aligned}$$

$$U^{-} = E_0(1-\varsigma)^{5/3} + \frac{1}{2}N\mu_B B(1-\varsigma)$$

where $E_0 = \frac{3}{10}N\varepsilon_F$

The total internal energy is

$$\begin{aligned}
U_{tot} &= U^{+} + U^{-} \\
&= E_0(1+\varsigma)^{5/3} - \frac{1}{2}N\mu_B B(1+\varsigma) \\
&\quad + E_0(1-\varsigma)^{5/3} + \frac{1}{2}N\mu_B B(1-\varsigma) \\
&= E_0(1+\varsigma)^{5/3} + E_0(1-\varsigma)^{5/3} - N\mu_B B\varsigma
\end{aligned}$$

The net magnetic moment is

$$\begin{aligned}
M &= \mu_B N^{+} - \mu_B N^{-} \\
&= \mu_B \frac{N}{2}[(1+\varsigma)-(1-\varsigma)] \\
&= \mu_B N\varsigma
\end{aligned}$$

The condition for the minimum of U_{tot};

$$\frac{\partial U_{tot}}{\partial \varsigma} = \frac{5}{3}E_0(1+\varsigma)^{2/3} - \frac{5}{3}E_0(1-\varsigma)^{2/3} - N\mu_B B$$

In the limit of $\varsigma \to 0$,

$$\begin{aligned}
\frac{\partial U_{tot}}{\partial \varsigma} &\approx \frac{5}{3}E_0(1+\frac{2}{3}\varsigma) - \frac{5}{3}E_0(1-\frac{2}{3}\varsigma) - N\mu_B B \\
&= \frac{20}{9}E_0\varsigma - N\mu_B B
\end{aligned}$$

From the minimum condition we have

$$\frac{\partial U_{tot}}{\partial \varsigma} = \frac{20}{9} E_0 \varsigma - N\mu_B B = 0$$

or

$$\varsigma = \frac{9}{20} \frac{N\mu_B B}{E_0} = \frac{3}{2} \frac{\mu_B B}{\varepsilon_F}$$

The corresponding net magnetic moment is

$$M = \mu_B N \varsigma = N\mu_B \frac{3}{2} \frac{\mu_B B}{\varepsilon_F} = \frac{3N{\mu_B}^2 B}{2\varepsilon_F}$$

which leads to the Pauli spin magnetization of the conduction electrons.

3. Conduction electron ferromagnetism (Stoner's model): ((Kittel problem 14-6))

[V: exchange interaction between parallel spins. $U = 0$.]

We approximate the effect of exchange interaction among the conduction electrons if we assume that electrons with parallel spins interact with each other with energy -V, and V is positive, while electrons with antiparallel spins do not interact with each other.

(a) Show that the total energy of the spin-up band is

$$U^{+} = E_0 (1+\varsigma)^{5/3} - \frac{1}{8} VN^2 (1+\varsigma)^2 - \frac{1}{2} N\mu_B B(1+\varsigma)$$

find a similar expression for U^{-}.

(b) Minimize the total energy and solve for ς in the limit $\varsigma \ll 1$. Show that the net magnetic moment is

$$M = \frac{3N{\mu_B}^2}{2\varepsilon_F - \frac{3}{2}VN}$$

So that the exchange interaction enhances the susceptibility.

(c) Show that with $B = 0$ the total energy is unstable at $\varsigma = 0$ when $V > (4\varepsilon_F / 3N)$. If this is satisfied, a ferromagnetic state ($\varsigma \neq 0$) will have a lower energy than the paramagnetic state. Because of the assumption $\varsigma \ll 1$, this is a sufficient condition for ferromagnetism, but it may not be a necessary condition. It is known as the **Stoner condition**.

((**Solution**))

(a) The total energy of the spin-up band U^{+} is given by

$$U^{+}=E_0(1+\varsigma)^{5/3}-\frac{1}{2}V(N^{+})^2-\frac{1}{2}N\mu_B B(1+\varsigma)$$
$$=E_0(1+\varsigma)^{5/3}-\frac{1}{8}VN^2(1+\varsigma)^2-\frac{1}{2}N\mu_B B(1+\varsigma)$$

Similarly, the total energy of the spin-down band U^{-} is given by

$$U^{-}=E_0(1-\varsigma)^{5/3}-\frac{1}{8}VN^2(1-\varsigma)^2+\frac{1}{2}N\mu_B B(1-\varsigma)$$

(b)

$$U_{tot}=U^{+}+U^{-}$$
$$=E_0(1+\varsigma)^{5/3}-\frac{1}{8}VN^2(1+\varsigma)^2-\frac{1}{2}N\mu_B B(1+\varsigma)$$
$$+E_0(1-\varsigma)^{5/3}-\frac{1}{8}VN^2(1-\varsigma)^2+\frac{1}{2}N\mu_B B(1-\varsigma)$$

$$\frac{\partial U_{tot}}{\partial\varsigma}=\frac{5}{3}E_0(1+\varsigma)^{2/3}-\frac{1}{4}VN^2(1+\varsigma)-\frac{1}{2}N\mu_B B$$
$$-\frac{5}{3}E_0(1-\varsigma)^{2/3}+\frac{1}{4}VN^2(1-\varsigma)-\frac{1}{2}N\mu_B B$$

or

$$\frac{\partial U_{tot}}{\partial\varsigma}=\frac{5}{3}E_0[(1+\varsigma)^{2/3}-(1-\varsigma)^{2/3}]-\frac{1}{4}VN^2[(1+\varsigma)-(1-\varsigma)]-N\mu_B B$$
$$\approx\frac{5}{3}E_0[1+\frac{2}{3}\varsigma-(1-\frac{2}{3}\varsigma)]-\frac{1}{2}VN^2\varsigma-N\mu_B B$$
$$=(\frac{20}{9}E_0-\frac{1}{2}VN^2)\varsigma-N\mu_B B$$

for $\varsigma<<1$. When $\dfrac{\partial U_{tot}}{\partial\varsigma}=0$, we have

$$\varsigma = \frac{N\mu_B B}{\frac{20}{9}E_0 - \frac{1}{2}VN^2} = \frac{3\mu_B B}{2\varepsilon_F - \frac{3}{2}VN}$$

The net magnetic moment M is obtained as

$$\begin{aligned} M &= \mu N^+ - \mu N^- \\ &= \frac{1}{2}\mu_B N(1+\varsigma) - \mu_B N(1-\varsigma) \\ &= \mu_B N\varsigma \end{aligned}$$

or

$$M = N\mu_B\varsigma = \frac{3N\mu_B{}^2 B}{2\varepsilon_F - \frac{3}{2}VN}$$

(c) For $B = 0$,

$$\begin{aligned} U_{tot}(\varsigma) &= E_0[(1+\varsigma)^{5/3} + (1-\varsigma)^{5/3}] - \frac{1}{8}VN^2[(1+\varsigma)^2 + (1-\varsigma)^2] \\ &= 2E_0(1+\frac{5}{9}\varsigma^2) - \frac{1}{4}VN^2(1+\varsigma^2) + O(\varsigma^4) \end{aligned}$$

Then we have

$$\begin{aligned} U_{tot}(\varsigma) - U_{tot}(\varsigma = 0) &= E_0\frac{10}{9}\varsigma^2 - \frac{1}{4}VN^2\varsigma^2 \\ &= (\frac{10}{9}E_0 - \frac{1}{4}VN^2)\varsigma^2 \\ &= (\frac{1}{3}\varepsilon_F - \frac{1}{4}VN)N\varsigma^2 \end{aligned}$$

where $E_0 = \frac{3}{10}N\varepsilon_F$. For $V > \frac{4\varepsilon_F}{3N}$, we have

$$U_{tot}(\varsigma) < U_{tot}(\varsigma = 0)$$

The paramagnetic state is unstable compared to the ferromagnetic state. The Taylor expansion of $U_{tot}(\varsigma)$ at $B = 0$ in the power of ς;

$$U_{tot}(\varsigma) = E_0[(1+\varsigma)^{5/3} + (1-\varsigma)^{5/3}] - \frac{1}{8}VN^2[(1+\varsigma)^2 + (1-\varsigma)^2]$$
$$= \left(\frac{3}{5}\varepsilon_F - \frac{1}{4}NV\right) + \left(\frac{1}{3}\varepsilon_F - \frac{1}{4}NV\right)\varsigma^2 + \frac{1}{81}\varepsilon_F\varsigma^4$$

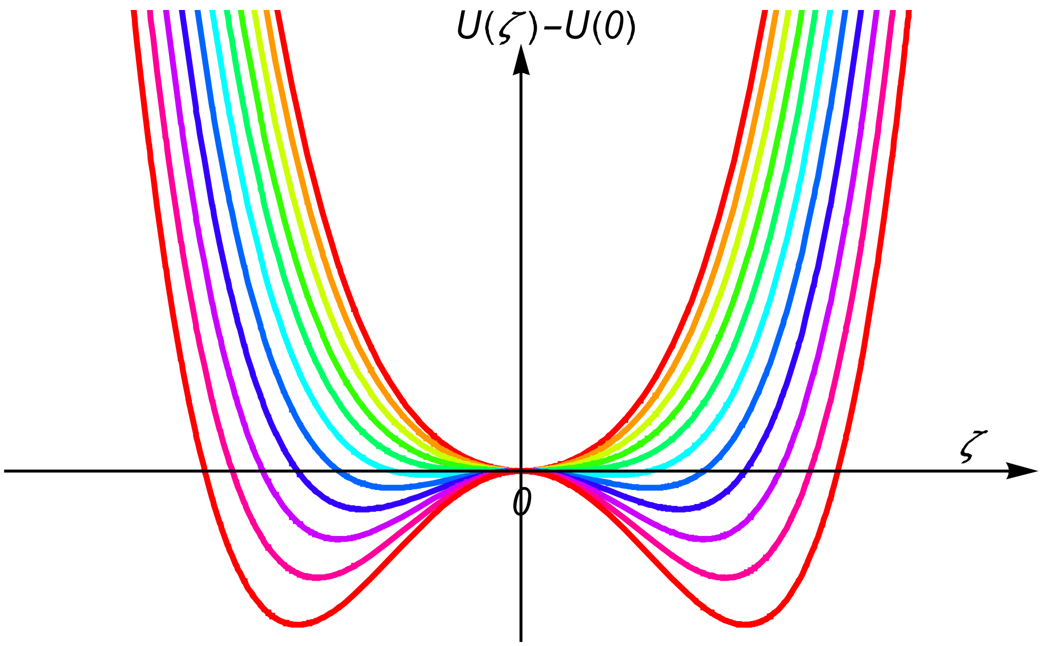

Fig. The total energy as a function of the order parameter ς. $V > V_0$ for the ferromagnetic state (local minimum at positive finite value of ς). $V < V_0$ for the paramagnetic state (minimum at ς=0).

3. S.H. Simon (The Oxford Solid State, Oxford, 2013)

[V= 0: exchange interaction between parallel spins. $U \neq 0$: Coulomb interaction between antiparallel spins.]

We consider adding a Hubbard interaction term

$$Un_+n_-$$

with $U \geq 0$, where

$$n_+ = N_+ = \frac{N}{2}(1+\varsigma), \qquad n_- = N_- = \frac{N}{2}(1-\varsigma)$$

$$n_+ - n_- = N\varsigma$$

((**Problem**)) **S.H. Simon (The Oxford Solid State, Oxford, 2013)**

Calculate the expectation value of this interaction term given that the up and down electrons from Fermi seas with n_+ and n_-.

Write this energy in terms of ς. Adding together the kinetic energy and interaction energy, determine the value of U for which it is favorable for ς to become non-zero.

$$\begin{aligned}
U_{total} &= E_{tot} + Un_+n_- \\
&= E_{tot} + \frac{1}{4}UN^2(1-\varsigma^2) \\
&= E_0(1+\varsigma)^{5/3} + E_0(1-\varsigma)^{5/3} + \frac{1}{4}UN^2(1-\varsigma^2)
\end{aligned}$$

In the limit of $\varsigma \to 0$, U_{tot} can be expanded as

$$U_{tot} = (2E_0 + \frac{1}{4}N^2U) + \left(\frac{10E_0}{9} - \frac{1}{4}N^2U\right)\varsigma^2 + \frac{10E_0}{243}\varsigma^4$$

$$\begin{aligned}
\frac{\partial U_{tot}}{\partial\varsigma} &= 2\left(\frac{10E_0}{9} - \frac{1}{4}N^2U\right)\varsigma + \frac{40E_0}{243}\varsigma^3 \\
&= 2\varsigma[\frac{20E_0}{243}\varsigma^2 - \left(\frac{1}{4}N^2U - \frac{10E_0}{9}\right)]
\end{aligned}$$

when $U > \frac{4\varepsilon_F}{3N}$, ς has a positive finite value,

$$\begin{aligned}
\varsigma^2 &= \frac{243}{20E_0}\left(\frac{1}{4}N^2U - \frac{10E_0}{9}\right) \\
&= \frac{81}{2\varepsilon_F}\left(\frac{1}{4}NU - \frac{1}{3}\varepsilon_F\right)
\end{aligned}$$

or

$$\varsigma = 9\sqrt{\left(\frac{NU}{8\varepsilon_F} - \frac{1}{6}\right)}$$

The net magnetic moment is

$$M = \mu_B N\varsigma = 9\mu_B N\sqrt{\left(\frac{NU}{8\varepsilon_F} - \frac{1}{6}\right)}$$

where

$$NU > \frac{4}{3}\varepsilon_F$$

4. Free energy approach for itinerant magnetism: $U \neq 0$

The interaction for the Hubbard model is given by

$$Un_\uparrow n_\downarrow = \frac{U}{4}(n_\uparrow + n_\downarrow)^2 - \frac{U}{4}(n_\uparrow - n_\downarrow)^2$$

The mean field approximation average is

$$U\langle n_\uparrow n_\downarrow\rangle = \frac{U}{4}\langle n_\uparrow + n_\downarrow\rangle^2 - \frac{U}{4}\langle n_\uparrow - n_\downarrow\rangle^2$$

The first term is the total density (fixed), while the second one is

$$H_{\text{int}} = \frac{U}{4}N^2 - \frac{U}{4}\left(\frac{M}{\mu_B}\right)^2$$

where the net magnetic moment is defined by

$$M = \mu_B\langle n_\uparrow - n_\downarrow\rangle$$

and the net number of electrons is

$$N = \langle n_\uparrow + n_\downarrow\rangle$$

We also recall the Pauli para-magnetism.

(i) Suppose that $U = 0$. The free energy is given by

$$F(M) = -MB + \frac{M^2}{2\chi_{Pauli}}$$

where

$$\chi_{Pauli} = \mu_B{}^2 D(\varepsilon_F).$$

From the condition that

$$\frac{\partial F(M)}{\partial M} = 0 = -B + \frac{M}{\chi_{Pauli}},$$

we have

$$M = \chi_{Pauli} B.$$

(ii) Next we set $B = 0$ and $U \neq 0$.

$$F(M) = \left(\frac{1}{2\chi_{Pauli}} - \frac{U}{4\mu_B}\right) M^2$$

The condition for ferromagnetism is given by

$$\frac{1}{2\chi_{Pauli}} - \frac{U}{4\mu_B{}^2} < 0$$

or

$$2 < UD(\varepsilon_F) \qquad \text{(Stoner criterion for the ferromagnetism)}$$

Note that for free electron Fermi gas model we have

$$D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}.$$

Using this, the Stoner criterion can be rewritten as

$$NU > \frac{4}{3}\varepsilon_F$$

5. Mean field theory for itinerant magnetism: Stoner criterion for ferromagnetism

A common model that provides a useful view of the way in which magnetism arises from the imbalance of spin magnetic moment of electrons from the Stoner model. In this model, we do not have the localized spins assumed in the Heisenberg model, but rather electrons described by Bloch states. We assume that there is an interaction between electron with the state $|\boldsymbol{k}\uparrow\rangle$ and electron with the state $|\boldsymbol{k}'\uparrow\rangle$,

$$U\sum_{\boldsymbol{k},\boldsymbol{k}'} n_{\boldsymbol{k}\uparrow} n_{\boldsymbol{k}'\downarrow}$$

where $n_{\boldsymbol{k}\uparrow}$ and $n_{\boldsymbol{k}'\downarrow}$ represent the occupation number corresponding to the states $|\boldsymbol{k}\uparrow\rangle$ and $|\boldsymbol{k}'\uparrow\rangle$. This interaction can be viewed as producing an internal magnetic field similar to the mean field theory.

The energy of the $+\mu_B$ state is

$$\varepsilon = \varepsilon_k - \mu_B B + U\langle n_\downarrow\rangle$$

The density of states is derived as

$$D_+(\varepsilon) = \frac{1}{2}D(\varepsilon + \mu_B B - U\langle n_\downarrow\rangle) = \frac{1}{2}a\sqrt{\varepsilon + \mu_B B - U\langle n_\downarrow\rangle}$$

where a is defined by

$$a = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

The energy of the $-\mu_B$ state is

$$\varepsilon = \varepsilon_k + \mu_B B + U\langle n_\uparrow\rangle$$

The density of states is derived as

$$D_{-}(\varepsilon)=\frac{1}{2}D(\varepsilon-\mu_B B-U\langle n_\downarrow\rangle)=\frac{1}{2}a\sqrt{\varepsilon-\mu_B B-U\langle n_\downarrow\rangle}$$

The net magnetic moment is given by

$$\begin{aligned}
M &= \mu_B(\langle n_\uparrow\rangle-\langle n_\downarrow\rangle)\\
&=\frac{\mu_B}{2}[\int_{-\mu_B B+U\langle n_\downarrow\rangle}^{\infty} f(\varepsilon)D(\varepsilon+\mu_B B-U\langle n_\downarrow\rangle)d\varepsilon\\
&\quad-\int_{\mu_B B+U\langle n_\downarrow\rangle}^{\infty} f(\varepsilon)D(\varepsilon-\mu_B B-U\langle n_\uparrow\rangle)d\varepsilon]
\end{aligned}$$

or

$$\begin{aligned}
M &= \mu_B(\langle n_\uparrow\rangle-\langle n_\downarrow\rangle)\\
&=\frac{\mu_B}{2}\int_0^\infty [f(\varepsilon-\mu_B B+U\langle n_\downarrow\rangle)-f(\varepsilon+\mu_B B+U\langle n_\uparrow\rangle)D(\varepsilon)d\varepsilon
\end{aligned}$$

Note that in the limit of $\frac{\mu_B B}{k_B T}\to 0$, we get

$$f(\varepsilon-\mu_B B+U\langle n_\downarrow\rangle)=f(\varepsilon)+(-\mu_B B+U\langle n_\downarrow\rangle)\frac{\partial f}{\partial\varepsilon}$$

$$f(\varepsilon+\mu_B B+U\langle n_\uparrow\rangle)=f(\varepsilon)+(\mu_B B+U\langle n_\uparrow\rangle)\frac{\partial f}{\partial\varepsilon}$$

using the Taylor expansion. Then we have

$$\begin{aligned}
M &= \frac{\mu_B}{2}\int_0^\infty [\mu_B B-U\langle n_\downarrow\rangle+(\mu_B B+U\langle n_\uparrow\rangle)](-\frac{\partial f}{\partial\varepsilon})D(\varepsilon)d\varepsilon\\
&=\frac{\mu_B}{2}[2\mu_B B-U\langle n_\downarrow\rangle+U\langle n_\uparrow\rangle)]D(\varepsilon_F)\\
&=\frac{\mu_B}{2}(2\mu_B B+\frac{MU}{\mu_B})D(\varepsilon_F)\\
&=(\mu_B{}^2 B+\frac{MU}{2})D(\varepsilon_F)
\end{aligned}$$

leading to the relation

$$M = \frac{\mu_B{}^2 D(\varepsilon_F)}{1 - \frac{UD(\varepsilon_F)}{2}} B$$

When $U = 0$, the susceptibility reduces to the Pauli susceptibility

$$\chi_{Pauli} = \mu_B{}^2 D(\varepsilon_F)$$

The instability when $UD(\varepsilon_F) > 2$ signals ferromagnetism. For $UD(\varepsilon_F) < 2$ ferromagnetism does not appear at any T.

6. Simon's summary

Hubbard model includes tight-binding hopping t and on-site "Hubbard" interaction U

(i) For partially filled band, the repulsive interaction (if strong enough) makes the system an (itinerant) ferromagnet: aligned spins can have lower energy because they do not double occupy sites, and therefore are lower energy with respect to U although it costs higher kinetic energy to align all the spins.

(ii) For a half-filled band, the repulsive interaction makes the Mott insulator antiferromagnetic: virtual hopping lowers the energy of anti-aligned neighboring spins.

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APPENDIX Free electron Fermi gas model (Pauli paramagnetism)

The density of states for free electron gas

$$D^{+}(\varepsilon)=\frac{V}{4\pi^{2}}\left(\frac{2m}{\hbar^{2}}\right)^{3/2}\sqrt{\varepsilon+\mu_{B}B}$$
$$=\frac{1}{2}a\sqrt{\varepsilon+\mu_{B}B}$$
$$=\frac{1}{2}D(\varepsilon+\mu_{B}B)$$

$$D^{-}(\varepsilon)=\frac{V}{4\pi^{2}}\left(\frac{2m}{\hbar^{2}}\right)^{3/2}\sqrt{\varepsilon-\mu_{B}B}$$
$$=\frac{1}{2}a\sqrt{\varepsilon-\mu_{B}B}$$
$$=\frac{1}{2}D(\varepsilon-\mu_{B}B)$$

since

$$\varepsilon=\frac{\hbar^{2}}{2m}k^{2}-\mu_{B}B,\qquad k=\left(\frac{2m}{\hbar^{2}}\right)^{1/2}\sqrt{\varepsilon+\mu_{B}B}$$

$$dk=\left(\frac{2m}{\hbar^{2}}\right)^{1/2}\frac{1}{2\sqrt{\varepsilon+\mu_{B}B}}d\varepsilon$$

$$D(\varepsilon)d\varepsilon=\frac{2V}{(2\pi)^{3}}4\pi k^{2}dk\left(\frac{2m}{\hbar^{2}}\right)^{3/2}\sqrt{\varepsilon}d\varepsilon$$
$$=\frac{V}{2\pi^{2}}\left(\frac{2m}{\hbar^{2}}\right)^{3/2}\sqrt{\varepsilon}d\varepsilon$$

or

$$D(\varepsilon)=\frac{V}{2\pi^{2}}\left(\frac{2m}{\hbar^{2}}\right)^{3/2}\sqrt{\varepsilon}=a\sqrt{\varepsilon}$$

$$a=\frac{V}{2\pi^{2}}\left(\frac{2m}{\hbar^{2}}\right)^{3/2}$$

(a) We consider the case when A T = 0 K,

$$
\begin{aligned}
U^{+} &= \frac{1}{2}\int_{-\mu_B B}^{\varepsilon_F} D(\varepsilon+\mu_B B)\varepsilon d\varepsilon \\
&= \frac{a}{15}(3\varepsilon_F - 2\mu_B B](\varepsilon_F+\mu_B B)^{3/2} \\
&= \frac{a}{15}[3(\varepsilon_F+\mu_B B) - 5\mu_B B](\varepsilon_F+\mu_B B)^{3/2} \\
&= \frac{a}{5}(\varepsilon_F+\mu_B B)^{5/2} - \frac{a}{3}\mu_B B(\varepsilon_F+\mu_B B)^{3/2}
\end{aligned}
$$

or

$$
U^{+} = \frac{a}{5}(\varepsilon_F+\mu_B B)^{5/2} - \frac{a}{3}\mu_B B(\varepsilon_F+\mu_B B)^{3/2}
$$

(b)

$$
\begin{aligned}
U^{-} &= \frac{1}{2}\int_{\mu_B B}^{\varepsilon_F} D(\varepsilon-\mu_B B)\varepsilon d\varepsilon \\
&= \frac{a}{15}(3\varepsilon_F + 2\mu_B B)(\varepsilon_F-\mu_B B)^{3/2} \\
&= \frac{a}{15}[3(\varepsilon_F-\mu_B B) + 5\mu_B B](\varepsilon_F-\mu_B B)^{3/2} \\
&= \frac{a}{5}(\varepsilon_F-\mu_B B)^{5/2} + \frac{a}{3}\mu_B B(\varepsilon_F-\mu_B B)^{3/2}
\end{aligned}
$$

or

$$
U^{-} = \frac{a}{5}(\varepsilon_F-\mu_B B)^{5/2} + \frac{a}{3}\mu_B B(\varepsilon_F-\mu_B B)^{3/2}
$$

where

$$
a\varepsilon_F^{\ 3/2} = \frac{3N}{2}\text{ .,} \qquad E_0 = \frac{1}{5}a\varepsilon_F^{\ 5/2} = \frac{3}{10}N\varepsilon_F
$$

(c) The concentration of spin-up electrons:

$$N^{+} = \frac{1}{2}\int_{-\mu_B B}^{\varepsilon_F} D(\varepsilon + \mu_B B)d\varepsilon$$

$$= \frac{a}{3}(\varepsilon_F + \mu_B B)^{3/2}$$

(d) The concentration of spin-down electrons

$$N^{-} = \frac{1}{2}\int_{\mu_B B}^{\varepsilon_F} D(\varepsilon - \mu_B B)d\varepsilon$$

$$= \frac{a}{3}(\varepsilon_F - \mu_B B)^{3/2}$$

(e) The total number of electrons:

$$N = \int_0^{\varepsilon_F} D(\varepsilon)d\varepsilon = a\int_0^{\varepsilon_F} \varepsilon^{1/2}d\varepsilon = \frac{2a}{3}\varepsilon_F^{3/2}$$

Based on these relations, the first part of U^{+} can be rewritten as

$$\frac{a}{5}(\varepsilon_F + \mu_B B)^{5/2} = \frac{a}{5}\varepsilon_F \varepsilon_F^{3/2}(1+x)^{5/2}$$

and N^{+} can be rewritten as

$$N^{+} = \frac{a}{3}\varepsilon_F^{3/2}(1+x)^{3/2} = \frac{N}{2}(1+x)^{3/2}$$

which is equal to $N^{+} = \frac{N}{2}(1+\varsigma)$

or

$$(1+x)^{3/2} = 1+\varsigma \qquad \text{or} \qquad 1+x = (1+\varsigma)^{2/3}$$

where

$$x = \frac{\mu_B B}{\varepsilon_F}$$

Similarly,

$$N^- = \frac{a}{3}\varepsilon_F{}^{3/2}(1-x)^{3/2} = \frac{N}{2}(1-x)^{3/2}$$

which is equal to $\quad N^- = \frac{N}{2}(1-\varsigma)$

or

$$(1-x)^{3/2} = 1-\varsigma \qquad \text{or} \qquad 1-x = (1-\varsigma)^{2/3}$$

Then we have

$$\begin{aligned} U^+ &= \varepsilon_F \frac{a}{5}\varepsilon_F{}^{3/2}(1+x)^{5/2} - \mu_B B N^+ \\ &= E_0(1+\varsigma)^{5/3} - \frac{1}{2}N\mu_B B(1+\varsigma) \end{aligned}$$

and

$$\begin{aligned} U^- &= \varepsilon_F \frac{a}{5}\varepsilon_F{}^{3/2}(1-x)^{5/2} + \mu_B B N^+ \\ &= E_0(1-\varsigma)^{5/3} + \frac{1}{2}N\mu_B B(1-\varsigma) \end{aligned}$$

where

$$E_0 = \frac{3}{10}N\varepsilon_F$$